Fields of u-invariant $2^r + 1$

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Abstract

In this article we provide the uniform construction of fields with all known u-invariants. We also obtain the new values for the u-invariant: 2^r+1, for $r > 3$. The main tools here are the new discrete invariant of quadrics (so-called, *elementary discrete invariant*), and the methods of [15] (which permit to reduce the questions of *rationality* of elements of the Chow ring over the base field to that over bigger field - the generic point of a quadric).

1 Introduction

The u-invariant of a field is defined as the maximal dimension of anisotropic quadratic form over it. For some time people were interested in the set of possible values of this invariant. Using elementary quadratic form theory it is easy to establish that u-invariant can not take values 3, 5 and 7. The conjecture of Kaplansky (1953) suggested that the only possible values are the powers of 2 (such examples were produced). This conjecture was disproved by A.Merkurjev in 1991, who constructed fields with all even u-invariants. The next challenge was to find out if the odd u-invariants are possible at all. The breakthrough here was made by O.Izhboldin who in 1999 constructed a field of u-invariant 9 - see [4]. Still the question of other possible values remained open. In the current article I would like to provide certain uniform method of constructing fields with various u-invariants. In particular, we get fields with even u-invariants without using the *index reduction formula* of Merkurjev, as well as fields of u-invariant $2^r + 1$, for all $r \geqslant 3$ (note, that O.Izhboldin conjectured the existence and had an idea of constructing fields with such u-invariants). In the case of the u-invariant 9 our construction is different from that of O.Izhboldin. I would say that it uses substantially more coarse invariants (like *generic discrete invariant of quadrics*), while the original construction used very subtle ones (like the *cokernel on the unramified cohomology*, etc. ...). So, I want to demonstrate that the u-invariant questions can be solved just with the help of "coarse" invariants. The costruction is based on the ideas from [15], which, in turn, are based on *symmetric operations* (see [13],[16]).

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2 Elementary discrete invariant

Everywhere below we will assume that the base-field k has characteristic 0. Although, many things work for odd characteristics as well, the use of algebraic cobordisms of M.Levine-F.Morel will require such assumption.

To each quadratic form q/k one can assign so-called *generic discrete invariant* $GDI(Q)$ - see [14], which is defined as the collection of subrings

$$GDI(Q,i) := image(\mathrm{CH}^*(G(Q,i))/2 \to \mathrm{CH}^*(G(Q,i)|_{\overline{k}})/2),$$

for all $0 \leqslant i \leqslant d := [\dim(Q)/2]$, where $G(Q,i)$ is the grassmannian of i-dimensional projective subspaces on Q.

For $J \subset I \subset \{0,\ldots,d\}$ let us denote the natural projection between partial flag varieties $F(Q,I) \to F(Q,J)$ as π with subindex I with J underlined inside it. Consider natural correspondences:

$$f_i : G(Q,i) \stackrel{\pi_{(0,\underline{i})}}{\longleftarrow} F(Q,0,i) \stackrel{\pi_{(\underline{0},i)}}{\longrightarrow} Q.$$

In $\mathrm{CH}^*(G(Q,i)|_{\overline{k}})$ we have special classes: $Z_j^{\boxed{i-d}} \in \mathrm{CH}^j, \dim(Q)-d-i \leqslant j \leqslant \dim(Q)-i$, and $W_j^{\boxed{i-d}} \in \mathrm{CH}^j, 0 \leqslant j \leqslant d-i$, defined by:

$$Z_j^{\boxed{i-d}} := (f_i)^*(l_{\dim(Q)-i-j});\quad W_j^{\boxed{i-d}} := (f_i)^*(h^{i+j}),$$

where $l_s \in \mathrm{CH}_s(Q|_{\overline{k}})$ is the class of a projective subspace of dimension s, and $h^s \in \mathrm{CH}^s(Q|_{\overline{k}})$ is the class of plane section of codimension s.

Let us denote as $z_j^{\boxed{i-d}}$ and $w_j^{\boxed{i-d}}$ the same classes in $\mathrm{CH}^*/2$. We will call classes $z_j^{\boxed{i-d}}$ *elementary*. Notice, that the classes $w_j^{\boxed{i-d}}$ always belong to $GDI(Q,i)$.

Let Tav_i be the tautological $(i+1)$-dimensional vector bundle on $G(Q,i)$.

Proposition 2.1 *For any* $0 \leqslant i \leqslant d$, *and* $\dim(Q) - d - i \leqslant j \leqslant \dim(Q) - i$,

$$c_\bullet(-Tav_i) = \sum_{j=0}^{d-i} W_j^{\boxed{i-d}} + 2 \sum_{d-i<j\leqslant\dim(Q)-i} Z_j^{\boxed{i-d}}.$$

Proof: Since $Tav_0 = \mathcal{O}(-1)$ on Q, the statement is true for $i=0$. Consider the correspondence

$$f_i : G(Q,i) \stackrel{\pi_{(0,\underline{i})}}{\longleftarrow} F(Q,0,i) \stackrel{\pi_{(\underline{0},i)}}{\longrightarrow} Q.$$

Notice, that $F(Q,0,i)$ is naturally identified with the projective bundle $\mathbb{P}_{G(Q,i)}(Tav_i)$, and the sheaf $\pi^*_{(\underline{0},i)}(Tav_0)$ is naturally identified with $\mathcal{O}(-1)$. Thus,

$$(\pi_{(0,\underline{i})})_*(\pi_{(\underline{0},i)})^*(c_\bullet(-Tav_0)) = (\pi_{(0,\underline{i})})_*(c_\bullet(-\mathcal{O}(-1))) = c_\bullet(-Tav_i).$$

$\square$

Remark: In particular, for $i = d$ we get another proof of [14, Theorem 2.5(3)].

Definition 2.2 *Define the elementary discrete invariant* $EDI(Q)$ *as the collection of subsets* $EDI(Q,i)$ *consisting of those* j *that* $z_j^{\boxed{i-d}} \in GDI(Q,i)$.

One can visualize $EDI(Q)$ as the coordinate $d \times d$ square, where some integral nodes are marked, each row corresponds to particular grassmannian, and the codimension of a "node" is decreasing up and right. The lower row corresponds to a quadric itself, and the upper one - to the last grassmannian. The south-west corner is marked if and only if Q is isotropic.

Example 2.3 *The $EDI(Q)$ for the 10-dimensional excellent form looks as:*

$$
\begin{array}{ccccc}
\bullet & \bullet & \bullet & \bullet & \circ \\
\bullet & \bullet & \circ & \bullet & \circ \\
\bullet & \bullet & \circ & \circ & \circ \\
\circ & \bullet & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ
\end{array}
$$

The following statement puts serious constraints on possible markings.

Proposition 2.4 *Let $0 \leqslant i < d$, and $j \in EDI(Q,i)$. Then $j, j-1 \in EDI(Q,i+1)$.*

This can be visualized as:

$$
\begin{tikzcd}
\bullet & \bullet \\
& \bullet \arrow[ul] \arrow[u]
\end{tikzcd}
$$

Proof:
The Proposition easily follows from the next Lemma. Let us temporarily denote $\pi_{(\underline{i},i+1)}$ as α, and $\pi_{(i,\underline{i+1})}$ as β.

Lemma 2.5

$$
\begin{aligned}
&\alpha^*(Z_j^{\boxed{i-d}}) = \beta^*(Z_j^{\boxed{i+1-d}}) + c_1(\mathcal{O}(1)) \cdot \beta^*(Z_{j-1}^{\boxed{i+1-d}});\\
&\alpha^*(W_j^{\boxed{i-d}}) = \beta^*(W_j^{\boxed{i+1-d}}) + c_1(\mathcal{O}(1)) \cdot \beta^*(W_{j-1}^{\boxed{i+1-d}}), 0 \leqslant j < d-i;\\
&\alpha^*(W_{d-i}^{\boxed{i-d}}) = 2\beta^*(Z_{d-i}^{\boxed{i+1-d}}) + c_1(\mathcal{O}(1)) \cdot \beta^*(W_{d-i-1}^{\boxed{i+1-d}}),
\end{aligned}
$$

where $\mathcal{O}(1)$ is the standard sheaf on the projective bundle

$$F(Q,i,i+1) = \mathbb{P}_{G(Q,i+1)}(Tav_{i+1}^{\vee}).$$

Proof: By definition, $Z_j^{\boxed{i-d}}, W_j^{\boxed{i-d}}$ have the form $(\pi_{(0,\underline{i})})_*(\pi_{(\underline{0},i)})^*(x)$, for certain $x \in \mathrm{CH}^*(Q|_{\overline{k}})$. Since the square

$$
\begin{tikzcd}
F(Q,i,i+1) \arrow[d, "\pi_{(\underline{i},i+1)}"'] & F(Q,0,i,i+1) \arrow[l, "\pi_{(0,\underline{i},\underline{i+1})}"'] \arrow[d, "\pi_{(\underline{0},\underline{i},i+1)}"] \\
G(Q,i) & F(Q,0,i) \arrow[l, "\pi_{(0,\underline{i})}"]
\end{tikzcd}
$$

is transversal cartesian, $(\pi_{(\underline{i},i+1)})^*$ of these elements is equal to

$$(\pi_{(0,\underline{i},\underline{i+1})})_*(\pi_{(\underline{0},\underline{i},i+1)})^*(\pi_{(\underline{0},i)})^*(x) = (\pi_{(0,\underline{i},\underline{i+1})})_*(\pi_{(\underline{0},i,\underline{i+1})})^*(\pi_{(\underline{0},i+1)})^*(x).$$

Variety $F(Q,0,i,i+1)$ is naturally a divisor D on the transversal product $F(Q,0,i+1) \times_{G(Q,i+1)} F(Q,i,i+1)$ with $\mathcal{O}(D) = \pi^*_{(\underline{0},i,i+1)}(\mathcal{O}(h)) \otimes \pi^*_{(0,\underline{i},\underline{i+1})}(\mathcal{O}(1))$, where $\mathcal{O}(h)$ is the sheaf given by the hyperplane section on Q. Then

$$\begin{aligned}(\pi_{(0,\underline{i},\underline{i+1})})_*(\pi_{(\underline{0},i,\underline{i+1})})^*(\pi_{(\underline{0},i+1)})^*(x) = \\ c_1(\mathcal{O}(1)) \cdot (\pi_{(i,\underline{i+1})})^*(\pi_{(0,\underline{i+1})})_*(\pi_{(\underline{0},i+1)})^*(x) + \\ (\pi_{(i,\underline{i+1})})^*(\pi_{(0,\underline{i+1})})_*(\pi_{(\underline{0},i+1)})^*(h \cdot x).\end{aligned}$$

It remains to plug in the appropriate x. $\square$

Notice, that the projectors giving the decomposition:

$$\mathrm{CH}^*(\mathbb{P}_{G(Q,i+1)}(Tav^{\vee}_{i+1}))/2 = \oplus_{l=0}^{i} c_1(\mathcal{O}(1))^l \cdot \mathrm{CH}^*(G(Q,i+1))/2$$

are defined over the base field. Thus, the element of this group is defined over k if and only if all of it's coordinates are. Since the cycle $z_j^{\boxed{i-d}}$ is defined over k, by Lemma 2.5 the cycles $z_j^{\boxed{i+1-d}}$ and $z_{j-1}^{\boxed{i+1-d}}$ are defined too. $\square$

If we have a codimension 1 subquadric P of a quadric Q, the EDI's of them are related by the following.

Proposition 2.6 *Let $d_P := [\dim(P)/2]$ and $d_Q := [\dim(Q)/2]$. Then*

$$z_j^{\boxed{i-d_Q}}(Q) \textit{ is defined} \Rightarrow z_j^{\boxed{i-d_P}}(P) \textit{ is defined};$$

$$z_j^{\boxed{i-d_P}}(P) \textit{ is defined} \Rightarrow z_j^{\boxed{i+1-d_Q}}(Q) \textit{ is defined}.$$

Proof: Consider the natural embedding: $e: G(P,i) \to G(Q,i)$. Then, it follows from the definition that $e^*(z_j^{\boxed{i-d_Q}}) = z_j^{\boxed{i-d_P}}$. To prove the second statement just observe that Q is a codimension 1 subquadric in P', where $p' = p \perp \mathbb{H}$, and $z_j^{\boxed{i-d_P}}(P)$ is defined $\Rightarrow z_j^{\boxed{i-d_{P'}}}(P')$ is defined. $\square$

Unfortunately, the Steenrod operations, in general, do not act on the $EDI(Q,i)$, since they do not preserve *elementary classes*. But they act in the lower and the upper row: for the quadric itself, and for the last grassmannian. Also, it follows from [14, Main Theorem 5.8] that $EDI(Q,d)$ carries the same information as $GDI(Q,d)$. The same is true about $EDI(Q,0)$ and $GDI(Q,0)$ by the evident reasons.

The action of the Steenrod operations on the *elementary classes* can be described as follows.

Proposition 2.7 *Let* $0 \leqslant i \leqslant d$, *and* $\dim(Q)-d-i \leqslant j \leqslant \dim(Q)-i$, *then*

$$S^m(z_j^{\boxed{i-d}}) = \sum_{l=0}^{d-i} \binom{j-l}{m-l} z_{j+m-l}^{\boxed{i-d}} \cdot w_l^{\boxed{i-d}},$$

where elementary classes of codimension more than $\dim(Q)-i$ *are assumed to be* 0.

Proof: We know that $S^{\bullet}(l_s) = (1+h)^{\dim(Q)-s+1} l_s$. Since $(\pi_{\underline{0},i})^*(\mathcal{O}(1))$ is the sheaf $\mathcal{O}(1)$ on $F(Q,0,i) = \mathbb{P}_{G(Q,i)}(Tav_i)$,

$$S_{\bullet}(\pi_{(\underline{0},i)})^*(l_{\dim(Q)-i-j}) = c_{\bullet}(-T_{F(Q,0,i)}) \cdot (1+H)^{i+j+1} \cdot (\pi_{(\underline{0},i)})^*(l_{\dim(Q)-i-j}),$$

where $H = c_1(\mathcal{O}(1))$. Since $S_{\bullet}$ commutes with the push-forward morphisms,

$$S^{\bullet}(z_j^{\boxed{i-d}}) = S^{\bullet}(\pi_{(0,\underline{i})})_*(\pi_{(\underline{0},i)})^*(l_{\dim(Q)-i-j}) =$$
$$(\pi_{(0,\underline{i})})_*(c_{\bullet}(-T_{fiber}) \cdot (1+H)^{i+j+1} \cdot (\pi_{(\underline{0},i)})^*(l_{\dim(Q)-i-j})).$$

Since $c_{\bullet}(-T_{fiber}) = c_{\bullet}(-Tav_i \otimes \mathcal{O}(1))$, in the light of Proposition 2.1, $(\,mod\,2)$ this is equal to $\sum_{l=0}^{d-i} w_l^{\boxed{i-d}}(1+H)^{-i-1-l}$. Thus,

$$S^{\bullet}(z_j^{\boxed{i-d}}) = (\sum_{l=0}^{d-i} w_l^{\boxed{i-d}})(\pi_{(0,\underline{i})})_*(\pi_{(\underline{0},i)})^*((1+h)^{j-l} l_{\dim(Q)-i-j}) =$$
$$\sum_{r \geqslant 0} \sum_{l=0}^{d-i} \binom{j-l}{r-l} z_{j+r-l}^{\boxed{i-d}} w_l^{\boxed{i-d}}.$$

$\square$

Remark: In particular, for $i = d$ we get a new proof of [14, Theorem 4.1].

The following fact is well-known (see, for example, [2]). We will give an independent proof below.

Proposition 2.8 *The ring* $\mathrm{CH}^*(G(Q,i)|_{\overline{k}})$ *is generated by the classes*

$$Z_j^{\boxed{i-d}}, \dim(Q) - d - i \leqslant j \leqslant \dim(Q) - i, \;\; \textit{and } W_j^{\boxed{i-d}}, 0 \leqslant j \leqslant d - i.$$

Proof: For $0 \leqslant l \leqslant i$, let us denote the pull back of $Z_j^{\boxed{l-d}}$ to $F(Q, 0, \ldots, i)$ by the same symbol. On this flag variety we have natural line bundles $\mathcal{L}_k := Tav_k/Tav_{k-1}$. Let us denote $h_k := c_1(\mathcal{L}_k^{-1})$.

Lemma 2.9 *Suppose that* $Q|_E$ *is completely split. Then the ring* $\mathrm{CH}^*(F(Q, 0, \ldots, i)|_E)$ *is generated by* $W_j^{\boxed{l-d}}, 0 \leqslant l \leqslant i, 1 \leqslant j \leqslant d - l$, *and* $Z_j^{\boxed{l-d}}, 0 \leqslant l \leqslant i, \dim(Q) - d - l \leqslant j \leqslant \dim(Q) - l$.

Proof: Induction on i. For $i = 0$ the statement is evident.

Statement 2.10 *Let* $\pi : Y \to X$ *be smooth morphism,* X*-smooth. For* $x \in X^{(r)}$, Y_x *be the fiber over the point* x. *Let* ζ *denote the generic point of* X, *and* $s_x : \mathrm{CH}^*(Y_\zeta) \to \mathrm{CH}^*(Y_x)$ *be the specialization map. Let* $N \subset \mathrm{CH}^*(Y)$ *be a subgroup. Suppose:*

(a) *the map* $N \to \mathrm{CH}^*(Y|_\zeta)$ *is surjective;*

(b) *all the maps* s_x *are surjective.*

Then $\mathrm{CH}^*(Y) = N \cdot \pi^*(\mathrm{CH}^*(X))$.

Proof: On $\mathrm{CH}^*(Y)$ we have decreasing filtration $F^\bullet$, where F^r consists of classes, having a representative with the image under π of codimension $\geqslant r$. This gives the surjection:

$$\oplus_r \oplus_{x \in X^{(r)}} \mathrm{CH}^*(Y_x) \to gr_{F^\bullet} \mathrm{CH}^{r+*}(Y).$$

Let $[x] \in \mathrm{CH}^r(X)$ be the class represented by the closure of x. Clearly, the image of $\pi^*([x]) \cdot N$ covers the image of $\mathrm{CH}^*(Y_x)$ in F^r/F^{r+1}. $\square$

Consider the projection

$$\pi_{(\underline{0},\ldots,\underline{i-1},i)} : F(Q,0,\ldots,i-1,i) \to F(Q,0,\ldots,i-1).$$

Let $Q_{\{i\},x}/E(x)$ be the fiber of this projection over the point x. It is a completely split quadric of dimension $\dim(Q)-2i$. Thus, the condition (b) of the Statement 2.10 is satisfied. Since

$$Z_j^{\boxed{i-d}}|_{Q_{\{i\},\zeta}} = l_{\dim(Q)-2i-j}, \qquad W_j^{\boxed{i-d}}|_{Q_{\{i\},\zeta}} = h^j$$

we can take N additively generated by $Z_j^{\boxed{i-d}}, \dim(Q)-d-i \leqslant j \leqslant \dim(Q)-2i$, and $W_j^{\boxed{i-d}}, 0 \leqslant j \leqslant d-i$. Then the condition (a) will be satisfied too. The induction step follows. $\square$

Lemma 2.5 implies that the $Z_j^{\boxed{l-d}}, W_j^{\boxed{l-d}}$, for $l < i$ are expressible in terms of $Z_k^{\boxed{i-d}}, W_k^{\boxed{i-d}}$ and $h_m, 0 \leqslant m \leqslant i$. Let $A \subset \mathrm{CH}^*(G(Q,i))$ be the subring generated by $Z_j^{\boxed{i-d}}, W_j^{\boxed{i-d}}$. Since $F(Q,0,\ldots,i)$ is a variety of complete flags of subspaces of the vector bundle Tav_i on $G(Q,i)$, the ring $\mathrm{CH}^*(F(Q,0,\ldots,i))$ is isomorphic to

$$\mathrm{CH}^*(G(Q,i))[h_0,\ldots,h_i]/(\sigma_r(h) - c_r(Tav_i^{\vee}), 1 \leqslant r \leqslant i+1).$$

But $c_r(Tav_i^{\vee}) \in A$, by Proposition 2.1. Since A and $h_m, 0 \leqslant m \leqslant i$ generate $\mathrm{CH}^*(F(Q,0,\ldots,i))$, A must coinside with $\mathrm{CH}^*(G(Q,i))$. $\square$

In particular, since the cycles $W_j^{\boxed{i-d}}$ are defined over k, we have:

Corollary 2.11 *The graded part of* $\mathrm{CH}^*(G(i,Q)|_{\overline{k}})$ *of degree less or equal* $(d-i)$ *consists of classes which are defined over* k*.*

Notice, that for $i = d$, $\mathrm{CH}^*(G(Q,d)|_{\overline{k}})/2$ is generated as a ring by $z_j^{\boxed{0}}$, and moreover, $GDI(Q,d)$ is always generated as a ring by the subset of $z_j^{\boxed{0}}$ contained in it - see [14, Main Theorem 5.8].

3 Generic points of quadrics and Chow groups

In this section I would like to remind the principal result of [15]. Let Q be smooth projective quadric, Y be smooth quasiprojective variety, and $\overline{y} \in \mathrm{CH}^m(Y|_{\overline{k}})/2$. This will be our main tool in the construction of fields with various u-invariants.

Theorem 3.1 ([15, Corollary 3.5],[16, Theorem 4.3].)
Suppose $m < [\dim(Q) + 1/2]$. Then

$$\overline{y}|_{\overline{k(Q)}} \text{ is defined over } k(Q) \;\; \Leftrightarrow \;\; \overline{y} \text{ is defined over } k.$$

Example 3.2 *Let $\alpha = \{a_1, \ldots, a_n\} \in \mathrm{K}^M_n(k)/2$ be a nonzero pure symbol, and Q_α be the respective anisotropic Pfister quadric. Then in $EDI(Q_\alpha)$ the marked nodes will be exactly those ones which live above the main (N-W to S-E) diagonal.*

$$\begin{array}{cccccc}
\circ & \bullet & \bullet & \cdots & \bullet & \bullet \\
\circ & \circ & \bullet & \cdots & \bullet & \bullet \\
\circ & \circ & \circ & \cdots & \bullet & \bullet \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\circ & \circ & \circ & \cdots & \circ & \bullet \\
\circ & \circ & \circ & \cdots & \circ & \circ
\end{array}$$

Really, over own generic point $k(Q_\alpha)$ quadric Q_α becomes hyperbolic, and so, all the elementary cycles are defined there. Then the cycles above the main diagonal got to be defined already over the base field, since their codimension is smaller than $[\dim(Q_\alpha) + 1/2]$. On the other hand, the N-W corner could not be defined over k, since otherwise all the elementary cycles on the last grassmannian of Q_α would be defined over k, but the product of all these cycles is the class of a rational point on this grassmannian (see [14]). Since Q_α is not hyperbolic over k, this is impossible. The rest of the picture follows from Proposition 2.4.

The proof of Theorem 3.1 uses Algebraic Cobordisms of M.Levine-F.Morel. Let me say few words about the latter.

3.1 Algebraic Cobordisms

In [7, 8, 9, 5] M.Levine and F.Morel have constructed the universal oriented generalized cohomology theory Ω^* on the category of smooth quasiprojective varieties over the field k of characteristic 0, called *Algebraic Cobordism*.

For any smooth quasiprojective X over k, the additive group $\Omega^*(X)$ is generated by the classes $[v : V \to X]$ of projective maps from smooth varieties V subject to certain relations, and the upper grading is the codimensional one. There is natural morphism of theories $pr : \Omega^* \to \mathrm{CH}^*$. The main properties of Ω^* are:

(1) $\Omega^*(\mathrm{Spec}(k)) = \mathbb{L} = \mathrm{MU}(pt)$ - the Lazard ring, and the isomorphism is given by the topological realization functor;

(2) $\mathrm{CH}^*(X) = \Omega^*(X)/\mathbb{L}^{<0} \cdot \Omega^*(X)$.

On Ω^* there is the action of the Landweber-Novikov operations. Let $R(\sigma_1, \sigma_2, \ldots) \in \mathbb{L}[\sigma_1, \sigma_2, \ldots]$ be some polynomial, where we assume $\deg(\sigma_i) = i$. Then $S^R_{L-N} : \Omega^* \to \Omega^{*+\deg(R)}$ is given by:

$$S^R_{L-N}([v : V \to X]) := v_*(R(c_1, c_2, \ldots) \cdot 1_V),$$

where $c_j = c_j(-T_V + v^*T_X)$.

If $R = \sigma_i$, we will denote the respective operation simply as S^i_{L-N}. The following statement follows from the definition of Steenrod and Landweber-Novikov operations - see P.Brosnan [1], A.Merkurjev [12], and M.Levine [6]

Proposition 3.3 *There is commutative square:*

$$\begin{array}{ccc} \Omega^*(X) & \xrightarrow{S^i_{L-N}} & \Omega^{*+i}(X) \\ \downarrow & & \downarrow \\ \mathrm{CH}^*(X)/2 & \xrightarrow[S^i]{} & \mathrm{CH}^{*+i}(X)/2, \end{array}$$

where S^i is the Steenrod operation ([18, 1]).

In particular, $pr \circ S^i_{L-N}(\Omega^m) \subset 2 \cdot \mathrm{CH}^{i+m}$, if $i > m$, and $pr \circ (S^m_{L-N} - \square)(\Omega^m) \in 2 \cdot \mathrm{CH}^{2m}$. This implies that $(\,\mathrm{modulo}\, 2 - \mathrm{torsion}\,)$ we have well defined maps $\frac{pr \circ S^i_{L-N}}{2}$ and $\frac{pr \circ (S^m_{L-N} - \square)}{2}$. In reality, these maps can be lifted to a well defined, so-called, *symmetric operations* $\Phi^{t^{i-m}} : \Omega^m \to \Omega^{m+i}$ - see [16]. Since over algebraically closed field all our varieties are cellular, and thus, the Chow groups of them are torsion-free, we will not need such subtleties, but we will keep the notation from [16], and denote our maps as $\phi^{t^{i-m}}$.

3.2 Beyond the Theorem 3.1

Below we will need to study the relation between the rationality of $\overline{y}$ and $\overline{y}|_{\overline{k(Q)}}$ for $\mathrm{codim}(\overline{y})$ slightly bigger than $[\dim(Q)+1/2]$. The methods involved are just the same as are employed for the proof of Theorem 3.1.

Let Y be smooth quasiprojective variety, Q be smooth projective quadric. Let $v \in \mathrm{CH}^*(Y \times Q)/2$ be some element, and $w \in \Omega^*(Y \times Q)$ be it's arbitrary lifting. Over $\overline{k}$, quadric Q becomes a cellular variety with basis of Chow groups and Cobordisms given by the set $\{l_i, h^i\}_{0 \leqslant i \leqslant [\dim(Q)/2]}$ of projective subspaces and plane sections. This implies that

$$\mathrm{CH}^*(Y \times Q|_{\overline{k}}) = \oplus_{i=0}^{[\dim(Q)/2]} (\mathrm{CH}^*(Y|_{\overline{k}}) \cdot l_i \oplus \mathrm{CH}^*(Y|_{\overline{k}}) \cdot h^i), \text{ and}$$

$$\Omega^*(Y \times Q|_{\overline{k}}) = \oplus_{i=0}^{[\dim(Q)/2]} (\Omega^*(Y|_{\overline{k}}) \cdot l_i \oplus \Omega^*(Y|_{\overline{k}}) \cdot h^i)$$

- see [17, Section 2]. In particular,

$$\overline{v} = \sum_{i=0}^{[\dim(Q)/2]} (\overline{v}^i \cdot h^i + \overline{v}_i \cdot l_i), \quad \text{and} \quad \overline{w} = \sum_{i=0}^{[\dim(Q)/2]} (\overline{w}^i \cdot h^i + \overline{w}_i \cdot l_i).$$

Proposition 3.4 *Let Q be smooth projective quadric of dimension $\geqslant 4n-1$, Y be smooth quasiprojective variety, and $v \in \mathrm{CH}^{2n+1}(Y \times Q)/2$ be some element. Then the class*

$$\overline{v}^0 + S^1(\overline{v}^1) + \overline{v}^1 \cdot \overline{v}_{\dim(Q)-2n} + \overline{v}^0 \cdot \overline{v}_{\dim(Q)-2n-1}$$

in $\mathrm{CH}^{2n+1}(Y|_{\overline{k}})/(2; 2-torsion)$ is defined over k.

Corollary 3.5 *Let Q be smooth projective quadric of dimension $\geqslant 4n-1$, Y be smooth quasiprojective variety, and $\overline{y} \in \mathrm{CH}^{2n+1}(Y|_{\overline{k}})/2$ is defined over $k(Q)$. Then, either*

(*a*) $z_{2n+1}^{\boxed{-[\dim(Q)/2]}}(Q|_{k(Y)})$ *is defined; or*

(*b*) *for certain $\overline{v}^1 \in \mathrm{CH}^{2n}(Y|_{\overline{k}})/2$, and for certain divisor $\overline{v}_{\dim(Q)-2n} \in \mathrm{CH}^1(Y|_{\overline{k}})/2$, the element*

$$\overline{y} + S^1(\overline{v}^1) + \overline{v}^1 \cdot \overline{v}_{\dim(Q)-2n}$$

in $\mathrm{CH}^{2n+1}(Y|_{\overline{k}})/(2; 2-torsion)$ is defined over k.

Proof: Since $\overline{y}$ is defined over $k(Q)$, there is $x \in \mathrm{CH}^{2n+1}(Y|_{k(Q)})/2$ such that $\overline{x} = \overline{y}|_{\overline{k(Q)}}$. Using the surjection $\mathrm{CH}^*(Y \times Q) \twoheadrightarrow \mathrm{CH}^*(Y|_{k(Q)})$ lift the x to an element $v \in \mathrm{CH}^{2n+1}(Y \times Q)/2$. Then $\overline{v} = \sum_{i=0}^{[\dim(Q)/2]}(\overline{v}^i \cdot h^i + \overline{v}_i \cdot l_i)$, where $\overline{v}^0|_{\overline{k(Q)}} = \overline{y}|_{\overline{k(Q)}}$. But for any extension of fields $F/\overline{k}$ (with smaller one algebraically closed), the restriction morphism on Chow groups (with any coefficients) is injective by the specialization arguments. Thus, $\overline{v}^0 = \overline{y}$. It remains to apply the Proposition 3.4, and observe that if $\overline{v}_{\dim(Q)-2n-1} \in \mathrm{CH}^0(Y|_{\overline{k}}) = \mathbb{Z}/2 \cdot 1$ is nonzero, then the cycle $l_{\dim(Q)-2n-1} = z_{2n+1}^{\boxed{-[\dim(Q)/2]}}$ is defined over $k(Y)$. Really, this cycle is just equal to $\overline{v}|_{\overline{k(Y)}}$. $\square$

Remark: One can get rid of factoring $(2 - torsion)$ in the statements above by using genuine *symmetric operations* (see [16], cf. [15]) instead of Landweber-Novikov operations. As was explained above, for our purposes it is irrelevant.

Before proving the Proposition let us study a bit some special power series. Denote as $\gamma(t) \in \mathbb{Z}/2[[t]]$ the power series $1 + \sum_{i\geqslant 0} t^{2^i}$. Then $\gamma(t)$ satisfies the equation:

$$\gamma^2 - \gamma = t,$$

and generates the quadratic extension of $\mathbb{Z}/2(t)$. In particular, for any $m \geqslant 0$, $\gamma^m = a_m\gamma + b_m$ for certain unique $a_m, b_m \in \mathbb{Z}/2(t)$. The following statement is clear.

Observation 3.6 (1) $a_{m+1} = a_m + b_m, \quad b_{m+1} = ta_m$

(2) *a_m and b_m are polynomials in t of degree $\leqslant [m-1/2]$ and $[m/2]$, respectively.*

For the power series $\beta(t)$ let us denote as $(\beta)_{\leqslant l}$ the polynomial $\sum_{j=0}^{l} \beta_j t^j$, and as $(\beta)_{>l}$ - the remaining part $\beta - (\beta)_{\leqslant l}$.

Lemma 3.7

$$a_m = (\gamma^m)_{\leqslant [m/2]} = (\gamma^m)_{\leqslant [m-1/2]}$$

Proof: Let $m = 2^k + m_1$, where $0 \leqslant m_1 < 2^k$. Then $\gamma^m = \gamma^{2^k} \cdot \gamma^{m_1} = (a_{m_1}\gamma + b_{m_1}) + O(t^{2^k}) = (1 + \sum_{i=0}^{k-1} t^{2^i})a_{m_1} + b_{m_1} + O(t^{2^k})$. Observation

3.6 implies that $(\gamma^m)_{\leqslant[m/2]} = (1+\sum_{i=0}^{k-1} t^{2^i})a_{m_1} + b_{m_1}$. On the other hand, $\gamma^{2^k} = \gamma + (\sum_{j=1}^{k-1} t^{2^j})$, thus γ^m is equal to

$$(\gamma + (\sum_{j=1}^{k-1} t^{2^j}))(a_{m_1}\gamma + b_{m_1}) =$$
$$a_{m_1}\gamma + a_{m_1}t + (\sum_{j=1}^{k-1} t^{2^j})a_{m_1}\gamma + b_{m_1}\gamma + (\sum_{j=1}^{k-1} t^{2^j})b_{m_1} =$$
$$((1+\sum_{i=0}^{k-1} t^{2^i})a_{m_1} + b_{m_1})\gamma + (ta_{m_1} + (\sum_{j=1}^{k-1} t^{2^j})b_{m_1}).$$

Hence, $a_m = ((1+\sum_{i=0}^{k-1} t^{2^i})a_{m_1} + b_{m_1}) = (\gamma^m)_{\leqslant[m/2]}$. The second equality follows from Observation 3.6(2) $\square$

Lemma 3.7 implies that

$$\gamma^m = (\gamma^m)_{\leqslant[m-1/2]} \cdot \gamma + t(\gamma^{m-1})_{\leqslant[m-2/2]}.$$

Lemma 3.8

$$(\gamma^m)_{>[m/2]} = t^m(1+mt) + O(t^{m+2})$$

Proof: Use induction on m and on the number of 1's in the binary presentation of m. For $m = 2^k$ the statement is clear. Let now $m = 2^k + m_1$, where $0 < m_1 < 2^k$. We have: $(\gamma^m)_{>[m/2]} = ((\gamma^{m+1})_{\leqslant[m/2]})_{>[m/2]} + t((\gamma^m)_{\leqslant[m-1/2]} \cdot \gamma^{-1})_{>([m/2]-1)} = t(a_m \cdot \gamma^{-1})_{>[m/2]-1}$.

$a_m = (\gamma^m)_{\leqslant[m/2]} = (\gamma^{2^k} \cdot \gamma^{m_1})_{\leqslant[m/2]} = (\gamma^{m_1})_{\leqslant[m/2]} = (a_{m_1}\gamma + b_{m_1})_{\leqslant[m/2]}$, and since degrees of a_{m_1} and b_{m_1} are no more than $[m_1/2]$, this expression should be equal to $\gamma^{m_1} + a_{m_1}t^{2^k} + O(t^{2^{k+1}})$. Then

$$a_m \cdot \gamma^{-1} = \gamma^{m_1-1} + a_{m_1}\gamma^{-1}t^{2^k} + O(t^{2^{k+1}}) =$$
$$(a_{m_1-1}\gamma + b_{m_1-1}) + a_{m_1}\gamma^{-1}t^{2^k} + O(t^{2^{k+1}}) =$$
$$(a_{m_1-1}(1+\sum_{i=0}^{k-1} t^{2^i}) + b_{m_1-1}) + t^{2^k}(a_{m_1-1} + a_{m_1}\gamma^{-1}) + O(t^{2^{k+1}}).$$

Since the degree of a_{m_1-1} is no more than $[m_1/2] - 1$, using Observation 1, we get:

$$(a_m \cdot \gamma^{-1})_{>[m/2]-1} = t^{2^k}(a_{m_1-1} + a_{m_1}\gamma^{-1}) + O(t^{2^{k+1}}) =$$
$$t^{2^k}(\gamma^{m_1-2} + a_{m_1-1}\gamma^{-1}) + O(t^{2^{k+1}}) =$$
$$t^{2^k}\gamma^{-1}(\gamma^{m_1-1} + (\gamma^{m_1-1})_{\leqslant[m_1-1/2]}) + O(t^{2^{k+1}}).$$

Consequently, $(\gamma^m)_{>[m/2]} = t^{2^k+1}(\gamma^{m_1-1})_{>[m_1-1/2]} \cdot \gamma^{-1} + O(t^{2^{k+1}})$. And, by the inductive assumption, this is equal to

$$t^{2^k+1}(t^{m_1-1}(1+(m_1-1)t))\gamma^{-1} + O(t^{2^k+m_1+2}) = t^m(1+mt) + O(t^{m+2}).$$

□

Corollary 3.9

$$(a_m \cdot \gamma^{-1})_{>[m/2]-1} = t^{m-1}(1+mt) + O(t^{m+1})$$

Observe now that $\gamma^{-1}(t) = \sum_{i\geqslant 0} t^{2^i-1}$. Denote as $\delta(t)$ the polynomial $a_{2n+1}(t)$. Then

$$\delta(t)\gamma^{-1}(t) = \alpha(t) + t^{2n} + t^{2n+1} + O(t^{2n+2}), \tag{1}$$

where $\delta(t)$ and $\alpha(t)$ are polynomials of degree $\leqslant n$. Observation 3.6(1) shows that $\delta = 1 + t + \ldots$.

Proof of Proposition 3.4: The case of $\dim(Q) \geqslant 4n-1$ can be reduced to that of $\dim(Q) = 4n-1$ by considering arbitrary subquadric $Q' \subset Q$ of dimension $4n-1$, and restricting v to $Y \times Q'$. So, we will assume that $\dim(Q) = 4n-1$.

Let $Q_s \xrightarrow{e_s} Q$ be arbitrary smooth subquadric of Q of dimension s. Denote as $w(s)$ the class $(id \times e_s)^*(w) \in \Omega^{2n+1}(Q_s \times Y)$. Then

$$\overline{w(s)} = \sum_{0\leqslant i\leqslant \min(2n-1,s)} \overline{w}^i \cdot h^i + \sum_{4n-s-1\leqslant j\leqslant 2n-1} \overline{w}_j \cdot l_{j-4n+s+1}.$$

Let $\pi_{Y,s} : Q_s \times Y \to Y$ be the natural projections.

Consider the element

$$u := (\pi_{Y,2n+1})_*\phi^{t^0}(w(2n+1)) + \sum_{p=n+1}^{2n+1} \delta_{2n+1-p}\phi^{t^{2p-(2n+1)}}(\pi_{Y,p})_*(w(p))$$

in $\mathrm{CH}^{2n+1}(Y)/2$, where δ_j are the coefficients of the power series δ above. Let us compute $\overline{u}$. Since we are computing modulo 2-torsion, it is sufficient to compute $2\overline{u}$, which is equal to the Chow-trace of

$$(\pi_{Y,2n+1})_*(S^{2n+1}_{L-N} - \square)(\overline{w}(2n+1)) + \sum_{p=n+1}^{2n+1} \delta_{2n+1-p} S^p_{L-N}(\pi_{Y,p})_*(\overline{w}(p)).$$

Using multiplicative properties of the Landweber-Novikov operations:

$$S^a_{L-N}(x \cdot y) = \sum_{b+c=a} S^b_{L-N}(x) S^c_{L-N}(y),$$

and Proposition 3.3, we get (*modulo* 4):

$$pr(\pi_{Y,2n+1})_* S^{2n+1}_{L-N}(\overline{w}(2n+1)) = \sum_{j=0}^{2n-1} \binom{j}{2n-j+1} \cdot 2 \cdot S^j(\overline{v}^j) +$$
$$pr(\binom{2n+1}{1} \cdot S^{2n}_{L-N}(\overline{w}_{2n-1}) + \binom{2n+2}{0} \cdot S^{2n+1}_{L-N}(\overline{w}_{2n-2})).$$

Codimension of $\overline{v}^j$ is $2n+1-j$, thus either $\binom{j}{2n-j+1}$ is zero, or $S^j(\overline{v}^j)$ is, and our expression is equal to $pr(S^{2n}_{L-N}(\overline{w}_{2n-1}) + S^{2n+1}_{L-N}(\overline{w}_{2n-2}))$. Also, (*modulo* 4),

$$pr(\pi_{Y,2n+1})_* \square(\overline{w}(2n+1)) = 2 \cdot pr(\overline{w}^0 \overline{w}_{2n-2} + \overline{w}^1 \overline{w}_{2n-1}).$$

In the same way, (*modulo* 4),

$$prS^p_{L-N}(\pi_{Y,p})_*(\overline{w}(p)) = \sum_{j=0}^{min(2n-1,p)} \binom{-(p+2-j)}{p-j} \cdot 2 \cdot S^j(\overline{v}^j) +$$
$$pr(\sum_{i=0}^{p-2n} \binom{-(i+1)}{i} S^{p-i}_{L-N}(\overline{w}_{i+4n-1-p})).$$

Observe, that the second sum is empty for $p < 2n$, is equal (*modulo* 4), to $prS^{2n}_{L-N}(\overline{w}_{2n-1})$ for $p = 2n$, and to $prS^{2n+1}_{L-N}(\overline{w}_{2n-2})$ for $p = 2n+1$ (we used the fact that $prS^m_{L-N}(x)$ is divisible by 2, if $m > \mathrm{codim}(x)$ - see Proposition 3.3).

Since the coefficient $\binom{-(l+2)}{l}$ is odd if and only if $l = 2^k - 1$, for some k, the first sum is equal to:

$$2 \sum_{j=0}^{min(2n-1,p)} (\gamma^{-1})_{p-j} \cdot S^j(\overline{v}^j).$$

Taking into account that $\delta(t) = 1 + t + \ldots$, we get:

$$pr \sum_{p=n+1}^{2n+1} \delta_{2n+1-p} S^p_{L-N}(\pi_{Y,p})_*(\overline{w}(p)) =$$

$$2 \sum_{p=n+1}^{2n+1} \sum_{j=0}^{min(2n-1,p)} \delta_{2n+1-p}(\gamma^{-1})_{p-j} \cdot S^j(\overline{v}^j) +$$

$$(prS^{2n}_{L-N}(\overline{w}_{2n-1}) + prS^{2n+1}_{L-N}(\overline{w}_{2n-2})) =$$

$$2 \sum_{j=0}^{2n-1} (\delta \cdot \gamma^{-1})_{2n+1-j} S^j(\overline{v}^j) + (prS^{2n}_{L-N}(\overline{w}_{2n-1}) + prS^{2n+1}_{L-N}(\overline{w}_{2n-2})) =$$

$$2(\overline{v}^0 + S^1(\overline{v}^1)) + (prS^{2n}_{L-N}(\overline{w}_{2n-1}) + prS^{2n+1}_{L-N}(\overline{w}_{2n-2})),$$

in the light of Corollary 3.9.

Putting things together (and again using the fact that $prS^m_{L-N}(x)$ is divisible by 2, if $m > \mathrm{codim}(x)$), we obtain:

$$2\overline{u} = 2(\overline{v}^0 + S^1(\overline{v}^1) + \overline{v}^1 \cdot \overline{v}_{2n-1} + \overline{v}^0 \cdot \overline{v}_{2n-2}).$$

Since u is defined over the base-field k, the Proposition is proven. $\square$

There is another result which extends a bit Theorem 3.1.

Proposition 3.10 ([15, Statement 3.8]) *Let Y be smooth quasiprojective variety, Q smooth projective quadric over k. Let $\overline{y} \in \mathrm{CH}^m(Y|_{\overline{k}})/2$. Suppose $z^{\boxed{0}}_{[\dim(Q)+1/2]}(Q)$ is defined. Then for $m \leqslant [\dim(Q)+1/2]$,*

$$\overline{y}|_{\overline{k(Q)}} \text{ is defined over } k(Q) \ \Leftrightarrow\ \overline{y} \text{ is defined over } k.$$

Proposition 3.10 extends Theorem 3.1 in the direction of the following:

Conjecture 3.11 ([15, Conjecture 3.11]) *In the notations of Theorem 3.1, suppose $z^{\boxed{\dim(Q)-l-d}}_{l}(Q)$ is defined. Then for any $m \leqslant l$,*

$$\overline{y}|_{\overline{k(Q)}} \text{ is defined over } k(Q) \ \Leftrightarrow\ \overline{y} \text{ is defined over } k.$$

This conjecture is known for $l = [\dim(Q)+1/2], \dim(Q)-1, \dim(Q)$.

3.3 Some auxilary facts

For our purposes it will be important to be able (under certain conditions) to get rid of the last term in the formula from Proposition 3.4. For this we will need the following facts.

Proposition 3.12 *Let R be smooth quadric, $d_R := [\dim(R)/2]$, $0 \leqslant i \leqslant d_R$, and $f : G(R,i) \stackrel{\alpha}{\leftarrow} F(R,0,i) \stackrel{\beta}{\rightarrow} R$ be the natural correspondence. Let $z_{\dim(R)-i}^{\boxed{i-d_R}}$ is defined. Let $t \in \mathrm{CH}_{\dim(R)-i}(G(R,i))/2$ be such that $f_*(t) = 1 \in \mathrm{CH}^0(R)/2$. Then $f_*(t \cdot z_{\dim(R)-i}^{\boxed{i-d_R}}) = l_i \in \mathrm{CH}_i(R)/2$.*

Proof: Really, by the definition, $z_{\dim(R)-i}^{\boxed{i-d_R}} = f^*(l_0) = \alpha_*\beta^*(l_0)$. By the projection formula,

$$f_*(t \cdot z_{\dim(R)-i}^{\boxed{i-d_R}}) = \beta_*\alpha^*(t \cdot z_{\dim(R)-i}^{\boxed{i-d_R}}) = \beta_*\alpha^*\alpha_*(\alpha^*(t) \cdot \beta^*(l_0)).$$

Again, by the projection formula, $\beta_*(\alpha^*(t) \cdot \beta^*(l_0)) = l_0$. Thus, $\alpha^*(t) \cdot \beta^*(l_0)$ is a zero-cycle of degree 1 on $F(R,0,i)$, and $\alpha_*(\alpha^*(t) \cdot \beta^*(l_0))$ is a zero cycle of degree 1 on $G(R,i)$. Proposition follows. □

Let $v \in \mathrm{CH}^m(Y \times Q)/2$ be some element. Then

$$\overline{v} = \sum_{i=0}^{d} (\overline{v}^i \cdot h^i + \overline{v}_i \cdot l_i).$$

Statement 3.13 *Suppose $z_m^{\boxed{\dim(Q)-m-d}}(Q)$ is defined. Then for any v as above, there exists $u \in \mathrm{CH}^m(Y \times Q)/2$ such that $\overline{u}^0 = \overline{v}^0$, and $\overline{u}_{\dim(Q)-m} = 0$.*

Proof: If $\overline{v}_{\dim(Q)-m} = 0$, there is nothing to prove. Otherwise, the class $l_{\dim(Q)-m} \in \mathrm{CH}_{\dim(Q)-m}(Q|_{k(Y)})/2$ is defined. Indeed, let

$$\rho_X : \mathrm{CH}^*(Y \times X)/2 \twoheadrightarrow \mathrm{CH}^*(X|_{k(Y)})/2$$

be the natural restriction. Then $\rho_Q(\overline{v}) = l_{\dim(Q)-m}$ plus $\lambda \cdot h^m$, if $2m = \dim(Q)$ (notice, that $\overline{v}_{\dim(Q)-m} \in \mathrm{CH}^0$). Anyway, this implies that over $k(Y)$ variety $G(Q, \dim(Q)-m)$ has a zero-cycle of degree 1, and thus, a rational point.

Let $x \in \mathrm{CH}_{\dim(Y)}(G(Q, \dim(Q) - m) \times Y)/2$ be arbitrary lifting of the class of a point on $G(Q, \dim(Q) - m)|_{k(Y)}$ with respect to $\rho_{G(Q,\dim(Q)-m)}$. Let

$$f : G(Q, \dim(Q) - m) \stackrel{\alpha}{\leftarrow} F(Q, 0, \dim(Q) - m) \stackrel{\beta}{\rightarrow} Q$$

be the natural correspondence. Consider $u' := (f \times id)_*(x) \in \mathrm{CH}^m(Q \times Y)/2$. Proposition 3.12 implies that the (defined over k) cycle

$$u" := \pi_Y^*(\pi_Y)_*((h^{\dim(Q)-m} \times 1_Y) \cdot (f \times id)_*(x \cdot z_m^{\boxed{\dim(Q)-m-d}}(Q)))$$

satisfy: $\overline{u"}^0 = \overline{u'}^0$, and (evidently) $\overline{u"}_{\dim(Q)-m} = 0$. Since $\overline{u'}_{\dim(Q)-m} = 1 = \overline{v}_{\dim(Q)-m}$, it remains to take: $u := v - u' + u"$. □

4 Even u-invariants

The fields of any given u-invariant were constructed by A.Merkurjev in [10] using his *index-reduction formula* for central simple algebras. Namely, A.Merkurjev showed that over the generic point of a quadric Q the index of a central simple algebra A can drop at most by the factor 2, and the latter happens if and only if $C_0(q)$ can be mapped to A. In particular, division algebra of index 2^t will stay divison over the field $k(Q)$, for any q of dimension $> 2(t+1)$. So, if over a base field k we have a form $p \in I^2$ of dimension $2t+2$ whose C_0 has index 2^t, then it will have the same index over $k(Q)$ for any form q of bigger dimension, and thus, $p|_{k(Q)}$ will stay anisotropic. It remains to construct Merkurjev tower of fields making all forms of dimension $> 2(t+1)$ isotropic.

Let me give another construction, which does not use *index reduction formula*. Instead, I will use the class $z_{\dim(P)-d}^{\boxed{0}}$ - the North-West corner.

Let p be form of dimension $2t+2$ such that $z_t^{\boxed{0}}(P)$ is not defined over k. One can use generic form - see [15, Statement 3.6]. Then, by Theorem 3.1, for any form q of dimension $> \dim(p)$, $z_t^{\boxed{0}}(P|_{k(Q)})$ is not defined as well. But then Proposition 2.4 implies that $z_{\dim(P)}^{\boxed{-d}}(P|_{k(Q)})$ is not defined. That is, $p|_{k(Q)}$ is anisotropic. It remains to construct Merkurjev tower of fields.

5 Odd u-invariants

Let us analize a bit the above construction. Instead of working with the cycle $z_{\dim(P)}^{\boxed{-[\dim(P)/2]}}$ - the class of a rational point on a quadric P, we worked with the (smaller codimensional!) cycle $z_{[\dim(P)+1/2]}^{\boxed{0}}$, and used the fact that rationality of the former implies rationality of the latter (Proposition 2.4).

Unfortunately, for odd-dimensional forms we can not use the class $z_{[\dim(P)+1/2]}^{\boxed{0}}$. Really, if p is any such form, then for $q := p \perp \langle det_{\pm}(p)\rangle$, $z_{[\dim(P)+1/2]}^{\boxed{0}}(P|_{k(Q)})$ will be defined, since rationality of this class is equivalent to the rationality of $z_{[\dim(Q)+1/2]}^{\boxed{0}}(Q|_{k(Q)})$ - observe that

$$G(Q, [\dim(Q)/2]) = G(P, [\dim(P)/2]) \coprod G(P, [\dim(P)/2]),$$

and the rationality of the latter follows from the rationality of the class $z_{\dim(Q)}^{\boxed{-[\dim(Q)/2]}}(Q|_{k(Q)})$ (isotropity of $Q|_{k(Q)}$). So, even if we start from the form, where our class is not defined, over generic point of some bigger-dimensional form it will become rational, and we can not control anisotropity of P.

But observe that the rationality of $z_{\dim(P)}^{\boxed{-[\dim(P)/2]}}$ implies rationality not just of $z_{[\dim(P)+1/2]}^{\boxed{0}}$, but of all West edge $z_{[\dim(P)+1/2]-s}^{\boxed{-s}}$, $0 \leqslant s \leqslant [\dim(P)/2]$. So, let us use these other cycles.

Let the form p has dimension $2^r + 1$. In this case, one can use previous to the last grassmannian, and the class $z_{2^{r-1}+1}^{\boxed{-1}}$ on it.

Theorem 5.1 *Let* $\dim(p) = 2^r + 1$*,* $r \geqslant 3$*, and* $EDI(P)$ *looks as*

$$\begin{array}{cccc} ? & \circ & \cdots & \circ \\ \circ & \circ & \cdots & \circ \\ \cdots & \cdots & \cdots & \cdots \\ \circ & \circ & \cdots & \circ \end{array}$$

Let $\dim(q) > \dim(p)$*. Then* $EDI(P|_{k(Q)})$ *has the same property.*

In particular, $p|_{k(Q)}$ *is anisotropic.*

Corollary 5.2 *For any* $r \geqslant 3$ *there is a field of* u*-invariant* $2^r + 1$*.*

Proof: Start with *generic form* p over $k = k_0(x_1, \ldots, x_{2^r+1})$. Then $EDI(P)$ is empty. This follows from Proposition 2.4 and [15, Statement 3.6]. Let F be the limit of the Merkurjev tower of fields which makes all forms of dimension $> \dim(p)$ isotropic. Then it follows from Theorem 5.1 that $p|_F$ is anisotropic. Thus, $u(F) = 2^r + 1$. $\square$

Proof of Theorem 5.1:

Let $d := [\dim(P)/2] = 2^{r-1} - 1$. It follows from Theorem 3.1 that the cycles $z_j^{\boxed{0}}(P|_{k(Q)})$, $1 \leqslant j \leqslant d$ are not defined. That is, we have $\circ$'s to the right of "?". In the light of Proposition 2.4 it remains only to treat the case of $z_{d+2}^{\boxed{-1}}(P|_{k(Q)})$ (that is, the node just below the "?").

Suppose this cycle is defined over $k(Q)$. We clearly can assume that $\dim(q) = \dim(p) + 1 = 2^r + 2$. Let us denote $G(P, d-1)$ temporarily as Y. We have $y \in \mathrm{CH}^{d+2}(Y|_{k(Q)})/2$ such that $\overline{y} = z_{d+2}^{\boxed{-1}} \in \mathrm{CH}^{d+2}(Y|_{\overline{k(Q)}})/2$. Let us lift it to $v \in \mathrm{CH}^{d+2}(Y \times Q)/2$ via the natural projection $\mathrm{CH}^{d+2}(Y \times Q)/2 \overset{\rho_Y}{\twoheadrightarrow} \mathrm{CH}^{d+2}(Y|_{k(Q)})/2$.

Statement 5.3 *There exists such $v \in \mathrm{CH}^{d+2}(Y \times Q)/2$ that $\overline{v}^0 = \overline{y}$, and $\overline{v}_d = 0 \in \mathrm{CH}_d(Q|_{\overline{k(Y)}})/2$.*

Proof: Let v be arbitrary lifting of y with respect to ρ_Y. If $\overline{v}_d = 0$, there is nothing to prove. Otherwise, let us show that $z_{2^{r-1}+1}^{\boxed{-1}}(Q)$ is defined over k.

Suppose $\overline{v}_d \neq 0$, then it is equal to $1 \in \mathrm{CH}^0(Y|_{\overline{k}})/2$. Then $\overline{\rho_Q(v)} = l_d \in \mathrm{CH}_d(Q|_{\overline{k(Y)}})/2 = \mathbb{Z}/2 \cdot l_d$. Thus, $z_{d+2}^{\boxed{-[\dim(Q)/2]}}(Q|_{k(Y)})$ is defined. By Proposition 2.4, $z_3^{\boxed{-2}}(Q|_{k(Y)})$ is defined too (it lives in the same column above). We want to show that $z_3^{\boxed{-2}}(Q)$ is defined.

Consider the two towers of fibrations:

$$\mathrm{Spec}(k) \leftarrow P \leftarrow \ldots \leftarrow F(P, 0, 1, \ldots, d-1);$$
$$\mathrm{Spec}(k) \leftarrow Q \leftarrow \ldots \leftarrow F(Q, 0, 1, \ldots, d-1),$$

with the generic fibers - quadrics $P = P_1, \ldots, P_d$, $Q = Q_1, \ldots, Q_d$ of dimension $2d+1, 2d-1, \ldots, 3$, and $2d+2, 2d, \ldots, 4$, respectively. Let us denote $k_{a,b} := k(F(P, 0, \ldots, a-1) \times F(Q, 0, \ldots, b-1))$. Then

$$k_{a+1,b} = k_{a,b}(P_a) \quad \text{and} \quad k_{a,b+1} = k_{a,b}(Q_b).$$

Since we have embeddings of fields

$$k \subset k(Y) = k(G(P, d-1)) \subset k(F(P, 0, \ldots, d-1)) = k_{d,0},$$

$z_3^{\boxed{-2}}(Q|_{k_{d,0}})$ is defined. Then by Proposition 2.4, $z_1^{\boxed{0}}(Q|_{k_{d,0}})$ is defined. By Theorem 3.1, $z_1^{\boxed{0}}(Q)$ and $z_2^{\boxed{0}}(Q) = (z_1^{\boxed{0}}(Q))^2 = S^1(z_1^{\boxed{0}}(Q))$ are defined.

It follows from Corollary 2.11 that for arbitrary elements $\overline{\alpha}, \overline{\beta} \in \mathrm{CH}^*(G(Q, d-1)|_{\overline{k_{a-1,0}}})/2$ of codimension 2 and 1, respectively, the class $S^1(\overline{\alpha}) + \overline{\alpha} \cdot \overline{\beta}$ is defined over any field, where Q is defined, in particular, over $k_{a-1,0}$. It follows from Corollary 3.5 that

$$z_3^{\boxed{-2}}(Q|_{k_{a,0}}) \text{ is defined } \Rightarrow \begin{cases} \text{either} & z_3^{\boxed{-2}}(Q|_{k_{a-1,0}}) \text{ is defined;} \\ \text{or} & z_3^{\boxed{-[\dim(P_a)/2]}}(P_a|_{k_{a-1,d}}) \text{ is defined,} \\ & \text{and } [\dim(P_a)/2] \leqslant 3. \end{cases}$$

Let us show that the second case is impossible. Really, by Proposition 2.4,

$$z_3^{\boxed{-[\dim(P_a)/2]}}(P_a|_{k_{a-1,d}}) \text{ is def. } \Rightarrow z_{3-[\dim(P_a)/2]}^{\boxed{0}}(P_a|_{k_{a-1,d}}) \text{ is def.}$$

Since $3 - [\dim(P_a)/2] \leqslant 2$, $\dim(Q_b) \geqslant 4$, and $z_2^{\boxed{0}}(Q_d)$ is defined, by Proposition 3.10 and Theorem 3.1,

$$z_{3-[\dim(P_a)/2]}^{\boxed{0}}(P_a|_{k_{a-1,b}}) \text{ is def. } \Rightarrow z_{3-[\dim(P_a)/2]}^{\boxed{0}}(P_a|_{k_{a-1,b-1}}) \text{ is def.}$$

Then $z_{3-[\dim(P_a)/2]}^{\boxed{0}}(P_a/k_{a-1,0})$ is defined, and $z_{3-[\dim(P_a)/2]}^{\boxed{0}}(P)$ is defined (by Theorem 3.1). This contradicts to the conditions of our Theorem (here we are using the fact that $r \geqslant 3$). Thus,

$$z_3^{\boxed{-2}}(Q|_{k_{a,0}}) \text{ is defined } \Rightarrow z_3^{\boxed{-2}}(Q|_{k_{a-1,0}}) \text{ is defined,}$$

and, consequently, $z_3^{\boxed{-2}}(Q)$ is defined. By Proposition 2.4, $z_3^{\boxed{-1}}(Q)$ is defined too. Proposition 2.7 implies that

$$z_{2^{r-1}+1}^{\boxed{-1}}(Q) = S^{2^{r-2}} S^{2^{r-3}} \ldots S^2(z_3^{\boxed{-1}}(Q))$$

is also defined over k. Since $z^{\boxed{-1}}_{2^{r-1}+1}(Q)$ is defined, everything follows from Statement 3.13. $\square$

Consider $v \in \mathrm{CH}^{d+2}(Y \times Q)/2$ satisfying the conditions of Statement 5.3. As above, $\overline{v} = \sum_{i=0}^{2^{r-1}} (\overline{v}^i \cdot h^i + \overline{v}_i \cdot l_i)$. Then, by Proposition 3.4, the class $\overline{v}^0 + S^1(\overline{v}^1) + \overline{v}^1\overline{v}_{2^{r-1}} + \overline{v}^0\overline{v}_{2^{r-1}-1}$ is defined over k. But $\overline{v}_{2^{r-1}-1} = 0$. Thus on Y we have the class $\overline{v}^0 + S^1(\overline{v}^1) + \overline{v}^1\overline{v}_{2^{r-1}}$ defined over k.

Now it is time to use the specific of Y and v. Our Y is a grassmannian $G(P, d-1)$. In particular, it is a geometrically cellular variety, and the map $\mathrm{CH}^*(Y|_{\overline{k}})/2 \to \mathrm{CH}^*(Y|_{\overline{k(Q)}})/2$ is an isomorphism. Thus, $\overline{v}^0 = z^{\boxed{-1}}_{d+2}$. On the other hand, $\overline{v}_{2^{r-1}} = \overline{v}_{d+1}$ belongs to $\mathrm{CH}^1(Y|_{\overline{k}})/2$, and so is equal either to 0, or to $w^{\boxed{-1}}_1$. So, on $G(P, d-1)$ we have class either of the form $z^{\boxed{-1}}_{d+2} + S^1(\overline{v}^1)$, or of the form $z^{\boxed{-1}}_{d+2} + S^1(\overline{v}^1) + \overline{v}^1 w^{\boxed{-1}}_1$ defined over k. The following Statement shows that such class should be nonzero.

Statement 5.4 *Let R be a smooth projective quadric of dimension $4n-1$, $d = [\dim(R)/2] = 2n-1$. Then $z^{\boxed{-1}}_{\dim(R)-d+1}(R)$ belongs to the image of neither of two maps: $S^1, (S^1 + w^{\boxed{-1}}_1 \cdot \ )$:*

$$\mathrm{CH}^{\dim(R)-d}(G(R, d-1))/2 \to \mathrm{CH}^{\dim(R)-d+1}(G(R, d-1))/2$$

Proof: We can assume that $k = \overline{k}$. Consider the natural projections:

$$G(R, d-1) \xleftarrow{\alpha} F(R, d-1, d) \xrightarrow{\beta} G(R, d).$$

The map β provides $F(R, d-1, d)$ with the structure of the projective bundle $\mathbb{P}_{G(R,d)}(Tav_d^\vee)$, and the Chern classes of Tav_d are divisible by 2 - see Proposition 2.1 (and [14]). Thus, $\mathrm{CH}^*(F(R, d-1, d))/2 = \mathrm{CH}^*(G(R, d))/2[h]/(h^{d+1})$, where $h = c_1(\mathcal{O}(1))$. By Lemma 2.5,

$$\alpha^*(z^{\boxed{-1}}_{\dim(R)-d+1}) = \beta^*(z^{\boxed{0}}_{\dim(R)-d}) \cdot h, \ \text{ and } \ h = \alpha^*(w^{\boxed{-1}}_1).$$

The first fact now is simple, since

$$S^1(\alpha^*(z^{\boxed{-1}}_{\dim(R)-d+1})) = S^1(\beta^*(z^{\boxed{0}}_{\dim(R)-d}) \cdot h) = \beta^*(z^{\boxed{0}}_{\dim(R)-d}) \cdot h^2,$$

by Proposition 2.7, and the latter element is nonzero. Thus, even $\alpha^*(z^{\boxed{-1}}_{\dim(R)-d+1})$ can not be in the image of S^1, since $S^1\circ S^1=0$.

To prove the second fact, observe that

$$\alpha^*(z^{\boxed{-1}}_{\dim(R)-d+1})=\beta^*(z^{\boxed{0}}_{\dim(R)-d})\cdot h=(S^1+h\cdot\ \ )(\beta^*(z^{\boxed{0}}_{\dim(R)-d})).$$

Let $u\in \mathrm{CH}^{\dim(R)-d}(G(R,d-1))/2$ be such that $(S^1+w^{\boxed{-1}}_1\cdot\ \ )(u)=z^{\boxed{-1}}_{\dim(R)-d+1}$. Then $(S^1+h\cdot\ \ )(\beta^*(z^{\boxed{0}}_{\dim(R)-d})-\alpha^*(u))=0$. Since d is odd, the differential $(S^1+h\cdot\ \ )$ acts without cohomology on $\mathrm{CH}^*(G(R,d))/2[h]/(h^{d+1})$. Consequently, $(\beta^*(z^{\boxed{0}}_{\dim(R)-d})-\alpha^*(u))=(S^1+h\cdot\ \ )(w)$, for some $w\in \mathrm{CH}^{\dim(R)-d-1}(F(R,d-1,d))/2$. This implies

$$\alpha_*\beta^*(z^{\boxed{0}}_{\dim(R)-d})=\alpha_*(S^1+h\cdot\ \ )(w),$$

since $\alpha_*\alpha^*=0$. Notice that $\alpha: F(R,d-1,d)\to G(R,d-1)$ is a conic bundle with relative tangent sheaf $\alpha^*(\mathcal{O}(w^{\boxed{-1}}_1))=\mathcal{O}(h)$. Thus, $\alpha_*(S^1+h\cdot\ \ )(w)=S^1(\alpha_*(w))$, and $\alpha^*\alpha_*(\beta^*(z^{\boxed{0}}_{\dim(R)-d}))=S^1(\alpha^*\alpha_*(w))$. But $\alpha^*\alpha_*(\beta^*(z^{\boxed{0}}_{\dim(R)-d}))=h^{\dim(R)-d-1}=h^d$, and this element is not in the image of S^1, as one can easily see. The contradiction shows that u as above does not exist. $\square$

It follows from the Statement 5.4 that in $\mathrm{CH}^{d+2}(G(P,d-1)|_{\overline{k}})/2$ we have nonzero class t defined over k. Then $\alpha^*(t)\in \mathrm{CH}^{d+2}(F(P,d-1,d))/2$ will be also nonzero class defined over k. But the subring of k-rational classes in $\mathrm{CH}^*(F(P,d-1,d)|_{\overline{k}})/2$ is $GDI(P,d)[h]/(h^{d+1})$, and by the main result of [14], $GDI(P,d)$ as a ring is generated by the elementary classes $z^{\boxed{0}}_j$ contained in it. By the conditions of our Theorem, among such classes only $z^{\boxed{0}}_{d+1}$ could be defined over k. Then the degree $=(d+2)$ component of the subring of k-rational classes in $\mathrm{CH}^*(F(P,d-1,d))/2$ is contained in $\mathbb{Z}/2\cdot(\beta^*(z^{\boxed{0}}_{d+1})\cdot h)=\mathbb{Z}/2\cdot\alpha^*(z^{\boxed{-1}}_{d+2})$. Thus, if t is nonzero, it got to be $z^{\boxed{-1}}_{d+2}$. But this class is not defined over k by the condition of the Theorem. And the contradiction shows that the class $z^{\boxed{-1}}_{d+2}$ is not defined over $k(Q)$ as well. Theorem 5.1 is proven. $\square$

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