LEVELS OF QUATERNION ALGEBRAS

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Abstract. The level of a ring R with $1 \neq 0$ is the smallest positive integer s such that -1 can be written as a sum of s squares in R, provided -1 is a sum of squares at all. D.W. Lewis showed that any value of type 2^n or $2^n + 1$ can be realized as level of a quaternion algebra, and he asked whether there exist quaternion algebras whose levels are not of that form. Using function fields of quadratic forms, we construct such examples.

1. Introduction

A famous result by E. Artin and O. Schreier [1] says that if F is a field (of characteristic $\neq 2$) then F has at least one ordering if and only if -1 cannot be written as a sum of squares in F. In this situation, the field F is said to be *formally real*, or *real* for short. Naturally, one might then ask how many squares are actually needed to write -1 as a sum of squares in a nonreal field F. This has lead to the definition of the level which we formulate for arbitrary (possibly nonassociative) rings with $1 \neq 0$:

Definition 1.1. Let R be a ring with $1 \neq 0$. Let *level* $s(R)$ is defined as follows:

(1) If -1 is not a sum of squares in R, then $s(R) = \infty$.
(2) If -1 is a sum of squares in R, then

$$s(R) = \min\{n \mid \exists x_1, \cdots, x_n \in R : \ -1 = x_1^2 + \cdots + x_n^2\}$$

In the early 1930s, Van der Waerden asked which values can arise as level of a field. At the time, all fields where the level was known and finite had level 1, 2 or 4. H. Kneser [7] proved in 1934 that the only possible finite values were of the form 1, 2, 4, 8 or certain multiples of 16, though there were still no known fields of finite level > 4. The complete solution to the level question was finally given by Pfister [15] who showed that the level of a field, if finite, must always be a 2-power, and that all these values could in fact be realized.

The level question for integral domains was solved in 1980 by Z.D. Dai, T.Y. Lam and C.K. Peng [2] who proved that any positive integer can occur as level of an integral domain, more precisely, they showed that the integral domain

$$R = \mathbb{R}[X_1, \cdots, X_n]/(1 + X_1^2 + \cdots + X_n^2)$$

has level n. The proof is topological in nature and invokes the Borsuk-Ulam theorem. Incidentally, the quotient field $F = \mathrm{Quot}(R)$ has level 2^k where k is such

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that $2^k \leq n < 2^{k+1}$, yielding Pfister's original examples of fields whose level is a prescribed 2-power.

Levels of noncommutative rings have been studied by various authors, see Lewis's survey [13] and the list of references there. There are variations of the notion of level in the noncommutative case such as the product level, where squares are replaced by products of squares, and for rings R with involution σ such as the hermitian level, where squares are replaced by "hermitian squares" $x\sigma(x)$. But we will only consider the level as defined above.

Most of the study of levels in the noncommutative case concerns central simple algebras, and quaternion algebras in particular. Obviously, we may assume that the centers of these algebras are fields of characteristic different from 2 in order for the level problem to be of any interest. Lewis [11] constructed examples of quaternion division algebras whose levels take any given prescribed value of the form 2^k or $2^k + 1$, while indicating that it is not known whether other values can be realized.

This has then been formulated explicitly as a question by Leep [10, Question (2)], and again by Lewis [13, Open Question 1]:

Question. What integers can occur as $s(D)$ for a quaternion division algebra D ? In particular, can $s(D)$ take values that are not of the form 2^k or $2^k + 1$?

In this note, we will give a partial answer to that question by showing that there are infinitely many values not of the form 2^k or $2^k + 1$ that occur as level of quaternion division algebras. More precisely,

Theorem 1.2. *Let m be a nonnegative integer. Then there exists a quaternion division algebra D with $m+1 \leq s(D) \leq m+1+[m/3]$ (where $[x]$ ($x \in \mathbb{R}$) denotes the largest integer $\leq x$).*

For example, for $m = 5$, we deduce the existence of a quaternion division algebra D with $s(D) \in \{6, 7\}$. For $m = 17$ and $m = 23$, we get such algebras D, D' with $18 \leq s(D) \leq 23$ and $24 \leq s(D') \leq 31$. More generally, for any $k \geq 4$, we can in a similar way find disjoint intervals I and I' inside $[2^k + 2, 2^{k+1} - 1]$ and construct quaternion division algebras D and D' with $s(D) \in I$ and $s(D') \in I'$, showing that in each interval bounded by large enough consecutive 2-powers, there exist at least two values not of the form 2^k or $2^k + 1$ that can be realized as level of quaternion division algebras.

Our method of proof uses function fields of quadrics and facts about isotropy of quadratic forms over such function fields. The arguments as such in our proofs are rather elementary. However, at one point we do have to invoke a very deep result by Karpenko and Merkurjev, Theorem 2.2. The drawback of our elementary approach is that we can only give bounds for the level of the constructed quaternion algebra and not its exact value. It should also be noted that our methods are in spirit similar to those employed by Laghribi and Mammone [8] who gave a construction of quaternion algebras of level 2^k resp. $2^k + 1$ different from Lewis's original construction of such algebras.

We should note that James O'Shea [14] has used similar methods to study the level of octonion algebras and to construct octonion algebras of prescribed level.

The paper is structured as follows. In the next section, we collect results from quadratic form theory that will be needed in our construction. In section 3, we motivate and sketch the idea underpinning our construction and explain its limitations. The construction itself will be given in section 4, where we also derive explicit bounds on the level of the produced quaternion algebras. In section 5, we sketch a result by O'Shea [14] concerning quaternion algebra whose so-called sublevels are bounded in a similar way.

2. Some facts about quadratic forms

Throughout this article, fields are assumed to be of characteristic $\neq 2$. All what we need from the algebraic theory of quadratic forms over fields can be found in Lam's book [9]. By 'quadratic form' over a field F (or 'form' for short), we will always mean a finite-dimensional nonsingular quadratic form defined over F. Recall that a form over F is called anisotropic if it does not represent 0 nontrivially over F, and hyperbolic if it is isometric to an orthogonal sum of hyperbolic planes $\mathbb{H} = \langle 1, -1\rangle$.

An orthogonal sum $\varphi \perp \cdots \perp \varphi$ of n copies of a form φ over F will be denoted by $n \times \varphi$, whereas $a\varphi$ with $a \in F^*$ denotes the form φ scaled by the factor a.

By Witt decomposition, any form φ decomposes up to isometry in a unique way as an orthogonal sum $\varphi \cong \varphi_{an} \perp \varphi_h$ with φ_{an} anisotropic and φ_h hyperbolic. The Witt index of φ is defined to be $i_W(\varphi) = \frac{1}{2}\dim\varphi_h$, and it is in fact nothing else but the dimension of a maximal totally isotropic subspace of φ.

A form π is called an n-fold Pfister form if π is isometric to a tensor product of n binary forms: $\pi \cong \langle 1, -a_1\rangle \otimes \cdots \otimes \langle 1, -a_n\rangle$, $a_i \in F^*$. We write $\langle\langle a_1, \cdots, a_n\rangle\rangle$ for short.

If K/F is a field extension and φ is a form over F, we denote by φ_K the form obtained from φ by scalar extension to K.

The function field $F(\varphi)$ of a quadratic form φ over F is defined as follows. If $\dim\varphi \geq 3$, then $F(\varphi)$ is the function field of the projective quadric $\varphi = 0$. If $\varphi \cong \langle a, b\rangle$ is anisotropic (i.e. $\varphi \not\cong \mathbb{H}$), we put $F(\varphi) = F(\sqrt{-ab})$. If $\varphi \cong \mathbb{H}$ or if $\dim\varphi = 1$, we put $F(\varphi) = F$.

It is well known that for $\dim\varphi \geq 2$, $F(\varphi)$ can be realized as a purely transcendental extension of transcendence degree $\dim\varphi - 2$, followed by a quadratic extension, and that $F(\varphi)/F$ is purely transcendental if and only if φ is isotropic.

If φ is an anisotropic form over F of dimension ≥ 2, then φ is isotropic over $F(\varphi)$, and the first Witt index $i_1(\varphi)$ of φ is defined to be $i_W(\varphi_{F(\varphi)})$. It is a well known fact due to the generic nature of the function field that for any field extension K/F with φ_K isotropic, one has $i_W(\varphi_K) \geq i_1(\varphi)$.

Our construction uses in a crucial way the isotropy behaviour of a quadratic form when passing to the function field of another form.

The next lemma is folklore.

Lemma 2.1. *Let φ and ψ be forms over F, and let $F(x)$ be the rational function field in the variable x over F. Then $i_W(\varphi \perp x\psi) = i_W(\varphi) + i_W(\psi)$. In particular, $\varphi \perp x\psi$ is anisotropic over $F(x)$ if and only if φ and ψ are anisotropic over F.*

Proof. Using a standard degree argument, one readily checks that $\varphi \perp x\psi$ is anisotropic over $F(x)$ if and only if φ and ψ are anisotropic over F (this is in fact [9, Exercise 3, p. 313]). The statement about the Witt indices for arbitrary φ and ψ now follows by applying Witt decomposition to φ and ψ over F. □

The next result is a deep theorem due to Karpenko and Merkurjev [6]. It is the crucial ingredient in our construction.

Theorem 2.2 (Karpenko-Merkurjev)**.** *Let φ and ψ be anisotropic forms over F. Suppose that $\varphi_{F(\psi)}$ is isotropic. Then $\dim\varphi - i_1(\varphi) \geq \dim\psi - i_1(\psi)$, and equality holds if and only if $\psi_{F(\varphi)}$ is isotropic as well.*

The next lemma is also well-known, we include the straighforward proof for the reader's convenience.

Lemma 2.3. *Let φ be a form over F and let ψ be a subform of φ, i.e. $\varphi \cong \psi \perp \tau$ for some form τ.*

(i) *If φ is isotropic and $\dim\psi > \dim\varphi - i_W(\varphi)$, then ψ is isotropic as well.*

(ii) *Let φ be an anisotropic form over F and let ψ be a subform of F with $\dim\psi > \dim\varphi - i_1(\varphi)$. Then $\dim\varphi - i_1(\varphi) = \dim\psi - i_1(\psi)$.*

Proof. (i) Consider ψ as the restriction of φ to a sub-vector space W of dimension $\dim\psi$ of the underlying vector space V of φ. Now any maximal totally isotropic subspace of φ is of dimension $i_W(\varphi)$ and intersects therefore nontrivially with W by an easy dimension count. Therefore, W contains a nonzero isotropic vector.

(ii) By assumption on $\dim\psi$ and by (i), using that $i_1(\varphi) = i_W(\varphi_{F(\varphi)})$, we have that ψ becomes isotropic over $F(\varphi)$. On the other hand, ψ and therefore also φ becomes isotropic over $F(\psi)$. The result now follows from Theorem 2.2 □

In certain situations, one can say a little more about i_1. This will allow us to refine the main theorem slightly.

Lemma 2.4. *Let π be an anisotropic n-fold Pfister form over F and let φ be another form over F.*

(i) *There exist forms φ_0 and φ_h such that $\pi\otimes\varphi_0$ is anisotropic, φ_h is hyperbolic, and $\pi\otimes\varphi \cong \pi\otimes\varphi_0 \perp \pi\otimes\varphi_h$. In particular, $i_W(\pi\otimes\varphi) = 2^{n-1}\dim\varphi_h = 2^n i_W(\varphi_h)$.*

(ii) *Suppose $\pi\otimes\varphi$ is anisotropic, and let ψ be a subform of $\pi\otimes\varphi$ and $\dim\psi > 2^n(\dim\varphi - 1)$. Then $\dim\psi - i_1(\psi) \leq 2^n(\dim\varphi - 1)$.*

Proof. (i) This is well-known, see e.g. [18, Theorem 2].

(ii) By (i) and the definition of i_1, we have $\dim(\pi\otimes\varphi) - i_1(\pi\otimes\varphi) \leq 2^n\dim\varphi - 2^n < \dim\psi$. By Lemma 2.3(ii), we have

$$\dim\psi - i_1(\psi) = \dim(\pi\otimes\varphi) - i_1(\pi\otimes\varphi) \leq 2^n(\dim\varphi - 1)\ . \qquad \square$$

We also need a few facts about orderings and signatures. Recall that an ordering P on a field F is a subset $P \subsetneq F$ such that $P+P \subseteq P$, $P\cdot P \subseteq P$ and $P\cup(-P) = F$. One can readily check that if P is an ordering of F, then $-1 \notin P$, $P\cap(-P) = \{0\}$, and every sum of squares is in P. In fact, by Artin-Schreier, F is real if and only if F has an ordering.

If P is an ordering on F and if K/F is a field extension, then we say that P extends to K if there is an ordering Q on K such that $Q\cap F = P$.

Given an ordering P on F, we obtain a total ordering "$\leq_P$" on F by defining $a \geq_P b$ if $a - b \in P$.

Let $\varphi \cong \langle a_1, \cdots, a_n\rangle$ be a quadratic form over a real field F, and let P be an ordering on F. The signature of φ at P is defined by

$$\mathrm{sgn}_P(\varphi) = |\{i \mid a_i >_P 0\}| - |\{i \mid a_i <_P 0\}|$$

It is an invariant for the isometry class of φ (this is nothing but Sylvester's law of inertia). We say that φ is indefinite at P if $\dim\varphi > |\,\mathrm{sgn}_P(\varphi)|$.

Using Witt decomposition and the fact that $\mathrm{sgn}_P(\mathbb{H}) = 0$, it is obvious that $\dim \varphi_{an} \geq |\,\mathrm{sgn}_P(\varphi)|$.

Lemma 2.5. *Let φ be a quadratic form over a real field F, $\dim \varphi \geq 2$, and let P be an ordering on F. Then P extends to $F(\varphi)$ if and only if φ is indefinite at P.*

In this situation and if ψ is another form over F, then $\dim(\psi_{F(\varphi)})_{an} \geq |\,\mathrm{sgn}_P(\psi)|$.

Proof. The first part is due to Elman-Lam-Wadsworth [3, Theorem 3.5] and independently Knebusch [4, Lemma 10].

The second part follows readily from the remark preceding the proposition, using the fact that if Q is an ordering on $F(\varphi)$ extending P, then $\mathrm{sgn}_P(\psi) = \mathrm{sgn}_Q(\psi_{F(\varphi)})$. □

3. Some observations about the levels of quaternion algebras

Recall that a quaternion algebra over F is a 4-dimensional central simple F-algebra. If Q is a quaternion algebra, one can find $a, b \in F^*$ and an F-basis $\{1, i, j, k\}$ of Q such that $i^2 = a$, $j^2 = b$, and $ij = -ji = k$, in which case we denote Q by $(a, b)_F$.

$Q = (a, b)_F$ is a division algebra if and only if the Pfister form $\langle\langle a, b\rangle\rangle$ is anisotropic over F, and $(a, b)_F \cong (c, d)_F$ as F-algebras if and only if $\langle\langle a, b\rangle\rangle \cong \langle\langle c, d\rangle\rangle$ (see [9, Ch.III, Theorems 2.5, 2.7]).

If Q is not division, then $Q \cong M_2(F)$ and we have $s(Q) = 1$. Indeed,

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} .$$

Remark 3.1. This also readily implies that for any split even-dimensional central simple algebra A, i.e. $A \cong M_{2n}(F)$ for some $n \geq 1$, we get $s(A) = 1$. In the odd-dimensional split case, the situation is more complicated. The full result for split algebras $M_m(F)$, $m \geq 2$, reads

$$s(M_m(F)) = \begin{cases} 1 & \text{if } m \geq 2 \text{ is even} \\ \min\{3, s(F)\} & \text{if } m \geq 3 \text{ is odd} \end{cases}$$

This follows readily from a more general result due to Richman [17] that states that every element $x \in M_m(F)$ is a sum of two squares provided x is not of type cI_m with m odd and $c \in F^*$ not a sum of two squares. Such elements cI_m can be written as sums of three but not two squares (here, I_m denotes the $m \times m$ identity matrix).

Let us from now on assume that $Q = (a, b)_F$ is a division quaternion algebra over F (with F-basis $\{1, i, j, k\}$ as above). We start with some elementary observations.

Let $\zeta \in Q$. Then $\zeta = x + yi + zj + wk$ with $x, y, z, w \in F$, and we call x the *scalar part* of ζ, and $\zeta' = yi + zj + wk$ its *pure part.* If $x = 0$, then $\zeta = \zeta'$ is called a *pure quaternion.* One readily computes that

$$\zeta^2 = x^2 + ay^2 + bz^2 - abw^2 + 2x\zeta' ,$$

in particular, the scalar part of ζ^2 is represented by the quadratic form $\langle 1, a, b, -ab\rangle$, whereas $\zeta'^2 = ay^2 + bz^2 - abw^2$ is an element in F represented by the form $\langle a, b, -ab\rangle$. This immediately leads to the following observation:

Lemma 3.2. (i) *If -1 is a sum of n squares in Q, then -1 is represented by $n \times \langle 1, a, b, -ab\rangle$ over F, i.e., $\langle 1\rangle \perp (n \times \langle 1, a, b, -ab\rangle)$ is isotropic over F.*

(ii) -1 *is a sum of* n *squares of pure quaternions in* Q *if and only if* -1 *is represented by* $n\times\langle a,b,-ab\rangle$ *over* F*, i.e.,* $\langle 1\rangle\perp(n\times\langle a,b,-ab\rangle)$ *is isotropic over* F*.*

For $n\geq 1$, let us put

$$\begin{aligned}\psi_n &:= \langle 1\rangle\perp(n\times\langle 1,a,b,-ab\rangle)\ ,\\ \varphi_n &:= \langle 1\rangle\perp(n\times\langle a,b,-ab\rangle)\ .\end{aligned}$$

Note that $\dim\psi_n = 4n+1$ and $\dim\varphi_n = 3n+1$.

The previous lemma readily implies the following criterion.

Corollary 3.3. *Let* $n>m\geq 0$ *be integers. Let* $Q=(a,b)_F$ *and* φ_n *and* ψ_m *be as above. If* φ_n *is isotropic and* ψ_m *is anisotropic, then* $s(Q)\in[m+1,n]$*.*

Remark 3.4. With this criterion at hand, one might try the following 'generic' construction of quaternion algebras of a given level in $[m+1,n]$.

Start with a quaternion division algebra $(a,b)_F$ over a suitably chosen F (in which -1 is not a square) such that φ_n and ψ_m are anisotropic ($m<n$). Let now $K_n = F(\varphi_n)$.

The 4-dimensional form $\langle\langle a,b\rangle\rangle$, which is anisotropic over F since $(a,b)_F$ is division, will stay anisotropic over K_n. Indeed, otherwise $\langle\langle a,b\rangle\rangle$ would become hyperbolic over K_n, thus φ_n would be similar to a subform of $\langle\langle a,b\rangle\rangle$ over F (see, e.g., [9, Ch. X, Cor. 4.9]). This is impossible for dimension reasons if $n\geq 2$, and it is also impossible for $n=1$ by comparing determinants because $\det\varphi_1 = \det\langle 1,a,b,-ab\rangle = -1 = -\det\langle\langle a,b\rangle\rangle\in F^*/F^{*2}$ with -1 not a square in F. In particular $Q_{K_n}=(a,b)_{K_n}$ will be division.

Clearly, $(\varphi_n)_{K_n}$ is isotropic, in particular, $s(Q_{K_n})\leq n$. If one can now show that ψ_m stays anisotropic over K_n, then by the previous corollary, $s(Q_{K_n})\in[m+1,n]$.

If this reasoning worked for $m=n-1$, we could deduce that $s(Q_{K_n})=n$. And although this construction might indeed yield a quaternion algebra of level n, we cannot, unfortunately, employ the above reasoning for $m=n-1$ in general as the following result shows (which generalizes Proposition 2.5 in [8]).

Proposition 3.5. *Keep the notations as above. Let* $n=2^rs$ *be a positive integer with* s *odd, and let* $t\geq 1$ *be maximal such that* $s\equiv -1 \bmod 2^t$*. Assume that* $t\geq 3$ *or* $r\geq 2$*, and that* φ_n *is isotropic. Then so is* ψ_{n-1}*.*

Proof. Assume first that $t\geq 3$ and let $\tau=(n+2^r)\times\langle 1,a,b,-ab\rangle$. Then 2^{r+t} divides $n+2^r$ (but not 2^{r+t+1} by the maximality of t), and we can write $\tau\cong 2^{r+t}\times\sigma$ for some form σ. Note that $\varphi_n\subset\tau$, so τ is isotropic. If the $(r+t)$-fold Pfister $\langle\langle -1,\cdots,-1\rangle\rangle$ is isotropic and thus hyperbolic, then so is τ and we clearly have $i_W(\tau)\geq 2^{r+t}$. This also holds if the $(r+t)$-fold Pfister $\langle\langle -1,\cdots,-1\rangle\rangle$ is anisotropic by Lemma 2.4. But we also have that $\psi_{n-1}\subset\tau$, and

$$\dim\tau-\dim\psi_{n-1} = 4(n+2^r)-(4(n-1)+1) = 2^{r+2}+3 < 2^{r+3}\leq 2^{r+t}\leq i_W(\tau)\ ,$$

hence ψ_{n-1} is isotropic by Lemma 2.3.

Suppose now that $r\geq 2$. Then we put $\tau=n\times\langle 1,a,b,-ab\rangle$. This time, τ is divisible by the r-fold Pfister $\langle\langle -1,\cdots,-1\rangle\rangle$, φ_n and ψ_{n-1} are subforms of τ, and similarly as before we get $\dim\tau-\dim\psi_{n-1}=4n-(4(n-1)+1)=3<2^r\leq i_W(\tau)$, and again, ψ_{n-1} is isotropic by Lemma 2.3. □

So, for example, if φ_7 is isotropic ($r = 0$, $s = 7$, $t = 3$), then so is ψ_6, and for any n divisible by 4, if φ_n is isotropic, then so is ψ_{n-1}.

We do not know in general whether the construction described in Remark 3.4 will actually yield $s(Q_{K_n}) = n$ (except for small n or values of type $n = 2^k + 1$ for certain F, a, b, see Remark 4.4 below). However, the above reasoning to obtain $s(Q_{K_n}) \in [m+1, n]$ will work (for certain $a, b \in F^*$ over a suitably chosen F) if we make n sufficiently large compared to m, using Theorem 2.2. We explain this construction in the following section.

4. The construction

Throughout this section, let F_0 be any real field, and let $F = F_0(a, b)$ be the rational function field in the two variables a, b over F_0. Let $Q = (a, b)_F$ and let φ_n, ψ_m be as above, with integers $n > m \geq 0$.

Lemma 4.1. *Let $K_n = F(\varphi_n)$.*

(i) *Q is a division algebra over F and stays a division algebra over K_n.*
(ii) *φ_n and ψ_m are anisotropic.*
(iii) *$i_1(\varphi_n) = 1$.*

Proof. It follows easily from F_0 being real together with Lemma 2.1 that $\langle\langle a, b\rangle\rangle$, φ_n and ψ_m are anisotropic. In particular, $Q = (a, b)_F$ is a division algebra and will stay so over K_n as explained in Remark 3.4.

To show (iii), let P be any ordering on F with $a <_P 0$ and $b <_P 0$ (such orderings always exist). Then $-ab <_P 0$ and we get

$$|\operatorname{sgn}_P(\varphi_n)| = |\operatorname{sgn}_P((n \times \langle a, b, -ab\rangle) \perp \langle 1\rangle)| = 3n - 1 < 3n + 1 = \dim \varphi_n \ ,$$

so φ_n is indefinite at P and it follows from Lemma 2.5 that $3n-1 \geq \dim((\varphi_n)_{K_n})_{an} \geq |\operatorname{sgn}_P(\varphi_n)| = 3n - 1$, therefore $\dim((\varphi_n)_{K_n})_{an} = 3n - 1 = \dim \varphi_n - 2$ and thus $i_1(\varphi_n) = 1$. □

Corollary 4.2. *Let $n > m \geq 0$ be integers. If $4m + 1 - i_1(\psi_m) < 3n$, then $s(Q_K) \in [m+1, n]$.*

Proof. We have $\dim \varphi_n - i_1(\varphi_n) = 3n + 1 - i_1(\varphi_n) = 3n$ by Lemma 4.1(iii), and $\dim \psi_m - i_1(\psi_m) = 4m + 1 - i_1(\psi_m)$. By Theorem 2.2, if $4m + 1 - i_1(\psi_m) < 3n$, then φ_m stays anisotropic over $K_n = F(\varphi_n)$. It follows from Remark 3.4 that $s(Q_{K_n}) \in [m+1, n]$. □

Note that $i_1(\psi_m) \geq 1$ for $m \geq 1$, and hence, we can always construct Q_{K_n} as above with $s(Q_{K_n}) \in [m+1, n]$ whenever $4m < 3n$ (this also clearly holds for $m = 0$). Thus, we obtain Theorem 1.2 as corollary:

Corollary 4.3. *Let $n = m + 1 + [m/3]$. Then $s(Q_{K_n}) \in [m+1, m+1+[m/3]]$.*

Remark 4.4. This corollary obviously allows us to get levels 1, 2, 3. It should furthermore be noted that it is possible to show that in the above situation, ψ_{2^k} stays anisotropic over $K_n = F(\varphi_n)$ with $n = 2^k + 1$, so that we then get $s(Q_{K_n}) = 2^k + 1$. This was shown in [8] and it yields a construction of quaternion algebras of levels of the form $2^k + 1$ that is different from Lewis's original one.

The preceding corollary can be refined somewhat by getting more information on $i_1(\psi_m)$ for certain values of m.

Corollary 4.5. *Let $m = 2^k m'$ with m' odd, and let $\ell \geq 1$ be maximal such that $m' \equiv -1 \bmod 2^\ell$. Let $n = m + 1 + [\frac{m-2^k(2^\ell-4)}{3}]$. Then $s(Q_{K_n}) \in [m+1, n]$.*

Before we prove this, let us remark that if $\ell = 1$, then this yields an interval with a larger upper bound than Corollary 4.3. If $\ell = 2$, it yields the same interval, and if $\ell > 2$ it yields an interval with strictly smaller upper bound.

Proof. In view of the preceding remark, we may assume that $\ell \geq 2$. Write $m = 2^k(r2^\ell - 1)$ with r odd. Consider the form $\tau \cong (m + 2^k) \times \langle 1, a, b, -ab\rangle$ (which is clearly anisotropic). Since $m + 2^k = r2^{k+\ell}$, it follows that τ is divisible by the $(k+\ell)$-fold Pfister form $\langle\langle -1, \cdots, -1\rangle\rangle$. Hence, by Lemma 2.4, we have $i_1(\tau) \geq 2^{k+\ell}$.

Now clearly $\psi_m \subset \tau$. Furthermore, $2^{k+2} \leq 2^{k+\ell}$ since we assumed $\ell \geq 2$. Hence

$$\dim \psi_m = 4m + 1 > 4(m + 2^k) - 2^{k+\ell} = \dim \tau - 2^{k+\ell} \geq \dim \tau - i_1(\tau)$$

and thus, by Lemma 2.4, we get $\dim \psi_m - i_1(\psi_m) = \dim \tau - i_1(\tau) \leq 4m - 2^k(2^\ell - 4)$. Thus, if $n = m+t$ is such that $3(m+t) > 4m - 2^k(2^\ell - 4)$, i.e. $3t > m - 2^k(2^\ell - 4)$, we have that $s(Q_{K_n}) \in [m + 1, n]$. The smallest integer t for which this holds is $t = 1 + [\frac{m-2^k(2^\ell-4)}{3}]$, and with this t and $n = m + t$ we get $s(Q_{K_n}) \in [m + 1, n] = [m + 1, m + 1 + [\frac{m-2^k(2^\ell-4)}{3}]]$. □

Example 4.6. For $m = 94 = 2(3.2^4 - 1)$, we have $k = 1$ and $\ell = 4$. Corollary 4.3 yields a quaternion algebra whose level is in the interval $[95, 95+[95/3]] = [95, 126]$, whereas Corollary 4.5 yields a level in $[95, 95 + [(95 - 2(16 - 4))/3]] = [95, 118]$.

5. Some remarks on the sublevel

The sublevel of a unitary ring R is defined as follows:

(1) If 0 is not a sum of nonzero squares in R, then $\underline{s}(R) = \infty$.
(2) If 0 is a sum of nonzero squares in R, then

$$\underline{s}(R) = \min\{n \mid \exists x_1, \cdots, x_n, x_{n+1} \in R, x_i^2 \neq 0 \;:\; 0 = x_1^2 + \cdots + x_{n+1}^2\}$$

Generally, $\underline{s}(R) \leq s(R)$, and one clearly has equality if R is a field. For a quaternion division algebra $Q = (a, b)_F$ over a field F (of characteristic $\neq 2$), it is still an open question how $s(Q)$ and $\underline{s}(Q)$ are related to each other in general. In all cases where the precise values are known, one has $s(Q) \in \{\underline{s}(Q), \underline{s}(Q) + 1\}$.

Using a similar argument as before, one readily sees that if $\widehat{\psi}_m \cong m \times \langle 1, a, b, -ab\rangle$ is anisotropic, then $\underline{s}(Q) \geq m$. On the other hand, if φ_n (as before) is isotropic, then one readily sees that 0 can be written in a nontrivial way as a sum of $n + 1$ squares and so $\underline{s}(Q) \leq n$. It is now not surprising that similar techniques as in our construction of certain new levels can be used to generate new sublevels, and it was O'Shea [14] who adapted our method accordingly and gave explicit details. We quickly sketch one of his results since all the tools are in place.

If, as in the previous section, $F = F_0(a, b)$ with real F_0 and variables a, b, we see that $\widehat{\psi}_m$ and φ_n are anisotropic. Again, let $K_n = F(\varphi_n)$. If we write $m = 2^k m'$ with m' odd, then it follows from Lemma 2.4 that $i_1(\widehat{\psi}_m) \geq 2^k$. Recall that $i_1(\varphi_n) = 1$. Thus, by Theorem 2.2, we have that $\widehat{\psi}_m$ stays anisotropic over K_n if $\dim \varphi_n - i_1(\varphi_n) = 3n > 4m - 2^k \geq \dim \widehat{\psi}_m - i_1(\widehat{\psi}_m)$. This is satisfied whenever $n \geq m + 1 + [(m - 2^k)/3]$, so we obtain

Proposition 5.1 (O'Shea [14]). *Let Q and $m = 2^k m'$ (m' odd) be as above. Put $n = m + 1 + [(m - 2^k)/3]$ and let K_n be as above. Then $m \leq \underline{s}(Q_{K_n}) \leq s(Q_{K_n}) \leq m + 1 + [(m - 2^k)/3]$.*

For example, $m = 5$ yields a quaternion algebra of sublevel in $[5, 7]$ This shows that the sublevel of a quaternion division algebra can take values other than 3 or powers of 2, and it is clear that the above construction yields infinitely many new values that are not of that type, thus partially answering Open Question 2 in [13].

Remark 5.2. If we are only interested in the level, then Proposition 5.1 does not yield smaller intervals with given lower bound than those obtained previously. For if $k = 0$, then this proposition yields the interval $[m', m' + 1 + [(m' - 1)/3]]$, whereas Corollary 4.3 (with $m + 1 = m'$ there) gives $[m', m' + [(m' - 1)/3]]$ (and Corollary 4.5 might still yield smaller upper bounds depending on the 2-adic value of $m' - 1$). If $k > 0$, then one obtains the same interval as in Corollay 4.5 by replacing the tuple (m, k, r, ℓ) in the statement there by the tuple $(m - 1, 0, m', k)$, using that $m = (m - 1) + 1 = (2^k m' - 1) + 1$.

For further results on levels and sublevels of quaternion and octonion algebras obtained using variations of our methods, we refer to [14].

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