Galois cohomology and forms of algebras over Laurent polynomial rings

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1 Introduction

The main thrust of this work is the study of two seemingly unrelated questions: Non-abelian Galois cohomology of Laurent polynomial rings on the one hand, while on the other, a class of infinite dimensional Lie algebras which, as rough approximations, can be thought off as higher nullity analogues of the affine Kac-Moody Lie algebras.

Though the algebras in question are in general infinite dimensional over the given base field (say the complex numbers), they can be thought as being finite *provided that the base field is now replaced by a ring* (in this case the centroid of the algebras, which turns out to be a Laurent polynomial ring). This leads us to the theory of reductive group schemes as developed by M. Demazure and A. Grothendieck [SGA3]. Once this point of view is taken, Algebraic Principal Homogeneous Spaces[1] and their accompanying non-abelian étale cohomology, arise naturally. It is this geometrical approach to infinite dimensional Lie theory that is one of the central themes of our work.

To illustrate these ideas, let us briefly look at the case of affine Kac-Moody Lie algebras over an algebraically closed field k of characteristic 0. Let $\widehat{\mathcal{L}}$ be such an algebra, and let $\mathcal{L}$ be the derived algebra of $\widehat{\mathcal{L}}$, modulo its centre. The Lie algebra $\widehat{\mathcal{L}}$ can be recovered from $\mathcal{L}$ (by taking the universal central extension and then attaching a derivation), and we will now concentrate on the Lie algebra $\mathcal{L}$ itself. Recall that the centroid of $\mathcal{L}$ is the subring $\mathrm{Ctd}_k(\mathcal{L}) \subset \mathrm{End}_k(\mathcal{L})$ comprised of elements that commute with the Lie bracket of $\mathcal{L}$. The k–Lie algebra $\mathcal{L}$ is infinite dimensional, but by viewing now $\mathcal{L}$ as an algebra over its centroid in the natural way, we find ourselves back on the finite world: There exists a finite dimensional simple Lie algebra $\mathfrak{g}$, and a finite Galois extension S of $\mathrm{Ctd}_k(\mathcal{L})$, such that $\mathcal{L} \otimes_{\mathrm{Ctd}_k(\mathcal{L})} S$ and $\mathfrak{g} \otimes_k S$ are isomorphic as S–Lie algebras. Since the centroid of an affine algebra can be identified with the Laurent polynomial ring $k[t^{\pm 1}]$, we see that $\mathcal{L}$ is a twisted forms, for the étale topology on $\mathrm{Spec}(k[t^{\pm 1}])$, of the $k[t^{\pm 1}]$–algebras $\mathfrak{g} \otimes_k k[t^{\pm 1}]$. Accordingly, we can attach to $\mathcal{L}$ a torsor $\mathbf{X}_{\mathcal{L}}$ whose isomorphism class lives in $H^1_{\acute{e}t}\big(k[t^{\pm 1}], \mathbf{Aut}(\mathfrak{g})\big)$.

[1]Also called Torsors. This terminology is due to Giraud.

More generally, we could take $\widehat{\mathcal{L}}$ to be an Extended Affine Lie Algebra (EALA for short), and $\mathcal{L}$ its centreless core ([AABFP], [N1] and [N2]). This is a beautiful class of infinite dimensional Lie algebras with striking connections to the classical finite dimensional landscape (simple Lie algebras of course, but also Jordan algebras, alternative algebras, quadratic forms...). The centroids are now Laurent polynomial rings $R_n = k[t_1^{\pm 1}, \cdots t_n^{\pm 1}]$ in finitely many variables. Just as in the affine Kac-Moody case, the resulting algebras $\mathcal{L}$ are twisted forms of algebras of the form $\mathfrak{g} \otimes_k R_n$.

These examples hint towards a possible deep connection between some aspects of contemporary infinite dimensional Lie theory, and the Galois cohomology of the ring $R_n = k[t_1^{\pm 1}, \cdots t_n^{\pm 1}]$. The present paper sets out to explore this possibility.

1.1 Notation and conventions

To help the reader we list below most of the notation and conventions used throughout the paper.

k denotes a field which, with the sole exception of §5, is *assumed to be of characteristic* 0. As usual, $\overline{k}$ will denote an algebraic closure of k. The tensor product $\otimes_k$ will be denoted by the unadorned symbol $\otimes$.

k*–alg* denotes the category of commutative associative unital k-algebras.

$R_n = k[t_1^{\pm 1}, \cdots t_n^{\pm 1}]$, $R_{n,d} = k[t_1^{\pm \frac{1}{d}}, t_2^{\pm \frac{1}{d}}, \cdots t_n^{\pm \frac{1}{d}}]$ and $R_{n,\infty} = \varinjlim_d R_{n,d}$.

$K_n = k(t_1^{\pm 1}, \cdots t_n^{\pm 1})$, $K_{n,d} = k(t_1^{\pm \frac{1}{d}}, t_2^{\pm \frac{1}{d}}, \cdots t_n^{\pm \frac{1}{d}})$ and $K_{n,\infty} = \varinjlim_d K_{n,d}$.

$F_n = k((t_1))((t_2))...((t_n))$.

$\pi_1(R_n)$ denotes the algebraic fundamental group of $\mathrm{Spec}(R_n)$ at the geometric point $\mathrm{Spec}(\overline{F_n})$ for some (fixed) algebraic closure $\overline{F_n}$ of F_n.

$(\zeta_n)_{n \geq 1}$ is a set of compatible primitive n-roots of unity, i.e. $\zeta_{\ell n}^{\ell} = \zeta_n$ (in the case when k is algebraically closed of characteristic 0).

For a given R in k*–alg*, by an R*–group* we will understand an affine group scheme over $\mathrm{Spec}(R)$. If $\mathbf{G}$ is an R–group, the pointed set of non-abelian Čech cohomology on the étale site of $X = \mathrm{Spec}(R)$ with coefficients in $\mathbf{G}$, is denoted by $H^1_{\acute{e}t}(X, \mathbf{G})$, or also by $H^1_{\acute{e}t}(R, \mathbf{G})$ (accordingly with customary usage depending on the context). At times, specially during proofs in order to cut down on notation, we write H^1 instead of $H^1_{\acute{e}t}$. When $\mathbf{G}$ is smooth, the set $H^1_{\acute{e}t}(R, \mathbf{G})$ measures the isomorphism classes of principal homogeneous spaces (torsors) over X under $\mathbf{G}$ (see Ch. IV §1 of [M] for basic definitions and references).

Given an R–group $\mathbf{G}$ and a morphisms $R \to S$ in k–alg, we let $\mathbf{G}_S$ denote the S-group $\mathbf{G} \times_{\mathrm{Spec}(R)} \mathrm{Spec}(S)$ obtained by base change. For convenience, we will under these circumstances denote most of the times $H^1_{\acute{e}t}(S, \mathbf{G}_S)$ by $H^1_{\acute{e}t}(S, \mathbf{G})$.

The expression *linear algebraic group (defined) over* k, is to be understood in the sense of Borel [Bo]. For a k-group $\mathbf{G}$, this is equivalent to requiring that $\mathbf{G}$ be smooth.[2] The connected component of the identity of such group $\mathbf{G}$, will be denoted by $\mathbf{G}^0$.

A reductive R–group is to be understood in the sense of [SGA3]. In particular, a reductive k–group is a reductive *connected* algebraic group defined over k in the sense of Borel.

1.2 Brief description of contents

Section 2. This section is devoted to the étale cohomology of Laurent polynomials ring $k[t_1^{\pm 1}, t_2^{\pm 1}]$ in two variables, and connections with Serre's Conjecture II for the corresponding function field $k(t_1, t_2)$ (the analogue of Conjecture I was dealt with in [P2]).

Section 3. Loop torsors, the main topic of this section, are the torsors that are of interest in infinite dimensional Lie theory. One of the main results of this section is the existence of an invariant, the Witt-Tits index, that can be attached to loop torsors. In the case of two variables, we defined another invariant with values in the Brauer group. One of the major results of the paper (Theorem 3.17) asserts that this Brauer invariant is fine enough to classify inner loop torsors. It is not however fine enough to distinguish torsors in general (§3.6), which leads to the failure of the analogue of Serre's Conjecture II for the ring $k[t_1^{\pm 1}, t_2^{\pm 1}]$.

Section 4. As explained in the Introduction, the study of Extended Affine Lie Algebras is intimately related to the study of $k[t_1^{\pm 1}, \cdots t_n^{\pm 1}]$–forms of finite dimensional simple Lie algebras. This section contains general results about forms of arbitrary finite dimensional algebras (mostly assumed to be central and perfect) over rings.

Section 5. This is detailed study of the nature of forms of algebras in the case when the base ring is a Laurent polynomial ring in finitely many variables. Particular emphasis is put on the case when the base algebra is a finite dimensional simple Lie algebra, and the ensuing connections with Extended Affine Lie Algebras.

Section 6. This section contains several conjectures related to the Galois cohomology of $k[t_1^{\pm 1}, t_2^{\pm 1}]$, and the classification of Extended Affine Lie Algebras in nullity 2.

[2] A smooth k–group is affine (by our convention on k–groups), and algebraic (since smooth schemes are by definition locally of finite type).

2 Torsors over $\mathbf{k[t_1^{\pm 1}, t_2^{\pm 1}]}$ and Serre's Conjecture II

Throughout this section k is assumed to be algebraically closed. Torsors over $k[t_1^{\pm 1}]$ were studied in [P2], and behave according to an analogue of Serre's Conjecture I (Steinberg's theorem). We now look at the case of $k[t_1^{\pm 1}, t_2^{\pm 1}]$. The situation here is much more delicate, and some of the results perhaps unexpected. We shall come back to this section when we look at Extended Affine Lie Algebras of nullity 2 in §6.

We set $R = k[t_1^{\pm 1}, t_2^{\pm 1}]$, $K = k(t_1, t_2)$, and $F = k((t_1))((t_2))$.

2.1 Cohomology of finite modules

We start by collecting some basic facts about the étale cohomology of R.

Proposition 2.1. *1.* $\mathrm{Gal}(F) \simeq \pi_1(R) \simeq \widehat{\mathbf{Z}} \times \widehat{\mathbf{Z}}$.

2. $H^1_{ét}(R, \boldsymbol{\mu}_n) \simeq H^1_{ét}(F, \boldsymbol{\mu}_n) \simeq (\mathbb{Z}/n\mathbb{Z})^2$.

3. $H^2_{ét}(R, \boldsymbol{\mu}_n) \simeq H^2_{ét}(F, \boldsymbol{\mu}_n) \simeq \mathbb{Z}/n\mathbb{Z}$.

4. $\mathrm{Br}(R) \simeq \mathbb{Q}/\mathbb{Z}$ *and* ${}_n\mathrm{Br}(R) \simeq \mathbb{Z}/n\mathbb{Z}$. *The canonical maps* $\mathrm{Br}(R) \to \mathrm{Br}(F)$ *and* ${}_n\mathrm{Br}(R) \to_n \mathrm{Br}(F)$ *are isomorphisms.*

5. Let S/R be a finite connected étale cover of degree d. The restriction map $\mathrm{Res}_{S/R} : H^2(R, \boldsymbol{\mu}_n) \to H^2(S, \boldsymbol{\mu}_n)$ is mutiplication by d, and the corestriction map $\mathrm{Cor}_{S/R} : H^2(S, \boldsymbol{\mu}_n) \to H^2(R, \boldsymbol{\mu}_n)$ is the identity.

6. Given $i \in \mathbb{Z}$, the class $[i] \in \mathbb{Z}/n\mathbb{Z} \cong_n \mathrm{Br}(R)$ is represented by the R–Azumaya algebra $A(i, n)$ with presentation

$$T_1^n = t_1,\; T_2^n = t_2^i,\; T_2 T_1 = \zeta_n\, T_1 T_2.$$

Proof. (1) Consider the Galois covering $R_{2,d} = R[t_1^{\pm 1/d}, t_2^{\pm 1/d}]$ of $R = R_2$, with Galois group $(\mathbf{Z}/d\mathbf{Z})^2$ generated by τ_i $(i = 1, 2)$ defined by

$$\tau_i(t_j^{1/d}) = (\zeta_d)^{\delta_{i,j}}\; t_j^{1/d}.$$

The crucial point is that every connected finite étale cover of R is dominated by one of these Galois extensions. Thus $\pi_1(R_2) = \varprojlim_d \mathrm{Gal}(R_{2,d}/R) = \varprojlim_d (\mathbb{Z}/d\mathbb{Z})^2 = (\widehat{\mathbb{Z}})^2$ (see [GP1, cor. 3.2] and [GP2] for details).

(2) , (3) and (4) : see [GP2, §3 and §4.2]. Because of (4), the calculations of (5) can be carried on F instead of R. One can now reason as in proposition 6.3.9 of [GS]. By (3) and (4) we have $\mathbb{Z}/n\mathbb{Z} \simeq H^2(R, \boldsymbol{\mu}_n) \simeq {}_n\mathrm{Br}(R) \simeq {}_n\mathrm{Br}(F)$. So for establishing (6), it suffices to show that the element $[i] \in \mathbb{Z}/n\mathbb{Z}$ is represented by the F–algebra $A(i, n) \otimes_R F$. But this is Proposition 5.7.1 of [GS]. □

Remark 2.2. Note that the isomorphisms of (1) and (2) above depend on the choice of a compatible set of primitive roots of unity $(\zeta_n)_{n\geq 1}$.

Remark 2.3. Let $\mathrm{Inv} : \mathrm{Br}(R) \to \mathbb{Q}/\mathbb{Z}$ be the group isomorphism constructed in Part (4) of the above Proposition. Let A be an R–Azumaya algebra of degree n, and write $\mathrm{Inv}(A) = \frac{p}{q}$ with p, q coprime. Then the R–algebras A and $A(p,q)$ are Brauer equivalent. We claim that q divides n. Since $\mathrm{Br}(R) \simeq \mathrm{Br}(F)$, the statement can be checked for $A \otimes_R F$. But this central simple F-algebra has period (exposant) q, and we do know that q divides the degree n of $A \otimes_R F$ (cf. [GS], § 4.5).

Remark 2.4. The class of $1 \in \mathbf{Z}/n\mathbf{Z} \simeq H^2(R, \mathbf{Z}/n\mathbf{Z}) \simeq {}_n\mathrm{Br}(R)$ is nothing but the cup–product $\chi_1 \cup \chi_2$, where $\chi_i : \widehat{\mathbf{Z}} \times \widehat{\mathbf{Z}} \to \mathbf{Z}/n\mathbf{Z}$ stand for the unique continuous character satisfying $\chi_i(n_1, n_2) = n_i \bmod (n)$, for all $n_1, n_2, \in \mathbb{Z}$ ([GS] Remark 6.3.8).

2.2 Serre's conjecture II

By theorem 2.1 of [CTGP], the cohomology and classification of semisimple groups over a two-dimensional geometric field with no factors of type E_8, is known. By adding some extra assumptions to the nature of the geometric field (assumptions which hold in the case of $k(t_1, t_2)$ that we are interested in), the groups with factors of type E_8 can also be covered. More precisely:

Theorem 2.5. *Let K be a two-dimensional function field over k. Let $\mathbf{G}$ be a semisimple, simply connected K–group, and $\mathbf{G} = \prod_{i=1}^{r} \mathcal{R}_{K_i/K}(\mathbf{G}_i)$ the decomposition of $\mathbf{G}$ in almost simple factors. Assume that $K_i = L_i(x_i)$, where L_i is a field of transcendence degree one over k for all $i = 1, ..., r$. Let*

$$1 \to \boldsymbol{\mu} \to \mathbf{G} \to \mathbf{G}^{\mathrm{ad}} \to 1$$

be the central isogeny associated to the centre $\boldsymbol{\mu}$ of $\mathbf{G}$. Then the boundary map

$$\delta : H^1(K, \mathbf{G}^{\mathrm{ad}}) \to H^2(K, \boldsymbol{\mu})$$

is bijective.

Proof. Since groups of type E_8 have trivial centre, we can consider separately groups with no E_8 factors, and groups of type E_8. As it has already been mentioned, for groups $\mathbf{G}$ with no E_8 factors, the Theorem at hand is a special case of theorem 2.1 of [CTGP]. Let $\mathbf{E}_8$ be the Chevalley k–group of type E_8. This group is simply connected and we have $\mathbf{E}_8 \cong \mathbf{Aut}(\mathbf{E}_8)$. Therefore by Shapiro's formula, it will suffice to show that $H^1(K_i, \mathbf{E}_8) = 1$. Let L be an extension of k of transcendence degree one. For any closed point x of the projective line $\mathbb{P}^1_L$, the completion of $L(t)$ at x is isomorphic to $L(x)((\pi_x))$. The field $L(x)$ is a finite extension of L, so $\mathrm{cd}(L(x)) \leq 1$. By Bruhat-Tits theory, we have $H^1(L(x)((\pi_x)), \mathbf{E}_8) = 1$ ([BT], théorème 4.7). In other words, all classes of $H^1(L(t), \mathbf{E}_8)$ are unramified with respect to the closed points of the projective line $\mathbf{P}^1_L$. By Harder's lemma ([H], Lemma 4.1.3), the map $H^1(\mathbb{P}^1_L, \mathbf{E}_8) \to H^1(L(t), \mathbf{E}_8)$ is then surjective. According to the theorem of Grothendieck-Harder (cf. [Gi1], th. 3.8.a), we have the following exact sequence of pointed sets

$$1 \to H^1_{Zar}(\mathbb{P}^1_L, \mathbf{E}_8) \to H^1(\mathbb{P}^1_L, \mathbf{E}_8) \xrightarrow{ev_\infty} H^1(L, \mathbf{E}_8),$$

where ev_∞ stands for the pull-back map defined by the point at infinity. But $H^1(L, \mathbf{E}_8) = 1$ by Steinberg's theorem. This implies that $H^1_{Zar}(\mathbb{P}^1_L, \mathbf{E}_8) \cong H^1(\mathbb{P}^1_L, \mathbf{E}_8)$. As Zariski torsors are rationally trivial, we conclude that $H^1(L(t), \mathbf{E}_8) = 1$ as desired. $\square$

Based on this last Theorem, an optimistic outcome for the $\mathbf{G}$–cohomology of R takes the following form.

Question 2.6. *Let $\mathbf{G}$ be a semisimple R–group. Let $1 \to \boldsymbol{\mu} \to \widetilde{\mathbf{G}} \to \mathbf{G} \to 1$ be its universal covering. Is the boundary map $H^1_{ét}(R, \mathbf{G}) \to H^2_{ét}(R, \boldsymbol{\mu})$ bijective ?*

In particular, if $\mathbf{G}$ is simply connected, the question is whether $H^1_{ét}(R, \mathbf{G})$ vanishes, namely a variant of Serre's conjecture II.

2.3 Some evidence

Theorem 2.7. *Let $\mathbf{G}$ be a semisimple R–group, and $1 \to \boldsymbol{\mu} \to \widetilde{\mathbf{G}} \to \mathbf{G} \to 1$ its universal covering.*

1. *The boundary map $H^1_{ét}(R, \mathbf{G}) \to H^2_{ét}(R, \boldsymbol{\mu})$ is surjective.*
2. *If $\mathbf{G}$ is split, then $H^1_{ét}(R, \widetilde{\mathbf{G}}) = 1$.*

Proof. (1) Let $\alpha \in H^2(R, \boldsymbol{\mu})$. We consider the restriction maps

$$H^2(R, \boldsymbol{\mu}) \to H^2(K, \boldsymbol{\mu}) \to H^2(F, \boldsymbol{\mu}).$$

By Theorem 2.5 $\delta_K : H^1(K, \mathbf{G}) \to H^2(K, \boldsymbol{\mu})$ is an isomorphism, so there exist a non-empty open subvariety $U \subset \mathrm{Spec}(R)$, and a $\gamma \in H^1(U, \mathbf{G})$, such that $\delta_U(\gamma) = \alpha_U \in H^2(U, \boldsymbol{\mu})$. For every point $x \in \mathrm{Spec}(R)^{(1)} \setminus U$, the completed field $\widehat{K}_x$ is a field of Laurent series over the residue field $k(x)$, which is the field of functions of a k–curve. So

$$H^1(\widehat{K}_x, \mathbf{G}) \cong H^2(\widehat{K}_x, \boldsymbol{\mu})$$

(Theorem 2.5). On the other hand, consider the residue map

$$\partial_x : H^2(\widehat{K}_x, \boldsymbol{\mu}) \to H^1(k(x), \boldsymbol{\mu}(-1))$$

where $\boldsymbol{\mu}(-1)$ stands for the Tate twist $\boldsymbol{\mu}(-1) = \mathrm{Hom}(\widehat{\mathbb{Z}}, \boldsymbol{\mu}) \simeq \mathrm{Hom}(\boldsymbol{\mu}_d, \boldsymbol{\mu})$ for some $d >> 0$ (see [GMS] §I.7.8). Since $\mathrm{cd}(k(x)) = 1$, ∂_M is an isomorphism. We summarize the previous facts in the commutative diagram

$$\begin{array}{ccccc} H^1(U, \mathbf{G}) & \xrightarrow{\delta} & H^2(U, \boldsymbol{\mu}) & & \\ \downarrow & & \downarrow & & \\ H^1(\widehat{K}_x, \mathbf{G}) & \xrightarrow[\simeq]{\delta} & H^2(\widehat{K}_x, \boldsymbol{\mu}) & \xrightarrow[\simeq]{\partial_x} & H^1(k(x), \boldsymbol{\mu}(-1)). \end{array}$$

We have $\partial_x(\alpha_U) = 0$, so $\delta(\widehat{\gamma}_x) = 0$. It follows that $\gamma_{\widehat{K}_x} = 1$. By Harder's lemma [H, lemma 4.1.3], the class γ extends on codimension 1 points of $\mathrm{Spec}(R) \setminus U$. The variety $\mathrm{Spec}(R)$ is a smooth affine surface, so γ extends to $\mathrm{Spec}(R)$ [CTS, théorème 6.13], i.e there exists some class $\widetilde{\gamma} \in H^1(R, \mathbf{G})$ such that $\widetilde{\gamma}_U = \gamma$. By construction, $\delta_F(\widetilde{\gamma}_F) = \alpha_F \in H^2(F, \boldsymbol{\mu})$. Since the restriction $H^2(R, \boldsymbol{\mu}) \to H^2(F, \boldsymbol{\mu})$ is an isomorphism (Proposition 2.1(4)), we conclude that $\delta_R(\gamma) = \alpha$.
(2) Assume that $\mathbf{G}$ is split and simply connected. In particular, $\mathbf{G}$ is defined over k. We see R as the localisation of $k[t_1, t_2]$ at $t_1 t_2 \neq 0$. Let $\gamma \in H^1(R, \mathbf{G})$. By Bruhat-Tits ([BT], cor 3.15), we have $H^1(k(t_1)((t_2)), \mathbf{G}) = 1$ and $H^1(k(t_2)((t_1)), \mathbf{G}) = 1$. Harder's lemma implies that γ extends at the generic points of the subvarieties $t_1 = 0$ and $t_2 = 0$ of $\mathbf{A}^2_k$. In other words, there exists an open subvariety $U \subset \mathbf{A}^2_k$ such that $\mathrm{codim}_X(U) = 2$ and γ extends to U. Again by [CTS, théorème 6.13], γ extends to a class $\widetilde{\gamma}_{\mathbf{A}^2_k}$. Since $H^1(\mathbf{A}^2_k, \mathbf{G}) = 1$ ([Rg], *cf.* [CTO], proposition 2.4), it follows that $\gamma = \widetilde{\gamma}_U = 1 \in H^1(R, \mathbf{G})$ as desired. □

Corollary 2.8. *Every semisimple R–group scheme $\mathbf{G}$ of type $\mathbf{E}_8$, $\mathbf{F}_4$ or $\mathbf{G}_2$ is split. Furhermore $H^1(R, \mathbf{G}) = 1$.*

Remark 2.9. Serre's conjecture on rationally trivial torsors proven by Colliot-Thélène and Ojanguren ([CTO], théorème 3.2) permits to give a shorter proof of the second statement of Theorem 2.7. Taking into account Theorem 2.5, all R–torsors under $\mathbf{G}$ are rationally trivial. Thus $H^1_{Zar}(R, \widetilde{\mathbf{G}}) = H^1(R, \widetilde{\mathbf{G}})$, and therefore $H^1(R, \widetilde{\mathbf{G}}) = 1$ by [GP2, corollary 2.3].

One more piece of evidence towards a positive answer of Question 2.6 comes from reduced norms (in analogy with the theorem of Merkurjev and Suslin which characterizes fields of cohomological dimension 2 by the surjectivity of their reduced norms. See [Se], II.3.2).

Lemma 2.10. *Let $A(i,n)$ be the standard Azumaya R–algebra defined in Proposition 2.1. Then the reduced norm map ([K2], I.7.3.1.5) $\mathrm{N} : A(i,n)^\times \to R^\times$ is surjective.*

Proof. The algebra $A = A(i,n)$ has presentation $T_1^n = t_1$, $T_2^n = t_2^i$, $T_2T_1 = \zeta_n T_1 T_2$. In particular, it contains $R[T_1]$ as a maximal commutative R–subalgebra. As in the field case, the restriction of the reduced norm to $R[T_1]^\times$ is nothing but the norm map $\mathrm{N}_1 : R[T_1]^\times \to R^\times$ of the cyclic Galois R–algebra $R[T_1]$. Since $\mathrm{N}_1(T_1) = (-1)^{n-1}t_1$, we have $(-1)^{n-1}t_1.R^{\times n} \subset \mathrm{N}(A^\times)$. So the group $R^\times/\mathrm{N}(A^\times)$ is cyclic generated by the class of t_2. Similarly, $R[T_2]$ is also a cyclic Galois R–algebra and $\mathrm{N}_2(T_2) = (-1)^{n-1}t_2$. We conclude that $A(i,n)^\times \to R^\times$ is surjective. □

3 Loop torsors

Throughout this section the base field k is assumed to be algebraically closed and of characteristic 0.

Let $\mathbf{G}$ be a reductive k–group. We fix a maximal torus $\mathbf{T}$ of $\mathbf{G}$, and a base Δ of the root system $\Phi = \Phi(\mathbf{G},\mathbf{T})$. Let $\mathfrak{t}$ denote the Lie algebra of $\mathbf{T}$. For any subset $I \subset \Delta \subset \Phi(\mathbf{G},\mathbf{T})$ we set

$$\mathfrak{t}_I = \bigcap_{a\in I} \ker(a) \subset \mathfrak{t},$$

and let $\mathbf{T}_I$ be the subtorus of $\mathbf{T}$ with Lie algebra $\mathfrak{t}_I$. Define $\mathbf{L}_I = \mathcal{Z}_{\mathbf{G}}(\mathbf{T}_I)$; this is a Levi subgroup of the standard parabolic group $\mathbf{P}_I$.

In what follows, we will often encounter the following situation: $\mathbf{H}$ is a subgroup of $\mathbf{G}$, and $\boldsymbol{x} = (x_1, x_2, ..., x_n)$ is an n–tuple of elements of $\mathbf{H}$. We summarize this by simply saying that "$\boldsymbol{x}$ lies in $\mathbf{H}$".[3]

Let $\boldsymbol{x} = (x_1, x_2, ..., x_n)$ be an n–tuple of commuting elements of *finite order* of $\mathbf{G}$. Let d be an integer such that $x_1^d = \cdots = x_n^d = 1$. Recall the Galois covering $R_{n,d} = R[t_1^{\pm 1/d}, \cdots, t_2^{\pm 1/d}]$ of R_n, with Galois group $(\mathbf{Z}/m\mathbf{Z})^n$ generated by τ_i $(i = 1, ..., n)$ defined by

$$\tau_i(t_j^{1/d}) = (\zeta_d)^{\delta_{i,j}}\ t_j^{1/d}.$$

This enables us to define the cocycle $\alpha(\boldsymbol{x}) \in Z^1(\mathrm{Gal}(R_{n,d}/R_n), \mathbf{G}(R_{n,d}))$ as follows[4]

$$\alpha(\boldsymbol{x}) : \mathrm{Gal}(R_{n,d}/R_n) \to \mathbf{G}(k) \to \mathbf{G}(R_{n,d}),\ \tau_1^{i_1} \cdots \tau_n^{i_n} \mapsto x_1^{-i_1} x_2^{-i_2} \cdots x_n^{-i_n}.$$

As it is customary, we denote the class of $\alpha(\boldsymbol{x})$ in $H^1(R_n, \mathbf{G})$ by $[\alpha(\boldsymbol{x})]$. Observe that this class is independent of the choice of the common period d.

[3]Strictly speaking, the x_i are elements of $\mathbf{H}(k)$. This abuse of notations, whenever harmless, will be used throughout.

[4]The somehow unnatural appearance of inverses in the definition of $\alpha(\boldsymbol{x})$, is consequence of the way $\alpha(\boldsymbol{x})$ arises from multiloop algebras. See §6 below for details.

Classes of the form $[\alpha(\boldsymbol{x})]$ as above are called *loop classes*. They form a subset $H^1_{loop}(R_n, \mathbf{G}) \subset H^1(R_n, \mathbf{G})$. The elements of these classes are called *loop torsors*.

There is another useful way of looking at loop torsors in terms of the algebraic fundamental group of R. Recall that $\pi_1(R_n) \simeq (\widehat{\mathbb{Z}})^n$ under our fixed choice of compatible roots of unity, and that $H^1_{ct}(\pi_1(R_n), \mathbf{G}(R_{n,\infty})) \xrightarrow{\sim} H^1(R_n, \mathbf{G})$ (see [GP1, cor. 3.2], or [GP2, cor. 2.15]) The cocycle $\alpha(\boldsymbol{x})$ corresponding to $\boldsymbol{x}$ can thus be thought as the unique (continuous) map $\alpha(\boldsymbol{x}) : (\widehat{\mathbb{Z}})^n \to \mathbf{G}(k)$ for which

$$\alpha(\boldsymbol{x}) : (i_1, ..., i_n) \mapsto x_1^{-i_1} x_2^{-i_2} \cdots x_n^{-i_n}$$

for all $(i_1, ..., i_n) \in \mathbb{Z}^n$.

3.1 The Witt-Tits index of a loop torsor

The following Theorem is the crucial ingredient which will allow us to attach an index set $I \subset \Delta$ to a loop torsor. This result is also related to the work of Reichstein-Youssin [RY] linking non-toral abelian sugroups and the essential dimension of $\mathbf{G}$. Our proof is a higher dimensional version in characteristic 0 of [Gi2] (e.g. proposition 3).

Theorem 3.1. *Let $\boldsymbol{x} = (x_1, x_2, ..., x_n)$ be an n–tuple of commuting elements of finite order of $\mathbf{G}$. For a subset $I \subset \Delta$, the following conditions are equivalent:*

1. $\boldsymbol{x}$ normalizes a parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ of type I,

2. $[\alpha(\boldsymbol{x})_{K_n}] \in \operatorname{im}\big(H^1(K_n, \mathbf{P}_I) \to H^1(K_n, \mathbf{G})\big)$,

2^{bis}. The twisted group ${}_{\alpha(\boldsymbol{x})}\mathbf{G}_{K_n}$ admits a K_n–parabolic subgroup of type I.

3 $[\alpha(\boldsymbol{x})_{F_n}] \in \operatorname{im}\big(H^1(F_n, \mathbf{P}_I) \to H^1(F_n, \mathbf{G})\big)$,

$3^{\text{ bis}}$. The twisted group ${}_{\alpha(\boldsymbol{x})}\mathbf{G}_{F_n}$ admits a F_n–parabolic subgroup of type I.

In particular, if $\mathbf{G}$ is semisimple, then $\boldsymbol{x}$ is irreducible (i.e it does not lie in any proper parabolic subgroup of $\mathbf{G}$), if and only if the corresponding twisted F_n–group ${}_{\alpha(\boldsymbol{x})}\mathbf{G}_{F_n}$ is anisotropic.

Proof. By a theorem of Chevalley ([Bo] theorem 11.6) $\mathbf{P}_I$ is its own normalizer in $\mathbf{G}$. The equivalence (2) $\Longleftrightarrow$ (2^{bis}) and (3) $\Longleftrightarrow$ (3^{bis}) then follows from a classical lemma ([Se], III.2.2, lemme 1).

(1) $\Longrightarrow$ (2): Up to conjugacy by a suitable element $g \in \mathbf{G}(k)$, we may assume that the x_i's lie in $\mathbf{P}_I(k)$. Because $\operatorname{Gal}(K_n)$ acts trivially on g, the $g^{-1}\alpha(\boldsymbol{x})g$ define a cocycle which is cohomologous to $\alpha(\boldsymbol{x})$. Now (2) is clear.

(2) $\Longrightarrow$ (3): Obvious.

(3) $\Longrightarrow$ (1): Consider the k–variety $\mathbf{Y} = \mathbf{G}/\mathbf{P}_I$ of parabolic subgroups of $\mathbf{G}$ of type I, as well as the corresponding twisted F_n–variety

$$\mathbf{X} = {}_{\alpha(\boldsymbol{x})}\mathbf{Y}_{F_n}.$$

Our hypothesis is that $\mathbf{X}(F_n) \neq \emptyset$. We have

$$\mathbf{X}(F_n) = \Big\{ y \in \mathbf{Y}(F_{n,m}) \mid \alpha(\boldsymbol{x})(\sigma).{}^{\sigma}y = y \ \text{ for all } \sigma \in \mathrm{Gal}(F_{n,m}/F_n) \Big\}.$$

Since $\mathbf{Y}$ is complete, we have

$$\mathbf{Y}(F_{n-1,m}[[\sqrt[m]{t_n}]]) = \mathbf{Y}(F_{n,m}).$$

Now, by specializing at $t_n = 0$, we obtain

$$\Big\{ y \in \mathbf{Y}(F_{n-1,m}) \mid \alpha(\boldsymbol{x})(\sigma).{}^{\sigma}y = y \ \text{ for all } \sigma \in \mathrm{Gal}(F_{n,m}/F_n) \Big\} \neq \emptyset,$$

where the Galois action of $\mathrm{Gal}(F_{n,m}/F_n)$ on $\mathbf{Y}(F_{n-1,m})$, is induced by the canonical projection $\mathrm{Gal}(F_{n,m}/F_n) \to \mathrm{Gal}(F_{n-1,m}/F_{n-1})$. Repeating the same process, we finally get

$$\Big\{ y \in \mathbf{Y}(k) \mid \alpha(\boldsymbol{x})(\sigma).{}^{\sigma}y = y \ \text{ for all } \sigma \in \mathrm{Gal}(F_{n,m}/F_n) \Big\} \neq \emptyset,$$

and therefore

$$\Big\{ y \in \mathbf{Y}(k) \mid \alpha(\boldsymbol{x})(\sigma).y = y \ \forall \sigma \in \mathrm{Gal}(F_{n,m}/F_n)) \Big\} \neq \emptyset$$

since $\mathrm{Gal}(F_{n,m}/F_n)$ acts trivially on $\mathbf{Y}(k)$. But this means that all of the x_i normalize a k-parabolic subgroup of type I, hence (1). $\square$

Example 3.2. For the split group E_8 and its standard non-toral abelian subgroup $(\mathbb{Z}/2\mathbb{Z})^9$, the corresponding loop torsor is studied by Chernousov-Serre [CS]. Our result gives another proof that the associated twisted group defined over $k(t_1, ..., t_9)$ is anisotropic.

Since assertions (2) and (3) of Theorem 3.1 are satisfied for a unique maximal index I, namely the Witt-Tits index [5] of ${}_{\alpha(\boldsymbol{x})}\mathbf{G}_{K_n}$, we get the following interesting fact.

[5] Let K/k be a field and $[z] \in H^1(K, \mathbf{G})$. The twisted group ${}_z\mathbf{G}$ admits a single $\mathbf{G}(K)$–conjugacy classe of minimal parabolic subgroups, and any such minimal $\mathbf{P}$ is $\mathbf{G}(\overline{K})$–conjugated to a unique minimal standard parabolic subgroup $\mathbf{P}_I \subset \mathbf{G}$. This I is called the Witt-Tits index of ${}_z\mathbf{G}$, and it depends only on $[z] \in H^1(K, \mathbf{G})$ ([BoT], § 6.5). In terms of Galois cohomology, $\mathbf{P}_I$ is the unique minimal standard parabolic subgroup of $\mathbf{G}$ such that $[z] \in \mathrm{Im}(H^1(K, \mathbf{P}_I) \to H^1(K, \mathbf{G}))$.

Corollary 3.3. *The minimal elements (with respect to inclusion) of the set of parabolic subgroups of* $\mathbf{G}$ *normalized by* $x_1, ..., x_n$ *are all conjugate under* $\mathbf{G}(k)$*. The type* $I(\boldsymbol{x})$ *of this conjugacy class is the Witt-Tits index of the* F_n*–group* ${}_{\alpha(\boldsymbol{x})}\mathbf{G}_{F_n}$*.*

We call $I(\boldsymbol{x})$ the *Witt-Tits index* of $\boldsymbol{x}$. As we shall see later, this invariant plays a crucial role in the classification of loop torsors.

Remark 3.4. One can also define an index in the linearly reductive case by taking into account the star action of $\boldsymbol{x}$ on Δ. More precisely, assume $\mathbf{F}$ is a linear algebraic group whose connected component $\mathbf{F}^0 := \mathbf{G}$ is reductive. We have an exact sequence of algebraic groups

$$1 \to \mathbf{G}^{ad} \to \mathbf{Aut}(\mathbf{G}) \to \mathbf{Out}(\mathbf{G}) \to 1$$

([SGA3] XXV théorème 1.3). Let $\boldsymbol{x}$ be an n–tuple of commuting elements of finite order of $\mathbf{F}$. The above exacts sequence yields an action of $< \boldsymbol{x} >$ on $\mathbf{Out}(\mathbf{G})$. This is the first part of the invariant attached to $\boldsymbol{x}$. The second part is defined as in the connected case, by replacing $\mathbf{P}_I$ by $\mathbf{N}_{\mathbf{F}}(\mathbf{P}_I)$ in Theorem 3.1. If $\mathbf{G}$ is semisimple and adjoint, then $\mathbf{Out}(\mathbf{G})$ can be identified with the group $\mathrm{Aut}(\Delta)$ of automorphisms of Δ. By the uniqueness of type for outer forms, the action of $\boldsymbol{x}$ on $\mathrm{Aut}(\Delta)$ leaves the index set I stable.

3.2 Reducibility

Recall that a subgroup $\mathbf{H}$ of a reductive k–group $\mathbf{G}$ is called $\mathbf{G}$–*irreducible* if $\mathbf{H}$ is not contained in any proper parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$.

Recall also that, by definition, a reductive k–group is connected.[6] There is a weaker notion, that of linearly reductive subgroup $\mathbf{H}$ of $\mathbf{G}$, that we now need (see [BMR] for details). A linear algebraic group $\mathbf{H}$ is *linearly reductive* if every rational representation of $\mathbf{H}$ is semisimple. Because our algebraically closed base field is of characteristic 0, for $\mathbf{H}$ to be linearly reductive it is necessary and sufficient that $\mathbf{H}^0$ be reductive.

Lemma 3.5. *Let* $\mathbf{G}$ *be a reductive* k*–group, and* $\mathbf{H}$ *a subgroup of* $\mathbf{G}$*. If* $\mathbf{S}$ *is a maximal torus of* $\mathcal{Z}_{\mathbf{G}}(\mathbf{H})$*, then* $\mathbf{S} = \mathcal{Z}(\mathcal{Z}_{\mathbf{G}}(\mathbf{S}))^0$ *and* $\mathcal{Z}_{\mathbf{G}}(\mathbf{S})$ *is a Levi subgroup of a parabolic subgroup of* $\mathbf{G}$*.*

Proof. Let $\mathbf{S}'$ be the identity component of the centre of $\mathcal{Z}_{\mathbf{G}}(\mathbf{S})$. Clearly $\mathbf{S} \subset \mathbf{S}'$. Since $\mathbf{H} \subset \mathcal{Z}_{\mathbf{G}}(\mathbf{S})$, we have $\mathbf{S}' \subset \mathcal{Z}_{\mathbf{G}}(\mathbf{H})$. Since $\mathbf{S}$ is a maximal torus of $\mathcal{Z}_{\mathbf{G}}(\mathbf{H})$, we have $\mathbf{S} = \mathbf{S}'$. So $\mathbf{S} = \mathcal{Z}(\mathcal{Z}_{\mathbf{G}}(\mathbf{S}))^0$. That $\mathcal{Z}_{\mathbf{G}}(\mathbf{S})$ is the Levi subgroup of a parabolic subgroup of $\mathbf{G}$ is well known ([Bo] proposition 20.4). □

[6]See the conventions in §1.1

Recall that if $\mathbf{H}$ is linearly reductive, then the centralizer $\mathcal{Z}_{\mathbf{G}}(\mathbf{H})$ of $\mathbf{H}$ in $\mathbf{G}$ is also linearly reductive ([R], proposition 10.1.5). Corollary 3.5 of [BMR] and its proof state the following.

Theorem 3.6. *Let $\mathbf{H}$ be a linearly reductive subgroup of a reductive k–group $\mathbf{G}$, and let $\mathbf{S}$ be a maximal torus of $\mathcal{Z}_{\mathbf{G}}(\mathbf{H})$. Then*

1. *$\mathbf{H}$ is irreducible in $\mathcal{Z}_{\mathbf{G}}(\mathbf{S})$.*
2. *If $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$ for which $\mathcal{Z}_{\mathbf{G}}(\mathbf{S})$ is a Levi subgroup, then $\mathbf{P}$ is a minimal element (with respect to inclusion) of the set of parabolic subgroups of $\mathbf{G}$ that contain $\mathbf{H}$.*
3. *If $\mathbf{P}$ is a minimal element (with respect to inclusion) of the set of parabolic subgroups of $\mathbf{G}$ that contain $\mathbf{H}$, then $\mathbf{H}$ is $\mathbf{L}$–irreducible for some Levi subgroup $\mathbf{L}$ of $\mathbf{P}$.*

Corollary 3.7. *Let $\mathbf{H}$ be a linearly reductive subgroup of a reductive k–group $\mathbf{G}$. For a parabolic parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$, the following conditions are equivalent:*

1. *$\mathbf{P}$ is a minimal element of the set of parabolic subgroups containing $\mathbf{H}$.*
2. *There exists a maximal torus $\mathbf{S}$ of $\mathcal{Z}_{\mathbf{G}}(\mathbf{H})$ such that $\mathcal{Z}_{\mathbf{G}}(\mathbf{S})$ is a Levi subgroup of $\mathbf{P}$.*

Proof. (2) $\Longrightarrow$ (1): This is given by the second assertion of Theorem 3.6.
(1) $\Longrightarrow$ (2): Let $\mathbf{L}$ be a Levi subgroup of $\mathbf{P}$ for which $\mathbf{H}$ is $\mathbf{L}$–irreducible (Theorem 3.6(3)). For $\mathbf{S} := \mathcal{Z}(\mathbf{L})^0$, we have $\mathbf{L} = \mathcal{Z}_{\mathbf{G}}(\mathbf{S})$ ([Bo] proposition 11.23 and corollary 14.19), and $\mathbf{S} \subset \mathcal{Z}_{\mathbf{G}}(\mathbf{H})$ (since $\mathbf{H} \subset \mathbf{L}$). We claim that $\mathbf{S}$ is a maximal torus of $\mathcal{Z}_{\mathbf{G}}(\mathbf{H})$. Let $\mathbf{S}' \subset \mathcal{Z}_{\mathbf{G}}(\mathbf{H})$ be a torus containing $\mathbf{S}$. Since $\mathbf{S}$ commutes with $\mathbf{S}'$, we have $\mathbf{S}' \subset \mathcal{Z}_{\mathbf{G}}(\mathbf{S}) = \mathbf{L}$. If $\mathbf{S} \subsetneq \mathbf{S}'$, then $\mathbf{S}'$ is not central in $\mathbf{L}$. Thus $\mathcal{Z}_{\mathbf{L}}(\mathbf{S}')$ is a Levi subgroup of a proper parabolic subgroup $\mathbf{Q}$ of $\mathbf{L}$ (*ibid.* proposition 20.6). But since $\mathbf{H}$ commutes with $\mathbf{S}'$, we have $\mathbf{H} \subset \mathcal{Z}_{\mathbf{L}}(\mathbf{S}') \subset \mathbf{Q}$. This contradicts our assumption on $\mathbf{L}$–irreducibility.

□

The minimality of parabolic subgroups containing $\mathbf{H}$ can be tested on elements of finite order of $\mathbf{H}$.

Corollary 3.8. *Let $\mathbf{H}$ be a linearly reductive subgroup of a reductive group $\mathbf{G}$. There exists elements $x_1, ..., x_n \in \mathbf{H}$ of* finite order *such that*

1. *$\mathcal{Z}_{\mathbf{G}}(\mathbf{H}) = \mathcal{Z}_{\mathbf{G}}(x_1, ..., x_n)$.*

2. The abstract group $< x_1, ..., x_n >$ is linearly reductive (i.e. the Zariski closure of $< x_1, ..., x_n >$ in $\mathbf{G}$ is linearly reductive).

In this case, the minimal elements of the set of parabolic subgroups containing $\mathbf{H}$ are precisely the minimal elements of the set of parabolic subgroups containing $\{x_1, ..., x_n\}$.

Proof. Let $\mathbf{H}_{ss}$ (respectively $\mathbf{H}_{fin}$), denote the subset of $\mathbf{H}(k)$ consisting of those elements which are semisimple (respectively of finite order).

We claim that $\mathcal{Z}_{\mathbf{G}}(\mathbf{H}) = \cap_{x \in \mathbf{H}_{ss}} \mathcal{Z}_{\mathbf{G}}(x)$. For since the finite constant group $\mathbf{H}/\mathbf{H}^0$ is diagonalizable, there exists a set of coset representatives of $\mathbf{H}^0$ in $\mathbf{H}$ comprised of semisimple elements. This, together with the fact that the semisimple elements of $\mathbf{H}^0$ are dense in $\mathbf{H}^0$ ([Bo] theorem 11.10), establishes the claim.

If $x \in \mathbf{H}_{ss}$, then $\overline{< x >} \subset \mathbf{H}$ is a closed subgroup of a torus of $\mathbf{G}$. In particular, the elements of finite order of $\overline{< x >}$, form a dense subset of $\overline{< x >}$. Together with our previous claim, this shows that $\mathcal{Z}_{\mathbf{G}}(\mathbf{H}) = \cap_{x \in \mathbf{H}_{fin}} \mathcal{Z}_{\mathbf{G}}(x)$. Given that each $\mathcal{Z}_{\mathbf{G}}(x)$ is closed and $k[\mathbf{G}]$ is noetherian, we conclude that $\mathcal{Z}_{\mathbf{G}}(\mathbf{H}) = \mathcal{Z}_{\mathbf{G}}(x_1) \cap .... \cap \mathcal{Z}_{\mathbf{G}}(x_m) := \mathcal{Z}_{\mathbf{G}}(x_1, ..., x_m)$ for some $x_1, ..., x_m \in \mathbf{H}_{fin}$.

Observe that $\mathcal{Z}_{\mathbf{G}}(x_1, ..., x_m) = \mathcal{Z}_{\mathbf{G}}(x_1, ..., x_m, x)$ for all $x \in \mathbf{H}_{fin}$. To finish the proof of (1) and (2) therefore, it will suffice to show that there exists $x_{m+1}, ..., x_n \in \mathbf{H}_{fin}$ such that the abstract group $< x_1, ..., x_n >$ is linearly reductive.

Let $\mathbf{H}_1 = \overline{< x_1, ..., x_m >}$. If $\mathbf{H}_1^0$ is reductive we are done. If not, the $\mathbf{H}_1^0 = \mathbf{U}_1.\mathbf{L}_1$ where $1 \neq \mathbf{U}_1$ (resp. $\mathbf{L}_1$) is the unipotent radical (resp. a Levi subgroup) of $\mathbf{H}_1^0$. Since $\mathbf{H}^0$ has trivial unipotent radical, there exists $x_{m+1} \in \mathbf{H}_{fin}^0$ such that $x_{m+1}\mathbf{U}_1 x_{m+1}^{-1} \neq \mathbf{U}_1$ (recall that $\mathbf{H}_{fin}^0$ is dense in $\mathbf{H}^0$). Let $\mathbf{H}_2 = \overline{< x_1, ..., x_m, x_{m+1} >}$. If $\mathbf{H}_1^0 = \mathbf{H}_2^0$, these two groups would have the same unipotent radical, namely $\mathbf{U}_1$. But they do not: x_{m+1} does not normalize $\mathbf{U}_1$. Thus $\mathbf{H}_1^0 \subsetneq \mathbf{H}_2^0$. One now considers the unipotent radical of $\mathbf{H}_2^0$ and repeats the above argument. Since the dimension of the resulting groups $\mathbf{H}_1^0 \subsetneq \mathbf{H}_2^0 \subsetneq ...$ are bounded by $\mathbf{H}^0$, there exists elements $x_{m+1}, ..., x_n \in \mathbf{H}_{fin}$ as desired.

Finally, let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$ which is minimal among those containing $\{x_1, ..., x_n\}$. Then $\mathbf{P}$ is minimal among those parabolic subgroups containing the group $\mathbf{K} = \overline{< x_1, ..., x_n >}$. Since $\mathbf{K}$ is linearly reductive, there exists a maximal torus $\mathbf{S}$ of $\mathcal{Z}_{\mathbf{G}}(\mathbf{K})$ such that $\mathbf{L} = \mathcal{Z}_{\mathbf{G}}(\mathbf{S})$ is a Levi subgroup of $\mathbf{P}$ (Corollary 3.7). By construction, $\mathcal{Z}_{\mathbf{G}}(\mathbf{K}) = \mathcal{Z}_{\mathbf{G}}(\mathbf{H})$. In particular, $\mathbf{S}$ commutes with $\mathbf{H}$, and therefore $\mathbf{H} \subset \mathbf{L}$. □

3.3 Almost commutative subgroups

Following [BFM], we say that a subgroup $\mathbf{H}$ of $\mathbf{G}$ is *almost commutative* if $(\mathbf{H}.\mathcal{Z}(\mathbf{G}))/\mathcal{Z}(\mathbf{G})$ is abelian. The last assertion of the following result generalises lemma 2.1.2 of *loc. cit.* about compact Lie groups (see also [KS]).

Proposition 3.9. *Let* $\mathbf{H}$ *be an almost commutative subgroup of* $\mathbf{G}$ *which is linearly reductive, and let* $\mathbf{S}$ *be a maximal torus of* $\mathcal{Z}_{\mathbf{G}}(\mathbf{H})$.

1. *The* k*–parabolic subgroups of* $\mathbf{G}$ *which are minimal among those containing* $\mathbf{H}$ *are all conjugate under* $\mathbf{G}(k)$*. Furthermore, the type* $I(\mathbf{H})$ *of this conjugacy class is the Witt–Tits index of some twisted inner form of* $\mathbf{G}$.

2. *If* $J \subset \Delta$ *is such that* $\mathbf{S}$ *is conjugated to* $\mathbf{T}_J$*, then* $J = I(\mathbf{H})$.

Proof. (1) After replacing $\mathbf{G}$ by $\mathbf{G}^{ad}$, we may assume that $\mathbf{H}$ is an abelian subgroup of $\mathbf{G}$. Corollary 3.8 provides an n–tuple $\boldsymbol{x} = (x_1, ..., x_n)$ of elements of finite order of $\mathbf{G}$, such that the parabolic subgroups which are minimal among those containing $\mathbf{H}$, are precisely those which are minimal among those containing $\{x_1, .., x_n\}$. We can now apply Corollary 3.3.1.
(2) Up to conjugacy, we may assume that $\mathbf{T}_J$ is a maximal torus of $\mathcal{Z}_{\mathbf{G}}(\mathbf{H})$. Corollary 3.7 implies that $\mathbf{P}_J$ is a minimal parabolic subgroup containing $\mathbf{H}$. By (1), we get that $J = I(\mathbf{H})$.

□

Lemma 3.10. *Let* $\lambda : \widetilde{\mathbf{G}} \to \mathbf{G}$ *be an isogeny of reductive* k*–groups. Let* $\mathbf{H}$ *be an almost commutative subgroup of* $\mathbf{G}$ *which is linearly reductive, and set* $\widetilde{\mathbf{H}} = \lambda^{-1}(\mathbf{H})$. *Then*

1. $\widetilde{\mathbf{H}}$ *is an almost commutative and linearly reductive subgroup of* $\widetilde{\mathbf{G}}$.

2. *The morphism* $\mathcal{Z}_{\widetilde{\mathbf{G}}}(\widetilde{\mathbf{H}})^0 \to \mathcal{Z}_{\mathbf{G}}(\mathbf{H})^0$ *is a central isogeny.*

3. *Via the natural identification of* Δ *with a base of the root system* $\Phi(\widetilde{\mathbf{G}}, \widetilde{\mathbf{T}})$*, we have* $I(\widetilde{\mathbf{H}}) = I(\mathbf{H})$.

Proof. We begin by recalling certain relevant facts about isogenies (see [Bo]§22 for details). Because λ is separable, it is central. In particular, for a subgroup $\mathbf{S}$ of $\mathbf{G}$ to be a maximal torus, it is necessary and sufficient that $\lambda^{-1}(\mathbf{S})$ be a maximal torus of $\widetilde{\mathbf{G}}$. Let $\widetilde{\mathbf{T}} = \lambda^{-1}(\mathbf{T})$. This is a maximal torus of $\widetilde{\mathbf{G}}$, and the standard map $\lambda^* : \Phi(\mathbf{G}, \mathbf{T}) \to \Phi(\widetilde{\mathbf{G}}, \widetilde{\mathbf{T}})$ is an isomorphism.

(1) The isogeny λ maps the unipotent radical of $\widetilde{\mathbf{H}}^0$ injectively into the unipotent radical of $\mathbf{H}^0$, which is trivial by assumption. This shows that $\widetilde{\mathbf{H}}^0$ is reductive, hence that $\mathbf{H}$ is linearly reductive. Given that λ is central, that $\widetilde{\mathbf{H}}$ is almost commutative follows from the almost commutativity of $\mathbf{H}$, together with $\mathcal{Z}(\widetilde{\mathbf{G}}) = \cap \widetilde{\mathbf{T}} = \cap \lambda^{-1}(\mathbf{T}) =$

$\lambda^{-1}(\cap \mathbf{T}) = \lambda^{-1}(\mathcal{Z}(\mathbf{G}))$ (the intersections being taken over all maximal torus of $\widetilde{\mathbf{G}}$ and $\mathbf{G}$ respectively).

(2) The induced map $\lambda : \mathcal{Z}_{\widetilde{\mathbf{G}}}(\widetilde{\mathbf{H}})^0 \to \mathcal{Z}_{\mathbf{G}}(\mathbf{H})^0$ has finite central kernel, so only surjectivity must be checked. It is enough to do this at the Lie algebra level. But here the situation is clear. Indeed

$$Lie(\mathcal{Z}_{\widetilde{\mathbf{G}}}(\widetilde{\mathbf{H}})) = Lie(\widetilde{\mathbf{G}})^{\mathrm{Ad}\,\widetilde{\mathbf{H}}},\ Lie(\mathcal{Z}_{\mathbf{G}}(\mathbf{H})) = Lie(\mathbf{G})^{\mathrm{Ad}\,\mathbf{H}},$$

and

$$\mathrm{Ad}\,\lambda(\widetilde{x})(d\lambda(\widetilde{y})) = d\lambda(\mathrm{Ad}\,\widetilde{x}(\widetilde{y}))$$

for all $\widetilde{x}$ and $\widetilde{y}$ in $\widetilde{\mathbf{G}}$, were $d\lambda := Lie(\lambda)$ is the differential of λ ([DG] II §4.1 and §5 5.7).

(3) By (2) and the opening remarks on isogenies, the maximal tori of $\mathcal{Z}_{\widetilde{\mathbf{G}}}(\widetilde{\mathbf{H}})^0$ are precisely of the form $\lambda^{-1}(\mathbf{S})$ for $\mathbf{S}$ a maximal torus of $\mathcal{Z}_{\mathbf{G}}(\mathbf{H})^0$. Now (3) follows from Proposition 3.9.

□

3.4 Almost commuting families of elements of finite order

Recall that an n–tuple $\boldsymbol{x} = (x_1, ..., x_n)$ of elements of $\mathbf{G}$ is said to *almost commutes* if $[x_i, x_j] \in \mathcal{Z}(\mathbf{G})$ for $i, j = 1, .., n$. This is equivalent to require that the group $< \boldsymbol{x} >$ generated by the x_i be almost commutative. Define the *rank* of such a family $\boldsymbol{x}$ to be the dimension of the centralizer of $\mathcal{Z}_{\mathbf{G}}(\boldsymbol{x}) = \mathcal{Z}_{\mathbf{G}}(x_1, ..., x_n)$ of $\boldsymbol{x}$ in $\mathbf{G}$. Notice that $\boldsymbol{x}$ is of rank zero if and only if $\mathcal{Z}_{\mathbf{G}}(\boldsymbol{x})$ is a finite group.

If in addition the x_i are all of finite order, there is a more subtle invariant of $\boldsymbol{x}$ that can be defined in terms of any maximal torus $\mathbf{S}$ of $\mathcal{Z}_{\mathbf{G}}(\boldsymbol{x})$.

Proposition 3.11. *Let $\boldsymbol{x} = (x_1, ..., x_n)$ be an almost commuting n–tuple of elements of finite order of a semisimple k–group $\mathbf{G}$.*

1. *The group $< \boldsymbol{x} >$ generated by the x_i is finite. In particular, this group is linearly reductive and all its elements are semisimple.*

2. *There is an unique subset $I = I(\boldsymbol{x}) \subset \Delta$ for which any maximal torus $\mathbf{S}$ of $\mathcal{Z}_{\mathbf{G}}(x)$ is conjugated to $\mathbf{T}_I$.*

3. *Let $\mathbf{S}$ be a maximal torus of $\mathcal{Z}_{\mathbf{G}}(x)$. Then $\boldsymbol{x}$ belongs to $Z_{\mathbf{G}}(\mathbf{S})$ and its image in $\mathcal{Z}_{\mathbf{G}}(\mathbf{S})/\mathbf{S}$ is a rank zero n–tuple of the semisimple k–group $\mathcal{Z}_{\mathbf{G}}(\mathbf{S})/\mathbf{S}$.*

4. *Let* $\widetilde{\mathbf{G}} \to \mathbf{G}$ *be the universal covering of* $\mathbf{G}$. *There exist liftings* $\widetilde{\boldsymbol{x}} = (\widetilde{x}_1, ..., \widetilde{x}_n)$ *of* $\boldsymbol{x}$ *in* $\widetilde{\mathbf{G}}$, *i.e. such that* $\lambda(\widetilde{x}_i) = x_i$ *for* $i = 1, ..., n$. *Moreover,* $< \widetilde{\boldsymbol{x}} >$ *is finite and* $I(\widetilde{\boldsymbol{x}}) = I(\boldsymbol{x})$.

5. *If* $\mathbf{S}$ *is a maximal torus of* $\mathcal{Z}_{\mathbf{G}}(\boldsymbol{x})$, *there exists a maximal torus* $\mathbf{T}$ *of* $\mathbf{G}$ *containing* $\mathbf{S}$ *such that* $< \boldsymbol{x} > \subset \mathbf{N}_{\mathbf{G}}(\mathbf{T})$. *Furthermore,* $\mathbf{S} = \big(\mathbf{T} \cap \mathcal{Z}_{\mathbf{G}}(< \boldsymbol{x} >)\big)^0$.

Proof. (1) If $\mathbf{G}$ is adjoint, then $\boldsymbol{x}$ is a commuting family of elements of finite order, and the result is clear. The general case follows from the fact the kernel of the canonical morphism $\mathbf{G} \to \mathbf{G}^{ad} = \mathbf{G}/\mathcal{Z}(\mathbf{G})$ is finite and central.

(2) This is nothing but Proposition 3.9 applied to the case $\mathbf{H} =< \boldsymbol{x} >$.[7]

(3) Since $\mathbf{S}$ is the connected centre of $\mathcal{Z}_{\mathbf{G}}(\mathbf{S})$, the group $\mathcal{Z}_{\mathbf{G}}(\mathbf{S})/\mathbf{S}$ is semisimple (possibly trivial). By construction, any maximal torus of $\mathcal{Z}_{\mathbf{G}}(\mathbf{S})/\mathbf{S}$ containing the images $x_i\mathbf{S}$ of the x_i's must be finite. By (2) the resulting n–tuple is of rank zero.

(4) Liftings do exist since the map $\widetilde{\mathbf{G}}(k) \to \mathbf{G}(k)$ is surjective (k is algebraically closed). Since the universal covering $\widetilde{\mathbf{G}} \to \mathbf{G}$ is an isogeny, the remaining assertions of (3) follow from (1) and (2) with the help of Lemma 3.10.

(5) We have $< \boldsymbol{x} > \subset \mathcal{Z}_{\mathbf{G}}(\mathbf{S})$. By the main theorem of [BM], there exists a maximal torus $\mathbf{T}$ of $\mathcal{Z}_{\mathbf{G}}(\mathbf{S})$ such that $< \boldsymbol{x} > \subset \mathbf{N}_{\mathbf{G}}(\mathbf{T})$. Since $\mathbf{S}$ is the connected centre of $\mathcal{Z}_{\mathbf{G}}(\mathbf{S})$, we conclude that $\mathbf{T}$ contains $\mathbf{S}$. Finally $(\mathbf{T} \cap \mathcal{Z}_{\mathbf{G}}(< \boldsymbol{x} >))^0$ is a torus of $\mathcal{Z}_{\mathbf{G}}(< \boldsymbol{x} >)^0$ containing $\mathbf{S}$. But $\mathbf{S}$ is a maximal torus of $\mathcal{Z}_{\mathbf{G}}(< \boldsymbol{x} >)^0$, hence $\mathbf{S} = \big(\mathbf{T} \cap \mathcal{Z}_{\mathbf{G}}(< \boldsymbol{x} >)^0\big)$.

□

3.5 Almost commuting pairs and their invariants

We now concentrate on the case $n = 2$. For convenience, as in §2, we set $R = k[t_1^{\pm 1}, t_2^{\pm 1}]$, $K = k(t_1, t_2)$, $F = k((t_1))((t_2))$.

Proposition 3.12. *If a semisimple* k–*group* $\mathbf{G}$ *has a rank zero pair of almost commuting elements, then* $\mathbf{G}$ *is of type* $A_{r_1} \times \cdots \times A_{r_l}$.

Proof. Let $\boldsymbol{x} = (x_1, x_2)$ be an almost commuting rank zero pair of elements of $\mathbf{G}$. The group $\mathcal{Z}_{\mathbf{G}}(\boldsymbol{x})$ is finite, both x_1, x_2 are of finite order. By Lemma 3.11, we may assume that $\mathbf{G}$ is simply connected.

First case: $k = \mathbb{C}$: Let $\mathbf{G}_{an}$ be the real anisotropic form of $\mathbf{G}$. The group $\mathbf{G}_{an}(\mathbb{R})$ is a maximal compact subgroup of $\mathbf{G}(\mathbb{C})$. The finite group $\mathcal{Z}_{\mathbf{G}}(\boldsymbol{x})$ of $\mathbf{G}(\mathbb{C})$ lies in a maximal compact subgroup of $\mathbf{G}(\mathbb{C})$. By Cartan's theorem, up to conjugacy by an

[7] According to [BFM], the uniqueness of the index set I "is clear". This uniqueness is not clear to us however, at least not without the arguments explained herein.

element of $\mathbf{G}(\mathbb{C})$, we may assume that $\mathcal{Z}_{\mathbf{G}}(\boldsymbol{x}) \subset \mathbf{G}_{an}(\mathbb{R})$. Then $\boldsymbol{x}$ is a rank zero pair of commuting elements of the compact Lie group $\mathbf{G}_{an}(\mathbb{R})$. Proposition 4.1.1 of [BFM] then states that $\mathbf{G}_{an}$ is a product of groups of type A.

Second case : k is a subfield of $\mathbb{C}$: Since the notion of rank zero pair is algebraic, it follows that the proposition holds also in this case.

General case: The field k contains the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ in k. The group $\mathbf{G}$ is defined over $\mathbb{Q}$. Since the elements of $\mathcal{Z}_{\mathbf{G}}(\boldsymbol{x})$ are of finite order, it follows that $\mathcal{Z}_{\mathbf{G}}(\boldsymbol{x}) \subset \mathbf{G}(\overline{\mathbb{Q}}) \subset \mathbf{G}(k)$. Therefore $\boldsymbol{x}$ is a rank zero pair of $\mathbf{G}(\overline{\mathbb{Q}})$ and the second case shows that $\mathbf{G}$ is product of groups of type A. □

For adjoint groups, the last result has the following useful consequence.

Lemma 3.13. *Assume that $\mathbf{G}$ is adjoint and simple. Let $\boldsymbol{x} = (x_1, x_2)$ be a commuting pair of elements of $\mathbf{G}$ of rank zero. Then $\mathbf{G} = \mathbf{PGL}_n$, and there exists $d \geq 1$ relatively prime to n for which $\boldsymbol{x}$ is $\mathbf{G}(k)$–conjugated to the pair $(\operatorname{Ad} Y_1, \operatorname{Ad} Y_2^d)$, where Y_1 and Y_2 in $\mathbf{GL}_n(k)$ are given by*

$$(3.1) \qquad Y_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & & 0 \\ & \cdots & & 1 & 0 \end{pmatrix}, \qquad Y_2 = \begin{pmatrix} 1 & 0 & \cdots & & \\ 0 & \zeta_n & 0 & \cdots & 0 \\ 0 & \cdots & & & 0 \\ 0 & \cdots & & 0 & \zeta_n^{n-1} \end{pmatrix}.$$

Proof. The group $\mathbf{G}$ is adjoint simple and, by the last Lemma, of type A. Thus $\mathbf{G} = \mathbf{PGL}_n$ for some $n \geq 2$. Let $\boldsymbol{x} = (x_1, x_2)$ be a rank zero pair of $\mathbf{PGL}_n(k)$. Let (X_1, X_2) be any lift of (x_1, x_2) to a pair of elements of $\mathbf{GL}_n(k)$. Then the commutator of these two elements satisfies $[X_2, X_1] = \operatorname{diag}(\zeta_n, ..., \zeta_n)^d$, for some $d \geq 0$.

By looking at the infinitesimal centralizer of x_1 and x_2 ([Bo], corollary to theorem 9.2), we see that the centraliser (under the adjoint action) of $\{x_1, x_2\}$ in $\mathfrak{sl}_n(k)$ is trivial. From this it follows that the centralizer of $\{X_1, X_2\}$ in $M_n(k) = k \operatorname{Id} \oplus \mathfrak{sl}_n(k)$ consists only of the homotheties $k \operatorname{Id}$. So X_1 and X_2 do not commute (for otherwise $x_1 = 1 = x_2$ and (x_1, x_2) is not a rank zero pair). This forces $d \notin n\mathbf{Z}$. Let m be the order of d in $\mathbf{Z}/n\mathbf{Z}$. We have $(\zeta_n^d)^m = 1$, so $[X_1^m, X_2] = 1$. Thus X_1^m commutes with X_1 and X_2, and therefore $X_1^m = z_1 \operatorname{Id} \in k^\times \operatorname{Id}$. Similarly, $X_2^m = z_2 \operatorname{Id} \in k^\times \operatorname{Id}$. Choose $a_i \in k$ such that $a_i^n = z_i^{-1}$, and set $Z_i = a_i X_i$. Then

$$Z_1^m = 1, \; Z_2^m = 1, \text{ and } Z_2 Z_1 = \omega Z_1 Z_2$$

where $\omega = \zeta_n^d$. Since ω is a primitive m–root of unity, the k–algebra $\{Z_1, Z_2\}$ is a non trivial quotient of the standard cyclic central simple k–algebra $(1,1)_\omega$ ([GS], §2.5) of degree m. Since $(1,1)_\omega$ is simple, $\{Z_1, Z_2\} \simeq (1,1)_\omega \simeq M_m(k)$. But the centralizer of

$\{Z_1, Z_2\}$ in $M_n(k)$ is k, so $m = n$ by the double centralizer theorem (cf. [Sc], corollary 8.4.8). So d is relatively prime to n. On the other hand, the elements Y_1, Y_2 of (3.1) above satisfy

$$Y_1^n = 1,\ (Y_2^d)^m = 1, \text{ and } Y_2^d\, Y_1 = \zeta_n^d\, Y_1\, Y_2^d. \tag{3.2}$$

We can now apply lemma 1.2 of [Ro] (with $\omega = \zeta_n^d$ as primitive n–root of unity), to conclude that (Z_1, Z_2) is $\mathbf{GL}_n(k)$–conjugated to (Y_1, Y_2^d). Since $\mathrm{Ad}\, Z_i = \mathrm{Ad}\, X_i$, the Lemma now follows by pushing into $\mathbf{PGL}_n(k)$. □

Remark 3.14. One checks that $\det(Y_i) = (-1)^{n+1}$. When n is even, replacing Y_i by $\widetilde{Y}_i := \zeta_{2n} Y_i$ produces elements of $\mathbf{SL}_n(k)$ with the desired properties.

These examples of pairs enables us to recover the following useful result:

Lemma 3.15. *Let $\mathbf{G}$ be a semisimple k–group. If $c \in \mathcal{Z}(\mathbf{G})$, then $c = [x_1, x_2]$ for some elements of finite order x_1 and x_2 of $\mathbf{G}$.*

Proof. It is well known that $\mathbf{G}$ admits a semisimple subgroup $\mathbf{H}$ of type $A_{r_1} \times ... \times A_{r_l}$ such that $\mathrm{rank}(\mathbf{H}) = \mathrm{rank}(\mathbf{G})$. Such a subgroup $\mathbf{H}$ contains a maximal torus of $\mathbf{G}$, so it contains $\mathcal{Z}(\mathbf{G})$. This reduces the problem to the case of a group $\mathbf{G}$ of type $A_{r_1} \times ... \times A_{r_l}$. Let $\widetilde{\mathbf{G}}$ be the simply connected covering of $\mathbf{G}$. Since the map $\mathcal{Z}(\widetilde{\mathbf{G}}) \to \mathcal{Z}(\mathbf{G})$ is surjective, it will suffice to establish the Lemma for $\widetilde{\mathbf{G}}$. Furthermore, since $\widetilde{\mathbf{G}}$ is the product of almost simple semisimple simply connected group, we may also assume that $\widetilde{\mathbf{G}}$ is almost simple, i.e $\widetilde{\mathbf{G}} = \mathbf{SL}_n$. But then, from the calculations in the proof of previous Proposition, and with the notation of the last Remark, we obtain $\mathcal{Z}(\mathbf{SL}_n) = \{[\widetilde{Y}_1^m, \widetilde{Y}_2] : 0 \le m < n\}$. □

Let $\boldsymbol{x} = (x_1, x_2)$ be a commuting pair of elements of *finite order* of $\mathbf{G}$, and $[\alpha(\boldsymbol{x})] \in H^1(R, \mathbf{G})$ the corresponding loop class (see the beginning of this section). Consider now the simply connected covering

$$1 \to \boldsymbol{\mu} \to \widetilde{\mathbf{G}} \xrightarrow{\lambda} \mathbf{G} \to 1$$

of $\mathbf{G}$. By Proposition 3.11(4), any lifting $\widetilde{\boldsymbol{x}} = (\widetilde{x}_1, \widetilde{x}_2)$ of $\boldsymbol{x}$ to $\widetilde{\mathbf{G}}$ is a pair of almost commuting elements of finite order. Since the exact sequence $1 \to \boldsymbol{\mu}(k) \to \widetilde{\mathbf{G}}(k) \to \mathbf{G}(k) \to 1$ is central, the commutator $[\widetilde{x}_1, \widetilde{x}_2] \in \widetilde{\mathbf{G}}$ does not depend on the lifting $\widetilde{\boldsymbol{x}}$. We denote this commutator by $\mu(\boldsymbol{x})$. As we shall see, $\mu(\boldsymbol{x})$ is an important invariant encoded in the cohomology class $\alpha(\boldsymbol{x})$.

Proposition 3.16. *The image of $[\alpha(\boldsymbol{x})]$ under the connecting map*

$$\delta : H^1_{ét}(R, \mathbf{G}) \to H^2_{ét}(R, \boldsymbol{\mu}) \cong \boldsymbol{\mu}$$

is given by the formula

$$\delta([\alpha(\boldsymbol{x})]) = \mu(\boldsymbol{x})^{-1}.$$

Proof. ¿From $\widetilde{x}_1\widetilde{x}_2 = \mu(\boldsymbol{x})\widetilde{x}_2\widetilde{x}_1$ it easily follows that

$$\widetilde{x}_1^{p_1}\,\widetilde{x}_2^{p_2} = \mu(\boldsymbol{x})^{p_1p_2}\,\widetilde{x}_2^{p_2}\,\widetilde{x}_1^{p_1}. \tag{3.3}$$

for all p_1 and p_2 in $\mathbb{Z}$. Let $\widetilde{\alpha} : \pi_1(R) \simeq \widehat{\mathbb{Z}} \times \widehat{\mathbb{Z}} \to \boldsymbol{\mu}$ be the unique (continuous) map satisfying $\widetilde{\alpha} : (i_1, i_2) \mapsto \widetilde{\alpha}_{(i_1,i_2)} = \widetilde{x}_1^{-i_1}\widetilde{x}_2^{-i_2}$ for all $(i_1, i_2) \in \mathbb{Z}\times\mathbb{Z}$. By the consideration explained in §5 above, we see that $\delta([\alpha(\boldsymbol{x})])$ corresponds, under the connecting map $\delta : H^1(R, \mathbf{G}) \to H^2(R, \boldsymbol{\mu})$, to the class of the 2–cocycle $c : \widehat{\mathbb{Z}} \times \widehat{\mathbb{Z}} \to \boldsymbol{\mu}$ given by

$$\begin{aligned}
c_{(i_1,i_2),(j_1,j_2)} :&= \widetilde{\alpha}_{(i_1,i_2)}{}^{(i_1,i_2)}\widetilde{\alpha}_{(j_1,j_2)}\widetilde{\alpha}^{-1}_{(i_1+j_1,i_2+j_2)} = \widetilde{\alpha}_{(i_1,i_2)}\widetilde{\alpha}_{(j_1,j_2)}\widetilde{\alpha}^{-1}_{(i_1+j_1,i_2+j_2)}\\
&= \widetilde{x}_1^{-i_1}\,\widetilde{x}_2^{-i_2}\,\widetilde{x}_1^{-j_1}\,\widetilde{x}_2^{-j_2}\Big(\widetilde{x}_1^{-i_1-j_1}\,\widetilde{x}_2^{-i_2-j_2}\Big)^{-1}\\
&= \widetilde{x}_1^{-i_1}\,\widetilde{x}_2^{-i_2}\,\widetilde{x}_1^{-j_1}\,\widetilde{x}_2^{i_2}\,\widetilde{x}_1^{i_1+j_1}\\
&= \boldsymbol{\mu}(\boldsymbol{x})^{-j_1i_2}.
\end{aligned}$$

The class of this cocycle in $H^2(\widehat{\mathbb{Z}} \times \widehat{\mathbb{Z}}, \boldsymbol{\mu})$ is precisely the cup product $-\theta_1 \cup \theta_2$ of the homomorphisms $\theta_1, \theta_2 : \widehat{\mathbb{Z}} \times \widehat{\mathbb{Z}} \to \boldsymbol{\mu}$ given by $(i_1, i_2) \mapsto \mu(\boldsymbol{x})^{i_1}$ and $(i_1, i_2) \mapsto \mu(\boldsymbol{x})^{i_2}$ respectively (see Remark 2.4). Thus $\delta(\alpha(\boldsymbol{x})) = \mu(\boldsymbol{x})^{-1} \in H^2(R, \boldsymbol{\mu}) \cong \boldsymbol{\mu}$ as stated. □

Theorem 3.17. *Let* $1 \to \boldsymbol{\mu} \to \widetilde{\mathbf{G}} \xrightarrow{\lambda} \mathbf{G} \to 1$ *be the simply connected covering of a semisimple* k*–group* $\mathbf{G}$*. Then the boundary map* $\delta : H^1(R, \mathbf{G}) \to H^2(R, \boldsymbol{\mu})$ *induces a bijection*

$$H^1_{loop}(R, \mathbf{G}) \xrightarrow{\sim} H^2_{ét}(R, \boldsymbol{\mu}).$$

In other words, to an R–loop torsor $\mathbf{X}$ under $\mathbf{G}$, we can attach a "Brauer invariant" in $H^2(R, \boldsymbol{\mu}) \subset \mathrm{Br}(R)$ which characterizes the isomorphism class of $\mathbf{X}$. The Brauer invariant can be easily computed (Proposition 3.16). We will come back to this in §6, when we classify inner multiloop algebras of nullity 2.

We begin with two preliminary results needed for the proof.

Lemma 3.18. *1. If* $\mathbf{G} = \mathbf{G}_1 \times \mathbf{G}_2$*, Theorem 3.17 holds for* $\mathbf{G}$ *iff it holds for* $\mathbf{G}_1$ *and* $\mathbf{G}_2$*;*

2. $\ker(H^1_{ét}(R, \mathbf{G}) \to H^2_{ét}(R, \boldsymbol{\mu})) = \ker(H^1_{ét}(R, \mathbf{G}) \to H^2(K, \boldsymbol{\mu}))$
$= \ker(H^1_{ét}(R, \mathbf{G}) \to H^1(K, \mathbf{G}))$*.*

3. Given $[z] \in H^1_{loop}(R, \mathbf{G})$*, assume that*

(a) $H^1_{loop}(R, \mathbf{G}^{ad}) \to H^2(R, \mathcal{Z}(\widetilde{\mathbf{G}}))$ *has trivial fiber at* $[z^{ad}]$*;*

(b) $\ker\Big(H^1(R, {}_z\widetilde{\mathbf{G}}_R) \to H^1(R, {}_z\mathbf{G}^{ad}_R)\Big) = 1$*.*

Then $H^1_{loop}(R,\mathbf{G}) \to H^2(R,\boldsymbol{\mu})$ *has trivial fiber at* $[z]$.

Proof. (1) This is obvious.
(2) By Proposition 2.1(1) (or by general properties of the Brauer group. See [M, Ch. IV cor 2.6], $H^2(R,\boldsymbol{\mu})$ injects in $H^2(K,\boldsymbol{\mu})$. On the other hand, $H^1(K,\mathbf{G}) \simeq H^2(K,\boldsymbol{\mu})$ by Theorem 2.5. Now (2) follows.
(3) Consider the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & & & 1 & & \\
 & & & & & & \downarrow & & \\
 & & 1 & & & & \mathcal{Z}(\mathbf{G}) & & \\
 & & \downarrow & & & & \downarrow & & \\
1 & \longrightarrow & \boldsymbol{\mu} & \longrightarrow & \widetilde{\mathbf{G}} & \xrightarrow{\lambda} & \mathbf{G} & \longrightarrow & 1 \\
 & & \downarrow & & \wr\downarrow & & p\downarrow & & \\
1 & \longrightarrow & \mathcal{Z}(\widetilde{\mathbf{G}}) & \longrightarrow & \widetilde{\mathbf{G}} & \xrightarrow{\widetilde{p}} & \mathbf{G}^{ad} & \longrightarrow & 1 \\
 & & \downarrow & & & & \downarrow & & \\
 & & \mathcal{Z}(\mathbf{G}) & & & & 1 & & \\
 & & \downarrow & & & & & & \\
 & & 1 & & & & & &
\end{array} \tag{3.4}$$

Passing to cohomology yields

$$\begin{array}{ccccc}
H^1_{loop}(R,\mathbf{G}) & \subset & H^1(R,\mathbf{G}) & \longrightarrow & H^2(R,\boldsymbol{\mu}) \\
p_*\downarrow & & p_*\downarrow & & \downarrow \\
H^1_{loop}(R,\mathbf{G}^{ad}) & \subset & H^1(R,\mathbf{G}^{ad}) & \longrightarrow & H^2(R,\mathcal{Z}(\widetilde{\mathbf{G}})).
\end{array}$$

By diagram chasing and assumption (a), it will suffice to establish the triviality of the fiber of p_* at $[z] \in H^1_{loop}(R,\mathbf{G})$. This fiber is controled by the diagram of torsion maps of the twisted R–group ${}_z\mathbf{G}_R := {}_z\mathbf{G}$ ([DG], III.4.3.4), namely

$$\begin{array}{ccccc}
 & & H^1(R,\mathbf{G}) & \xrightarrow{p_*} & H^1(R,\mathbf{G}^{ad}) \\
 & & \theta_z\downarrow\wr & & \theta_z\downarrow\wr \\
H^1(R,\mathcal{Z}(\mathbf{G})) & \longrightarrow & H^1(R,{}_z\mathbf{G}_R) & \xrightarrow{{}_zp_*} & H^1(R,{}_z\mathbf{G}^{ad}_R)
\end{array}$$

In other words, $p_*^{-1}([z]) \cong {}_zp_*^{-1}(1)$. Since $\mathcal{Z}(\widetilde{\mathbf{G}})$ and $\mathcal{Z}(\mathbf{G})$ are both constant and finite, the map $H^1(R,\mathcal{Z}(\widetilde{\mathbf{G}})) \to H^1(R,\mathcal{Z}(\mathbf{G}))$ is surjective. We then see that in the commutative diagram

$$\begin{array}{ccccc}
H^1(R, \mathcal{Z}(\widetilde{\mathbf{G}})) & \longrightarrow & H^1(R, {}_z\widetilde{\mathbf{G}}_R) & \xrightarrow{{}_z\widetilde{p}_*} & H^1(R, {}_z\mathbf{G}_R^{ad}) \\
\downarrow & & \downarrow & & \downarrow \wr \\
H^1(R, \mathcal{Z}(\mathbf{G})) & \longrightarrow & H^1(R, {}_z\mathbf{G}_R) & \xrightarrow{{}_z p_*} & H^1(R, {}_z\mathbf{G}_R^{ad}).
\end{array}$$

the map $\ker({}_z\widetilde{p}_*) \to \ker({}_z p_*)$ is surjective. Since by assumption (b) we have $\ker({}_z\widetilde{p}_*) = 1$, we conclude that ${}_z p_*^{-1}(1) = 1$, hence that $p_*^{-1}([z]) = [z]$ as desired. □

We also need the following fairly general cohomological result.

Lemma 3.19. *Let $I \subset \Delta$ be a subset, and A and object in k–alg. Then.*

1. *The canonical map $H^1(A, \mathbf{L}_I) \to H^1(A, \mathbf{P}_I)$ is bijective.*
2. *If $\mathrm{Pic}(A) = 0$, the canonical map $H^1(A, \mathbf{L}_I) \to H^1(A, \mathbf{L}_I/\mathbf{T}_I)$ is injective.*

Proof. (1) This is a special case of corollary 2.3 of exp. XXVI of [SGA3].
(2) The sequence of algebraic groups $1 \to \mathbf{T}_I \to \mathbf{L}_I \to \mathbf{L}_I/\mathbf{T}_I \to 1$ is central. According to § III.4.5.3 of [DG], the fibers of $H^1(A, \mathbf{L}_I) \to H^1(A, \mathbf{L}_I/\mathbf{T}_I)$ are suitable quotients of $H^1(A, \mathbf{T}_I)$. Since $\mathbf{T}_I$ is a split torus, the hypothesis $\mathrm{Pic}(A) = 0$ implies that $H^1(A, \mathbf{T}_I) = 0$. Thus the map $H^1(A, \mathbf{L}_I) \longrightarrow H^1(A, \mathbf{L}_I/\mathbf{T}_I)$ is injective. □

Proof of Theorem 3.17: We may asume without loss of generality that $\mathbf{G} \neq 1$.

To establish the injectivity of the map $H^1_{loop}(R, \mathbf{G}) \to H^2(R, \boldsymbol{\mu})$, we will reason according to the index I of the relevant pairs. Since by Theorem 2.5, the map $H^1(K, \mathbf{G}) \to H^2(K, \boldsymbol{\mu})$ is a bijection, it will suffice to show that the map $H^1_{loop}(R, \mathbf{G}) \to H^1(K, \mathbf{G})$ is injective. Suppose then that we are given two pairs $\boldsymbol{x} = (x_1, x_2)$ and $\boldsymbol{x}' = (x'_1, x'_2)$ of commuting elements of finite order of $\mathbf{G}$ for which $[\alpha(\boldsymbol{x})]_K = [\alpha(\boldsymbol{x}')]_K \in H^1(K, \mathbf{G})$. Since the twisted groups ${}_{\alpha(\boldsymbol{x})}\mathbf{G}_K$ and ${}_{\alpha(\boldsymbol{x}')}\mathbf{G}_K$ are then isomorphic, they have same Witt-Tits index I. Theorem 3.1 implies that $I(\boldsymbol{x}) = I(\boldsymbol{x}') = I$.

Step 1: Injectivity for $\mathbf{G}$ adjoint when $\boldsymbol{x}$, $\boldsymbol{x}'$ are of rank zero:

By Lemma 3.18 we reduce to the case of a simple adjoint group. Since $\mathbf{G}$ is not trivial, we have $\mathbf{G} = \mathbf{PGL}_n$ with $n \geq 2$. According to Lemma 3.13, we may assume (up to conjugacy) that $\boldsymbol{x} = (\mathrm{Ad}(Y_1), \mathrm{Ad}(Y_2)^d)$, $\boldsymbol{x}' = (\mathrm{Ad}(Y_1), \mathrm{Ad}(Y_2)^{d'})$, with d, d' prime to n. The corresponding twisted K–algebras $A(d, n) \otimes_R K$ and $A(d', n) \otimes_R K$ are isomorphic if and only if $d' = d$ modulo n. Thus $\mathrm{Ad}(Y_2)^d = \mathrm{Ad}(Y_2)^{d'}$, and therefore $\boldsymbol{x}$ and $\boldsymbol{x}'$ are conjugated under $\mathbf{G}(k)$. Hence $[\alpha(\boldsymbol{x})] = [\alpha(\boldsymbol{x}')] \in H^1_{loop}(R, \mathbf{G})$ as desired.

Step 2: Injectivity when $\boldsymbol{x}$, $\boldsymbol{x}'$ are of rank zero: Again the simple factors of $\mathbf{G}^{ad}$ are all of type A. Assumption (a) of Lemma 3.18.(4) holds by step 1. Let us check that assumption (b), namely that $H^1(R, {}_z\widetilde{\mathbf{G}}_R) \to H^1(R, {}_z\mathbf{G}^{ad}_R)$ has trivial kernel for any $[z] \in H^1_{loop}(R, \mathbf{G}^{ad})$ corresponding to a rank zero pair. To check this, we may assume that $\mathbf{G}^{ad}$ is simple, i.e. $\mathbf{G} = \mathbf{PGL}_n$ with $n \geq 2$. As in Step 1, the class $[z]$ corresponds to the R–Azumaya algebra $A(d,n)$ for some d prime to n. The corresponding twisted adjoint and simply connected groups are ${}_z\mathbf{G}^{ad}_R = \mathbf{PGL}_1(A(d,n))$ and ${}_z\widetilde{\mathbf{G}}_R = \mathbf{SL}_1(A(d,n))$ respectively (the latter being the R–group scheme of elements of reduced norm 1 of $A(d,n)$). The exact sequence of twisted R–groups

$$1 \to \boldsymbol{\mu}_n \to \mathbf{SL}_1(A(d,n)) \to \mathbf{PGL}_1(A(d,n)) \to 1$$

gives rise to the exact sequence of pointed sets

$$1 \to \boldsymbol{\mu}_n(R) \to \mathbf{SL}_1(A(d,n))(R) \to \mathbf{PGL}_1(A(d,n))(R) \to$$

$$\to R^\times/(R^\times)^{\times n} \to H^1(R, \mathbf{SL}_1(A(d,n))) \to H^1(R, \mathbf{PGL}_1(A(d,n))).$$

Since $\mathrm{Pic}(R) = 1$, we have $\mathbf{PGL}_1(A(d,n))(R) = \mathbf{GL}_1(A(d,n))(R)/R^\times$, and the boundary map $\mathbf{PGL}_1(A(d,n))(R) \to R^\times/R^{\times n}$ is nothing but the reduced norm modulo $R^{\times n}$. But since this reduced norm map is surjective (Lemma 2.10), the map $H^1(R, \mathbf{SL}_1(A(d,n))) \to H^1(R, \mathbf{PGL}_1(A(d,n)))$ is injective. This completes the proof that assumption (b) of Lemma 3.18 holds, hence also the proof of Step 2. Accordingly, the map $H^1_{loop}(R, \mathbf{G}) \to H^2(R, \boldsymbol{\mu})$ has trivial fiber at loop classes corresponding to rank zero pairs.

Step 3: Injectivity, the general case: By Proposition 3.11(2) we may assume, after conjugating $\boldsymbol{x}$ and $\boldsymbol{x}'$ by two elements of $\mathbf{G}(k)$, that both $\boldsymbol{x}$ and $\boldsymbol{x}'$ are pairs of $\mathbf{L}_I = \mathcal{Z}_{\mathbf{G}}(\mathbf{T}_I)$ inducing rank zero pairs $\overline{\boldsymbol{x}}$ and $\overline{\boldsymbol{x}}'$ of $\mathcal{Z}_{\mathbf{G}}(\mathbf{T}_I)/\mathbf{T}_I$. Consider the exact sequence of groups

$$1 \to \mathbf{T}_I \to \mathbf{L}_I \to \mathbf{L}_I/\mathbf{T}_I \to 1.$$

Recall that $H^1(K, \mathbf{P}_I)$ injects in $H^1(K, \mathbf{G})$ ([BoT], théorème 4.13.a). The map $H^1(K, \mathbf{L}_I) \to H^1(K, \mathbf{G})$ is therefore injective by Lemma 3.19(1). Using again this same Lemma, but now for R, we obtain the following commutative diagram of pointed sets.

$$\begin{array}{ccccc}
H^1_{loop}(R, \mathbf{G}) & \subset & H^1(R, \mathbf{G}) & \longrightarrow & H^1(K, \mathbf{G}) \\
\uparrow & & \uparrow & & \cup \\
H^1_{loop}(R, \mathbf{L}_I) & \subset & H^1(R, \mathbf{L}_I) & \longrightarrow & H^1(K, \mathbf{L}_I) \\
\cap & & \cap & & \cap \\
H^1_{loop}(R, \mathbf{L}_I/\mathbf{T}_I) & \subset & H^1(R, \mathbf{L}_I/\mathbf{T}_I) & \longrightarrow & H^1(K, \mathbf{L}_I/\mathbf{T}_I)
\end{array}$$

The elements $[\alpha(\boldsymbol{x})]$ and $[\alpha(\boldsymbol{x'})]$ actually belong to $H^1_{loop}(R, \mathbf{L}_I)$, and map to $[\alpha(\overline{\boldsymbol{x}})]$ and $[\alpha(\overline{\boldsymbol{x'}})]$ in $H^1_{loop}(R, \mathbf{L}_I/\mathbf{T}_I)$. By chasing the above diagram, we see that $[\alpha(\overline{\boldsymbol{x}})]_K = [\alpha(\overline{\boldsymbol{x'}})]_K$ in $H^1(K, \mathbf{L}_I/\mathbf{T}_I)$, and that the rank zero pairs $\overline{\boldsymbol{x}}$ and $\overline{\boldsymbol{x'}}$ of $\mathbf{L}_I/\mathbf{T}_I$ have the same image in $H^1(K, \mathbf{L}_I/\mathbf{T}_I)$. By Step 2 then $[\alpha(\overline{\boldsymbol{x}})] = [\alpha(\overline{\boldsymbol{x'}})]$ in $H^1_{loop}(R, \mathbf{L}_I/\mathbf{T}_I)$. A second diagram chase enables us to conclude that $[\alpha(\boldsymbol{x})] = [\alpha(\boldsymbol{x'})]$ in $H^1_{loop}(R, \mathbf{L}_I)$.

Step 4: surjectivity: Given $c \in \boldsymbol{\mu} \simeq H^2(R, \boldsymbol{\mu})$, Lemma 3.15 provides an almost commuting pair $\boldsymbol{x} = (x_1, x_2)$ of elements of finite order of $\widetilde{\mathbf{G}}(k)$ such that $\mu(\boldsymbol{x}) = c^{-1}$. By Proposition 3.16 we have $\delta\big(\alpha(\boldsymbol{x})\big) = \mu(\boldsymbol{x})^{-1} = c$. □

3.6 Failure in the anisotropic case

The following example is analogous to the construction by Ojanguren-Sridharan of reduced rank one projective quaternion modules over the real affine plane, which are not free [OS].

Proposition 3.20. *Let $n \geq 2$ be an integer, and let $A = A(1, n)$ the standard Azumaya algebra over $R = k[t_1^{\pm 1}, t_2^{\pm 1}]$ with generators T_1, T_2, and relations $T_1^n = t_1$, $T_2^n = t_2$ and $T_2 T_1 = \zeta_n T_1 T_2$. The equation*

$$(1 + T_1)\lambda = (1 + T_2)\mu$$

defines an invertible A–module $\mathcal{L}$ which is not free. The Azumaya algebra $\mathrm{End}_A(\mathcal{L})$ has same class than A in the Brauer group, but is not isomorphic to A.

The proof is based on a valuation argument on the division algebra A_K. We equip the additive group $\Gamma := \mathbb{R} \oplus \mathbb{R}$ with the lexicographical order, and define a valuation $v : R \setminus \{0\} \to \Gamma$ by

$$v\Big(\sum_{i,j} a_{i,j}\, t_1^i t_2^j\Big) = \mathrm{Min}_\Gamma \{\, (j, i) \;\; | \;\; a_{i,j} \neq 0\} \in \Gamma.$$

We denote by $P \mapsto \overline{P}$ the specialisation map $k[t_1, t_2] \to k$ at the point $(0, 0)$. As before, $\mathrm{N} : A \to R$ denotes the reduced norm of A ([K2], I.7.3.1.5).

Lemma 3.21. *Given a non zero element*

$$x = \sum a_{i,j}\, T_1^i T_2^j \in A, \;\; a_{i,j} \in k,$$

define $v_A(x) = \frac{1}{n} \mathrm{Min}_\Gamma \{\, (j, i) \;\; | \;\; a_{i,j} \neq 0\} \in \Gamma$.

1. *v_A is a valuation on A which extends v on R. Furthermore*

$$v_A(x) = \frac{1}{n}\, v(\mathrm{N}(x)).$$

2. If $v_A(x) \geq 0$*, then* $\overline{\mathrm{N}(x)} = {a_{0,0}}^n$.

For valuation theory on division algebras, we refer the reader to the nice survey by Wadsworth [W2].

Proof. (1) The completion of $K = k(t_1, t_2)$ with respect to v is the field $F = k((t_1))((t_2))$. Extend the function v_A to $A_F^\times$ by the formula

$$w\Big(\sum_{j=-p}^{\infty} \sum_{i=p_j}^{\infty} a_{i,j} T_1^i T_2^j\Big) = \mathrm{Inf}_\Gamma \Big\{ (j,i) \ \mid \ a_{i,j} \neq 0 \Big\} \in \Gamma.$$

Since A_F is a cyclic division algebra, example 2.7 of [W2] states that the map w is a valuation on A_F.[8] So v_A is a valuation. The formula $v_A = \frac{1}{n} v \circ \mathrm{N}$ was proven independently by Ershov [E1][E2] and Wadsworth [W1].
(2) This is a special case of a formula due to Ershov ([E1], Corollary 2). □

Proof of Proposition 3.20: Put $\zeta = \zeta_n$ and $\epsilon = \prod_{i=0}^{n-1} \zeta^i = (-1)^{n-1}$. The module $\mathcal{L}$ is the kernel of the map $f : A \oplus A \to A$, $(\lambda, \mu) \mapsto (1+T_1)\lambda - (1+T_2)\mu$. Since $f(1+T_2, 1+T_1) = T_1T_2 - T_2T_1 = (1-\zeta)T_1T_2$ is invertible, f is split surjective. Thus $\mathcal{L}$ is a projective module of reduced rank 1. Assume that $\mathcal{L}$ is free, i.e $\mathcal{L} = (\lambda_0, \mu_0)A$ with $(\lambda_0, \mu_0) \in A^2$. By taking reduced norms, we have

$$\mathrm{N}(\lambda_0)\mathrm{N}(1+T_1) \ = \ \mathrm{N}(\mu_0)\mathrm{N}(1+T_2). \tag{3.5}$$

For a non zero element $a \in A$, by Galois descent, there exists a unique element $\mathrm{N}'(a) \in A$ such that $\mathrm{N}'(a)a = a\mathrm{N}'(a) = \mathrm{N}(a) \in A$. Since

$$f\Big(\mathrm{N}'(1+T_1)(1+T_2), \mathrm{N}(1+T_1)\Big) = \mathrm{N}(1+T_1)(1+T_2) - (1+T_2)\mathrm{N}(1+T_1) = 0,$$

we have

$$\Big(\mathrm{N}'(1+T_1)(1+T_2), \mathrm{N}(1+T_1)\Big) = (\lambda_0, \mu_0)\alpha, \quad \alpha \in A. \tag{3.6}$$

By comparing norms on (3.6), we obtain

$$\mathrm{N}(\lambda_0) \mid \mathrm{N}(1+T_1)^{n-1}\mathrm{N}(1+T_2) \text{ and } \mathrm{N}(\mu_0) \mid \mathrm{N}(1+T_1)^n.$$

[8] Schilling's theorem states actually that w is the unique valuation on A_F extending v (*loc. cit*, corollary 2.2).

Similarly, since $f\Big(\mathrm{N}(1+T_2), \mathrm{N}'(1+T_1)(1+T_2)\Big) = 0$, we have $\mathrm{N}(\lambda_0) \mid \mathrm{N}(1+T_2)^n$ and $\mathrm{N}(\mu_0) \mid \mathrm{N}(1+T_1)\mathrm{N}(1+T_2)^{n-1}$. We also have

$$\mathrm{N}(1+T_1) = \prod_{i=0,\dots,n-1} (1+\zeta^i T_1) = 1 + \epsilon t_1,$$

and similarly $\mathrm{N}(1+T_2) = 1+\epsilon t_2$. The above yield the following three identities.

1. $\mathrm{N}(\lambda_0)\,(1+\epsilon t_1) = \mathrm{N}(\mu_0)\,(1+\epsilon t_2)$;
2. $\mathrm{N}(\lambda_0) \mid (1+\epsilon t_2)^n$ and $\mathrm{N}(\lambda_0) \mid (1+\epsilon t_1)^{n-1}\,(1+\epsilon t_2)$;
3. $\mathrm{N}(\mu_0) \mid (1+\epsilon t_1)^n$ and $\mathrm{N}(\mu_0) \mid (1+\epsilon t_1)\,(1+\epsilon t_2)^{n-1}$.

Thus $\mathrm{N}(\lambda_0) = (1+\epsilon t_2)u$ and $\mathrm{N}(\mu_0) = (1+\epsilon t_1)u$ for some $u \in R^\times$. Since the reduced norm $\mathrm{N} : A^\times \to R^\times$ is surjective (Lemma 2.10), we may henceforth assume with no loss of generality that $\mathrm{N}(\lambda_0) = 1+\epsilon t_2$ and $\mathrm{N}(\mu_0) = 1+\epsilon t_1$. We have $v(1+\epsilon t_1) = 0$, so Lemma 3.21 shows that $v_A(\mu_0) = 0$ and similarly we have $v_A(\lambda_0) = 0$. Hence

$$\lambda_0 = a_{0,0} + \sum_{(i,j)>(0,0)} a_{i,j}\, T_1^i T_2^j, \quad a_{i,j} \in k,$$

$$\mu_0 = b_{0,0} + \sum_{(i,j)>(0,0)} b_{i,j}\, T_1^i T_2^j, \quad b_{i,j} \in k.$$

with $a_{0,0}, b_{0,0} \neq 0$. Lemma 3.21(2) enables us to specialize at $(0,0)$ the equality $\mathrm{N}(\lambda_0) = 1+\epsilon t_2$. This yields ${a_{0,0}}^n = 1$ and ${b_{0,0}}^n = 1$. On the other hand, the equation $(1+T_1)\lambda_0 = (1+T_2)\mu_0$ now implies that $a_{0,0} = b_{0,0}$. Thus, after multiplying λ_0 and μ_0 by the n–root of unity $a_{0,0}^{-1}$, we may assume that $a_{0,0} = b_{0,0} = 1$ while still keeping the identities $\mathrm{N}(\lambda_0) = 1+\epsilon t_2$ and $\mathrm{N}(\mu_0) = 1+\epsilon t_1$.

We now look at the behavior at infinity by considering the valuation v_∞ on R which is the highest bidegree with respect to t_2, t_1. We extend v_∞ to a valuation $v_{\infty,A}$ on A as in Lemma 3.21. Since $v_\infty(\mathrm{N}(\lambda_0)) = v_\infty(1+\epsilon t_2) = (1,0)$, it follows that $v_{\infty,A}(\lambda_0) = \frac{1}{n}(1,0)$. Hence $\lambda_0 = 1 + \zeta_n^i T_2$ for some i. Similarly $\mu_0 = 1+\zeta_n^j T_1$. In all cases, we have $(1+T_1)\lambda_0 \neq (1+T_2)\mu_0$ which is a contradiction.

We conclude that $\mathcal{L}$ is not free. The Azumaya algebra $\mathrm{End}_A(\mathcal{L})$ has the same class than A in the Brauer group. Since $\mathrm{Pic}(R) = 0$, lemma 4.1.(3) of [GP2] enables to conclude that $\mathrm{End}_A(\mathcal{L})$ is not isomorphic to A. □

Corollary 3.22. *There exists an R–Azumaya algebra[9] $\mathcal{M}$ which is a non trivial Zariski form of $A(1,n)$. Moreover, $[\mathcal{M}]$ and $[A(1,n)] \in H^1(R, \mathbf{PGL}_n)$ have same connecting invariant $1 \in \mathbf{Z}/n\mathbf{Z} \cong H^2(R, \boldsymbol{\mu}_n)$.*

[9] The Margaux algebra.

Remark 3.23. By twisting by A the exact sequence $1 \to \boldsymbol{\mu}_n \to \mathbf{SL}_n \to \mathbf{PGL}_n \to 1$, the last assertion can be rephrased by saying that $H^1\big(R, \mathbf{SL}_1(A(1,n))\big) \neq 1$. This shows that, in contrast to the nullity 1 case of $k[t_1^{\pm 1}]$, the analogue of Serre conjecture II for $k[t_1^{\pm 1}, t_2^{\pm 1}]$ fails (see Question 2.6).

4 Twisted forms of algebras over rings

In this section the base field k is of *arbitrary characteristic*. Throughout A will denote a *finite dimensional* k–algebra (not necessarily unital or associative; for example a Lie algebra). All rings are assumed to be commutative and unital.

4.1 Multiplication algebras and centroids

Let R be a ring. For an arbitrary R–algebra $\mathcal{L}$, recall that the *multiplication algebra* $\mathrm{Mult}_R(\mathcal{L})$ of $\mathcal{L}$, is the unital subalgebra of $\mathrm{End}_R(\mathcal{L})$ generated by $\{1, l_x, r_x\}$, where l_x (resp. r_x) denotes the left (resp. right) multiplication operator by the element $x \in \mathcal{L}$. The abelian group $\mathcal{L}$ has a natural left $\mathrm{Mult}_R(\mathcal{L})$–module structure, and an algebra structure thereof if the ring $\mathrm{Mult}_R(\mathcal{L})$ is commutative.

The *centroid* $C_R(\mathcal{L})$ of $\mathcal{L}$, is the centralizer of $\mathrm{Mult}_R(\mathcal{L})$ in $\mathrm{End}_R(\mathcal{L})$. Thus $C_R(\mathcal{L})$ is the subalgebra of $\mathrm{End}_R(\mathcal{L})$ consists of all the endomorphisms of the R–module $\mathcal{L}$ that commute with right and left multiplication by elements of $\mathcal{L}$, i.e.

$$C_R(\mathcal{L}) = \{\chi \in \ \mathrm{End}_R(\mathcal{L}) : \ \chi(xy) = \chi(x)y = x\chi(y) \text{ for all } x, y \text{ in } \mathcal{L}\}.$$

For $r \in R$, define $\lambda_r \in \ \mathrm{End}_R(\mathcal{L})$ by $\lambda_r(x) = rx$. Then $\lambda_r \in C_R(\mathcal{L})$, and the map $\lambda_{\mathcal{L}} : r \to \lambda_r$ is a ring homomorphism from R into $C_R(\mathcal{L})$. Recall that $\mathcal{L}$ is called *central* if $\lambda_{\mathcal{L}}$ is an isomorphism. If $\mathcal{L}$ is a faithful R–module, the map $\lambda_{\mathcal{L}}$ is injective and we may, and at times will, identify R with a subring $\lambda_{\mathcal{L}}(R)$ of $C_R(\mathcal{L})$.

Let $\mathcal{L}' = \{\sum x_i y_i : x_i, y_i \in \mathcal{L}\}$ (finite sums of course). $\mathcal{L}'$ is a two-sided ideal of $\mathcal{L}$, and we recall that $\mathcal{L}$ is called *perfect* if $\mathcal{L} = \mathcal{L}'$.

Lemma 4.1. *Let $\mathcal{L}$ be an R–algebra. Let $R_0 \to R$ by any (unital) ring homomorphism, and denote by $\mathcal{L}_0$ the resulting R_0–algebra structure on $\mathcal{L}$. Then.*

1. *$\mathcal{L}$ is perfect as an R–algebra if and only if $\mathcal{L}_0$ is perfect as an R_0–algebra.*
2. *Assume $\mathcal{L}$ is a perfect R–algebra. Then $C_R(\mathcal{L})$ is commutative and the canonical map $C_R(\mathcal{L}) \to C_{R_0}(\mathcal{L}_0)$ is a ring isomorphism.*

Proof. (1) is clear. That $C_R(\mathcal{L})$ is commutative is proved in[J, Ch. X lemma 1]. The remaining point is that, because $\mathcal{L}$ is perfect, any endomorphism of the additive underlying group of $\mathcal{L}$ that commutes with right and left multiplication, is automatically R–linear (see [ABP2, lemma 4.1] for details). ☐

Remark 4.2. We will be interested in looking at $\mathcal{L}$ not only as an R–algebra, but also as a k–algebra. The loop algebra attached to an affine Kac-Moody case is a good example to have in mind. These are infinite dimensional algebras over $\mathbb{C}$, but much can be gained by looking at them as algebras over their centroids (which are Laurent polynomial rings). In view of this last result, if $\mathcal{L}$ is perfect, its centroid is independent of which of these two base rings one uses.

4.2 Forms of simple algebras over rings

Recall that A is a finite dimensional k–algebra. Throughout R denotes an object of k–alg.

Definition 4.3. An *R–form of A* is an algebra $\mathcal{L}$ over R for which there exists a faithfully flat and finitely presented extension S/R in k–alg such that

$$\mathcal{L} \otimes_R S \simeq_S A \otimes S \tag{4.1}$$

(isomorphism of S–algebras).

Remark 4.4. Since $A \otimes S \simeq (A \otimes R) \otimes_R S$, the R–algebra $\mathcal{L}$ is nothing but an R–form (trivialized by $\mathrm{Spec}(S)$ in the $f.p.p.f.$ topology of $\mathrm{Spec}(R)$) of the R–algebra $A \otimes R$. Since $\mathrm{Spec}(R)$ is affine, the isomorphism classes of such R–algebras are parametrized by $H^1_{fppf}(R, \mathbf{Aut}(A_R))$; the pointed set of Čech cohomology on the (small) $f.p.p.f.$ site of $\mathrm{Spec}(R)$ with coefficients on $\mathbf{Aut}(A_R)$. The R–group sheaf $\mathbf{Aut}(A_R)$ is in fact an affine R–group scheme (because A is finite dimensional). We have $\mathbf{Aut}(A_R) = \mathbf{Aut}(A)_R$. If $\mathbf{Aut}(A)$ is smooth (for example if $\mathrm{char}(k) = 0$), then S in (4.1) may be assumed to be an étale cover. (see [SGA3] and [M] for details).

Example 4.5. (1) If A is the matrix algebra of rank n, the R–forms of A are the Azumaya algebras over R of constant rank n.

(2) If $\mathfrak{g}$ is a finite dimensional simple Lie algebra over an algebraically closed field k of characteristic zero, then the $k[t^{\pm 1}]$–forms of $\mathfrak{g}$ are precisely the affine Kac-Moody Lie algebras (derived modulo their centres) over k. This is consequence of an analogue of Serre Conjecture I for the Dedekind ring $k[t^{\pm 1}]$. See [P2] for details.

Lemma 4.6. *Let $\mathcal{L}$ be an R–form of a finite dimensional perfect and central finite dimensional k–algebra A.*

1. *$\mathcal{L}$ is perfect. In particular, its centroid $C_R(\mathcal{L})$ is commutative and coincides with $C_k(\mathcal{L})$.*

2. *As an R–module, $\mathcal{L}$ is faithfully projective (in particular of finite type).*

3. *The canonical map $\lambda_{\mathcal{L}} : R \to C_R(\mathcal{L})$ is a ring isomorphism. In particular, $\mathcal{L}$ is central as an R–algebra.*

Proof. (1) We have

$$\begin{aligned}(\mathcal{L}/\mathcal{L}') \otimes_R S \simeq \mathcal{L} \otimes_R S/\mathcal{L}' \otimes_R S \simeq A \otimes S/(\mathcal{L} \otimes_R S)' \\ \simeq A \otimes S/(A \otimes S)' = 0\end{aligned}$$

(the first isomorphism because S/R is flat, the last equality because A, hence $A \otimes S$, is perfect). Since S/R is faithfully flat, $\mathcal{L}/\mathcal{L}' = 0$. The assertions about the centroid now follow from Lemma 4.1.

(2) After a faithfully flat base change $\mathcal{L}$ becomes free of finite rank. By descent properties then, $\mathcal{L}$ is a projective R–module of finite type. To see that this module is faithful, observe that if $r \in R$ annihilates $\mathcal{L}$, then the image s of r in S annihilates the S–module $\mathcal{L} \otimes_R S \simeq A \otimes S$. Thus $s = 0$ (the k–algebra A, being central, cannot be zero dimensional). Since S/R is faithfully flat, $r = 0$ as desired.

(3) Since $\mathcal{L}$ is faithful, the canonical map $\lambda_{\mathcal{L}} : R \to C_R(\mathcal{L})$ is injective, and we may thereof identify R with a subring $\lambda_{\mathcal{L}}(R)$ of the (commutative) ring $C_R(\mathcal{L})$. Since $\mathcal{L}$ is projective of finite type, the canonical map $\phi : \mathrm{End}_R(\mathcal{L}) \otimes_R S \to \mathrm{End}_S(\mathcal{L} \otimes_R S) \simeq \mathrm{End}_S(A \otimes S)$ is an S–algebra isomorphism. Clearly $\phi\big(C_R(\mathcal{L}) \otimes_R S\big) \subset C_S(A \otimes S) \simeq S$ (the latter by [ABP2.5] lemma 2.3(a)). It follows that $\phi(\lambda_{\mathcal{L}}(R) \otimes_R S) = \phi\big(C_R(\mathcal{L}) \otimes_R S\big)$. Thus the inclusion $\lambda_{\mathcal{L}}(R) \otimes_R S \subset C_R(\mathcal{L}) \otimes_R S$ is an equality (being an equality after applying ϕ). By faithful flatness, $\lambda_{\mathcal{L}}(R) = C_R(\mathcal{L})$ as desired. □

Remark 4.7. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be R–forms of a finite dimensional perfect and central k–algebra A. If $\phi : \mathcal{L}_1 \to \mathcal{L}_2$ is an isomorphism of k –algebras, then $C(\phi) : C_k(\mathcal{L}_1) \to C_k(\mathcal{L}_2)$ given by $\chi \mapsto \phi\chi\phi^{-1}$ is an isomorphism in *k–alg*. By part (3) of the last Lemma, there exists a unique $\widehat{\phi} \in \mathrm{Aut}_k(R)$ such that the diagram

$$\begin{array}{ccc} R & \xrightarrow{\widehat{\phi}} & R \\ \lambda_{\mathcal{L}_1} \downarrow & & \downarrow \lambda_{\mathcal{L}_2} \\ C_k(\mathcal{L}_1) & \xrightarrow[C(\phi)]{} & C_k(\mathcal{L}_2) \end{array}$$

commutes. We have

$$\phi(rx) = \widehat{\phi}(r)\phi(x)$$

for all $r \in R$ and $x \in \mathcal{L}_1$. To say that ϕ is R–linear, is to say that $\widehat{\phi} = 1$. If $\psi : \mathcal{L}_2 \to \mathcal{L}_3$ is another isomorphism as above, then $\widehat{\psi \circ \phi} = \widehat{\psi} \circ \widehat{\phi}$.

¿From the foregoing we obtain the following useful exact sequence of groups

$$1 \to \operatorname{Aut}_R(\mathcal{L}) \to \operatorname{Aut}_k(\mathcal{L}) \xrightarrow{\wedge} \operatorname{Aut}_k(R) \tag{4.2}$$

for any R–form $\mathcal{L}$ of A. This last map need not be surjective, and even when it is, the sequence need not split. Notwithstanding all of these problems, (4.2) is enough to completely describe the group $\operatorname{Aut}_k(\mathcal{L})$ of automorphisms of the k–algebra $\mathcal{L}$ in many interesting cases (see Examples 4.11 and 5.5 below, and also [PPS]).

Remark 4.8. Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra over k, with k algebraically closed of characteristic zero. Let $R_n = k[t_1^{\pm 1}, ..., t_n^{\pm 1}]$. As we saw in Example 4.5(2) , the case $n = 1$ corresponds to the affine Kac-Moody algebras. In general, there is a delicate connection between R_n–forms of $\mathfrak{g}$ and centerless cores of Extended Affine Lie Algebras (EALA's for short). As their name suggest, these algebras are higher nullity analogues of the affine algebras (see [AABFP] for details).

Neher has described ([N1] and [N2]) a precise procedure for building up a (tame) EALA out of centreless cores. It is not our intention to go into details about this construction, and the affine case will suffice to illustrate the spirit of how this goes. The centreless core $\mathcal{L}$ is in this case a "loop algebra" $L(\mathfrak{g}, \sigma)$ (see §6.2). One builds an EALA out of this by considering the universal central extension of $\mathcal{L}$ (which happens to be one dimensional), and tacking on a degree derivation. The algebra obtained is in this case an affine Kac-Moody Lie algebra.

The above showcases the crucial role that centreless cores play in the theory of EALAs. One of the main Theorems of [ABFP] shows that centreless cores which are finitely generated as modules over their centroids,[10] are always $k[t_1^{\pm 1}, ..., t_n^{\pm 1}]$–forms of a finite dimensional simple Lie algebra $\mathfrak{g}$. Centreless cores thus fall within the present language of forms. We believe that the approach described herewith is a new useful tool for the study of EALAs.

Our approach also sheds insight into some of the fundamental results of EALA theory. For example, Neher has shown that the centerless cores $\mathcal{L}$ under consideration, are always free module of finite rank over their centroid. But since $\mathcal{L}$ is a form of some $\mathfrak{g}$ as above, the previous Lemma tells that the R–module $\mathcal{L}$ is finitely generated and projective. Given that in the present situation $R = k[t_1^{\pm 1}, ..., t_n^{\pm 1}]$, freeness follows from a well-known theorem of Quillen and of Suslin.

[10] The ones which are not are fully understood.

4.3 Invariance of type

Let $\alpha \in \mathrm{Aut}_k(R)$. Define a new R–module structure on $\mathcal{L}$ by $r \cdot x = \alpha(r)x$. We denote the resulting R–algebra structure by ${}_\alpha\mathcal{L}$ to avoid confusion. (The multiplication in ${}_\alpha\mathcal{L}$ coincides with that of $\mathcal{L}$. It is the R–module structure that has changed.)

Assume $i : R \to S$ is an $fppf$ base change for which there exists an S–algebra isomorphism $\psi : \mathcal{L} \otimes_R S \simeq A \otimes S$. Then $i_\alpha = i \circ \alpha : R \to S$ is also $fppf$, and there exists a unique S–algebra isomorphism

$$\psi_\alpha : {}_\alpha\mathcal{L} \otimes_R S \to A \otimes S$$

satisfying $\psi_\alpha(x \otimes s) = \psi(x \otimes s)$ (where on the left hand $\otimes_R$ we view S as an R–algebra via i_α). Thus ${}_\alpha\mathcal{L}$ is also an R–form of A, said to be *obtained from $\mathcal{L}$ by twisting by* α. Note that $\mathrm{id} : {}_\alpha\mathcal{L} \to \mathcal{L}$ is a k–algebra isomorphism, and that $\widehat{\mathrm{id}} = \alpha$ (see Remark 4.7).

Theorem 4.9. [Invariance of type] *Let $\mathcal{L}_1$ (resp. $\mathcal{L}_2$) be an R–form of some finite dimensional perfect central k–algebra A_1 (resp. A_2). If $\mathcal{L}_1$ and $\mathcal{L}_2$ are isomorphic as k-algebras, there exists a finite field extension K/k for which $A_1 \otimes K$ and $A_2 \otimes K$ are isomorphic as K–algebras. In particular, if k is algebraically closed then A_1 and A_2 are isomorphic.*

Proof. Let $\phi : \mathcal{L}_1 \to \mathcal{L}_2$ be a k–algebra isomorphism, and let $\widehat{\phi} \in \mathrm{Aut}_k(R)$ be the corresponding automorphism at the centroid level (Remark 4.7). Denote $\widehat{\phi}^{-1}$ by α, and consider the twisted algebra ${}_\alpha\mathcal{L}_1$. Then the k–algebra isomorphism $\phi_\alpha = \phi \circ \mathrm{id} : {}_\alpha\mathcal{L}_1 \to \mathcal{L}_2$ satisfies $\widehat{\phi_\alpha} = \mathrm{id}$. It follows from this that we may (and henceforth do) assume with no loss of generality that our ϕ is an R–algebra isomorphism.

Consider the "switch" $\sigma : S_1 \otimes S_2 \to S_2 \otimes S_1$. Fix S_i–algebra isomorphisms $\psi_i : \mathcal{L}_i \otimes_R S_i \to A_i \otimes S_i$. The composite map

$${}^\sigma\psi_2 : \mathcal{L}_2 \otimes_R S_1 \otimes S_2 \xrightarrow{\mathrm{id}\otimes\sigma} \mathcal{L}_2 \otimes_R S_2 \otimes S_1 \xrightarrow{\psi_2\otimes\mathrm{id}} A_2 \otimes S_2 \otimes S_1 \xrightarrow{\mathrm{id}\otimes\sigma} A_2 \otimes S_1 \otimes S_2$$

is an isomorphism of $S_1 \otimes S_2$–algebras. Thus the composite map

$$\psi : A_1 \otimes S_1 \otimes S_2 \xrightarrow{\psi_1^{-1}\otimes\mathrm{id}} \mathcal{L}_1 \otimes_R S_1 \otimes S_2 \xrightarrow{\phi\otimes\mathrm{id}\otimes\mathrm{id}} \mathcal{L}_2 \otimes_R S_1 \otimes S_2 \xrightarrow{{}^\sigma\psi_2} A_2 \otimes S_1 \otimes S_2$$

is also an isomorphism of $S_1 \otimes S_2$–algebras.

Let $\mathfrak{m}$ be a maximal ideal of $S_1 \otimes S_2$, and let $F = (S_1 \otimes S_2)/\mathfrak{m}$ the corresponding quotient field. Our $\psi : A_1 \otimes S_1 \otimes S_2 \to A_2 \otimes S_1 \otimes S_2$ induces, upon reduction modulo $\mathfrak{m}$, an F–algebra isomorphism $\overline{\psi} : A_1 \otimes F \to A_2 \otimes F$. Since the A_i are finite dimensional, we may replace F by a subfield K which is finite dimensional over k. $\square$

4.4 Twisted automorphism groups

As before, A denotes a finite dimensional k–algebra, and $\mathcal{L}$ an R–form of A. Our next objective is to look in some detail at the group of automorphisms of $\mathcal{L}$. For simplicity we will henceforth assume that the extension S/R trivializing $\mathcal{L}$ is *finite and Galois*. By the Isotriviality Theorem of [GP1], this assumption is superfluous in the case we are interested in, namely Laurent polynomials in finitely many variables over a field of characteristic zero.[11]

Let $\mathbf{G} = \mathbf{Aut}(A)$. This is a linear algebraic group over k whose functor of points is given by

$$\mathbf{G}(S) = \mathrm{Mor}_{k-sch}\big(\mathrm{Spec}(S), \mathbf{G}\big) = \mathrm{Aut}_S(A \otimes S), \tag{4.3}$$

where the latter is the abstract group of automorphisms of the S–algebra $A \otimes S$. Each $\gamma \in \Gamma = \mathrm{Gal}(S/R)$ induces an automorphism γ^* of $\mathrm{Spec}(S)$ as a scheme over $\mathrm{Spec}(R)$, and a fortiori also as a scheme over $\mathrm{Spec}(k)$. For $\gamma \in \Gamma$ and $g \in \mathbf{G}(S) = \mathrm{Mor}_{k-sch}\big(\mathrm{Spec}(S), \mathbf{G}_R\big)$, define

$${}^{\gamma}g = g \circ \gamma^*. \tag{4.4}$$

This yields an action of Γ on $\mathbf{G}(S)$. (If one thinks of g as a matrix with entries in S, then ${}^{\gamma}g$ is nothing but the matrix obtained by applying γ to each entry of g).

We now look at the k–group functor $\mathbf{Aut}(\mathbf{G})$ of automorphisms of $\mathbf{G}$. By definition, for any S in *k–alg*

$$\mathbf{Aut}(\mathbf{G})(S) = \mathrm{Aut}(\mathbf{G}_S). \tag{4.5}$$

Some care is needed not to misunderstand this definition. The right hand side is the (abstract) group of automorphisms of the S–group $\mathbf{G}_S$ obtained by the base change S/k, and *not* the group of automorphisms of the group $\mathbf{G}(S)$.

We will henceforth assume that the k–group $\mathbf{Aut}(\mathbf{G})$ is *representable* (for example $\mathbf{G}$ reductive). Just as in (4.4) above, the group Γ acts on $\mathbf{Aut}(\mathbf{G})(S) = \mathrm{Mor}_{k-sch}\big(\mathrm{Spec}(S), \mathbf{Aut}(\mathbf{G})\big)$ by means of Γ^*.

Next consider the group homomorphism

$$\mathrm{int} : \mathbf{G} \to \mathbf{Aut}(\mathbf{G})$$

given by conjugation. Fix an element $g \in \mathbf{G}(S)$. Then the composite sequence

$$\mathrm{Spec}(S) \xrightarrow{\gamma^*} \mathrm{Spec}(S) \xrightarrow{g} \mathbf{G} \xrightarrow{\mathrm{int}} \mathbf{Aut}(\mathbf{G}),$$

[11] The industrious reader may rewrite this section in the general case.

together with the definition of the action of Γ on $\mathbf{Aut}(\mathbf{G})(S)$, readily yield the equality

$$(4.6) \qquad {}^{\gamma}\mathrm{int}\, g = \mathrm{int}\, {}^{\gamma}g \quad \text{for all} \quad g \in \mathbf{G}(S) \text{ and } \gamma \in \Gamma.$$

We now return to our twisted form $\mathcal{L}$. Up to R–isomorphism, we may assume that

$$(4.7) \qquad \mathcal{L} = \{x \in A \otimes S : \, u_\gamma \, {}^{\gamma}x = x \text{ for all } \gamma \in \Gamma\}$$

for some fixed cocycle $u = (u_\gamma)_{\gamma \in \Gamma} \in Z^1\big(\Gamma, \mathbf{G}(S)\big)$. From (4.6) it follows that $\widetilde{u}_\gamma :=$ int u_γ defines a cocycle $\widetilde{u} \in Z^1\big(\Gamma, \mathbf{Aut}(\mathbf{G})(S)\big)$. Let ${}_{\widetilde{u}}\mathbf{G}_R$ be the corresponding twisted group. This is an affine R–group scheme which becomes $\mathbf{G}_S = \mathbf{Aut}(A_S)$ after the base change S/R.

Proposition 4.10. *With the above notation and asumptions we have*

$$\mathrm{Aut}_R(\mathcal{L}) = ({}_{\widetilde{u}}\mathbf{G}_R)(R).$$

That is, the automorphisms of the twisted R–algebra $\mathcal{L}$, are precisely the R–points of the corresponding twisted group ${}_{\widetilde{u}}\mathbf{G}_R$.

Proof. By definition

$$({}_{\widetilde{u}}\mathbf{G}_R)(R) = \{\theta \in \mathrm{Aut}_S\,(A \otimes S) : \, \widetilde{u}_\gamma \, {}^{\gamma}\theta = \theta \text{ for all } \gamma \in \Gamma\}.$$

Thus, if $\theta \in {}_{\widetilde{u}}\mathbf{G}_R(R)$ and $x \in \mathcal{L} \subset A \otimes S$ (see (4.7)), we have by (4.4) that

$$\begin{aligned} u_\gamma \, {}^{\gamma}\big(\theta(x)\big) &= u_\gamma \, {}^{\gamma}\theta \, {}^{\gamma}x = u_\gamma \, {}^{\gamma}\theta u_\gamma^{-1} u_\gamma \, {}^{\gamma}x \\ &= (\widetilde{u}_\gamma \, {}^{\gamma}\theta)(x) = \theta(x), \end{aligned}$$

thereby showing that θ stabilizes $\mathcal{L}$.

Conversely, let $\tau \in \mathrm{Aut}_R(\mathcal{L})$. Then $\theta := \tau \otimes 1 \in \mathrm{Aut}_S(\mathcal{L} \otimes_R S) \simeq \mathrm{Aut}_S(A \otimes S) = \mathbf{G}(S)$. For $x \in \mathcal{L}$ we have as above, that $\theta(x) = (\widetilde{u}_\gamma \, {}^{\gamma}\theta)(x)$. Thus $\theta = \widetilde{u}_\gamma \, {}^{\gamma}\theta$ when restricted to $\mathcal{L}$. Since $\mathcal{L}$ spans $A \otimes S$ as an S–module, the result follows. $\square$

This last Proposition, together with Remark 4.7, can be put to good use to study the group of automorphisms of many important families of infinite dimensional Lie algebras.[12] We will illustrate how this goes via two interesting examples; quantum tori (immediately below) and affine Kac-Moody Lie algebras (Example 5.5).

[12]Most notably Extended Affine Lie Algebras.

Example 4.11. (Quantum tori) Assume k contains a primitive n–th root of unity ζ_n, and consider the quantum torus A_q where $q = \begin{pmatrix} 1 & \zeta_n \\ \zeta_n^{-1} & 1 \end{pmatrix}$. By definition, A_q is the associative unital k–algebra generated by $T_1^{\pm 1}$ and $T_2^{\pm 1}$ subject to the relations

$$T_i T_i^{-1} = 1 = T_i^{-1} T_i \quad \text{and} \quad T_2 T_1 = \zeta_n T_1 T_2.$$

The center of A_q is $R = k[t_1^{\pm 1}, t_2^{\pm 1}]$ where $t_\imath = T_i^n$. Thus A_q, as an R–algebra, is nothing but the cyclic algebra $(t_1, t_2, \zeta)_R$. In particular, A_q is an R–Azumaya algebra of constant rank n. Furthermore

$$A_q \otimes_R S \simeq M_n(S)$$

with $S = R[T_1]$.

Since the Picard group of R is trivial, every R–linear automorphism of A_q is inner. By Proposition 4.10, the R–group of such automorphisms is a twisted form ${}_{\tilde{u}}\mathbf{PGL}_{n,R}$ of $\mathbf{PGL}_{n,R}$. In particular $({}_{\tilde{u}}\mathbf{PGL}_{n,R})(R) \simeq A_q^\times / R^\times$ is a *finite group.* These are the R–linear automorphisms of the k–algebra A_q. To complete the picture we can use Remark 4.7. For example if ${k^\times}^n = k^\times$, an easy calculation shows that the canonical map

$$\widehat{\ }: \ \mathrm{Aut}_k(A_q) \to \ \mathrm{Aut}_k(R) \simeq (k^\times)^2 \rtimes \mathrm{GL}_2(\mathbb{Z})$$

is surjective if $n \leq 2$, and has image $(k^\times)^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$ if $n > 2$ (*cf.* [OP] and [N])

4.5 Graded considerations

The material in this section will play a crucial role in the Recognition Theorem for multiloop algebras in §6.4. The reader may want to postpone reading this section until then.

Let Λ be an abelian group (denoted additively). A Λ*–grading* on a k–algebra $\mathcal{B}$ is a collection of subspaces $\{\mathcal{B}^\lambda\}_{\lambda \in \Lambda}$ indexed by Λ such that

(i) The sum $\sum_{\lambda \in \Lambda} \mathcal{B}^\lambda$ is direct and equals $\mathcal{B}$.

(ii) $\mathcal{B}^\lambda \mathcal{B}^\mu \subset \mathcal{B}^{\lambda+\mu}$ for all λ, $\mu \in \Lambda$.

All of this information will be summarized by the expression " $\mathcal{B}$ is a Λ–graded k–algebra". For a Λ–graded k–algebra $\mathcal{B}$, the following two conditions are easily shown to be equivalent.

(GSI) $\mathcal{B} \neq \{0\}$ and the only homogeneous ideals of $\mathcal{B}$ are $\{0\}$ and $\mathcal{B}$.

(GSII) $\mathcal{B} \neq \{0\}$ and $\mathrm{Mult}_k(\mathcal{B})x = \mathcal{B}$ for all nonzero homogeneous x in $\mathcal{B}$.

We then say that $\mathcal{B}$ is *graded simple.*

Remark 4.12. Let $\mathcal{B}$ be graded simple. Then $\mathcal{B}$ is a monogenic $\mathrm{Mult}_k(\mathcal{B})$–module (in particular, finitely generated). By lemma 4.2.3 of [ABFP], the centroid of $\mathcal{B}$ is also Λ–graded. Thus, if $\chi \in C_k(\mathcal{B})\backslash\{0\}$, there exist unique $\lambda_1, \dots, \lambda_h \in \Lambda$ and $\chi_1, \dots, \chi_h \in C_k(\mathcal{B})$ such that

$$\begin{aligned}&\chi = \chi_1 + \cdots + \chi_n, \text{ and}\\ &\chi_i(\mathcal{B}^\lambda) \subset \mathcal{B}^{\lambda+\lambda_i} \quad \text{for all} \quad \lambda \in \Lambda \quad \text{and} \quad 1 \leq i \leq h.\end{aligned}$$

Definition 4.13. Let S be a Λ–graded object of k–*alg*. An S/R–form $\mathcal{L}$ of A is said to be Λ–*graded*, if there exists a Λ–graded structure on the k–algebra $\mathcal{L}$, and an S–algebra isomorphism $\psi : \mathcal{L} \otimes_R S \to A \otimes S$ which are *compatible* with each other, that is

$$\psi(\mathcal{L}^\lambda \otimes 1) \subset A \otimes S^\lambda$$

for all $\lambda \in \Lambda$.

Proposition 4.14. *Let A be a finite dimensional central simple k–algebra, and $S = \bigoplus_{\lambda \in \Lambda} S^\lambda$ a graded simple object of k–alg.*

1. *With the natural Λ–grading on $A \otimes S$ given by $(A \otimes S)^\lambda = A \otimes S^\lambda$, the k–algebra $A \otimes S$ is graded simple.*
2. *Let $\mathcal{L}$ be a Λ–graded S/R–form of A. Then $\mathcal{L}$ is a graded simple k–algebra. Furthermore, R is naturally identified with a graded k–subalgebra of S, and the canonical map $\lambda_{\mathcal{L}} : R \to C_k(\mathcal{L})$ is a Λ–graded isomorphism.*

Proof. (1) Let $x \in (A \otimes S)^\lambda$ be a nonzero element, and set $M = \mathrm{Mult}_k(A \otimes S)x$. We must show that $M = A \otimes S$. Write $x = \sum a_i \otimes s_i$ with $a_i \in A$ linearly independent, and $s_i \in S^\lambda$ nonzero. By Jacobson's Density theorem (see the proof of Ch.X, theorem 3 of [J]), there exists $\alpha \in \mathrm{Mult}_k(A)$ such that $\alpha a_1 = a_1$ and $\alpha a_i = 0$ if $i > 1$. This shows that $a_1 \otimes s_1 \in M$. Let $s \in S$. Since S is graded-simple, there exists $s' \in S$ such that $s's_1 = s$. Similarly since A is simple, there exist $a' \in A$ such that $a'a_1 \neq 0$. Then

$$a'a_1 \otimes s = (a' \otimes s')(a_1 \otimes s_1) \in M.$$

Thus $A \otimes s = \big(\mathrm{Mult}_k(A) \otimes 1\big)(a'a_1 \otimes s) \subset M$ for all $s \in S$. It follows that $M = A \otimes S$ as desired.

(2) Fix an S–algebra isomorphism $\psi : \mathcal{L} \otimes_R S \to A \otimes S$ which is compatible with the Λ–gradings. We begin with three general observations that will be used in the proof.

(a) Let $\alpha : R \to S$ be the underlying map to the R–algebra structure of S, and let $\beta : \mathcal{L} \to \mathcal{L} \otimes 1 \subset \mathcal{L} \otimes S$ the canonical map. Because α is faithfully flat, both α and β are injective.

(b) Every element of $A \otimes S$ is a sum of elements of the form $s\psi(y \otimes 1)$ with $y \in \mathcal{L}$ and $s \in S$.

(c) The associative unital k–algebra $\mathrm{Mult}_k(\mathcal{L})$ has a natural Λ–graded structure. Right and left multiplication by elements of $\mathcal{L}$ generate a graded (two-sided) ideal $\mathrm{Mult}_k(\mathcal{L})^+$ of $\mathrm{Mult}_k(\mathcal{L})$.

Fix a nonzero homogeneous element x of $\mathcal{L}$. Let $N = \mathrm{Mult}_k(\mathcal{L})^+ x$. This is a submodule (in fact an ideal) of the R–algebra $\mathcal{L}$ which is graded as a subspace of $\mathcal{L}$. We claim that $N \neq 0$. For otherwise, for all $s \in S$ and $y \in \mathcal{L}$ we have

$$\psi(x \otimes 1)(s\psi(y \otimes 1)) = s\psi(xy \otimes 1) = 0 = s\psi(yx \otimes 1) = (s\psi(y \otimes 1))\psi(x \otimes 1).$$

It now follows from (b) above, that $\psi(x \otimes 1)$ is killed by left and right multiplication by all elements of $A \otimes S$. Thus the one–dimensional space $k\psi(x \otimes 1) \subset A \otimes S$ is a nonzero graded ideal of $A \otimes S$. By part (1) we obtain $k\psi(x \otimes 1) = A \otimes S$. Since $A \otimes S$ is perfect and nonzero, we must have $x^2 \otimes 1 \neq 0$. By (a) then, $x^2 \neq 0$. But this is impossible since $x^2 \in N$. Thus $N \neq 0$ as claimed.

Let M be the S–submodule of $A \otimes S$ generated by $\psi(N \otimes 1)$. We claim that $M = A \otimes S$. By part (1), it will suffice to show that M is a nonzero graded ideal of $A \otimes S$. That $M \neq 0$ follows from (a) and the fact that $N \neq 0$. Our assumption on ψ implies that the subspace $\psi(N \otimes 1)$ of $A \otimes S$ is graded, hence M is also a graded subspace of $A \otimes S$. To show that M is an ideal of $A \otimes S$, observe that for $s \in S$ and $y \in \mathcal{L}$ we have

$$\begin{aligned}(s\psi(y \otimes 1))\psi(N \otimes 1) &= s\big(\psi(y \otimes 1)\psi(\mathrm{Mult}_k(\mathcal{L})^+ x \otimes 1)\big)\\ &\subset s\big(\mathrm{Mult}_k(\mathcal{L})^+ x \otimes 1\big) = sN \subset M.\end{aligned}$$

By (b) above M is a left ideal of $A \otimes S$. Similarly, M is a right ideal. The claim follows. As a consequence, we conclude that

$$\psi(N \otimes_R S) = A \otimes S = \psi(\mathcal{L} \otimes_R S).$$

Thus the canonical injective map $N \otimes_R S \hookrightarrow \mathcal{L} \otimes_R S$ is in fact an equality. By faithfully flat descent $N = \mathcal{L}$ as desired. This finishes the proof that $\mathcal{L}$ is graded–simple.

It remains to show that the induced map at the centroid level preserves the Λ–gradings (see Remark 4.12). Let $r \in R$. Write $\alpha(r) = \sum s_i$ with $s_i \in S^{\lambda_i}$. For $x \in \mathcal{L}^\lambda$

$$\psi(rx \otimes 1) = \psi(x \otimes \sum s_i) = \sum s_i \psi(x \otimes 1).$$

Since $rx \in \mathcal{L}$, we conclude by graded considerations that $s_i\psi(\mathcal{L}\otimes 1) \subset \psi(\mathcal{L}\otimes 1)$. Consider $\chi_i : \mathcal{L} \to \mathcal{L}$ defined by $\chi_i(x) = \beta^{-1}\psi^{-1}\lambda_{s_i}\psi\beta(x)$. Then

$$\begin{aligned}\chi_i(xy) &= \beta^{-1}\psi^{-1}\lambda_{s_i}\big(\psi(x\otimes 1)\psi(y\otimes 1)\big)\\ &= \beta^{-1}\psi^{-1}\big(\psi(x\otimes 1)s_i\psi(y\otimes 1)\big)\\ &= x\chi_i(y).\end{aligned}$$

Similarly $\chi_i(xy) = \chi_i(x)y$. Thus $\chi_i \in C_k(\mathcal{L})$. By Lemma 4.6.3 there exists a unique $r_i \in R$ such that $\chi_i = \lambda_{r_i}$. It follows from the definitions that $\alpha(r_i) - s_i$ kills $\psi(x\otimes 1)$. Since the S module $A\otimes S$ is free, we obtain $\alpha(r_i) = s_i$. This shows that R is identified, via α, with a graded k–subalgebra of S. Finally, one checks that $R^\lambda\mathcal{L}^\mu\otimes 1 \subset \mathcal{L}^{\lambda+\mu}\otimes 1$. This immediately shows that the isomorphism $\lambda_{\mathcal{L}} : R \to C_k(\mathcal{L})$ is Λ–graded. □

5 Forms of algebras over Laurent polynomial rings

Throughout this section k is assumed to be algebraically closed and of characteristic 0. Recall that $R_n = k[t_1^{\pm 1},\dots t_n^{\pm 1}]$, that $R_{n,d} = k[t_1^{\pm\frac{1}{d}}, t_2^{\pm\frac{1}{d}},\cdots t_n^{\pm\frac{1}{d}}]$, and that $R_{n,\infty} = \varinjlim_d R_{n,d}$. At the field level we have $K_{n,d} = k(t_1^{\pm\frac{1}{d}}, t_2^{\pm\frac{1}{d}},\cdots t_n^{\pm\frac{1}{d}})$ and $K_{n,\infty} = \varinjlim_d K_{n,d}$.

5.1 Multiloop Algebras

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Throughout this section A is a finite dimensional algebra over k. Recall that $(\zeta_n)_{n>0}$ in $k^\times$ is a compatible family of primitive roots of unity.

We begin by introducing the ingredients needed in the definition of multiloop algebras. Let $\boldsymbol{\sigma} = (\sigma_1,\dots,\sigma_n)$ be a commuting family of finite order automorphisms of the k–algebra A. Let d be a common period of the σ_i, i.e. $\sigma_i^d = 1$.

For each $(i_1,...,i_n) \in \mathbb{Z}^n$, consider the simultaneous eigenspaces

$$A_{i_1\dots i_n} := \{x \in A : \ \sigma_j(x) = \zeta_d^{i_j}x \text{ for all } \quad 1 \le j \le n\}$$

(which of course depend only on the i_j modulo the d). The *multiloop algebra* associated to this data is the k–subalgebra $\mathcal{L}$ of $A\otimes R_{n,\infty}$ defined as follows:

$$\mathcal{L} = \mathcal{L}(A,\boldsymbol{\sigma}) = \oplus A_{i_1\dots i_n}\otimes t_1^{i_1/d}\dots t_n^{i_n/d} \subset A\otimes R_{n,d} \subset A\otimes R_{n,\infty}. \tag{5.1}$$

Remark 5.1. Observe that $\mathcal{L}$ does not depend on the choice of period d, and that $\mathcal{L}$ has a natural R_n–algebra structure.

One easily verifies that

$$\mathcal{L} \otimes_{R_n} R_{n,d} \simeq_{R_{n,d}} A \otimes R_{n,d}.$$

Since $R_{n,d}/R_n$ is free of finite rank (hence $fppf$), $\mathcal{L}$ is an R_n–form of A which is trivialized by the extension $R_{n,d}/R_n$ (in the sense of §5).

From this last Remark it follows that to a multiloop algebra $\mathcal{L}$ as above, corresponds an R_n–torsor $\mathbf{X}_{\mathcal{L}}$ under the group $\mathbf{Aut}(A)$.[13] By descent theory, $\mathbf{X}_{\mathcal{L}}$ is representable by an affine R_n–scheme. The functor of points of $\mathbf{X}_{\mathcal{L}}$ is easily described: For all $S \in R_n\text{–}alg$

$$\mathbf{X}_{\mathcal{L}}(S) = \mathrm{Hom}_{S\text{-}alg}(\mathcal{L} \otimes_{R_n} S,\ A \otimes S).$$

The isomorphism class of the R_n–torsor $\mathbf{X}_{\mathcal{L}}$ will be denoted by $\big[\mathbf{X}_{\mathcal{L}}\big]$. Thus

$$\big[\mathbf{X}_{\mathcal{L}}\big] \in H^1_{\acute{e}t}\big(R_{n,d}/R_n,\ \mathbf{Aut}(A)\big) \subset H^1_{\acute{e}t}\big(R_n,\ \mathbf{Aut}(A)\big). \tag{5.2}$$

Recall the exact sequence of algebraic k–groups

$$1 \to \mathbf{Aut}^0(A) \to\ \mathbf{Aut}(A) \xrightarrow{\ \bar{}\ } \mathbf{Out}(A) \to 1, \tag{5.3}$$

where $\mathbf{Out}(A) = \mathbf{Aut}(A)/\mathbf{Aut}^0(A)$ is the finite constant k–group of connected components of $\mathbf{Aut}(A)$. Let $\mathrm{Out}(A) = \mathbf{Out}(A)(k)$, and let $^-:\ \mathrm{Aut}_k(A) = \mathbf{Aut}(A)(k) \to \mathrm{Out}(A)$ be the canonical map. The kernel of $^-$ is $\mathbf{Aut}^0(A)(k)$.

The extension $R_{n,d}/R_n$ is finite Galois. Its Galois group $\Gamma_{n,d}$ is henceforth identified with $(\mathbb{Z}/d\mathbb{Z})^n$ acting naturally on $R_{n,d}$ via our fixed choice of compatible roots of unity; namely ${}^{\bar{\mathbf{e}}}(t_i^{1/d}) = \zeta_d^{e_i} t_i^{1/d}$ for all $\mathbf{e} = (e_1, ..., e_n) \in \mathbb{Z}^n$. If we now let Γ acts on $\mathbf{Aut}(A)(R_{n,d}) = \mathrm{Aut}_{R_{n,d}-alg}(A \otimes R_{n,d})$ by conjugation, i.e. ${}^{\gamma}\sigma = (1 \otimes \gamma)\sigma(1 \otimes (-\gamma))$, we have a natural correspondence $H^1\big(\Gamma_{n,d}, \mathbf{Aut}(A)(R_{n,d})\big) \simeq H^1_{\text{ét}}\big(R_{n,d}/R_n,\ \mathbf{Aut}(A)\big)$. In terms of $H^1(\Gamma_{n,d}, \mathbf{Aut}(A)(R_{n,d}))$, the loop torsor $\mathbf{X}_{\mathcal{L}}$ corresponds to the cocycle $\alpha_{\mathcal{L}} \in Z^1\big(\Gamma_{n,d}, \mathbf{Aut}(A)(R_{n,d})\big)$ given by $(\alpha_{\mathcal{L}})_{\bar{\mathbf{e}}} = \sigma_1^{-e_1} \dots \sigma_n^{-e_n} \otimes \mathrm{id}\ \in\ \mathrm{Aut}_{R_{n,d}\text{-}alg}(A \otimes R_{n,d})$. We have a natural correspondence ([SGA1, Exp XI.5]).

$$H^1_{\acute{e}t}\big(R_n, \mathbf{Out}(A)\big) \simeq H^1_{ct}\big(\pi_1(R_n),\ \mathrm{Out}(A)\big) \tag{5.4}$$

Remark 5.2. Since the fundamental algebraic group $\pi_1(R_n)$ of $\mathrm{Spec}(R_n)$ is identified with $(\widehat{\mathbb{Z}})^n$ via our choice of compatible primitive roots of unity (Corollary 2.11 and Remark 2.13 of [GP2]), the set $H^1_{ct}\big(\pi_1(R_n),\ \mathrm{Out}(A)\big)$ is nothing but *the set of conjugacy classes of commuting n–tuples of elements of finite order automorphisms of* $\mathrm{Out}(A)$. This interpretation plays a crucial role in the classification of multiloop algebras by cohomological methods.

[13] Strictly speaking, the structure group is the affine R_n–group $\mathbf{Aut}(A \otimes R_n)$. As already mentioned, this harmless slight abuse of notation is used throughout.

Under the canonical map

$$H^1_{ét}\big(R_n,\,\mathbf{Aut}(A)\big) \to H^1_{ét}\big(R_n,\mathbf{Out}(A)\big) \simeq H^1_{ct}\big(\pi_1(R_n),\,\mathrm{Out}(A)\big)$$

arising from (5.3) and (5.4), our $[\mathbf{X}_{\mathcal{L}}] = [\alpha_{\mathcal{L}}]$ maps to the class $[\overline{\alpha}_{\mathcal{L}}]$ of the unique cocycle (= continuous homomorphism) $\overline{\alpha}_{\mathcal{L}} : \pi_1(R_n) \to \mathrm{Out}(A)$ satisfying

$$\mathbf{e} \mapsto \overline{\sigma}\,_1^{-e_1} \ldots \overline{\sigma}\,_n^{-e_n}$$

for all $\mathbf{e} = (e_1, ..., e_n) \in \mathbb{Z}^n$.

Define an action of the group $\mathrm{GL}_n(\mathbb{Z})$ on the set of n–tuples of commuting elements of finite order of $\mathrm{Aut}_k(A)$ as follows: for $\boldsymbol{a} = (a_{ij}) \in \mathrm{GL}_n(\mathbb{Z})$ and $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$ as above, set

$$({}^{\boldsymbol{a}}\boldsymbol{\sigma})_i = \prod_{j=1}^{n} \sigma_j^{a_{ij}} \quad \text{and} \quad {}^{\boldsymbol{a}}\boldsymbol{\sigma} = (({}^{\boldsymbol{a}}\boldsymbol{\sigma})_1, \ldots, ({}^{\boldsymbol{a}}\boldsymbol{\sigma})_n).$$

Non-abelian cohomology allows us to classify isomorphism classes of multiloop algebras *as algebras over* R_n. One is however interested in the classification of these as algebras over k. The following lemma is therefore most useful.

Lemma 5.3. *Let A be finite dimensional perfect and central algebra over k. If $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$ and $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_n)$ are two n–tuples of commuting automorphisms of finite order of A, the following conditions are equivalent:*

(1) $L(A, \boldsymbol{\sigma}) \simeq_k L(A, \boldsymbol{\tau})$.
(2) $L(A, {}^{\boldsymbol{a}}\boldsymbol{\sigma}) \simeq_{R_n} L(A, \boldsymbol{\tau})$ for some $\boldsymbol{a} \in \mathrm{GL}_n(\mathbb{Z})$.

Proof. Let $\phi : L(A, \boldsymbol{\sigma}) \to L(A, \boldsymbol{\tau})$ be an isomorphism of k–algebras. Identify R_n with the corresponding centroids (Lemma 4.6), and consider $\widehat{\phi} \in \mathrm{Aut}_k(R_n)$ such that $\phi(rx) = \widehat{\phi}(r)\phi(x)$ for all $r \in R_n$ and $x \in A$ (see Remark 4.7). We have $\widehat{\phi}(t_i) = \lambda_i t_1^{a_{1i}} \ldots t_n^{a_{ni}}$ for some $\boldsymbol{a} = (a_{ij}) \in \mathrm{GL}_n(\mathbb{Z})$ and some $\lambda_i \in k^\times$.

Identify $L(A, \boldsymbol{\sigma})$ with a k–subalgebra of $A \otimes R_{n,d}$ for a suitable d (Remark 5.1). Fix $\gamma_i \in k^\times$ such that $\gamma_i^d = \lambda_i$. Let α be the unique element of $\mathrm{Aut}_k(R_{n,d})$ satisfying $t_i^{1/d} \mapsto \gamma_i t_1^{a_{1i}/d} \ldots t_n^{a_{ni}/d}$. A straightforward calculation shows that the automorphism $\psi = 1 \otimes \alpha$ of the k–algebra $A \otimes R_{n,d}$, induces an isomorphism of $L(A, \boldsymbol{\sigma})$ onto $L(A, {}^{\boldsymbol{a}}\boldsymbol{\sigma})$. At the centroid level we have $\widehat{\psi} = \widehat{\phi}$. It follows that the k–linear isomorphism $\phi \circ \psi^{-1} : L(A, {}^{\boldsymbol{a}}\boldsymbol{\sigma}) \to L(A, \boldsymbol{\tau})$ is in fact R_n–linear.

The converse is clear since $L(A, \boldsymbol{\sigma})$ and $L(A, {}^{\boldsymbol{a}}\boldsymbol{\sigma})$ are evidently isomorphic as k–algebras as explained above. □

Corollary 5.4. *Let σ and τ be two finite order automorphisms of a finite dimensional perfect central k–algebra A. Then $L(A, \sigma) \simeq_k L(A, \tau)$ if and only if $L(A, \sigma) \simeq_{R_1} L(A, \tau)$ or $L(A, \sigma^{-1}) \simeq_{R_1} L(A, \tau)$.* □

5.2 The case of Lie algebras

The algebra A is now a finite dimensional (split) simple Lie algebra over k. To follow standard practices, we will denote A by $\mathfrak{g}$. The relevant exact sequence of algebraic groups is

$$1 \to \mathbf{G}^{ad} \to \mathbf{Aut}(\mathfrak{g}) \to \mathbf{Out}(\mathfrak{g}) \to 1, \tag{5.5}$$

where $\mathbf{Out}(\mathfrak{g})$ is the finite constant group corresponding to the (abstract) finite group $\mathrm{Out}(\mathfrak{g})$ of automorphisms of the Dynkin diagram of $\mathfrak{g}$ ([SGA3] Exp. XXV théorème 1.3). The sequence (5.5) is split. We fix a section $\mathbf{Out}(\mathfrak{g}) \to \mathbf{Aut}(\mathfrak{g})$ and identify thereof in what follows $\mathrm{Out}(\mathfrak{g})$ with a subgroup of $\mathbf{Aut}(\mathfrak{g})(k) = \mathrm{Aut}_k(\mathfrak{g})$.

Example 5.5. (Affine Kac-Moody Lie algebras) Let $\widehat{\mathcal{L}}$ be an affine Kac-Moody Lie algebra over k, and $\mathcal{L}$ be the derived algebra of $\widehat{\mathcal{L}}$ modulo its centre. As shown by Kac, there exists a finite dimensional simple Lie algebra $\mathfrak{g}$, and an automorphism π of the corresponding Dynkin diagram, such that

$$\mathcal{L} \simeq L(\mathfrak{g}, \pi). \tag{5.6}$$

As already mentioned, here and elsewhere $\pi \in \mathrm{Out}(\mathfrak{g})$ is viewed as an element of $\mathrm{Aut}_k(\mathfrak{g})$ via our *fixed* section of the split exact sequence (5.5) above. Thus loop algebras (in nullity 1) provides us with concrete realizations of the affine algebras.[14]

The structure of the group of automorphisms of an arbitrary symmetrizable Kac-Moody Lie algebra (derived modulo its centre), was determined in [PK]: It is generated by the "adjoint" Kac-Moody group, together with a "Cartan-like" subgroup $\widetilde{H}$, the symmetries of the extended Dynkin diagram, and the so-called Chevalley involution. In the case of an untwisted affine algebra , namely $\pi = 1$ in (5.6), it is known that the adjoint Kac-Moody group is nothing but $\mathbf{G}^{ad}(k[t^{\pm 1}])$ above.

The present cohomological viewpoint yields a new concrete realization of the automorphism group in *both* the twisted and untwisted affine case. To illustrate, consider the case of the twisted algebra $\mathcal{L}$ of type $BC^{(2)}_{n-2}$, $n > 2$ (type $A^{(2)}_{n-1}$ in Kac's notation). Here $\mathfrak{g} = \mathfrak{sl}_n(k)$, $\pi(X) = -X^{tr}$, $R = k[t^{\pm 1}]$, $S = k[t^{\pm 1/2}]$. The Galois group Γ of the extension S/R will throughout be identified with $\mathbb{Z}/2\mathbb{Z}$ via $^{\bar{1}}(t^{1/2}) = -t^{1/2}$. The cocycle $u \in Z^1\big(\Gamma, \mathrm{Aut}_S(\mathfrak{sl}_n \otimes S)\big)$ defining $\mathcal{L}$ is given by $u_{\bar{1}} = \pi^{-1} = \pi$. The R–linear automorphisms of $\mathcal{L}$ are the R–points of the twisted R–group that fits into the *split* exact sequence

$$1 \to {}_{\tilde{u}}\mathbf{PGL}_{n,R} \to \mathbf{Aut}_R(\mathcal{L}) \to \mathbb{Z}/2\mathbb{Z} \to 1.$$

[14] A priori, the affine algebras are given by generators and relations à la Chevalley. This presentation does not provide much insight into the nature of the algebras.

This sequence is obtained by first applying the base change R/k to (5.5), and then twisting by $\widetilde{u}$ as prescribed by Proposition 4.10.

We can give an explicit description of the R–points of the twisted group ${}_{\widetilde{u}}\,\mathbf{PGL}_{n,R}$. Indeed. The exact sequence of R–groups schemes

$$1 \to \mathbf{G}_{m,R} \to \mathbf{GL}_{n,R} \to \mathbf{PGL}_{n,R} \to 1$$

can be twisted by $\widetilde{u}$. The obstruction to the map ${}_{\widetilde{u}}\,\mathbf{GL}_{n,R}(R) \to {}_{\widetilde{u}}\,\mathbf{PGL}_{n,R}(R)$ being surjective lies on $H^1(R, {}_{\widetilde{u}}\mathbf{G}_{m,R})$. But this H^1 vanishes. One can see by general consideration [P2], or by direct computation by interpreting ${}_{\widetilde{u}}\mathbf{G}_{m,R}$ as a kernel of a Weil restriction, namely

$$1 \to {}_{\widetilde{u}}\mathbf{G}_{m,R} \to \mathcal{R}_{S/R}\mathbf{G}_{m,S} \overset{\mathrm{N}_{S/R}}{\to} \mathbf{G}_{m,R} \to 1,$$

and then passing to cohomology ($\mathrm{N}_{S/R}$ is surjective because k is algebraically closed, and $H^1(R, \mathcal{R}_{S/R}\mathbf{G}_{m,S}) = H^1(S, \mathbf{G}_{m,S}) = \operatorname{Pic}(S) = 0$ by Shapiro's Lemma).

The outcome is that R–linear inner automorphisms of our Kac-Moody algebra $\mathcal{L}$, i.e. the R–points of the image of ${}_{\widetilde{u}}\,\mathbf{PGL}_{n,R}$ inside $\mathbf{Aut}_R(\mathcal{L})$, are given by conjugation by a matrix $X \in \mathrm{GL}_n(S)$ satisfying $\widetilde{u}\,{}^{\overline{1}}X := u\,{}^{\overline{1}}Xu^{-1} = X$.

It remains to look at the map $\widehat{\ }: \operatorname{Aut}_k(\mathcal{L}) \to \operatorname{Aut}_k(R)$. By definition

$$\mathcal{L} = \{X \in \mathfrak{sl}_n \otimes S : -({}^{\overline{1}}X)^{tr} = X\}, \tag{5.7}$$

where $\Gamma = \mathbb{Z}/2\mathbb{Z}$ acts on the second coordinate of the matrix $X \in \mathfrak{sl}_n \otimes S$. On the other hand

$$\operatorname{Aut}_k(R) = \operatorname{Aut}_k(k[t^{\pm 1}]) \simeq k^\times \rtimes \mathbb{Z}/2\mathbb{Z}.$$

We claim that the map $\widehat{\ }$ is surjective. Clearly any element θ of $\operatorname{Aut}_k(R)$ induces an element $\widetilde{\theta}$ of $\operatorname{Aut}_k(\mathfrak{sl}_n \otimes S)$. So we must show that $\widetilde{\theta}$ stabilizes $\mathcal{L}$ for $\theta \in k^\times$ and $\theta \in \mathbb{Z}/2\mathbb{Z}$. If $\theta \in k^\times$, this is clear because k is algebraically closed. If θ is the generator of $\mathbb{Z}/2\mathbb{Z}$, we may assume $\theta(t) = t^{-1}$. Again using (5.7) one sees that $\widetilde{\theta}(\mathcal{L}) = \mathcal{L}$. We thus have the split exact sequence

$$1 \to \operatorname{Aut}_R(\mathcal{L}) \to \operatorname{Aut}_k(\mathcal{L}) \to \operatorname{Aut}_k(R) \to 1.$$

This finishes the description of the group of automorphisms of the affine Lie algebra $\mathcal{L}$.

We now return to our general discussion. In terms of cohomology, the affine Lie algebras as in (5.6) account for $H^1\big(R_1, \mathbf{Out}(\mathfrak{g})\big) \simeq \{\text{conjugacy classes in } \operatorname{Out}(\mathfrak{g})\}$. The classes of all R_1–forms of $\mathfrak{g}$ on the other hand, are measured by $H^1\big(R_1, \mathbf{Aut}(\mathfrak{g})\big)$. As it turns out, the canonical map $H^1\big(R_1, \mathbf{Aut}(\mathfrak{g})\big) \to H^1(R_1, \mathbf{Out}(\mathfrak{g}))$ is bijective. For this one needs to know that $H^1(R_1, -)$ vanishes for quasisplit R_1–groups of adjoint type. This was established in [P1] with the aid of Harder's work. More generally we have

Theorem 5.6. [P2] *Let* $\mathbf{G}$ *be a reductive group scheme over* $k[t^{\pm 1}]$*. Then* $H^1(k[t^{\pm 1}], \mathbf{G}) = 1$.

This Theorem shows that the $k[t^{\pm 1}]$–forms of $\mathfrak{g}$ are *precisely* the affine Kac-Moody algebras (derived modulo their centres) as it was mentioned in Example 4.5. In particular over $k[t^{\pm 1}]$, *all* forms of $\mathfrak{g}$ are loop algebras. By the invariance of type (Theorem 4.9), $\mathfrak{g}$ is unique up to isomorphism. For a fixed $\mathfrak{g}$, two $k[t^{\pm 1}]$–forms are isomorphic over k if and only if they are isomorphic over $k[t^{\pm 1}]$. This follows from Corollary 5.4, together with the fact that in $\mathrm{Out}(\mathfrak{g})$, every element is conjugate to its inverse. Combining all of the above we recover, by purely theoretical considerations, the existence of *exactly* 16 non-isomorphic classes of affine Lie algebras.[15]

Similar considerations apply to R_1–forms of an arbitrary finite dimensional k–algebras [P2].

We now turn our attention to the case $n = 2$. Here we find that some interesting and unexpected behavior arises. Theorem 5.6 ought to be thought as the validity of "Serre conjecture I" for $R_1 = k[t_1^{\pm 1}]$. Since Serre's Conjecture II holds for $K = k(t_1, t_2)$, one is lead to raise the following inevitable question.

Let $\mathbf{G}$ *be a semisimple group scheme over* $R_2 = k[t_1^{\pm}, t_2^{\pm 1}]$*. Assume* $\mathbf{G}$ *is of simply connected type. Is* $H^1(R_2, \mathbf{G})$ *trivial?*

Somehow surprisingly perhaps, the answer to this question is negative (as we have seen in §4.6). The next example recalls the reason for this failure, and the implications that this has for the classification of EALA by cohomological methods.

Example 5.7. ($\mathfrak{sl}_2$ in nullity 2) Let $\mathfrak{g} = \mathfrak{sl}_2 = \mathfrak{sl}_2(k)$. Then $\mathbf{Aut}(\mathfrak{g}) = \mathbf{PGL}_2$ and we have the exact sequence $1 \to \boldsymbol{\mu}_2 \to \mathbf{SL}_2 \to \mathbf{PGL}_2 \to 1$. Relativizing at $R = R_2 = k[t_1^{\pm 1}, t_2^{\pm 1}]$ and passing to cohomology yields

$$H^1_{ét}(R, \mathbf{SL}_2) \to H^1_{ét}(R, \mathbf{PGL}_2) \xrightarrow{\delta} H^2_{ét}(R, \boldsymbol{\mu}_2) \subset {}_2Br(R).$$

We have $H^2_{ét}(R, \boldsymbol{\mu}_2) \simeq \mathbb{Z}/2\mathbb{Z} = \{\overline{0}, \overline{1}\}$ (Proposition 2.1). The kernel of δ is trivial (straightforward for $\mathbf{SL}_2$, but also consequence of Theorem 2.7), so the fiber $\delta^{-1}(\overline{0})$ is comprised of one isomorphism class, namely the isomorphism class of the trivial R–Lie double loop algebra

$$\mathcal{L}_0 = \mathfrak{sl}_2 \otimes R = L(\mathfrak{sl}_2, \mathrm{id}, \mathrm{id}).$$

The Lie algebra $\mathcal{L}_0$ is the centreless core of an EALA of nullity 2.

[15] As far as we know, nowhere in the usual Kac-Moody literature it is actually shown that the algebras of these sixteen families are non isomorphic.

Since $\mathbf{PGL}_2$ is also the group of automorphisms of the matrix algebra $M_2(k)$, there is a natural correspondence between R–forms of $\mathfrak{g}$ and $M_2(k)$: Given an R–form A of $M_2(k)$, view A as a Lie algebra $\mathrm{Lie}(A)$ with bracket given by the commutator $[x,y] = xy - yx$. The derived Lie algebra $\mathrm{Lie}(A)'$ is then an R–form of $\mathfrak{sl}_2$.

We now apply this to the quaternion R–algebra $A = A(1,n)$, to obtain an R–Lie algebra $\mathcal{L}_1 = \mathrm{Lie}\,(A)'$. This is also a double loop algebra. In fact

$$\mathcal{L}_1 = L(\mathfrak{sl}_2, \sigma_1, \sigma_2),$$

where σ_1 and σ_2 in $\mathbf{Aut}(\mathfrak{sl}_2)(k)$ are given by conjugation by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ respectively. The Lie algebra $\mathcal{L}_1$ has anisotropic generic fiber, hence cannot be the centreless core of an EALA.

Because of the classification of commuting pair of elements of finite order of $\mathbf{PGL}_2(k)$, we know that $\mathcal{L}_0$ and $\mathcal{L}_1$ are the only two k–isomorphism classes of double loop algebras based on $\mathfrak{sl}_2$.

Following the same procedure, we can attach to the Margaux algebra $\mathcal{M}$ of §4.6 the Lie algebra

$$\mathcal{L}_2 = \mathrm{Lie}\,(\mathcal{M})'.$$

Recall that $\mathcal{M} = \mathrm{End}_A(M)$, where M is a rank one faithfully projective A–module which is not free. We have $\mathcal{M} \otimes_R S \simeq M_2(S)$ for $S = R[t_1^{1/2}]$. However A and $\mathcal{M}$ are not isomorphic R–algebras. Thus $\mathcal{L}_1$ and $\mathcal{L}_2$ are non-isomorphic R–forms of $\mathfrak{sl}_2$. The algebra $\mathcal{L}_2$ is neither a multiloop algebra, nor the centreless core of an EALA.

The algebras $\mathcal{L}_1$ and $\mathcal{L}_2$ are part of the fiber $\delta^{-1}(\overline{1})$ of the boundary map $\delta : H^1_{\acute{e}t}(R, \mathbf{PGL}_2) \to H^2_{\acute{e}t}(R, \boldsymbol{\mu}_2)$. This fiber, which is measured by $H^1_{\acute{e}t}\big(R, \mathbf{SL}_1(A)\big)$, has therefore at least two elements. However $\mathcal{L}_1$ is the *only* class of this fiber which is a multiloop algebra. In particular, $\mathcal{L}_1$ is not isomorphic to $\mathcal{L}_2$ *as a k–algebra*.

Remark 5.8. $\mathfrak{sl}_2 \otimes k[t_1^{\pm 1}]$ corresponds to the affine Kac-Moody Lie algebra $\widehat{\mathcal{L}}$ of type $A_1^{(1)}$. This algebra has Dynkin diagram $\circ$ <=> $\circ$. The nontrivial symmetry of this diagram "lifts" to an automorphism $\widehat{\sigma}$ of $\widehat{\mathcal{L}}$, which in turn induces an automorphism σ of the k–Lie algebra $\mathfrak{sl}_2 \otimes k[t_1^{\pm 1}]$. We have $\mathcal{L}_1 \simeq L(\mathfrak{sl}_2 \otimes k[t_1^{\pm 1}], \sigma)$. Thus our double loop algebra $\mathcal{L}_1 = L(\mathfrak{sl}_2, \sigma_1, \sigma_2)$ can be obtained as a single loop algebra of an affine algebra, but the automorphism σ of the affine algebra cannot be obtained from an automorphism of $\mathfrak{sl}_2$ by the base change $k[t_1^{\pm 1}]/k$. This point of view for the classification of double loop algebras (i.e. as single loop algebras of affine Kac-Moody algebras) will be described in [ABP3].

Remark 5.9. It is interesting to observe that unlike the nullity 1 case where inner automorphisms always lead to trivial loop algebras, two inner automorphisms may lead to non-trivial loop algebras. For type E_7, Theorem 3.17 shows that there must necessarily exists two commuting inner automorphisms whose corresponding loop algebra is not trivial. This was first empirically discovered by van de Leur with the aid of a computer [vdL].

If $\boldsymbol{x} = (x_1, ..., x_n)$ is an n–tuple of commuting elements of finite order of $\mathbf{G}^{ad}$, then $\operatorname{Ad}\boldsymbol{x} = (\operatorname{Ad}x_1, ..., \operatorname{Ad}x_n)$ is an n–tuple of commuting automorphisms of $\mathfrak{g}$. For convenience, we will denote the corresponding loop algebra $L(\mathfrak{g}, \operatorname{Ad}\boldsymbol{x})$ simply by $L(\mathfrak{g}, \boldsymbol{x})$.

Theorem 5.10. *Let* $\boldsymbol{x} = (x_1, x_2)$ *and* $\boldsymbol{y} = (y_1, y_2)$ *be two commuting pairs of elements of finite order of* $\mathbf{G}^{ad}$. *Then*

1. $L(\mathfrak{g}, \boldsymbol{x}) \simeq_{R-\mathrm{Lie}} L(\mathfrak{g}, \boldsymbol{y}) \Longleftrightarrow \mu(\boldsymbol{x}) = \mu(\boldsymbol{y})$.
2. $L(\mathfrak{g}, \boldsymbol{x}) \simeq_{k-\mathrm{Lie}} L(\mathfrak{g}, \boldsymbol{y}) \Longleftrightarrow \mu(\boldsymbol{x}) = \mu(\boldsymbol{y})^{\pm 1}$.

Proof. (1) This is consequence of the torsor interpretation of $L(\mathfrak{g}, \boldsymbol{x})$ and $L(\mathfrak{g}, \boldsymbol{y})$; namely that the boundary map $H^1(R, \mathbf{G}^{ad}) \to H^2(R, \boldsymbol{\mu})$ is bijective (Theorem 3.17), and determined by $\mu(\boldsymbol{x})$'s (Proposition 3.16).

(2) By Lemma 5.3, we are reduced to comparing $\mu(\boldsymbol{x})$ and $\mu({}^{\boldsymbol{a}}\boldsymbol{x})$. Consider then

$$ {}^{\boldsymbol{a}}\boldsymbol{x} = (x_1^{a_{11}} x_2^{a_{12}}\,,\, x_1^{a_{21}} x_2^{a_{22}}) \quad \text{where} \quad \boldsymbol{a} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}). $$

By definition

$$ \widetilde{x}_1^{a_{11}}\, \widetilde{x}_2^{a_{12}}\, \widetilde{x}_1^{a_{21}}\, \widetilde{x}_2^{a_{22}} = \mu({}^{\boldsymbol{a}}\boldsymbol{x})\, \widetilde{x}_1^{a_{21}}\, \widetilde{x}_2^{a_{22}}\, \widetilde{x}_1^{a_{11}}\, \widetilde{x}_2^{a_{12}}. $$

By repeated use of (3.3) we obtain $\mu({}^{\boldsymbol{a}}\boldsymbol{x}) = \mu(\boldsymbol{x})^{\det \boldsymbol{a}}$. Now (2) follows from (1) and Lemma 5.3. □

Remark 5.11. Clearly $\mu(x_1, x_2) = \mu(x_2, x_1)^{-1}$. As a consequence we recover the obvious fact that $L(\mathfrak{g}, x_1, x_2) \simeq_{k-\mathrm{Lie}} L(\mathfrak{g}, x_2, x_1)$. By contrast, we also obtain the following not entirely intuitive result: $L(\mathfrak{g}, x_1, x_2) \simeq_{R-\mathrm{Lie}} L(\mathfrak{g}, x_2, x_1)$ if and only if $\mu(\boldsymbol{x})$ is of period 2.

Remark 5.12. Similar considerations apply to any perfect and central finite dimensional algebra A for which $\mathbf{Aut}(A)^0$ is semisimple.

5.3 A characterization of multiloop algebras

The main theorem of [ABFP] asserts that centreless cores of EALAs which are modules of finite type over their centroids, are always multiloop algebras. The Lie algebra $\mathcal{L}_2$ described in Example 5.7, shows that for $n > 1$ there exists R_n–forms which *need not* be multiloop algebras. It thus seem important to give a criterion that distinguishes multiloop algebras among all forms.

The k–algebra $R_{n,\infty}$ has a very natural interpretation: As a k–space $R_{n,\infty}$ has basis $(t_1^{q_1}...t_n^{q_n})_{q_i\in\mathbb{Q}}$, and the multiplication is given by bilinear extension of $t_i^p t_i^q = t_i^{p+q}$. We give $R_{n,\infty}$ a $\mathbb{Q}^n$–grading in a natural fashion, by assigning $t_1^{q_1}\dots t_n^{q_n}$ degree $(q_1,\dots,q_n)$. Each $R_{n,d}$ is a graded subalgebra of $R_{n,\infty}$ whose homogeneous elements have degrees in $\frac{1}{d}\mathbb{Z}^n$. Note that a homogeneous element $r \in R_n$, when viewed as an element of $R_{n,d}$, is homogeneous of degree $d\lambda$ for some λ in the grading group $\frac{1}{d}\mathbb{Z}^n$ of $R_{n,d}$.

Theorem 5.13. [Recognition of multiloop algebras] *Let A be a finite dimensional simple algebra over an algebraically closed field k of characteristic 0. Let $n \geq 0$ be an integer. For a k–algebra $\mathcal{L}$ the following conditions are equivalent.*

1. *$\mathcal{L} \simeq_k L(A,\boldsymbol{\sigma})$ for some n–tuple $\boldsymbol{\sigma} = (\sigma_1,\dots,\sigma_n)$ of commuting finite order automorphisms of A.*

2. *There exists a $\mathbb{Q}^n$–grading of $\mathcal{L}$ and an $R_{n,\infty}$–algebra isomorphism $\psi : \mathcal{L}\otimes_{R_n} R_{n,\infty} \to A\otimes R_{n,\infty}$ which are compatible, namely $\psi(\mathcal{L}^\lambda\otimes 1) \subset A\otimes R_{n,\infty}^\lambda$ for all $\lambda \in \mathbb{Q}^n$.*

Proof. A multiloop algebra as in (1) clearly satisfies the conditions of (2). This follows at once form the very definition (5.1) of $L(A,\boldsymbol{\sigma})$, and the nature of the gradings.

Assume $\mathcal{L}$ and ψ are as in (2). We show that $\mathcal{L} \simeq L(A,\sigma_1,\dots,\sigma_n)$ by appealing to the realization theorem 8.3.2 of [ABFP]. To this end, it will suffice to give $\mathcal{L}$ a Λ–grading for which the following three conditions hold.

(a) Λ is finitely generated and torsion free.

(b) $\mathcal{L}$ is graded simple.

(c) There exists $d > 0$ such that $C(\mathcal{L}) = \underset{\lambda\in d\Lambda}{\oplus} C(\mathcal{L})^\lambda$. Furthermore, $C(\mathcal{L})^0 = k$.

Fix an $R_{n,\infty}$–algebra isomorphism $\psi : \mathcal{L}\otimes_{R_n} R_{n,\infty} \to A\otimes R_{n,\infty}$ such that $\psi(\mathcal{L}^\lambda\otimes 1) \subset A\otimes R_{n,\infty}^\lambda$ for all $\lambda \in \mathbb{Q}^n$.

Since A is finite dimensional, there exists $d_1 > 0$ such that $A\otimes 1 \subset \psi(\mathcal{L}\otimes R_{n,d_1})$. On the other hand, since $\mathcal{L}$ as an R_n–module is of finite type (Lemma 4.6(2)), there exists $d_2 > 0$ such that $\psi(\mathcal{L} \otimes 1) \subset A \otimes R_{n,d_2}$. If we now set $d = d_1 d_2$, the fact that ψ is $R_{n,\infty}$–linear shows that ψ induces, by restriction of the base ring, an $R_{n,d}$–algebra isomorphism

$$\psi : \mathcal{L} \otimes_{R_n} R_{n,d} \to A \otimes R_{n,d}$$

By assumption, for all $\lambda \in \mathbb{Q}^n$ we have

$$\psi(\mathcal{L}^\lambda \otimes 1) \subset A \otimes R^\lambda_{n,\infty} \cap A \otimes R_{n,d}.$$

As a consequence $\mathcal{L}^\lambda \otimes 1 \neq \{0\} \implies d\lambda \in \mathbb{Z}^n$. But since the extension $R_{n,d}/R_n$ is faithfully flat, the canonical map $\mathcal{L} \to \mathcal{L} \otimes_R R_{n,d}$ given by $x \to x \otimes 1$ is injective. In particular, then $\mathbb{Q}^n$–grading of $\mathcal{L}$ has support inside the subgroup $\Lambda = \frac{1}{d}\mathbb{Z}^n$.

As a Λ–graded algebra, we may identify $\mathcal{L}$ with a graded k–subalgebra of $A \otimes R_{n,d}$ (where $R_{n,d}$ is given the standard Λ–grading). Since $R_{n,d}$ is graded simple, $\mathcal{L}$ is graded simple by Proposition 4.14. This establishes (a) and (b) above.

Since $\mathcal{L}$ is graded simple, the centroid $C_k(\mathcal{L})$ of $\mathcal{L}$ inherits a Λ–grading

$$C_k(\mathcal{L}) = \bigoplus_{\lambda \in \Lambda} C_k(\mathcal{L})^\lambda,$$

where

$$C_k(\mathcal{L})^\lambda := \{\chi \in C_k(\mathcal{L}) : \chi(\mathcal{L}^\mu) \subset \mathcal{L}^{\lambda+\mu} \text{ for all } \mu \in \Lambda\}.$$

(see [ABFP] lemma 4.2.3). Let $r \in R_n$ be homogeneous. Then r, as an element of $R_{n,d}$, is homogeneous of degree $d\lambda$ for some $\lambda \in \Lambda$. If $x \in \mathcal{L}^\mu$ we have

$$\psi(rx \otimes 1) = \psi(x \otimes r) = r\psi(x \otimes 1) \in r(A \otimes R^\mu_{n,d}) \subset A \otimes R^{\mu+\lambda d}_{n,d}.$$

This forces $rx \in \mathcal{L}^{\mu+\lambda d}$, hence that $C_k(\mathcal{L}) = \bigoplus_{\lambda \in \Lambda} C_k(\mathcal{L})^{d\lambda}$. Finally, if $d\lambda = 0$, then r is a homogeneous element of R_n of degree 0, i.e. $r \in k$. Thus (c) above holds and the proof of the Theorem is now complete. □

Remark 5.14. By the Isotriviality Theorem [GP1, cor. 3.3] every R_n–form of A is split by $R_{n,\infty}$. The last Theorem therefore gives a way of recognizing multiloop algebras among all forms. The crucial ingredient is the existence of a $\mathbb{Q}^n$–grading compatible with that of $R_{n,\infty}$. Note how the defining relations of the algebra $\mathcal{M}$ of Example 5.7 "breaks" the compatibility between the gradings of the quaternion algebra $A(1, n)$ and $R_{n,\infty}$.

6 Conjectures

Throughout this section k is assumed to be algebraically closed and of characteristic 0. Let $R = k[t_1^{\pm 1}, t_2^{\pm 1}]$, and $K = k(t_1, t_2)$.

If the semisimple R–group $\mathbf{G}$ is isotropic (after the base change K/R), it is reasonable to expect that $H^1(R, \widetilde{\mathbf{G}}) = 1$. Indeed theorem 2.1 of [CTGP] states that the anisotropic kernel of a semisimple group defined over K is always of type A. This is somehow analogous to the fact that almost commuting rank zero pairs only appear in type A. In the anisotropic case however, this is not true: We have seen that $H^1(R, \mathbf{SL}_1(A(1,n)) \neq 1$ (Remark 3.23).

Conjecture 6.1. *Let $\mathbf{G}$ be a semisimple almost simple R–group with no factors of type A. Then the connecting map $H^1(R, \mathbf{G}) \to H^2(R, \boldsymbol{\mu})$ is bijective. In particular, $H^1_{loop}(R, \mathbf{G}) = H^1(R, \mathbf{G})$ (see Theorem 3.17).*

By Theorem 2.7(2), the conjecture is fully established for groups of type $\mathbf{G}_2$, $\mathbf{F}_4$ and $\mathbf{E}_8$ (because these groups are their own automorphism groups). The case of special orthogonal groups can be understood from the classification of R–quadratic forms.

Theorem 6.2. *(Parimala, [Pa])*

1. *Cancellation holds for rational R–isotropic forms : if q_1, q_2 are rationally isotropic R–forms such that $q_1 \perp q \cong q_2 \perp q$ for some quadratic R–form q, then $q_1 \cong q_2$.*

2. *Let q_1, q_2 be isotropic quadratic R–forms. If $q_1 \otimes_R K \cong q_2 \otimes_R K$, then $q_1 \cong q_2$.*

3. *Let q be a R–quadratic form of rank ≥ 5. Then q is diagonalizable and isotropic.*

Proof. 1) This is part of the proof of the classification of isotropic R–forms (proposition 3.4 of [Pa]).
2) Since the field K is of class C_2, the quadratic form q_K is isotropic. By theorem 3.5 of *loc. cit.*, there exists two $k[t_1^{\pm 1}]$–quadratic forms q_0 and q_1 such that $q = q_0 \perp \langle t_2 \rangle q_1$. Since quadratic forms over $k[t_1^{\pm 1}]$ are diagonalizable (Harder, cf. [Kn] §13.4.4), it follows that q is diagonalisable, i.e is a sum of rank one R–forms in the following list $\langle 1 \rangle$ $\langle t_1 \rangle$, $\langle t_2 \rangle$, $\langle t_1 t_2 \rangle$. Thus q contains an orthogonal summand $\langle z, z \rangle$ which is hyperbolic. We conclude that q is an isotropic R–quadratic form. $\square$

Corollary 6.3. *Let q be an R–quadratic form of rank ≥ 5. Then $H^1\big(R, \mathbf{Spin}(q)\big) = 1$ and $H^1\big(R, \mathbf{SO}(q)\big) \simeq \mathbf{Z}/2\mathbf{Z}$.*

Proof. Theorem 6.2 shows that q is diagonalisable and contains an hyperbolic summand. The exact sequence $1 \to \boldsymbol{\mu}_2 \to \mathbf{Spin}(q) \to \mathbf{SO}(q) \to 1$ of reductive R–groups induces the following exact commutative diagram of pointed sets

$$\begin{array}{ccccccc} \mathbf{SO}(q)(R) & \xrightarrow{Ns} & R^\times/(R^\times)^2 & \longrightarrow & H^1\big(R,\mathbf{Spin}(q)\big) & \longrightarrow & H^1\big(R,\mathbf{SO}(q)\big) \\ & & & & \downarrow & & \downarrow \\ & & & & 1 = H^1\big(K,\mathbf{Spin}(q)\big) & \longrightarrow & H^1\big(K,\mathbf{SO}(q)\big). \end{array}$$

Recall that $H^1\big(R,\mathbf{O}(q)\big)$ classifies isomorphism classes of quadratic modules of rank equal to rank(q). Parimala's theorem states that $H^1\big(R,\mathbf{O}(q)\big)$ injects in $H^1\big(K,\mathbf{O}(q)\big)$. By diagram chasing, it follows that the map

$$H^1\big(R,\mathbf{Spin}(q)\big) \to H^1\big(R,\mathbf{SO}(q)\big) \to H^1\big(R,\mathbf{O}(q)\big)$$

is trivial. On the other hand, since q is isotropic (over R), the sequence $1 \to \mathbf{SO}(q) \to \mathbf{O}(q) \to \mathbb{Z}/2\mathbb{Z} \to 1$ is split exact ([K2], proposition 5.2.2 page 225). It follows that the map $H^1(R,\mathbf{SO}(q)) \to H^1(R,\mathbf{O}(q))$ has trivial kernel, hence the triviality of the map $H^1(R,\mathbf{Spin}(q)) \to H^1(R,\mathbf{SO}(q))$. On the other hand, the spinor norm $\mathbf{SO}(q)(R) \to R^\times/(R^\times)^2$ contains even products of invertible values of q ([K2], page 232). Since q contains a hyperbolic summand, it follows that the spinor norm is surjective. By diagram chasing, we conclude that $H^1\big(R,\mathbf{Spin}(q)\big) = 1$. This implies that the kernel of the connecting map $H^1\big(R,\mathbf{SO}(q)\big) \to H^2(R,\mathbf{Z}/2\mathbf{Z}) = \mathbf{Z}/2\mathbf{Z}$ has trivial kernel. Now a classical twisting argument shows that the connecting map is indeed injective. Since it is also surjective by Theorem 2.7.(1), we conclude that $H^1\big(R,\mathbf{SO}(q)\big) \simeq \mathbf{Z}/2\mathbf{Z}$ as desired. □

In other words, Conjecture 6.1 holds for special orthogonal and spinor groups of quadratic forms of rank ≥ 5.

Our final conjecture states that, outside of type A, double loop algebras are completely determined by their Witt-Tits index.

Conjecture 6.4. *Let $\boldsymbol{x} = (x_1, x_2)$ and $\boldsymbol{y} = (y_1, y_2)$ be two commuting pairs of automorphisms of a finite dimensional simple Lie algebra $\mathfrak{g}$ over k. Assume $\mathfrak{g}$ is not of type A. Then $L(\mathfrak{g},\boldsymbol{x}) \otimes_R K$ is an isotropic finite dimensional simple Lie algebra over K. Furthermore,*

$$L(\mathfrak{g},\boldsymbol{x}) \simeq_{k-\text{Lie}} L(\mathfrak{g},\boldsymbol{y}) \Longleftrightarrow I(\boldsymbol{x}) = I(\boldsymbol{y}).$$

The assumption that $\mathfrak{g}$ is not of type A is necessary. For in type A_n, with $n >> 1$, there are anisotropic loop algebras which are not isomorphic (Theorem 5.10 and Proposition 2.1(6)). The conjecture holds if $\boldsymbol{x}$ and $\boldsymbol{y}$ are in $\mathbf{G}^{ad}$, for in this case the Brauer invariant and the Witt-Tits index determine each other.

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