Chow motives of generically split varieties

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Abstract

Let G be an anisotropic linear algebraic group over a field F which splits by a field extension of a prime degree. Let X be a projective homogeneous G-variety such that G splits over the function field of X. We prove that under certain conditions the Chow motive of X is isomorphic to a direct sum of twisted copies of an indecomposable motive $\mathcal{R}_X$. This covers all known examples of motivic decompositions of generically split projective homogeneous varieties (Severi-Brauer varieties, Pfister quadrics, maximal orthogonal Grassmannians) as well as provides new ones (exceptional varieties of types E_6 and E_8).

1 Introduction

The history of the subject of the present paper starts with a celebrated result of M. Rost about motivic decomposition of a Pfister quadric which became the crucial point in the proof of Milnor conjecture by V. Voevodsky. Briefly speaking, it says (see [Ro98]) that the Chow motive of a Pfister quadric X decomposes as a direct sum of (twisted) copies of a certain indecomposable motive $\mathcal{R}_X$. Hence, the motivic (and, hence, cohomological) behavior of X depends on a rather small object $\mathcal{R}_X$. Indeed, over the algebraic closure $\mathcal{R}_X$ splits as a direct sum $\mathbb{Z} \oplus \mathbb{Z}(r)$ of just two twisted copies of the motive of a point. As a consequence, it allows to compute and estimate several important cohomological invariants of X.

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One of the crucial properties of a Pfister quadric is that it becomes hyperbolic (totally isotropic) over its function field. In the present paper we deal only with varieties satisfying the analogous property which we call *generically split*. Namely, X is generically split if its Chow motive splits over the generic point of X as a direct sum of (twisted) copies of the motive of a point.

Note that the discussion about Pfister quadrics and Milnor conjecture essentially involves the prime $p = 2$ which is a minimal degree of a field extension that splits X. One may ask what happens for other primes. Or more precisely, is there some natural analog of a Pfister quadric which has the similar motivic decomposition. Observe that this question naturally arises in the context of Bloch-Kato conjecture (generalization of Milnor conjecture to other primes).

The goal of the paper is to show that any generically split projective homogeneous G-variety X provides such an analog, where G is an anisotropic simple linear algebraic group of inner type which splits by a field extension of a prime degree p. Observe that the property of being generically split depends on Tits indices of the group G (see [Ti66]). We prove (see Theorem 3.9) that the motive of such variety splits as a direct sum of (twisted) copies of a certain indecomposable motive $\mathcal{R}_X$. The prime p here is a torsion prime of G. In this way we obtain motivic analogs of a Pfister quadric for $p = 3$ (E_6/P_6) and $p = 5$ (E_8/P_8).

Our work was mostly motivated by the paper of N. Karpenko [Ka01], where he provided an elementary construction of the motive $\mathcal{R}_X$ for a Pfister quadric X. In fact, Theorem 3.9 can be viewed as a further generalization of Karpenko's and Rost's ideas and is based on Rost Nilpotence Theorem. The key idea is to reduce the problem of decomposing the motive of X to the problem of providing a certain family of algebraic cycles on X. The latter turns out to be a purely combinatorial problem related with properties of characteristic map studied by Grothendieck, Demazure, Karpenko and Merkurjev (see [Gr58], [De74], [KM05]).

The paper is organized as follows. In section 2 we provide general arguments concerning lifting of idempotents. In section 3 we study Chow motives of generically split varieties and prove the main result Theorem 3.9. In the next section 4 we provide properties of projective homogeneous varieties which will be extensively used in the applications. In particular, we relate the question of indecomposability of $\mathcal{R}_X$ with the question of existence of zero-cycles of degree one on X. In section 5 we provide applications of Theorem 3.9 to various examples of projective homogeneous varieties. Namely, to Severi-Brauer varieties, Pfister quadrics and their neighbors, maximal orthogonal Grassmannians, varieties of types G_2, F_4 and strongly inner E_6. In section 6 we decompose the motive of a variety of type E_8 which splits by an extension of degree $p = 5$. In the last section we reduce the computations of motives of general flag varieties to the examples of sections 5 and 6.

2 Decomposition of the diagonal

Let p be an integer (not necessary a prime) and $\mathbb{Z} \to \mathbb{Z}/p$ be a specialization map. The goal of the present section can be stated as follows. Given a ring R satisfying certain conditions, its specialization $\bar{R}$ modulo p and a family of pair-wise orthogonal idempotents q_i' in $\bar{R}$ such that $\sum_i q_i' = 1_{\bar{R}}$, produce a family of pair-wise orthogonal idempotents q_i in R such that $\bar{q}_i = q_i'$ and $\sum_i q_i = 1_R$.

2.1. Let A^* be a graded commutative ring which is a free $\mathbb{Z}$-module of rank N. By A^n we denote the n-th graded component of A (component of codimension n). Fix a homogeneous $\mathbb{Z}$-basis $\{e_k\}_{k=1\ldots N}$ of A which will be called standard. Assume there is a linear form $\deg\colon A \to \mathbb{Z}$ together with a (dual) homogeneous $\mathbb{Z}$-basis $\{e_l^\vee\}_{l=1\ldots N}$ of A such that

$$\deg(e_k e_l^\vee) = \delta_{k,l} \text{ for all } k \text{ and } l.$$

From now on fix an integer p and denote by $\bar{A}$ the reduction of A modulo p.

2.2 Definition. Given an element $\rho \in \bar{A}^r$ of codimension r a family $\{\alpha_i\}_{i\in\mathcal{I}}$ of homogeneous elements in $\bar{A}$ is called *ρ-balanced* if it satisfies the following conditions

(a) the set of indices $\mathcal{I}$ is equipped with an involution $i \mapsto i^\bullet$, $p \cdot \#\mathcal{I} = N$;

(b) for all $m = 0\ldots 2(p-1)$ and $i, j \in \mathcal{I}$

$$\deg(\rho^m \alpha_i \alpha_{j^\bullet}) \begin{cases} \in (\mathbb{Z}/p)^\times, & \text{if } m = p-1 \text{ and } i = j \\ = 0, & \text{otherwise.} \end{cases}$$

To simplify the notation denote $C_i = \deg(\rho^{p-1}\alpha_i\alpha_{i^\bullet}) \in (\mathbb{Z}/p)^\times$.

2.3 Lemma. *Let $\{\alpha_i\}_{i\in\mathcal{I}}$ be a ρ-balanced family of elements in $\bar{A}$. The elements $\rho^m\alpha_i$, $m = 0\ldots p-1$, $i \in \mathcal{I}$ form a homogeneous basis of $\bar{A}$ over $\mathbb{Z}/p$ called a ρ-basis.*

Proof. By 2.2.(b) the elements $\rho^m\alpha_i$ are linear independent. Indeed, if $x = \sum_{m,i} c_{m,i}\rho^m\alpha_i$ then $c_{m,i} = \frac{1}{C_i}\deg(x\rho^{p-1-m}\alpha_{i\bullet})$. Therefore, if $x = 0$ then $c_{m,i} = 0$ for all $i \in \mathcal{I}$ and $m = 0 \ldots p-1$. Let M be a quotient module of $\bar{A}$ by a submodule generated by $\{\rho^m\alpha_i\}$. Since $p \cdot \#\mathcal{I} = N$, $M/dM = 0$ for all prime divisors d of p. Therefore $M = 0$ and the lemma is proved. □

2.4. Let $\{\bar{e}_k\}$ be the reduction modulo p of the standard basis $\{e_k\}$. Consider a ρ-basis $\{\rho^m\alpha_i\}$ of $\bar{A}$. Let $\{\rho^m\alpha_i\}^{(n)} \subset \bar{A}^n$ be its component of codimension n. Let $D^{(n)} \in (\mathbb{Z}/p)^\times$ be the determinant of the matrix which expresses $\{\rho^m\alpha_i\}^{(n)}$ in terms of the components of the basis $\{\bar{e}_k\}$ of codimension n.

2.5 Definition. We say a ρ-balanced family $\{\alpha_i\}_{i\in\mathcal{I}}$ is *unimodular in codimension n* if $D^{(n)} \pm 1$. We say a ρ-balanced family is *unimodular* if it is unimodular in all codimensions n.

2.6 Lemma (Basis Lifting Property). *Assume there is a basis $\{f'_k\}$ of $\bar{A}^n$ such that the transition matrix from $\{\bar{e}_k\}^{(n)}$ to $\{f'_k\}$ has determinant ± 1. Then there exists a $\mathbb{Z}$-basis $\{f_k\}$ of A^n such that $\bar{f}_k = f'_k$, where $k = 1 \ldots l$.*

Proof. Since $\mathbb{Z}/p$ is a semi-local ring, the group $\mathrm{SL}_l(\mathbb{Z}/p)$ is generated by elementary matrices. Hence, the reduction map $\mathrm{SL}_l(\mathbb{Z}) \to \mathrm{SL}_l(\mathbb{Z}/p)$ is surjective and the lemma follows. □

2.7 Corollary. *Assume we are given an unimodular ρ-balanced family of elements in $\bar{A}$. Then the respective ρ-basis of $\bar{A}$ can be lifted to a homogeneous basis of A.*

Proof. Apply the lemma to all graded components of the ρ-basis. □

2.8 Remark. Observe that for $p = 2, 3, 4, 6$ any ρ-balanced family of elements is unimodular. Therefore any ρ-basis can be lifted to a homogeneous basis over $\mathbb{Z}$.

2.9. We endow the abelian group $A \otimes A$ with a (graded) ring structure by means of the composition product $(\alpha \otimes \beta) \circ (\alpha' \otimes \beta') \mapsto \deg(\alpha\beta') \cdot (\alpha' \otimes \beta)$ and denote the resulting ring by R. The identity element of R we call a *diagonal* and denote it by Δ_R. Recall that an element $q \in R$ is called an *idempotent* if $q \circ q = q$. We say two idempotents q and q' are *orthogonal* if $q \circ q' = q' \circ q = 0$.

2.10. Assume we are given an unimodular ρ-balanced family in $\bar{A}$. Let $\{u_{m,i}\}$ be a homogeneous $\mathbb{Z}$-basis of A obtained by lifting the ρ-basis $\{\bar{u}_{m,i} = \rho^m\alpha_i\}$

according to Corollary 2.7. Let $\{u^\vee_{m,i}\}$ be the respective dual basis, i.e., defined by

$$\deg(u_{m,i}u^\vee_{m',i'}) = \delta_{m,m'}\delta_{i,i'}. \tag{1}$$

By 2.2.(b) we obtain that

$$\bar{u}^\vee_{m,i} = \tfrac{1}{C_i}\rho^{p-1-m}\alpha_{i\bullet} \in \bar{A}. \tag{2}$$

2.11 Proposition. *Homogeneous elements* $q_i = \sum_{m=0}^{p-1} u_{m,i} \otimes u^\vee_{m,i}$*,* $i \in \mathcal{I}$*, are pair-wise orthogonal idempotents in* R *such that* $\sum_{i\in\mathcal{I}} q_i = \Delta_R$*. Moreover,*

$$\bar{q}_i = \tfrac{1}{C_i}\Big(\sum_{m=0}^{p-1} \rho^m \otimes \rho^{p-1-m}\Big)\cdot(\alpha_i \otimes \alpha_{i\bullet}) \tag{3}$$

Proof. The fact that they are pair-wise orthogonal idempotents follows from the definition of the composition product and (1). On the other hand, since Δ_R is the identity of R, $\Delta_R = \sum_{m,i} u_{m,i} \otimes u^\vee_{m,i}$. Formula (3) then follows from the definition of $u_{m,i}$ and (2). □

2.12 Remark. Consider an arbitrary ρ-basis. In each codimension n such that $D^{(n)} \neq \pm 1$ choose an element $\rho^{m_n}\alpha_{i_n}$ and replace it by $\frac{1}{D^{(n)}}\rho^{m_n}\alpha_{i_n}$. So obtained family of elements will form a unimodular homogeneous basis of $\bar{A}$. Let $\{v_{m,i}\}$ and $\{v^\vee_{m,i}\}$ be the respective liftings to A. By definition $\bar{v}_{m_n,i_n} = \frac{1}{D^{(n)}}\bar{u}_{m_n,i_n}$ and $\bar{v}^\vee_{m_n,i_n} = D^{(n)}\bar{u}^\vee_{m_n,i_n}$. Observe that by means of this procedure the idempotents $\bar{q}_i$ of (3) will not change.

3 Motives of generically split varieties

In the present section we apply the results of the previous section to Chow motives of generically split varieties. For the definition and properties of Chow motives we refer to the papers [Ma68] and [Ka01, sect.2].

3.1. Recall that a category of Chow motives is defined as follows. First, we define the category of *correspondences* $Corr(F)$ over a field F. Its objects are smooth projective varieties over F. For morphisms, called correspondences, we set $\mathrm{Mor}(X,Y) := \mathrm{CH}_{\dim X}(X \times Y)$. For any two correspondences $\alpha \in \mathrm{CH}(X \times Y)$ and $\beta \in \mathrm{CH}(Y \times Z)$ we define the composition product $\beta \circ \alpha \in \mathrm{CH}(X \times Z)$ as

$$\beta\circ\alpha = \mathrm{pr}_{13*}(\mathrm{pr}_{12}^*(\alpha)\cdot \mathrm{pr}_{23}^*(\beta)), \tag{4}$$

where pr_{ij} denotes the projection on the i-th and j-th factors of $X \times Y \times Z$ respectively and pr_{ij_*}, pr_{ij}^* denote the induced push-forwards and pull-backs for Chow groups. Taking the pseudo-abelian completion of $Corr(F)$ we obtain the category of *effective* Chow motives $Chow^{eff}(F)$. Its objects are pairs (X, p), where X is a smooth projective variety and $p \in \mathrm{Mor}(X, X)$ is an idempotent, that is, $p \circ p = p$. The morphisms between two objects (X, p) and (Y, q) are the compositions $q \circ \mathrm{Mor}(X, Y) \circ p$. Denote $\mathcal{M}(X) = (X, \mathrm{id})$. Following this definition the motive of a projective line splits as a direct sum $\mathcal{M}(\mathbb{P}^1) = \mathcal{M}(pt) \oplus L$ of the motive of a point and some motive L called *Lefschetz* motive. Formally inverting L one obtains the category of Chow motives $Chow(F)$.

By construction, $Chow(F)$ is a tensor additive category, where the tensor product is given by the usual product $(X, p) \otimes (Y, q) = (X \times Y, p \times q)$. For a given motive M denote by $M(k)$ its twist, i.e., the tensor product $M \otimes L^{\otimes k}$. For a cycle α denote by α^t the corresponding transposed cycle.

3.2 Definition. Let X be a smooth irreducible projective variety over a field F. We say L is a *splitting field* of X if the Chow motive $\mathcal{M}(X_L)$ of the scalar extension $X_L = X \times_F L$ is isomorphic to a direct sum of twisted Lefschetz motives and the natural map $\mathrm{res}_{L/F} : \mathcal{M}(X) \to \mathcal{M}(X_L)$ is an isomorphism after tensoring with $\mathbb{Q}$. We say X is *generically split* over F if its function field $K = F(X)$ is a splitting field of X.

3.3 Remark. According to Lemma 4.2 any projective homogeneous G-variety of a linear algebraic group G of inner type over F which splits over the generic point of X provides an example of a generically split variety.

Let L be a splitting field of X. In view of the notation of the previous section set $A = \mathrm{CH}(X_L)$ and denote by A_{rat} the image of the restriction map $\mathrm{res}_{L/F} \colon \mathrm{CH}(X) \to \mathrm{CH}(X_L)$. An element (cycle) of A_{rat} will be called *rational*.

3.4. According to [Ma68] and [KM05, Rem. 5.6] the ring $A = \mathrm{CH}(X_L)$ is a free $\mathbb{Z}$-module, there is Künneth formula $\mathrm{CH}(X_L \times X_L) = A \otimes A$ and Poincare duality. The latter means that for a given basis $\{e_k\}$ of A there is the dual basis $\{e_l^\vee\}$ with respect to the degree map $\deg : \mathrm{CH}(X_L) \to \mathbb{Z}$. Observe that the composition product introduced in 2.9 coincides with the composition of endomorphisms in $\mathrm{End}(\mathcal{M}(X_L))$, hence, $\mathrm{End}(\mathcal{M}(X_L)) = (A \otimes A)^{(\dim X)}$.

The following facts will be extensively used in the sequel

3.5 Lemma. *Let L be a splitting field of a variety X of finite degree over F. Then for any $\alpha \in A$ we have $[L:F]\cdot\alpha \in A_{rat}$.*

Proof. Choose an element α in $A = \mathrm{CH}(X_L)$. Since $\mathrm{res}_{L/F}\otimes\mathbb{Q}$ is surjective there exists an element $\beta \in \mathrm{CH}(X)$ and a non-zero integer m such that $\mathrm{res}_{L/F}(\beta) = m\alpha$. By projection formula

$$m \cdot \mathrm{cores}(\alpha) = \mathrm{cores}(\mathrm{res}(\beta)) = [L:F]\cdot\beta.$$

Applying res to both sides we obtain $m(\mathrm{res}(\mathrm{cores}(\alpha))) = m[L:F]\cdot\alpha$. Therefore $\mathrm{res}_{L/F}(\mathrm{cores}(\alpha)) = [L:F]\cdot\alpha$ and the lemma is proven. □

3.6 Lemma. *Let X be a generically split variety and L be a splitting field of X. Let $\mathrm{pr}_1 : (A\otimes A)^r \to A^r$ be the projection on the first summand in the direct sum decomposition $(A\otimes A)^r = (A^r\otimes A^0)\oplus\ldots\oplus(A^0\otimes A^r)$. Then for any $\rho\in A^r$ there exists a rational preimage of ρ by means of pr_1.*

Proof. Lemma follows from the commutative diagram

$$\begin{array}{ccccc}
\mathrm{CH}^r(X\times X) & \xrightarrow{\mathrm{res}_{L/F}} & \mathrm{CH}^r(X_L\times X_L) & \overset{\text{Künneth}}{=\!=\!=} & (A\otimes A)^r \\
\downarrow{\scriptstyle p_1^*} & & \downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1} \\
\mathrm{CH}^r(X_K) & \xrightarrow{\simeq} & \mathrm{CH}^r(X_{LK}) & \xrightarrow{\simeq} & A^r
\end{array}$$

where K is the quotient field of X, the first vertical arrow p_1^* is taken from localization sequence for Chow groups and, hence, is surjective and the isomorphisms come from the fact that L and K are splitting fields of X. □

3.7 Lemma (Rost Nilpotence)**.** *(see [VZ06]) Let X be a generically split variety and E/F be a field extension. Then the kernel of the natural map*

$$\mathrm{End}(\mathcal{M}(X)) \to \mathrm{End}(\mathcal{M}(X_E))$$

consists of nilpotent elements.

3.8 Corollary. *Let X be a generically split variety and E/F be a field extension. Given a direct summand M of $\mathcal{M}(X)$ and a family $q_i\in\mathrm{End}(M_E)$ of rational pair-wise orthogonal idempotents with $\sum_i q_i = \Delta_E$ there exist a family of pair-wise orthogonal idempotents $p_i\in\mathrm{End}(M)$ such that $\sum_i p_i = \Delta$ and $\mathrm{res}_{E/F}(p_i) = q_i$. Moreover, if M, N are direct summands of $\mathcal{M}(X)$ and φ is a rational isomorphism between the N_E and $M(i)_E$ for some i, then $N\simeq M(i)$.*

Now we are ready to state and prove the main result of this paper

3.9 Theorem. *Let X be a generically split variety. Assume there exists a splitting field L of X of a prime degree p together with an element $\rho \in \bar{A}^r = \mathrm{CH}^r(X_L)/p$ for some r such that*

(i) for all $s < r$ we have $\bar{A}^s = \bar{A}^s_{rat}$ and $\bar{A}^r = \langle \bar{A}^r_{rat}, \rho\rangle$;

(ii) there is a unimodular ρ-balanced family $\{\alpha_i\}_{i\in\mathcal{I}}$ in $\bar{A}_{rat}$.

Then the Chow motive of X with integral coefficients can be expressed as the direct sum

$$\mathcal{M}(X) \simeq \bigoplus_{i\in\mathcal{I}} \mathcal{R}(\operatorname{codim}\alpha_i),$$

where the motive $\mathcal{R}$ is indecomposable if and only if there are no zero-cycles of degree one on X.

Proof. By Lemma 3.5 all cycles divisible by p are rational. Hence, to prove that a certain cycle α is rational in $R = A \otimes A$ is the same as to prove it in $\bar{R} = \bar{A} \otimes_{\mathbb{Z}/p} \bar{A}$.

Consider two cycles $\rho \otimes 1 - \varepsilon(1 \otimes \rho)$, where $\varepsilon = \pm 1$. We claim that one of them is rational (cf. [Ka01, Lemma 5.1]). Indeed, since X is generically split by Lemma 3.6 there exists a rational preimage of $\rho \otimes 1$

$$\rho \otimes 1 + \delta(1 \otimes \rho) + \sum_k \mu_k \otimes \nu_k + 1 \otimes \gamma \in \mathrm{pr}_1^{-1}(\rho \otimes 1),$$

where $\delta \in \mathbb{Z}/p$, codimensions of μ_k and ν_k are less than r and $\gamma \in \bar{A}^r_{rat}$. By (i) the cycles μ_k and ν_k are rational. Therefore $\tau := \rho\otimes 1 + \delta(1\otimes\rho)$ is rational. If $\delta = -1$ take $\varepsilon = 1$. Otherwise the cycle $\tau + \tau^t = (1+\delta)(\rho \otimes 1 + 1 \otimes \rho)$ is rational and therefore we can take $\varepsilon = -1$. Hence, we obtain a rational cycle in $\bar{R}^{r(p-1)}$

$$\sigma = (\rho \otimes 1 - \varepsilon(1 \otimes \rho))^{p-1} = \sum_{m=0}^{p-1} \varepsilon^m (\rho^m \otimes \rho^{p-1-m}). \tag{5}$$

Applying Proposition 2.11 to the unimodular ρ-balanced family $\{\alpha_i\}$ in $\bar{A}_{rat}$, we obtain the family of pair-wise orthogonal idempotents q_i in R with $\bar{q}_i$ given by formula (3). Note that idempotents q_i are rational. Indeed, $q_i =$

$q'_i \circ q'_i$, where $q'_i = \sum_m \varepsilon^m (u_{m,i} \otimes u^\vee_{m,i})$ and the reduction $\bar{q}'_i = \frac{1}{C_i}\sigma \cdot (\alpha_i \otimes \alpha_{i^\bullet})$ is rational.

Note that the motives (X_L, q_i) and $(X_L, q_{i'})(\operatorname{codim}(\alpha_i) - \operatorname{codim}(\alpha_{i'}))$ are isomorphic by means of an isomorphism $\varphi_{i,i'}$ given by

$$\varphi_{i,i'} = \sum_m \varepsilon^m u_{m,i'} \otimes u^\vee_{m,i}. \tag{6}$$

Observe that $\bar{\varphi}_{i,i'} = \frac{1}{C_i}\sigma \cdot (\alpha_{i'} \otimes \alpha_{i^\bullet})$ and hence $\varphi_{i,i'}$ is rational.

Applying Corollary 3.8 to the idempotents q_i and isomorphisms $\varphi_{i,i'}$ we obtain the desired decomposition with $\mathcal{R} = (X, p_{i_0})$, where i_0 is a unique index such that $\operatorname{codim}(\alpha_{i_0}) = 0$.

Finally observe that the ring of endomorphisms $\operatorname{End}(\mathcal{R}_L)$ is additively generated by $u_{m,i_0} \otimes u^\vee_{m,i_0}$, $m = 0 \ldots p-1$. Assume there is a rational idempotent $q = \sum_m a_m (u_{m,i_0} \otimes u^\vee_{m,i_0})$. A straightforward computation shows that $a_m = 0, 1$ for all m. Consider the rational cycle $q \cdot q^t = (\sum_m a_m) \cdot (pt \otimes pt)$, where the class $pt = u_{p-1,i_0} \in \mathrm{CH}_0(X_L)$ is represented by a zero-cycle of degree one. The push-forward induced by the first projection sends this cycle to $(\sum_m a_m) \cdot pt$. Denote $a = \sum_m a_m$. Note that q is non-trivial if and only if $a \neq 0 \mod p$. Therefore, the motive $\mathcal{R}$ is decomposable if and only if the cycle $a \cdot pt$ is rational, where $a \neq 0 \mod p$. But the cycle $p \cdot pt$ is rational by Lemma 3.5 and since a and p are coprime, pt must be rational as well. Conversely, if pt is rational we can take $q = 1 \otimes pt \in \operatorname{End}(\mathcal{R}_L)$. The latter implies that the motive of a point $\mathbb{Z}$ is a direct summand in $\mathcal{R}$, i.e., $\mathcal{R}$ is decomposable. ☐

3.10 Remark. The fact that p is a prime integer was used in the proof two times. First, to prove that the cycle σ from (5) is rational. Indeed, to obtain the formula (5) we used the congruence $\binom{p-1}{i} = (-1)^i \mod p$ which holds only if p is prime. Second, to prove that the cycle pt is rational. In fact, we proved that the cycle $g.c.d.(a,p) \cdot pt$ is rational. Then using the fact that p is prime and $a < p$ we concluded that $g.c.d.(a,p) = 1$.

3.11 Remark. If one drops the assumption of unimodularity for the ρ-balanced family the proof still works by Remark 2.12, excluding the lifting of isomorphisms (6). So one produces a motivic decomposition $\mathcal{M}(X) \simeq \oplus_{i \in \mathcal{I}} \mathcal{R}_i$, where each $\mathcal{R}_i$ has a property that modulo p (but not integrally) it can be identified with $\mathcal{R}_i \otimes \mathbb{Z}/p \simeq \mathcal{R}_0(\operatorname{codim} \alpha_i) \otimes \mathbb{Z}/p$.

3.12 Proposition. *Let X and Y satisfy the hypothesis of Theorem 3.9 with the same splitting field L of a prime degree p and the same codimension r. Assume that one of two cycles $\rho_X \otimes 1 - \varepsilon(1 \otimes \rho_Y) \in \mathrm{CH}^r(X_L \times Y_L)$, $\varepsilon = \pm 1$ is rational. Then the motives $\mathcal{R}_X$ and $\mathcal{R}_Y$ appearing in the decomposition of $\mathcal{M}(X)$ and $\mathcal{M}(Y)$ respectively are isomorphic.*

Proof. Let $\{\alpha_i\}_{i\in\mathcal{I}}$ and $\{\alpha_j\}_{j\in\mathcal{J}}$ denote the respective ρ_X- and ρ_Y-balanced families. The explicit isomorphism between $\mathcal{R}_X = (X, p_{i_0})$ and $\mathcal{R}_Y = (Y, p_{j_0})$ is given by the cycle

$$\varphi_{i_0,j_0} = \sum_m \varepsilon^m u_{m,i_0} \otimes u^{\vee}_{m,j_0},$$

where $u_{m,i}$ and $u_{m,j}$ are liftings of the respective ρ_X- and ρ_Y-basis. Note that modulo p it coincides with the rational cycle

$$\bar{\varphi}_{i_0,j_0} = \tfrac{1}{C_{j_0}}(\rho_X \otimes 1 - \varepsilon(1 \otimes \rho_Y))^{p-1} \cdot (\alpha_{i_0} \otimes \alpha_{j_0^\bullet})$$

and so is rational. It remains to apply Corollary 3.8 to the isomorphism φ_{i_0,j_0}. □

3.13 Lemma. *Let X and Y satisfy the hypothesis of Theorem 3.9 with the same splitting field L of a prime degree p and the same codimension r. If $F(Y)$ is a splitting field of X and $F(X)$ is a splitting field of Y then the cycle $\rho_X \otimes 1 - \varepsilon(1 \otimes \rho_Y)$ is rational for some $\varepsilon \in (\mathbb{Z}/p)^\times$. In particular, for $p = 2$ and 3 one of two cycles $\rho_X \otimes 1 - \varepsilon(1 \otimes \rho_Y)$, $\varepsilon = \pm 1$ is rational.*

Proof. The proof is the same as in the beginning of the proof of Theorem 3.9 using a slightly modified version of Lemma 3.6, where instead of the product $A \otimes A = \mathrm{CH}(X_L) \otimes \mathrm{CH}(X_L)$ one considers $\mathrm{CH}(X_L) \otimes \mathrm{CH}(Y_L)$. □

4 Projective homogeneous varieties

In the present section we list several important properties of projective homogeneous varieties needed to apply Theorem 3.9. From now on we assume G is a simple linear algebraic group over a field F.

4.1 Lemma. *Let X be a projective homogeneous variety of a group of inner type over a field F then for any field extension E/F the natural map $\mathrm{res}_{E/F} : \mathcal{M}(X) \to \mathcal{M}(X_E)$ induces an isomorphism after tensoring with $\mathbb{Q}$.*

Proof. Consider an isomorphism induced by Chern character $\mathrm{CH}(X)\otimes\mathbb{Q}\simeq \mathrm{K}_0(X)\otimes\mathbb{Q}$. According to the results of paper [Pa94] the map $\mathrm{res}_{E/F}: K_0(X)\otimes\mathbb{Q}\to K_0(X_E)\otimes\mathbb{Q}$ is an isomorphism. To finish the proof apply Manin's identity principle. □

4.2 Lemma. *Let X be a projective homogeneous G-variety where G is a group of inner type over F which splits over the function field of X Then X is generically split.*

Proof. Follows from [CGM05, Corollary 7.6] and Lemma 4.1. □

4.3 Lemma. *Let G be a group of strongly inner type over F and X be a projective homogeneous G-variety. Then for any splitting field L of X the natural map $\mathrm{res}_{L/F}:\mathrm{Pic}(X)\to\mathrm{Pic}(X_L)$ is an isomorphism.*

Proof. Since for any field extension E/L the induced map $\mathrm{res}_{E/L}$ is an isomorphism we may assume $L=F_s$ is the separable closure of F. Consider an exact sequence from [MT95]

$$0\to\mathrm{Pic}(X)\xrightarrow{\mathrm{res}_{L/F}}\mathrm{Pic}(X_L)^{\Gamma}\xrightarrow{\alpha}\mathrm{Br}(F),\qquad(7)$$

where Γ denotes the absolute Galois group. For all groups of strongly inner type the image of α is trivial by [Ti71] and [MT95, Prop. 2.5]. Since G is of inner type, Γ acts trivially on $\mathrm{Pic}(X_L)$ and, hence, $\mathrm{Pic}(X_L)\simeq\mathrm{Pic}(X_L)^{\Gamma}$ and the lemma is proven. □

In the case of projective homogeneous varieties we can say more about the motive $\mathcal{R}$ appearing in Theorem 3.9. Namely,

4.4 Proposition. *Let X be a projective homogeneous G-variety such that over its function field the group G splits. Assume we are under the hypothesis of Theorem 3.9. If the motive $\mathcal{R}$ appearing in 3.9 is decomposable then it decomposes as the direct sum*

$$\mathcal{R}\simeq\bigoplus_{m=0}^{p-1}\mathbb{Z}(rm).$$

Proof. Indeed, if X possess a zero-cycle of degree one, then there exists a field extension E/F of degree l coprime to p such that X_E has a rational point. By [KR94, Thm. 3.10 and Cor. 3.9] since X is generically split, E is a splitting field of X and, therefore, the cycles $l\alpha$ are rational for all $\alpha\in\mathrm{CH}(X_E)$ by Lemma 3.5. Since l and p are coprime, all cycles in $\mathrm{CH}(X_E)$ are rational. By Rost Nilpotence $\mathcal{R}\simeq\bigoplus_{m=0}^{p-1}\mathbb{Z}(rm)$. □

4.5 Remark. We refer to [To04, Q. 0.2 and Section 9] for the discussion about existence of zero cycles of degree one on anisotropic projective homogeneous varieties.

5 Applications and Examples

As the first application of Theorem 3.9 we obtain the prime degree case of the main result of paper [Ka96] concerning motives of Severi-Brauer varieties

5.1 Corollary (Karpenko)**.** *Let D be a division algebra of a prime degree p over a field F and $\mathrm{SB}(M_n(D))$ be the Severi-Brauer variety of a central simple algebra $M_n(D)$. Then*

$$\mathcal{M}(\mathrm{SB}(M_n(D))) \simeq \bigoplus_{i=0}^{n-1} \mathcal{M}(\mathrm{SB}(D))(ip),$$

where $\mathcal{M}(\mathrm{SB}(D))$ is indecomposable.

Proof. The variety X is generically split and there is a splitting field L of X of degree p. Set $A = \mathrm{CH}(\mathbb{P}^{np-1})$. Set $\rho = h$ to be the class of a hyperplane section of $\mathbb{P}^{np-1}$, $\mathcal{I} = \{0, 1, \ldots, n-1\}$, $i^\bullet = n - 1 - i$ and $\alpha_i = h^{pi}$. Note that h^p is rational since $h^p = \mathrm{res}(j_*(1))$, where $j\colon \mathrm{SB}(M_{n-1}(D)) \hookrightarrow \mathrm{SB}(M_n(D))$ is the closed embedding and j_* is the induced push-forward.

Hence, $\{\alpha_i\}_{i\in\mathcal{I}}$ form a ρ-balanced family in $\bar{A}$. Note that the respective ρ-basis coincides with the standard basis of A (which consists of powers of h) and, hence, is unimodular. Applying Theorem 3.9 to the unimodular ρ-balanced family $\{\alpha_i\}$ we obtain the decomposition

$$\mathcal{M}(\mathrm{SB}(M_n(D))) \simeq \bigoplus_{i=0}^{n-1} \mathcal{R}(ip)$$

for some indecomposable motive $\mathcal{R}$.

To show that the motive $\mathcal{R}$ is isomorphic to the motive $\mathcal{M}(\mathrm{SB}(D))$ we apply Proposition 3.12 to $X = \mathrm{SB}(M_n(D))$, $Y = \mathrm{SB}(D)$ and the rational cycle $h_X \otimes 1 - 1 \otimes h_Y$ which generates the kernel of the map α of the exact sequence (7), where X is taken to be the product $\mathrm{SB}(M_n(D)) \times \mathrm{SB}(D)$. ☐

5.2 Remark. In view of Remark 3.10 to prove an arbitrary degree case it is enough to show that the cycle σ from (5) is rational. In [Ka00, Claim 7.8] it is shown that σ lies in the subring generated by rational cycles $(h_X \otimes 1 - 1 \otimes h_Y)$, $c_i(T_X) \otimes 1$ and $1 \otimes c_i(T_Y)$, where $c_i(T)$ is the i-th Chern class of the respective tangent bundle, and, therefore, is rational.

As the next application we obtain one of the results of paper [Ro98] about motives of Pfister quadrics

5.3 Corollary (Rost)**.** *Let X be a n-fold Pfister quadric or its maximal neighbor. Then*

$$\mathcal{M}(X) \simeq \bigoplus_{i=0}^{\dim X - r} \mathcal{R}(i),$$

where the motive $\mathcal{R}$ is decomposable if and only if X is isotropic, and in this case $\mathcal{R} \simeq \mathbb{Z} \oplus \mathbb{Z}(r)$, where $r = 2^{n-1} - 1$. The motive $\mathcal{R}$ is called Rost motive.

Proof. By [EKM, Cor. 9.10] and Lemma 4.2 the variety X is generically split. Moreover, it is split by a quadratic field extension L/F, i.e., in this case $p = 2$. Let $A = \mathrm{CH}(X_L)$ and $\bar{A}$ be its specialization modulo 2. Set $\rho \in \bar{A}^r$ to be the class of a maximal totally isotropic subspace of the respective quadratic space. Set $\mathcal{I} = \{0, 1, \ldots, \dim X - r\}$ and the involution $i \mapsto i^\bullet = \dim X - r - i$. By Lemma 4.3 all powers h^i, $i = 0 \ldots \dim X$ of a class of a hyperplane section $h \in \bar{A}^1$ are rational cycles in $\bar{A}$. Set $\alpha_i = h^i$ for $i \in \mathcal{I}$. Knowing the multiplicative structure of $\bar{A}$ (see [Ka01]) one immediately obtains that $\{\alpha_i\}_{i \in \mathcal{I}}$ form a unimodular ρ-balance family of elements. Applying Theorem 3.9 we finish the proof. □

The next example deals with maximal orthogonal Grassmannians

5.4 Corollary. *Let (V, q) be a quadratic space corresponding to a maximal n-fold Pfister neighbor q. Let X be the respective maximal orthogonal Grassmannian, i.e., the variety of maximal totally isotropic subspaces in (V, q). Then*

$$\mathcal{M}(X) \simeq \bigoplus_{i=0}^{\frac{r(r-1)}{2}} \mathcal{R}(i)^{\oplus a_i},$$

where the motive $\mathcal{R}$ is the Rost motive of the maximal n-fold Pfister neighbor q. In particular, $\mathcal{R}$ is decomposable if and only if q is isotropic, and in this case $\mathcal{R} \simeq \mathbb{Z} \oplus \mathbb{Z}(r)$, where $r = 2^{n-1} - 1$. The integers a_i are the coefficients at t^i in $\prod_{k=1}^{r-1}(1 + t^k)$.

Proof. According to Tits diagrams X is generically split and, moreover, splits by a quadratic field extension L which splits q. The Chow ring A of X_L has the following presentation (see [EKM])

$$A = \mathbb{Z}[e_1, \ldots, e_r]/\langle e_i^2 = e_{2i},\ 2i \le r;\ e_i^2 = 0,\ 2i > r\rangle,$$

where $\operatorname{codim} e_i = i$. For a subset $I = \{i_1, \ldots, i_k\} \subset \{1, \ldots, r\}$ we denote $e_I = e_{i_1} \ldots e_{i_k}$. The elements e_I form an additive basis of A. Moreover, for two cycles e_I, e_J with $\operatorname{codim} e_I + \operatorname{codim} e_J = \dim X = \frac{r(r+1)}{2}$ we have

$$e_I e_J = \begin{cases} pt, & J = \{1, \ldots, r\} \setminus I, \\ 0, & \text{otherwise.} \end{cases}$$

According to [EKM, Corollary 66.7] the cycles $e_1, \ldots, e_{r-1}$ are rational. Define $\rho = e_r$. Let $\mathcal{I}$ be the set of all subsets of $\{1, \ldots, r-1\}$ with involution being set-theoretical complement. Set $\alpha_I = e_I$, $I \in \mathcal{I}$. Observe that $\{\alpha_I\}_{I \in \mathcal{I}}$ form a ρ-balanced family of elements in $\bar{A}$. Applying Theorem 3.9 we obtain the desired motivic decomposition.

To show that $\mathcal{R}$ is the Rost motive apply Proposition 3.12 and Lemma 3.13 to the variety X and the Pfister quadric Y. □

5.5 Remark. In the case $r = 3$ the proof goes through for the maximal orthogonal Grassmannian corresponding to a quadratic form q with trivial even Clifford algebra $C_0(q)$.

Next we provide the proof of the main result of paper [Bo03] concerning motives of G_2-varieties.

5.6 Corollary (Bonnet). *Let G be a twisted form of a group of type* G_2. *Consider a projective homogeneous G-variety X corresponding to a maximal parabolic subgroup of G (there are two such varieties). Then*

$$\mathcal{M}(X) \simeq \bigoplus_{i=0}^{2} \mathcal{R}(i),$$

where $\mathcal{R}$ is the Rost motive of the maximal 3-fold Pfister neighbor. In particular, $\mathcal{R}$ is decomposable iff X is isotropic, and in this case $\mathcal{R} \simeq \mathbb{Z} \oplus \mathbb{Z}(3)$.

Proof. Again by Tits diagrams X is generically split and there is a quadratic field extension L/F which splits X, i.e., $p = 2$. Using Pieri formula 8.4 it can be shown that the Chow ring $\bar{A}$ of X_L modulo 2 is isomorphic to the Chow ring of the maximal neighbor of the 3-fold Pfister quadric given by the norm of the underlying octonion algebra. Moreover, since G is of strongly inner type over F, by Lemma 4.3 the subgroup of rational cycles of $\bar{A}$ is the same as for the Pfister neighbor. The proof then follows from the proof of Corollary 5.3. To show that the corresponding motives $\mathcal{R}$ are the same for both varieties of type G_2 use Proposition 3.12 and Lemma 3.13 together with the fact that G splits over the function fields of both. □

Now we give a short proof of the main result of paper [NSZ] about the motives of F_4-varieties.

5.7 Corollary. *Let G be a twisted form of a split group of type F_4 which splits by a cubic field extension. Let X be a projective homogeneous G-variety corresponding to a maximal parabolic subgroup of G given by the first or the last vertex of the Dynkin diagram. Then*

$$\mathcal{M}(X) \simeq \bigoplus_{i=0}^{7} \mathcal{R}(i),$$

where the motive $\mathcal{R}$ is the same for both varieties. $\mathcal{R}$ is decomposable if and only if X is isotropic, and in this case $\mathcal{R} \simeq \mathbb{Z} \oplus \mathbb{Z}(4) \oplus \mathbb{Z}(8)$.

Proof. By Tits diagrams X is generically split. Indeed, if $G_{F(X)}$ is not split, then $G_{F(X)}$ must have Tits index $F_{4,1}^{21}$ with anisotropic kernel $\mathrm{Spin}(q)$, where q is a quadratic form of dimension 7. By the hypothesis $\mathrm{Spin}(q)$ must split by a cubic field extension, contradiction.

Let L be a cubic field extension which splits G ($p = 3$). Since G is of strongly inner type over F by Lemma 4.3, the class of a hyperplane section h generating the Picard group $\mathrm{Pic}(X_L)$ is rational. Consider the Chow ring $\bar{A}$ modulo 3. Using Pieri formula 8.4 we show that $\langle h^i \rangle = \bar{A}^i$ for $i < 4$ and $h^4 = g_4^{(1)} + g_4^{(2)}$ is the sum of standard generators of $\bar{A}^4$. Set $\rho = g_4^{(1)}$. Define $\mathcal{I} = \{0, \ldots, 7\}$ and the involution $i^\bullet = 7 - i$, $i \in \mathcal{I}$. Set $\alpha_i = h^i$. Direct computations show that the elements $\{\alpha_i\}_{i\in\mathcal{I}}$ form a unimodular ρ-balanced family. Applying Theorem 3.9 we obtain the desired motivic decomposition.

As in the case of G_2 observe that the group G splits over the function fields of both varieties. □

Following the arguments of F_4-case one immediately obtains E_6-case, where E_6 is of strongly inner type. Namely,

5.8 Corollary. *Let G be a strongly inner form of a split group of type* E_6 *which splits by a cubic field extension. Let X be a projective homogeneous variety corresponding to a maximal parabolic subgroup of G given by the first or the last vertex of the Dynkin diagram. Then*

$$\mathcal{M}(X) \simeq \bigoplus_{i=0}^{8} \mathcal{R}(i),$$

where the motive $\mathcal{R}$ is the same as in F_4*-case. In particular, $\mathcal{R}$ is decomposable if and only if X is isotropic, and in this case $\mathcal{R} \simeq \mathbb{Z} \oplus \mathbb{Z}(4) \oplus \mathbb{Z}(8)$.*

Proof. By Tits diagrams X is generically split. Indeed, if $G_{F(X)}$ is not split, then $G_{F(X)}$ must have Tits index ${}^1E_{6,2}^{28}$ with anisotropic kernel $\mathrm{Spin}(q)$, where q is a quadratic form of dimension 8. By the hypothesis $\mathrm{Spin}(q)$ must split by a cubic field extension, contradiction.

Let L be a cubic field extension which splits G ($p = 3$). Since G is of strongly inner type over F by Lemma 4.3, the class of a hyperplane section h generating the Picard group $\mathrm{Pic}(X_L)$ is rational. Consider the Chow ring $\bar{A}$ modulo 3. Using Pieri formula 8.4 we show that $\langle h^i \rangle = \bar{A}^i$ for $i < 4$ and $h^4 = g_4^{(1)} + g_4^{(2)}$ is the sum of standard generators of $\bar{A}^4$. Set $\rho = g_4^{(1)}$. Define $\mathcal{I} = \{0, \ldots, 8\}$ and the involution $i^\bullet = 8 - i$, $i \in \mathcal{I}$. Set $\alpha_i = h^i$. Direct computations show that the elements $\{\alpha_i\}_{i \in \mathcal{I}}$ form a unimodular ρ-balanced family. It remains to apply Theorem 3.9.

The motive $\mathcal{R}$ is the same as in the F_4-case, since X splits over the function field of the F_4-variety Y and vice versa. Indeed, the underlying Jordan algebra splits over the function fields of X and Y. □

6 The case of E_8

Let G be a group of type E_8, which splits by a field extension of degree 5. Such a group exists according to [Ga06, A.6.]. Let X be a projective G-homogeneous variety of parabolic subgroups of type P_8. The variety X has dimension 57 and is the minimal possible E_8-variety in the sense of dimension. In the present section we prove

6.1 Theorem. *The Chow motive of X is isomorphic to*

$$\mathcal{M}(X) \simeq \bigoplus_{i=0\ldots 1, j=0\ldots 23} \mathcal{R}_{i,j},$$

where the motive $\mathcal{R}_{i,j}$ is decomposable if and only if X has a zero-cycle of degree one, and in this case $\mathcal{R}_{i,j}$ is isomorphic to the direct sum

$$\mathcal{R}_{i,j} \simeq (\mathbb{Z} \oplus \mathbb{Z}(6) \oplus \mathbb{Z}(12) \oplus \mathbb{Z}(18) \oplus \mathbb{Z}(24))(10i + j).$$

Moreover, considered with $\mathbb{Z}/5$-coefficients it is isomorphic to

$$\mathcal{R}_{i,j} \otimes \mathbb{Z}/5 \simeq \mathcal{R}_{0,0}(10i + j) \otimes \mathbb{Z}/5.$$

Proof. Let L be a splitting field of X of degree 5, i.e., $p = 5$. Since G is of strongly inner type over F, by Lemma 4.3 the Picard group of X_L is generated by a rational cycle h. Consider the Chow group $\bar{A} = \mathrm{CH}(X_L)/5$. Pieri formula 8.4 shows that $\bar{A}^i = \langle h^i \rangle$ for $i \leq 5$ and in codimension 6 it is generated by two elements. One of them is h^6 and, hence, is rational. According to 3.9.(i) we can take any other generator as a cycle ρ. In terms of the standard basis of $\bar{A}^6$ (see Appendix), $h^6 = g_6^{(1)} + g_6^{(2)}$ and we choose $\rho = g_6^{(1)}$.

We set $\mathcal{I} = \{0,1\} \times \{0,\ldots,23\}$ and the involution $(i,j)^\bullet = (1-i, 23-j)$. Define a ρ-balanced family $\{\alpha_{i,j}\}_{(i,j)\in\mathcal{I}}$ as $\alpha_{i,j} = c_{10}^i h^j$, where c_{10} is the 10-th Chern class of the tangent bundle over X_L. The choice of the 10-th Chern class is not accidental. Indeed, this is the next codimension (after 6) where the number of generators jumps by one. Direct computations (using formulae from Appendix) show that $\{\alpha_{(i,j)}\}_{(i,j)\in\mathcal{I}}$ is a ρ-balanced family. The respective ρ-basis is non-unimodular, since there are determinants which are equal to $\pm 2 \mod 5$. Applying Theorem 3.9 and Remark 3.11 we finish the proof of the Theorem. □

7 Chow motives of fibered spaces

The main result of paper [CPSZ] says that under certain restrictions the Chow motive of a twisted flag variety X can be expressed in terms of the motive of a 'minimal' flag, i.e., the one which corresponds to a maximal parabolic subgroup. These restrictions cover almost all twisted flag varieties corresponding to groups of types A_n and B_n together with some examples of types C_n, G_2 and F_4. In the present section using the result of Edidin and Graham [EG97] about cellular fibrations we extend this set of examples for groups of inner types C_n, D_n and exceptional groups. We also provide a shortened and uniform proof of the results obtained in [CPSZ]. More precisely, we prove the following

7.1 Theorem. *Let G_0 be a split simple linear algebraic group over a field F and $P \subset P'$ parabolic subgroups of G_0. For a cocycle $\xi \in Z^1(F, G_0)$ let $G = {}_\xi G_0$ denote the twisted form of G_0 and $X = {}_\xi(G_0/P)$ and $X' = {}_\xi(G_0/P')$ the respective twisted flag varieties. If G splits over the generic point of X', then the Chow motive $\mathcal{M}(X)$ of X is isomorphic (non-standardly) to a direct sum of twisted copies of the motive $\mathcal{M}(X')$, i.e.,*

$$\mathcal{M}(X) \simeq \bigoplus_{i\in\mathcal{I}} \mathcal{M}(X')(a_i).$$

Proof. Consider the standard map $\pi : X \to X'$. Since G splits over $F(X')$, the fiber of π over $F(X')$ splits, i.e., is isomorphic to P'/P. Moreover, there is an open subset of X' over which G splits. Since G acts transitively on X', the map π is a locally trivial fibration whose fiber is isomorphic to P'/P, i.e., has a decomposition into affine cells. To finish the proof apply the proposition below. □

7.2 Proposition. *Let $f : Y \to X$ be a smooth projective locally trivial fibration with X smooth and projective whose fiber $\mathcal{F}$ has a decomposition into affine cells. Then $\mathcal{M}(Y)$ is (non-standardly) isomorphic to $\mathcal{M}(X) \otimes \mathcal{M}(\mathcal{F})$.*

Proof. We follow the proof of [EG97, Prop. 1]. Define the morphism

$$\varphi : \bigoplus_{i\in\mathcal{I}} \mathcal{M}(X)(a_i) \to \mathcal{M}(Y)$$

to be the direct sum $\varphi = \bigoplus_{i\in\mathcal{I}} \varphi_i$, where each φ_i is given by the cycle $[\mathrm{pr}_Y^*(B_i)\cdot\Gamma_f] \in \mathrm{CH}(X\times Y)$ produced from the graph cycle Γ_f and the chosen (non-standard) basis $\{B_i\}_{i\in\mathcal{I}}$ of $\mathrm{CH}(Y)$ over $\mathrm{CH}(X)$. The realization of φ coincides exactly with an isomorphism of abelian groups $\mathrm{CH}(X)\otimes\mathrm{CH}(\mathcal{F}) \to \mathrm{CH}(Y)$ constructed in [EG97, Prop. 1]. By Manin's identity principle [Ma68] φ is an isomorphism and we are done. □

Let $P = P_\Theta$ be a standard parabolic subgroup corresponding to a subset Θ of the Dynkin diagram $\mathcal{D}$ and P_k, $k \in \mathcal{D}$, stands for the maximal parabolic subgroup $P_{\mathcal{D}\setminus\{k\}}$. Enumeration of roots follows Bourbaki. To simplify the notation we will write X_Θ for the form of G_0/P_Θ twisted by $\xi \in Z^1(F, G_0)$ and X_k for the form of G_0/P_k respectively. By definition all forms $G = {}_\xi G_0$ of G_0 are of inner type over F. Since we are interested in twisted flag varieties we may assume G_0 is adjoint. As an immediate consequence of the theorem we obtain that

7.3 Corollary. *For a group G of inner type $\mathcal{D}$ over F and twisted flag varieties X_Θ and X_k there is an isomorphism of Chow motives*

$$\mathcal{M}(X_\Theta) \simeq \bigoplus_{i\in\mathcal{I}} \mathcal{M}(X_k)(a_i),$$

where the number $k \in \mathcal{D}$ corresponds to a parabolic subgroup P_k containing P_Θ (i.e., $k \in \mathcal{D} \setminus \Theta$) and satisfies the following conditions depending on $\mathcal{D}$

- A_n: *k is prime to the degree of a splitting field of G (cf. [CPSZ, Thm. 2.1]);*
- B_n: *$k = n$ (cf. [CPSZ, Thm. 2.9]); if $G = \mathrm{PGO}^+(q)$, where q is a maximal Pfister neighbor, k is arbitrary;*
- C_n: *k is odd (cf. [CPSZ, Thm. 2.11]);*
- D_{2m+1}: *$k = 2m,\ 2m+1$;*
- D_{2m}: *$k = 2m-1,\ 2m$ in the case $G = \mathrm{PGO}^+(q)$, where q is a non-singular quadratic form over F; moreover, if q is a Pfister form, then k is arbitrary;*
- G_2: *k is arbitrary (cf. [CPSZ, Thm. 2.13]);*
- F_4: *$k \neq 4$ or G splits by a cubic field extension (cf. [CPSZ, Thm 2.14]);*
- E_6: $k = \begin{cases} 3,\ 5 & \\ 2,\ 4 & \text{if } G \text{ is of strongly inner type} \\ 1,\ 6 & \text{if } G \text{ is of strongly inner type and splits by a cubic field extension.} \end{cases}$
- E_8: *k is arbitrary in the case G splits by a field extension of degree 5.*

Proof. In order to apply the theorem to the fibration $X_\Theta \to X_k$ we need to assume that the group G splits over the generic point of X_k. The latter condition can be extracted from the Tits diagrams of G (see [Ti66]). Namely, it holds if the only Tits diagram of G containing a circled vertex numbered by k is the split one, i.e., having all vertices circled.

We provide the proof for the most complicated cases $G_0 = \mathrm{E}_6$ and E_8 only. All the other cases can be treated similarly and are left to the reader.

Analyzing Tits diagrams for a general E_6 we obtain immediately the cases $k = 3, 5$. For strongly inner forms of E_6 using the diagrams provided in [Ti90, 5.2] we obtain the cases $k = 2, 4$. In the cases $k = 1, 6$ the only possible non-split diagram is ${}^1\mathrm{E}_{6,2}^{28}$ whose anisotropic kernel is $\mathrm{Spin}(q)$, where q is a 3-fold Pfister form. But according to Springer's Theorem q can not split by a cubic field extension.

In the case of E_8 the anisotropic kernel of $G_{F(X_k)}$ is trivial since it can't split by a field extension of degree 5 by [To04, Thm. 5.1]. $\square$

Let S be a coefficient ring such that Krull-Schmidt theorem holds in the category of Chow motives of projective homogeneous G-varieties. For instance, S is a field or a discrete valuation ring (see [CM04, Cor. 9.7]). By $M \otimes S$ we denote a motive M considered with S-coefficients.

7.4 Corollary. *For a group G of inner type $\mathcal{D}$ over F there is an isomorphism of Chow motives with S-coefficients*

$$\mathcal{M}(X_\Theta) \otimes S \simeq \bigoplus_{i\in\mathcal{I}} (\mathcal{R} \otimes S)(a_i),$$

where an indecomposable motive $\mathcal{R}$ (depending only on G) and the subsets Θ are taken from the following list

- A_n: *For $G = \mathrm{PGL}_{n/d}(D)$, where D is a division algebra of index d, take $\mathcal{R} = \mathcal{M}(\mathrm{SB}(D))$ and Θ such that $gcd(\Theta \cup \{d\}) = 1$ (cf. [CPSZ, Prop. 2.4]).*
- C_n: *For $G = \mathrm{PGU}_{2n/d}(D, h)$, where D is a division algebra of index $d = 2^r$, with an anti-hermitian form h, take $\mathcal{R} = \mathcal{M}(\mathrm{SB}(D))$ and Θ such that $\mathcal{D} \setminus \Theta$ contains an odd number.*
- B_n *and D_n: Assume $G = \mathrm{PGO}^+(q)$, where q is a Pfister form or its maximal neighbor. Take $\mathcal{R}$ to be the Rost motive corresponding to q and arbitrary Θ.*

G_2: *For* $G = \mathrm{Aut}(\mathbb{O})$*, where* $\mathbb{O}$ *is an octonion algebra take* $\mathcal{R}$ *to be the Rost motive of a maximal Pfister neighbor of* $N_{\mathbb{O}}$ *and arbitrary* Θ*.*

F_4: *Assume* $G = \mathrm{Aut}(J)$*, where a Jordan algebra* J *splits by a cubic field extension. Take* $\mathcal{R}$ *to be the integral motive appearing in the decomposition of Cor. 5.7 and arbitrary* Θ*.*

E_6: *Assume* $G = \mathrm{Aut}(N_J)_{ad}$ *and* J *splits by a cubic field extension. Take the same* $\mathcal{R}$ *as in the* F_4*-case and arbitrary* Θ*.*

E_8: *Assume* G *splits by a field extension of degree* 5*. Take* $\mathcal{R}$ *to be an integral motive appearing in the decomposition of Theorem 6.1 and arbitrary* Θ*.*

Proof. Consider the variety X_k, where $k = 8$ if G has a type E_8 and $k = 1$ otherwise. The motive $M(X_k)$ with S-coefficients splits as a direct sum of copies of $\mathcal{R} \otimes S$ by the results of the previous section. Our conditions on G and Θ imply that G splits over $F(X_\Theta)$ and over $F(X_k)$. If $k \in \mathcal{D} \setminus \Theta$ we apply the previous corollary. Otherwise consider the flag $X_{\Theta\setminus\{k\}}$. According to the theorem the motive $\mathcal{M}(X_{\Theta\setminus\{k\}}) \otimes S$ has two different decompositions

$$\bigoplus_{i\in\mathcal{I}} \mathcal{M}(X_\Theta)(b_i) \otimes S \simeq \mathcal{M}(X_{\Theta\setminus\{k\}}) \otimes S \simeq \bigoplus_{j\in\mathcal{J}} \mathcal{M}(X_k)(b'_j) \otimes S \simeq \bigoplus_{l\in\mathcal{L}} \mathcal{R}(b''_l) \otimes S.$$

By Krull-Schmidt Theorem for motives with S-coefficients [CM04, Cor. 9.7] the motive $\mathcal{M}(X_\Theta) \otimes S$ has the desired decomposition. □

8 Appendix

8.1. Let G_0 be a split simple algebraic group of rank n defined over a field F. We fix a maximal split torus T in G_0 and a Borel subgroup B of G_0 containing T and defined over F. We denote by Φ the root system of G_0, by $\Pi = \{\alpha_1, \dots, \alpha_n\}$ the set of simple roots of Φ corresponding to B, by W the Weyl group, and by $S = \{s_1, \dots, s_n\}$ the corresponding set of fundamental reflections. Enumeration of roots follows Bourbaki.

Let $P = P_\Theta$ be a (standard) parabolic subgroup corresponding to a subset $\Theta \subset \Pi$, i.e., $P = BW_\Theta B$, where $W_\Theta = \langle s_\theta, \theta \in \Theta\rangle$. As P_i we denote the maximal parabolic subgroup $P_{\Pi\setminus\{\alpha_i\}}$.

Denote $W^\Theta = \{w \in W \mid \forall\, s \in \Theta \quad l(ws) = l(w)+1\}$, where l is the length function. The pairing $W^\Theta \times W_\Theta \to W$ with $(w, v) \mapsto wv$ is a bijection and $l(wv) = l(w) + l(v)$. It is easy to see that W^Θ consists of all representatives in the left cosets W/W_Θ which have minimal length.

8.2 (Standard basis)**.** Now consider the Chow ring of a projective homogeneous variety G_0/P_Θ. It is well known that $\mathrm{CH}(G_0/P_\Theta)$ is a free abelian group with a basis given by classes of varieties $X_w = \overline{PwP/P}$ that correspond to the elements $w \in W^\Theta$. We call such a basis standard. The codimension of the basis element $[X_w]$ equals $l(w_\theta) - l(w)$, where w_θ is the longest element of W_Θ. Standard generators of codimension n will be denoted by $g_n^{(i)}$.

There exists a natural injective pull-back homomorphism

$$\mathrm{CH}(G_0/P) \to \mathrm{CH}(G_0/B)$$

$$[X_w] \mapsto [X_{ww_\theta}]$$

The following results provide tools to perform computations in the Chow ring $\mathrm{CH}(G_0/P_\Theta)$.

8.3 (Poincaré duality)**.** In order to multiply two basis elements h and g of $\mathrm{CH}(G_0/P_\Theta)$ such that $\operatorname{codim} h + \operatorname{codim} g = \dim G_0/P_\Theta$ we use the following formula (see [Ko91, 1.4]):

$$[X_w] \cdot [X_{w'}] = \delta_{w, w_0 w' w_\theta} \cdot [X_1].$$

8.4 (Pieri formula)**.** In order to multiply two basis elements of $\mathrm{CH}(G_0/B)$ one of which is of codimension 1 we use the following formula (see [De74, Cor. 2 of 4.4]):

$$[X_{w_0 s_\alpha}][X_w] = \sum_{\beta\in\Phi^+,\, l(ws_\beta)=l(w)-1} \langle \beta^\vee, \overline{\omega}_\alpha\rangle [X_{ws_\beta}],$$

where α is a simple root and the sum runs through the set of positive roots $\beta \in \Phi^+$, s_α denotes the simple reflection corresponding to α and $\bar{\omega}_\alpha$ is the fundamental weight corresponding to α. Here $[X_{w_0 s_\alpha}]$ is the element of codimension 1.

8.5 (Characteristic map)**.** Let $\mathrm{P} = \mathrm{P}(\Phi)$ be the weight space. We denote as $\bar{\omega}_1, \ldots \bar{\omega}_l$ the basis of P consisting of fundamental weights. The symmetric algebra $S^*(\mathrm{P})$ is isomorphic to $\mathbb{Z}[\bar\omega_1,\dots\bar\omega_l]$. The Weyl group W acts on P, hence, on $S^*(\mathrm{P})$. Namely, for a simple root α_i,

$$w_{\alpha_i}(\bar\omega_j)=\begin{cases}\bar\omega_i-\alpha_i, & i=j,\\ \bar\omega_j, & \text{otherwise.}\end{cases}$$

We define a linear map $c\colon S^*(\mathrm{P})^{W_{P_\Theta}}\to \mathrm{CH}^*(G_0/P_\Theta)$ as follows. For a homogeneous W_{P_Θ}-invariant $u\in\mathbb{Z}[\bar\omega_1,\dots,\bar\omega_l]$

$$c(u)=\sum_{w\in W^\Theta,\, l(w)=\deg(u)}\Delta_w(u)[X_{w_0ww_\theta}],$$

where for $w=s_{i_1}\dots s_{i_k}$ we denote by Δ_w the composition of derivations $\Delta_{s_{i_1}}\circ\dots\circ\Delta_{s_{i_k}}$ and the derivation $\Delta_{s_i}\colon S^*(\mathrm{P})\to S^{*-1}(\mathrm{P})$ is defined by $\Delta_{s_i}(u)=\frac{u-s_i(u)}{\alpha_i}$.

8.6 (Tangent bundle)**.** Consider the tangent vector bundle T over G_0/P_Θ and observe that

$$c(T)=c(\prod_{\gamma\in\Sigma_u(P_\Theta)}(1+t\gamma)),$$

where $\Sigma_u(P_\Theta)$ is the set of (positive) roots lying in the unipotent radical of the parabolic subgroup P_Θ and $c(T)$ denotes the total Chern class. Since the tangent vector bundle is rational, the cycles $c_i(T)\in\mathrm{CH}^i(G_0/P_\Theta)$ are rational.

The formulae above were implemented in Maple package [NS]. Most of the computations in the paper were performed and checked using this package.

References

[Bo03] J.-P. Bonnet. Un isomorphisme motivique entre deux variétés homogènes projectives sous l'action d'un groupe de type G_2. Doc. Math. **8** (2003), 247–277.

[CPSZ] B. Calmès, V. Petrov, N. Semenov, K. Zainoulline. Chow motives of twisted flag varieties. Compositio Math. (2006), 18p., to appear

[CM04] V. Chernousov, A. Merkurjev. Motivic decomposition of projective homogeneous varieties and the Krull-Schmidt theorem, to appear in Transf. of Groups **11** (2006).

[CGM05] V. Chernousov, S. Gille, A. Merkurjev. Motivic decomposition of isotropic projective homogeneous varieties. Duke Math. J. **126** (2005), no. 1, 137–159.

[De74] M. Demazure. Désingularisation des variétés de Schubert généralisées. Ann. Sci. École Norm. Sup. (4) **7** (1974), 53–88.

[EG97] D. Edidin, W. Graham. Characteristic classes in the Chow ring. J. Algebraic Geom. 6 (1997), no. 3, 431–443.

[EKM] R. Elman, N. Karpenko, A. Merkurjev. Quadratic forms and algebraic cycles. Book in progress (2006), available from `http://www.math.uni-bielefeld.de/~karpenko/`

[Ful] W. Fulton. Intersection Theory. Second edition, Springer-Verlag, Berlin-Heidelberg, 1998.

[Ga06] S. Garibaldi. Cohomological Invariants. Lens lectures 2006. `http://www.mathcs.emory.edu/~skip/lens-ci/lens-ci.html`

[Gr58] A. Grothendieck, La torsion homologique et les sections rationnelles. Exposé 5 in Anneaux de Chow et applications.(French) Séminaire C. Chevalley; 2e année: 1958.

[Ka96] N. Karpenko. The Grothendieck Chow-motifs of Severi-Brauer varieties. St. Petersburg Math. J. **7** (1996), no. 4, 649-661.

[Ka00] N. Karpenko. Criteria of motivic equivalence for quadratic forms and central simple algebras. Math. Ann. **317** (2000), no. 3, 585-611.

[Ka01] N. Karpenko. Characterization of minimal Pfister neighbours via Rost projectors. J. Pure Appl. Algebra **160** (2001), 195–227.

[KM05] N. Karpenko, A. Merkurjev. Canonical p-dimension of algebraic groups. Advances in Mathematics, article in press `doi:10.1016/j.aim.2005.07.013`.

[Ko91] B. Köck. Chow Motif and Higher Chow Theory of G/P. Manuscripta Math. **70** (1991), 363–372.

[KR94] I. Kersten, U. Rehmann. General Splitting of Reductive Groups. Tohoku Math. J. **46** (1994), 35–70.

[Ma68] Y. Manin. Correspondences, motives and monoidal transformations. Math. USSR Sbornik **6** (1968), 439–470.

[MT95] A. Merkurjev, J.-P. Tignol. The multipliers of similitudes and the Brauer group of homogeneous varieties. J. Reine Angew. Math. **461** (1995), 13–47.

[NS] S. Nikolenko, N. Semenov. Chow ring structure made simple. Maple package. Preprint arXiv.org 2006 (math.AG/0606335).

[NSZ] S. Nikolenko, N. Semenov, K. Zainoulline. Motivic decomposition of anisotropic varieties of type F_4 into generalized Rost motives. Max-Planck-Institut für Mathematik, Preprint Series 2005 (90).

[Pa94] I. Panin. On the algebraic K-theory of twisted flag varieties. K-theory **8** (1994), 541–585.

[Ro98] M. Rost. The motive of a Pfister form. Preprint, 1998. `http://www.mathematik.uni-bielefeld.de/~rost/`

[Ti66] J. Tits. Classification of algebraic semisimple groups. In *Algebraic Groups and Discontinuous Subgroups (Proc. Symp. Pure Math.)*, Amer. Math. Soc., Providence, R.I., 1966.

[Ti71] J. Tits. Représentations linéaires irréductibles d'un groupe réductif sur un corps quelconque. (French) J. Reine Angew. Math. **247** (1971), 196–220.

[Ti90] J. Tits. Strongly inner anisotropic forms of simple algebraic groups. J. of Algebra **131** (1990), 648–677.

[To04] B. Totaro. Splitting fields for E_8-torsors. Duke Math. J. **121** (2004), no. 3, 425–455.

[VZ06] A. Vishik, K. Zainoulline. Motivic splitting lemma. Preprint (2006), 20pp.
`http://www.math.uni-bielefeld.de/~kirill/publications.html`