NEW VALUES FOR THE LEVEL AND SUBLEVEL OF COMPOSITION ALGEBRAS

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Abstract

Constructions of quaternion and octonion algebras, suggested to have new level and sublevel values, are proposed and justified. In particular, octonion algebras of level and sublevel 6 and 7 are constructed. In addition, Hoffmann's proof of the existence of infinitely many new values for the level of a quaternion algebra is generalised and adapted.

1 Introduction

In [Lew1], Lewis constructed quaternion algebras of level 2^k and $2^k + 1$, both of sublevel 2^k, for all integers $k \geq 0$. Laghribi and Mammone recovered these values as the level of a quaternion algebra using function field techniques (see [LM]). Pumplün employed their methodology in [Pu] to construct octonion algebras of level 2^k and $2^k + 1$. By constructing a quaternion algebra of sublevel 3, Krüskemper and Wadsworth produced the first example of a quaternion algebra whose sublevel was not a power of 2 (see [KW]). This construction was subsequently employed to prove the existence of an octonion algebra of sublevel 3, and indeed one of sublevel 5 was also produced (see [O'S]). Heretofore, these remained the only known values for the level and sublevel of quaternion and octonion algebras.

In Section 3.1, we introduce certain quaternion and octonion algebras, denoted $Q(n)$ and $O(n)$ respectively, which we conjecture to be of level and sublevel n, for all n.

In Section 3.1.1, we proceed to precisely determine the level and sublevel of these algebras for certain values of n. In particular, we produce the first example of a composition algebra whose known level is not equal to 2^k or $2^k + 1$ for some k.

Section 3.1.2 sees us outline the argument employed by Hoffmann [H3] to bound the level of $Q(n)$, proving the existence of infinitely many values for the level of $Q(n)$ which are not of the form 2^k or $2^k + 1$ for some k. We extend this argument to $O(n)$, and refine these bounds for particular n-values.

We conclude with Section 3.2, wherein we introduce alternative constructions of quaternion and octonion algebras, which we suggest to be of level and sublevel $l \cdot 2^k$. Some merits of these constructions are outlined.

2 Preliminaries

Let F denote a field of characteristic $\neq 2$, F_0 a formally real field and C a unital, not necessarily associative, F-algebra.

A map $*$ is called an *involution* on C if it is an anti-automorphism of period 2. We have $C = \mathrm{Sym}(C, *) \oplus \mathrm{Skew}(C, *)$, with $\mathrm{Sym}(C, *) = \{x \in C | x^* = x\}$ and $\mathrm{Skew}(C, *) = \{x \in C | x^* = -x\}$. An involution $*$ is called *scalar* if $x^*x \in F$ and $x^* + x \in F$ for all $x \in C$. For an algebra C with scalar involution $*$, we call $t_C(x) = x + x^*$ the *trace* of C and the quadratic form $T_C : C \to F$, $T_C(x) = t_C(x^2)$ the *trace form*.

The *Cayley-Dickson doubling process* is an algorithm for constructing new algebras with scalar involution from old ones. Applying the process to an algebra C with scalar involution $*$, together with a chosen scalar $\mu \in F^\times$, produces a new algebra, $\mathrm{Cay}(C, \mu)$, the *Cayley-Dickson double* of C, whose scalar involution we will also denote by $*$. We note that $\mathrm{Cay}(C, \mu)$ is the F-module $C \times C$, with multiplication defined by $(u, v)(u', v') = (uu' + \mu v'^* v, v'u + vu'^*)$ and involution given by $(u, v)^* = (u^*, -v)$, for $u, u', v, v' \in C$.

An algebra C is a *composition algebra* if there exists a nondegenerate quadratic form q on C which allows composition, that is $q(xy) = q(x)q(y)$ for all $x, y \in C$. Composition algebras are of rank 1, 2, 4 or 8. The composition algebras of rank 2 are the quadratic étale F-algebras; the composition algebras of rank 4 are the (non-commutative) quaternion algebras and those of rank 8 are the (non-commutative and non-associative) octonion algebras. For $a, b \in F^\times$, the *quaternion algebra* $Q = \left(\frac{a,b}{F}\right)$ over F is a 4-dimensional F-vector space with basis $\{1, i, j, k\}$, satisfying $i^2 = a, j^2 = b$ and $ij = -ji = k$. For $a, b, c \in F^\times$, the *octonion algebra* $\left(\frac{a,b,c}{F}\right)$ over F is defined as $\left(\frac{a,b,c}{F}\right) := \mathrm{Cay}\left(\left(\frac{a,b}{F}\right), c\right)$. We note that

$\mathrm{Cay}\left(\left(\frac{a,b}{F}\right),c\right) = \left(\frac{a,b}{F}\right) \oplus \left(\frac{a,b}{F}\right)e$ is an 8-dimensional F-vector space with basis $\{1,i,j,k,e,ie,je,ke\}$, satisfying $i^2 = a, j^2 = b$ and $e^2 = c$. Applying the Cayley-Dickson doubling process to a composition algebra C over F, yields another composition algebra (if the dimension of the new algebra is at most 8), or alternatively what is known as a *generalised Cayley-Dickson algebra*.

For $C = \left(\frac{a,b}{F}\right)$, we have $T_C \simeq 2\langle 1,a,b,-ab\rangle$, whereas for $C = \left(\frac{a,b,c}{F}\right)$, $T_C \simeq 2\langle 1,a,b,-ab,c,-ac,-bc,abc\rangle$. Since we are concerned with the isotropy of trace forms, we may disregard this scalar factor of 2. We define the *pure trace form* of a composition algebra C, denoted T_P, via the following relation: $T_C = \langle 1\rangle \perp T_P$.

For C a division algebra, we define the *level* of C, denoted $s(C)$, as the least integer n such that -1 is a sum of n squares in C. If no such integer exists, we say that $s(C) = \infty$. The *sublevel* of C, denoted $\underline{s}(C)$, is the least integer n for which 0 is a sum of $n+1$ squares of elements in C. If 0 is not expressible in this manner, we say that $\underline{s}(C) = \infty$. Note that $\underline{s}(C) \leq s(C)$.

In [Pf], Pfister showed that the level of a field, if finite, is a power of 2, and moreover that any prescribed power of 2 may be realised as the level of a field. This classification extends to the case where C is a quadratic étale F-algebra. A composition algebra is *split* if it contains a composition subalgebra which is isomorphic to $F \oplus F$, which is the case if and only if C contains zero divisors. Split quaternion and octonion algebras have level 1 (a consequence of a split quaternion algebra being isomorphic to M_2F). Hence, in order to complete the classification of the level of composition algebras, we consider quaternion and octonion algebras which are division.

For φ a regular n-dimensional quadratic form over F, we can consider φ as a homogeneous polynomial of degree 2, that is $\varphi(X) = \varphi(X_1,\ldots,X_n) = \sum_{i,j} \alpha_{ij}X_iX_j$, where $\alpha_{ij} \in F^{\times}$. For $n > 1$ and $\varphi \not\simeq \langle 1,-1\rangle$, the *function field* of φ, denoted $F(\varphi)$, is defined to be the quotient field of the integral domain

$$F[X]/(\varphi(X)) = F[X_1,\ldots,X_n]/(\varphi(X_1,\ldots,X_n)).$$

φ is said to be *isotropic* if there exists a non-zero vector x such that $\varphi(x) = 0$. By construction, every quadratic form is isotropic over its function field. The *Witt index* of φ, denoted $i_W(\varphi)$, is the dimension of a maximal totally isotropic subform of φ. The *first Witt index* of φ, denoted $i_1(\varphi)$, is the Witt index of φ over its function field. The *essential dimension* of φ is given by the following relation:

$\dim_{es}(\varphi) = \dim(\varphi) - i_1(\varphi) + 1$. The *value set* of φ, denoted $D_F(\varphi)$, is the set of elements of $F^\times$ which are represented by φ. By $D_F(\varphi)D_F(\psi)$ we will denote the usual set product of $D_F(\varphi)$ and $D_F(\psi)$.

We highlight the following recent progress towards classifying when one quadratic form becomes isotropic over the function field of another:

Theorem 2.1. *[I, Lemma 5.3] Let $\psi_0, \psi_1, \varphi_0, \varphi_1$ be anisotropic quadratic forms over F. For $\psi := \psi_0 \perp T\psi_1$ and $\varphi := \varphi_0 \perp T\varphi_1$, the following are equivalent:*

- *ψ isotropic over $F(T)(\varphi)$.*
- *ψ isotropic over $F((T))(\varphi)$.*
- *$D_{F'}(\varphi_0)D_{F'}(\varphi_1) \subseteq D_{F'}(\psi_0)D_{F'}(\psi_1)$ for every extension F' of F.*

Theorem 2.2. *[R, Theorem 5.1.5] ψ is isotropic over $F(\varphi) \iff D_{F'}(\varphi)D_{F'}(\varphi) \subseteq D_{F'}(\psi)D_{F'}(\psi)$ for every extension F' of F.*

In addition, Karpenko and Merkurjev employed advanced algebro-geometric techniques involving Chow groups to prove the following powerful result:

Theorem 2.3. *[KM, Theorem 4.1] Let φ and ψ be anisotropic over F and suppose that ψ is isotropic over $F(\varphi)$. Then*

(i) $\dim_{es}(\varphi) \leq \dim_{es}(\psi)$*;*

(ii) moreover, the equality $\dim_{es}(\varphi) = \dim_{es}(\psi)$ *holds if and only if φ is isotropic over $F(\psi)$.*

For an overview of function fields of quadrics or for further definitions and notation regarding quadratic forms, we refer the reader to [Lam].

3 Results

3.1 $Q(n)$ and $O(n)$

In this section, we will introduce certain families of quaternion and octonion algebras, before proceeding to investigate what values their level and sublevel may take.

3.1.1 Introducing $Q(n)$ and $O(n)$

As motivated above, we will alternate between considering quaternion and octonion algebras which are division. We will let C denote a quaternion or octonion algebra which is division. For those occasions in which we will need to discriminate between these alternatives, we will use Q to denote a quaternion division algebra and O an octonion division algebra. By T_P we will denote the pure trace form of the algebra in question, which should be clear from the context.

Since $D_F(n \times T_P) = \{\sum_{i=1}^{n} {p_i}^2 | p_i \in \mathrm{Skew}(C, *)\}$ for $*$ given by conjugation, if $\langle 1 \rangle \perp n \times T_P$ is isotropic then -1 is expressible as a sum of n squares of pure elements in C, yielding that $s(C) \leq n$.

For $n = 2^k - 1$, Lewis stated that the converse holds for quaternion division algebras (see [Lew2]). This result extends to composition algebras (see [O'S, Lemma 3.9]), allowing us to conclude that

$$s(C) \leq n \iff \langle 1 \rangle \perp n \times T_P \text{ is isotropic for } n = 2^k - 1.$$

Furthermore, for $n = 2^k$ Lewis showed that $s(Q) \leq n$ implies that $\langle 1 \rangle \perp n \times T_P$ or $(n+1) \times \langle 1 \rangle \perp (n-1) \times T_P$ is isotropic (see [Lew2, Lemma 3]), with Leep proving the converse in [Le, Theorem 2.2]. This equivalence also admits generalisation to the class of composition algebras:

$$s(C) \leq 2^k \iff \langle 1 \rangle \perp 2^k \times T_P \text{ or } (2^k+1) \times \langle 1 \rangle \perp (2^k - 1) \times T_P \text{ is isotropic} \quad (1)$$

In light of the above results, a natural question which then arose was whether $s(C) \leq n$ implies that $\langle 1 \rangle \perp n \times T_P$ is isotropic for all n. By producing a quaternion algebra of level 2^k such that $\langle 1 \rangle \perp 2^k \times T_P$ is anisotropic for all k, Koprowski [Kop] answered this question in the negative. Pumplün later showed that there also exist octonion algebras of level 2^k such that $\langle 1 \rangle \perp 2^k \times T_P$ is anisotropic for all k (see [Pu]).

More specifically, Koprowski's example and its octonion algebra analogue are directly relevant to (1), proving that if $s(C) \leq 2^k$ it remains necessary, for all values of k, to consider whether $(2^k + 1) \times \langle 1 \rangle \perp (2^k - 1) \times T_P$ becomes isotropic in consequence. As an interesting aside, we note that if $s(C) \leq 2^k$, it is actually the consideration of the isotropy of $\langle 1 \rangle \perp 2^k \times T_P$ which becomes redundant for k sufficiently large, as the following attests:

Theorem 3.1. *For Q a quaternion division algebra, $s(Q) \leq 2^k \iff (2^k+1) \times \langle 1 \rangle \perp (2^k - 1) \times T_P$ is isotropic, where $k \geq 2$. For O an octonion division algebra, $s(O) \leq 2^k \iff (2^k+1) \times \langle 1 \rangle \perp (2^k - 1) \times T_P$ is isotropic, where $k \geq 3$.*

Proof. Since $s(C) \leq 2^k \iff (2^k+1) \times \langle 1 \rangle \perp (2^k-1) \times T_P$ or $\langle 1 \rangle \perp 2^k \times T_P$ is isotropic, it suffices to prove that the isotropy of $\langle 1 \rangle \perp 2^k \times T_P$ implies that of $(2^k+1) \times \langle 1 \rangle \perp (2^k-1) \times T_P$, for k as above. Suppose $\langle 1 \rangle \perp 2^k \times T_P$ is isotropic, implying that $2^k \times T_C \simeq (2^k \times \langle 1 \rangle) \otimes T_C$ is isotropic. We know that $i_W(2^k \times T_C) \geq 2^k$ by [EL, Proposition 1.4]. For k as above, $2^k \times \langle 1 \rangle \perp (2^k-1) \times T_P$ is a subform of $2^k \times T_C$ of codimension $< 2^k$, and thus is also isotropic. ■

Since $s(C) = 2^k \Rightarrow \underline{s}(C) = 2^k$ ([Le, Theorem 2.5] for $C = Q$ and [O'S, Theorem 3.11] for $C = O$), Koprowski's example and its octonion algebra analogue prove that neither $s(C) = n$ nor $\underline{s}(C) = n$ implies that $\langle 1 \rangle \perp n \times T_P$ is isotropic for all n. Of course, this does not preclude the existence of composition algebras of level and sublevel n such that $\langle 1 \rangle \perp n \times T_P$ is isotropic, for all values of n.

As we have seen, the isotropy of $\langle 1 \rangle \perp n \times T_P$ encodes n as an upper bound for the level and sublevel of a composition algebra. Indeed, since $D_F(n \times T_P) = \{\sum_{i=1}^n {p_i}^2 | p_i \in \mathrm{Skew}(C, *)\}$, $\langle 1 \rangle \perp n \times T_P$ is the most natural quadratic form whose isotropy bounds the level. Thus it seems reasonable to suggest that if one were to construct composition algebras, subject to $\langle 1 \rangle \perp n \times T_P$ being isotropic, in a *suitably general fashion*, then their level and sublevel might actually equal n.

In defining $Q(n)$ and $O(n)$, we are aiming for just such a construction. In an attempt to make $Q(n)$ and $O(n)$ suitably general, the better to eradicate factors which could serve to reduce the level or sublevel below n, we choose the generators of our algebra to be transcendental over a formally real ground field, and extend to the function field of $\langle 1 \rangle \perp n \times T_P$, the natural method of generically encoding its isotropy. Hence, we let

$$Q(n) := \left(\frac{x, y}{F}\right) \otimes_F F\left(\langle 1 \rangle \perp n \times T_P\right)$$

and

$$O(n) := \left(\frac{x, y, z}{F}\right) \otimes_F F\left(\langle 1 \rangle \perp n \times T_P\right),$$

where T_P is the pure trace form of $\left(\frac{x,y}{F}\right)$ and $\left(\frac{x,y,z}{F}\right)$ respectively, and F denotes $F_0(x, y)$ and $F_0(x, y, z)$ respectively.

We posit the following conjecture:

Conjecture 3.2. *$s\left(Q(n)\right) = \underline{s}\left(Q(n)\right) = s\left(O(n)\right) = \underline{s}\left(O(n)\right) = n$ for all n.*

3.1.2 Determined Values

We begin our investigation of $Q(n)$ and $O(n)$ by noting that if the sublevel of these algebras equals n, then their level and sublevel must coincide.

We will now proceed to establish the first seven cases of the octonionic component of Conjecture 3.2. In doing so, we prove that 6 and 7 are realisable as both the level and sublevel of octonion algebras. We note that these are the first examples of composition algebras whose level is not of the form 2^k or $2^k + 1$ for some k.

Theorem 3.3. *$s\left(O(n)\right) = \underline{s}\left(O(n)\right) = n$, for $n \leq 7$.*

Proof. For ease of notation we let $\varphi := \langle 1 \rangle \perp n \times T_P$.

By construction $\underline{s}\left(O(n)\right) \leq n$.

Suppose $\underline{s}\left(O(n)\right) \leq n - 1$. Hence

$$\sum_{i=1}^{n} {c_i}^2 + \sum_{i=1}^{n} {p_i}^2 = 0 \tag{2}$$

and

$$\sum_{i=1}^{n} c_i p_i = 0, \tag{3}$$

where $c_i \in F\left(\varphi\right)$ and $p_i \in \mathcal{P}$, the $F\left(\varphi\right)$-vector space spanned by the pure octonions.

If $c_i = 0\ \forall\ i$, (2) implies that $n \times T_P$ is isotropic over $F\left(\varphi\right)$.

Alternatively, if there exists an i such that $c_i \neq 0$, (3) implies the existence of V, an $(n-1)$-dimensional $F\left(\varphi\right)$-subspace of $\mathcal{P}$, containing $p_1, \ldots, p_n$. Let $\beta : V \to F\left(\varphi\right)$, $p \mapsto p^2$. Hence, β is an $(n-1)$-dimensional subform of T_P and (2) implies that $\vartheta := n \times \left(\langle 1 \rangle \perp \beta\right)$ is isotropic over $F\left(\varphi\right)$.

Note that $n \times T_P$ and $8 \times \left(\langle -1 \rangle \perp T_P\right)$ are anisotropic over F by Springer's Theorem.

If $n \times T_P$ becomes isotropic over $F\left(\varphi\right)$, then the Pfister form $8 \times \left(\langle 1 \rangle \perp -T_P\right)$ becomes hyperbolic over $F\left(\varphi\right)$. Invoking the Cassels-Pfister Subform Theorem (see [S, p.155]), $\alpha(\varphi)$ should be a subform of $8 \times \left(\langle 1 \rangle \perp -T_P\right)$ for any $\alpha \in D_F\left(8 \times \left(\langle 1 \rangle \perp -T_P\right)\right)$. In particular, since $-x \in D_F\left(8 \times \left(\langle 1 \rangle \perp -T_P\right)\right)$, $-x(\varphi) \simeq \langle -x \rangle \perp n \times \langle -1, -xy, y, -xz, z, xyz, -yz \rangle$ should be a subform of $8 \times \left(\langle 1 \rangle \perp -T_P\right)$, implying that $n \times \langle -1 \rangle$ is also a subform, which is clearly false. Hence, $n \times T_P$ also remains anisotropic over $F(\varphi)$.

Thus, ϑ must be isotropic over $F(\varphi)$. By Theorem 2.2 we must have

$$D_{F'}(\varphi) D_{F'}(\varphi) \subseteq D_{F'}(\vartheta) D_{F'}(\vartheta) \text{ for every extension } F' \text{ of } F.$$

Let $F' = F$.

Since $T_P \simeq \langle x, y, -xy, z, -xz, -yz, xyz \rangle$, we have $x \in D_F(\varphi) D_F(\varphi)$. Suppose $\langle x \rangle$ is not a subform of β. Hence, all multiples of x in $D_F(\vartheta) D_F(\vartheta)$ are of the form $-x\left(\sum_{i=1}^m a_i^2\right)$, for $a_i \in F$ (note that $\langle (y)(-xy) \rangle \simeq \langle (z)(-xz) \rangle \simeq \langle (-yz)(xyz) \rangle \simeq \langle -x \rangle$). If x is positive with respect to some ordering on F, then $-x\left(\sum_{i=1}^m a_i^2\right)$ is necessarily negative, and vice-versa. Thus $x \notin D_F(\vartheta) D_F(\vartheta)$, which is a contradiction. Hence $\langle x \rangle$ is a subform of β.

Note that $y \in D_F(\varphi) D_F(\varphi)$. Now suppose that $\langle y \rangle$ is not a subform of β. Hence, all multiples of y in $D_F(\vartheta) D_F(\vartheta)$ are of the form $-y\left(\sum_{i=1}^m a_i^2\right)$, for $a_i \in F$ (note that $\langle (x)(-xy) \rangle \simeq \langle (z)(-yz) \rangle \simeq \langle (-xz)(xyz) \rangle \simeq \langle -y \rangle$). Again, if y is positive with respect to some ordering on F, then $-y\left(\sum_{i=1}^m a_i^2\right)$ is negative, and vice-versa. Thus $y \notin D_F(\vartheta) D_F(\vartheta)$, which is a contradiction. Hence $\langle y \rangle$ is a subform of β.

Similarly, since $-xy, z, -xz, -yz$ and $xyz \in D_F(\varphi) D_F(\varphi)$, $\langle -xy \rangle, \langle z \rangle, \langle -xz \rangle, \langle -yz \rangle$ and $\langle xyz \rangle$ must also be subforms of β.

Thus, $T_P \simeq \langle x \rangle \perp \langle y \rangle \perp \langle -xy \rangle \perp \langle z \rangle \perp \langle -xz \rangle \perp \langle -yz \rangle \perp \langle xyz \rangle$ is a subform of β, which contradicts the fact that β is $(n-1)$-dimensional.

Thus $\underline{s}(O(n)) \nleq n-1$, implying that $\underline{s}(O(n)) = n$. ∎

Showing that ϑ remains anisotropic over $F(\varphi)$ is a key step in the above proof. The machinery with which we successfully endeavoured to achieve this end is Roussey's Theorem 2.2. Hence, all the ingredients in the above proof lie within the "classical" theory of quadratic forms.

If we instead choose to invoke Theorem 2.3 of Karpenko and Merkurjev, whose proof relies upon deep algebro-geometric techniques, in order to prove anisotropy of forms over $F(\langle 1 \rangle \perp n \times T_P)$, we obtain a more succinct, if less elementary, argument:

Proof. As detailed above, to prove that $\underline{s}(O(n)) \nleq n-1$ it suffices to show that $n \times T_P$ and $n \times (\langle 1 \rangle \perp \beta)$ are anisotropic over $F(\langle 1 \rangle \perp n \times T_P)$. Since $i_1(\langle 1 \rangle \perp n \times T_P) =$

1 (see Lemma 3.8), the fact that $\dim(\langle 1\rangle \perp n \times T_P) > \dim(n \times T_P)$ implies that $n \times T_P$ is anisotropic over $F(\langle 1\rangle \perp n \times T_P)$ by Theorem 2.3. Similarly since $\dim(\langle 1\rangle \perp n \times T_P) > \dim(n \times (\langle 1\rangle \perp \beta))$, we can conclude that $n \times (\langle 1\rangle \perp \beta)$ is also anisotropic over $F(\langle 1\rangle \perp n \times T_P)$ by Theorem 2.3. Hence $\underline{s}(O(n)) \nleq n-1$. ■

Both of the above methods of proof apply to quaternion algebras as well, yielding:

Theorem 3.4. *$s(Q(n)) = \underline{s}(Q(n)) = n$ for $n \leq 3$.*

We note that the above results may be extended to their respective next cases. Indeed, the following result shows that Conjecture 3.2 is true for all $n = 2^k$:

Theorem 3.5. *$s(Q(n)) = \underline{s}(Q(n)) = s(O(n)) = \underline{s}(O(n)) = n$, for $n = 2^k$.*

Proof. We may assume that $k \geq 2$ by Theorem 3.3 and Theorem 3.4.

Suppose $\underline{s}\left(Q(2^k)\right) \leq 2^k - 1$. Thus, $\langle 1\rangle \perp (2^k-1) \times T_P$ is isotropic over $F(\varphi)$ by [O'S, Theorem 3.5], where $\varphi := \langle 1\rangle \perp 2^k \times T_P$. Since $\psi_{\langle x\rangle} := \langle 1\rangle \perp 2^k \times \langle x\rangle$ is a subform of φ, there exists an F-place from $F(\varphi)$ to $F\left(\psi_{\langle x\rangle}\right)$, by [K, Theorem 3.3]. Thus $\langle 1\rangle \perp (2^k-1) \times T_P$ is isotropic over $F\left(\psi_{\langle x\rangle}\right)$, by [K, Theorem 3.3]. Springer's Theorem implies that either $\langle 1\rangle \perp (2^k-1) \times \langle x\rangle$ or $(2^k-1) \times \langle 1, -x\rangle$ is isotropic over $F_0(x)\left(\psi_{\langle x\rangle}\right)$. However, $\langle 1\rangle \perp (2^k-1) \times \langle x\rangle$ is anisotropic over $F_0(x)\left(\psi_{\langle x\rangle}\right)$ by [H1, Theorem 1]. Hence, $(2^k-1) \times \langle 1, -x\rangle$ must be isotropic over $F_0(x)\left(\psi_{\langle x\rangle}\right)$, implying that $2^k \times \langle 1, -x\rangle$ is hyperbolic over $F_0(x)\left(\psi_{\langle x\rangle}\right)$. The Cassels-Pfister Subform Theorem implies that $\psi_{\langle x\rangle}$ is a subform of $2^k \times \langle 1, -x\rangle$, giving a desired contradiction. Hence $\underline{s}\left(Q(2^k)\right) = 2^k$.

Now suppose $\underline{s}\left(O(2^k)\right) \leq 2^k - 1$. Thus, $\langle 1\rangle \perp (2^k-1) \times T_P$ is isotropic over $F\left(\langle 1\rangle \perp 2^k \times T_P\right)$ by [O'S, Theorem 3.5]. Since $\psi_{\langle x,y,-xy\rangle} := \langle 1\rangle \perp 2^k \times \langle x, y, -xy\rangle$ is a subform of $\langle 1\rangle \perp 2^k \times T_P$, there exists an F-place from $F\left(\langle 1\rangle \perp 2^k \times T_P\right)$ to $F\left(\psi_{\langle x,y,-xy\rangle}\right)$, by [K, Theorem 3.3]. Thus $\langle 1\rangle \perp (2^k-1) \times T_P$ is isotropic over $F\left(\psi_{\langle x,y,-xy\rangle}\right)$, by [K, Theorem 3.3]. Springer's Theorem implies that either $\langle 1\rangle \perp (2^k-1) \times \langle x, y, -xy\rangle$ or $(2^k-1) \times \langle 1, -x, -y, xy\rangle$ is isotropic over $F_0(x,y)\left(\psi_{\langle x,y,-xy\rangle}\right)$. In proving the quaternion case above, we showed that $\langle 1\rangle \perp (2^k-1) \times \langle x, y, -xy\rangle$ is anisotropic over $F_0(x,y)\left(\psi_{\langle x,y,-xy\rangle}\right)$. Hence, $(2^k-1) \times \langle 1, -x, -y, xy\rangle$ must be isotropic over $F_0(x,y)\left(\psi_{\langle x,y,-xy\rangle}\right)$, implying that $2^k \times \langle 1, -x, -y, xy\rangle$ is hyperbolic over $F_0(x,y)\left(\psi_{\langle x,y,-xy\rangle}\right)$. The Cassels-Pfister Subform Theorem implies that $\psi_{\langle x,y,-xy\rangle}$ is a subform of $2^k \times \langle 1, -x, -y, xy\rangle$, furnishing a contradiction. Hence $\underline{s}\left(O(2^k)\right) = 2^k$. ■

Karpenko and Merkurjev's result, Theorem 2.3, provides another method of proof:

Proof. Since $i_1\left(\langle 1\rangle \perp 2^k \times T_P\right) = 1$ (see Lemma 3.7 and Lemma 3.8), the fact that $\dim\left(\langle 1\rangle \perp 2^k \times T_P\right) > \dim\left(\langle 1\rangle \perp (2^k-1) \times T_P\right)$ implies that $\langle 1\rangle \perp (2^k-1)\times T_P$ remains anisotropic over $F\left(\langle 1\rangle \perp 2^k \times T_P\right)$ by Theorem 2.3. Hence $\underline{s}\left(Q(2^k)\right) \nleq 2^k-1$ and $\underline{s}\left(O(2^k)\right) \nleq 2^k-1$. ∎

In their construction of a quaternion algebra of level 2^k+1 for $k \geq 1$, Laghribi and Mammone considered $Q(2^k+1)$, proving that $s\left(Q(2^k+1)\right) = 2^k+1$ [LM, Theorem 2.1]. Furthermore, Pumplün adapted their argument to show that $s\left(O(2^k+1)\right) = 2^k+1$ for $k \geq 1$ [Pu, Theorem 3.1]. We note that these results extend to the case where $k=0$, as this is merely the level component of Theorem 3.5 when $k=1$. We now present a simplified method of proving these results:

Theorem 3.6. *$s\left(Q(n)\right) = s\left(O(n)\right) = n$, for $n = 2^k+1$.*

Proof. Suppose $s\left(Q(2^k+1)\right) \leq 2^k$. Hence $(2^k+1)\times\langle 1\rangle \perp (2^k-1)\times T_P$ is isotropic over $F\left(\langle 1\rangle \perp (2^k+1)\times T_P\right)$ by Theorem 3.1 (we can assume that $k \geq 2$ by Theorem 3.4). Since $y(2^k+1)\times\langle 1,-x\rangle$ is a subform of $\langle 1\rangle \perp (2^k+1)\times T_P$, there exists an F-place from $F\left(\langle 1\rangle \perp (2^k+1)\times T_P\right)$ to $F\left((2^k+1)\times\langle 1,-x\rangle\right)$, by [K, Theorem 3.3]. Thus $(2^k+1)\times\langle 1\rangle \perp (2^k-1)\times T_P$ is isotropic over $F\left((2^k+1)\times\langle 1,-x\rangle\right)$, by [K, Theorem 3.3]. Springer's Theorem implies that either $(2^k+1)\times\langle 1\rangle \perp (2^k-1)\times\langle x\rangle$ or $(2^k-1)\times\langle 1,-x\rangle$ is isotropic over $F_0(x)\left((2^k+1)\times\langle 1,-x\rangle\right)$. However, both forms are anisotropic by [H1, Theorem 1]. Hence $s\left(Q(2^k+1)\right) \nleq 2^k$.

The same method applies in the octonion case. ∎

The values of n specified in the above results remain the only ones for which we can precisely determine $s\left(Q(n)\right), \underline{s}\left(Q(n)\right), s\left(O(n)\right)$ or $\underline{s}\left(O(n)\right)$. For other values of n, bounds may be placed on these quantities via Theorem 2.3, as we will now proceed to outline.

3.1.3 Bounding Undetermined Values

Given the similarities between sums of squares in quaternion and octonion algebras, that there exist octonion algebras whose level is not of the form 2^k or 2^k+1, as proved in Theorem 3.3, suggests that the same is true for quaternion algebras. Indeed, at a recent seminar in University College Dublin [H2], Detlev Hoffmann kindly communicated his method of showing the existence of infinitely many quaternion algebras whose level is neither 2^k nor 2^k+1 for some k, answering a question posed by David Leep in [Le]. The family of quaternion algebras that Hoffmann considers is $\{Q(n)\}$.

A key step in Hoffmann's argument is the following:

Lemma 3.7. *[H3, Lemma 4.1]* $i_1(\langle 1\rangle \perp n\times T_P) = 1$ *for all* n*, where* T_P *represents the pure trace form of* $Q(n)$*.*

Employing the same method of proof, one can obtain the corresponding result for $O(n)$, as we will now proceed to outline:

Lemma 3.8. $i_1(\langle 1\rangle \perp n\times T_P) = 1$ *for all* n*, where* T_P *represents the pure trace form of* $O(n)$*.*

Proof. For P an ordering of F_0, let P' represent an extension to F such that x, y and z are negative. Hence, $\langle 1\rangle \perp n\times T_P$ is indefinite with respect to P', implying that P' extends to $F(\langle 1\rangle \perp n\times T_P)$ by [ELW, Theorem 3.5 and Remark 3.6]. Now, since $\langle 1\rangle \perp n\times T_P$ has only one positive coefficient with respect to P', we can conclude that $i_1(\langle 1\rangle \perp n\times T_P) = 1$. ∎

Hoffmann invokes Lemma 3.7 to prove the following result:

Theorem 3.9. *[H3, Corollary 4.3] For* $n = m+1+\lfloor \frac{m}{3}\rfloor$*,* $s(Q(n)) \in [m+1, n]$*.*

The choice of $m+1+\lfloor \frac{m}{3}\rfloor$ for n is justifiable since it is the least value which ensures that the essential dimension of $\langle 1\rangle \perp n\times T_P$ is greater than or equal to the dimension of $\langle 1\rangle \perp m\times T_Q$, thereby ensuring that $s(Q) \nleq m$ via Theorem 2.3. For the sake of consistency, we reformulate the above result in the following manner:

$$s(Q(n)) \in \left[n - \left\lfloor \frac{n}{4}\right\rfloor, n\right] \text{ for all } n.$$

Of course, we may similarly invoke Lemma 3.8 to the same effect, allowing us to obtain the analogous bounds for $s(O(n))$:

Theorem 3.10. $s(O(n)) \in \left[n - \lfloor \frac{n}{8}\rfloor, n\right]$ *for all* n*.*

Proof. Since $i_1(\langle 1\rangle \perp n\times T_P) = 1$ for all n, if $\dim(\langle 1\rangle \perp n\times T_P) > \dim\left(\langle 1\rangle \perp m\times T_{O(n)}\right)$ then Theorem 2.3 implies that $\langle 1\rangle \perp m\times T_{O(n)}$ remains anisotropic over $F(\langle 1\rangle \perp n\times T_P)$, bounding $s(O(n))$ in $[m+1, n]$. Comparing dimensions, we see that $n - \lfloor \frac{n}{8} \rfloor - 1$ is the greatest value of m such that $\dim(\langle 1 \rangle \perp n \times T_P) > \dim\left(\langle 1 \rangle \perp m \times T_{O(n)}\right)$. Hence, $s(O(n)) \in \left[n - \lfloor \frac{n}{8} \rfloor, n\right]$. ∎

We note that the above result recovers the level component of Theorem 3.3.

In a similar fashion, we may further apply Hoffmann's methodology to considerations of the sublevel of $Q(n)$:

Theorem 3.11. *$\underline{s}(Q(n)) \in \left[n - \lfloor \frac{n+3}{4} \rfloor, n\right]$ for all n.*

Proof. Since $i_1(\langle 1 \rangle \perp n \times T_P) = 1$ for all n, if $\dim(\langle 1 \rangle \perp n \times T_P) > \dim\left((m+1) \times T_{Q(n)}\right)$ then Theorem 2.3 implies that $(m+1) \times T_{Q(n)}$ remains anisotropic over $F(\langle 1 \rangle \perp n \times T_P)$, bounding $\underline{s}(Q(n))$ in $[m+1, n]$. Comparing dimensions, we see that $n - \lfloor \frac{n+3}{4} \rfloor - 1$ is the greatest value of m such that $\dim(\langle 1 \rangle \perp n \times T_P) > \dim\left((m+1) \times T_{Q(n)}\right)$. Hence, $\underline{s}(Q(n)) \in \left[n - \lfloor \frac{n+3}{4} \rfloor, n\right]$. ∎

The corresponding argument yields the following bounds for the sublevel of $O(n)$:

Theorem 3.12. *$\underline{s}(O(n)) \in \left[n - \lfloor \frac{n+7}{8} \rfloor, n\right]$ for all n.*

Without placing any restrictions on the value that n may take, the above results represent the sharpest bounds on the level and sublevel of $Q(n)$ and $O(n)$ currently available. For a large class of values however, we may further reduce the intervals in which these quantities are known to lie.

We achieve this reduction by exploiting Theorem 2.3 more fully:

For $m = 2^k d$, $m \times T_C \simeq 2^k \times \langle 1 \rangle \otimes (d \times T_C)$. In consequence, $i_1(m \times T_C) \geq 2^k$ by [EL, Theorem 1.4]. Thus, invoking Theorem 2.3 in conjunction with this observation, we may increase the lower bounds on $\underline{s}(Q(n))$ and $\underline{s}(O(n))$ for certain n, and consequently those on $s(Q(n))$ and $s(O(n))$.

Theorem 3.13. *$s(Q(n))$ and $\underline{s}(Q(n)) \in [m, n]$ for $n \geq m + \left\lfloor \frac{m-2^k}{3} \right\rfloor + 1$, where k is the 2-adic order of m.*

Proof. [EL, Theorem 1.4] implies that $i_1\left(m \times T_{Q(n)}\right) \geq 2^k$. Comparing the essential dimensions of $\langle 1 \rangle \perp n \times T_P$ and $m \times T_{Q(n)}$, we see that $m + \left\lfloor \frac{m-2^k}{3} \right\rfloor + 1$ is the least value of n such that $\dim(\langle 1 \rangle \perp n \times T_P) - 1 > \dim\left(m \times T_{Q(n)}\right) - 2^k$. Hence, for $n \geq m + \left\lfloor \frac{m-2^k}{3} \right\rfloor + 1$, $m \times T_{Q(n)}$ is anisotropic over $F(\langle 1 \rangle \perp n \times T_P)$ by Theorem 2.3, implying that $\underline{s}(Q(n)) \in [m, n]$.

Since $\underline{s}(Q(n)) \leq s(Q(n))$, $s(Q(n)) \in [m, n]$ also. ∎

Arguing in a likewise manner, we obtain the following bounds for the level and sublevel of $O(n)$:

Theorem 3.14. *$s(O(n))$ and $\underline{s}(O(n)) \in [m, n]$ for $n \geq m + \left\lfloor \frac{m-2^k}{7} \right\rfloor + 1$, where k is the 2-adic order of m.*

Of course, if $n = 2^k + h$ for $h \geq 0$ sufficiently small, we may yet again increase the lower bounds on the levels and sublevels of $Q(n)$ and $O(n)$ relative to those derived from the above results:

Theorem 3.15. *$s(Q(n)), \underline{s}(Q(n)), s(O(n))$ and $\underline{s}(O(n)) \in \left[2^k, n\right]$, where 2^k is the largest 2-power $\leq n$.*

Proof. In the proof of Theorem 3.5, we showed that $\langle 1 \rangle \perp (2^k - 1) \times T_P$ is anisotropic over $F\left(\langle 1 \rangle \perp 2^k \times T_P\right)$. For $n \geq 2^k$, $\langle 1 \rangle \perp 2^k \times T_P$ is a subform of $\langle 1 \rangle \perp n \times T_P$, implying the existence of an F-place from $F(\langle 1 \rangle \perp n \times T_P)$ to $F\left(\langle 1 \rangle \perp 2^k \times T_P\right)$ by [K, Theorem 3.3]. Hence, $\langle 1 \rangle \perp (2^k - 1) \times T_P$ is anisotropic over $F(\langle 1 \rangle \perp n \times T_P)$ by [K, Proposition 3.1], implying the result. ■

Similarly, if $n = 2^k + h$ for $h \geq 1$ sufficiently small, we may interpret Theorem 3.6 as a further increase to earlier lower bounds on the levels of $Q(n)$ and $O(n)$:

Theorem 3.16. *$s(Q(n))$ and $s(O(n)) \in \left[2^k + 1, n\right]$, where 2^k is the largest 2-power $< n$.*

Proof. As in the proof of Theorem 3.6, $(2^k + 1) \times \langle 1 \rangle \perp (2^k - 1) \times T_P$ is anisotropic over $F\left(\langle 1 \rangle \perp (2^k + 1) \times T_P\right)$. Hence, as in the previous proof, it is also anisotropic over $F(\langle 1 \rangle \perp n \times T_P)$ for $n \geq 2^k + 1$. ■

For certain values of n, the lower bounds obtained for $s(Q(n)), \underline{s}(Q(n)), s(O(n))$ and $\underline{s}(O(n))$ in the above results are actually optimal with respect to the standard isotropy tests.

For example, if we consider the case where $n = 15$, we note that $s(Q(15))$ and $\underline{s}(Q(15)) \in [12, 15]$ as a consequence of Theorem 3.13. We cannot hope to show that $s(Q(15))$ or $\underline{s}(Q(15)) \neq 12$ via the standard methodology, since both $\langle 1 \rangle \perp 12 \times T_{Q(15)}$ and $13 \times T_{Q(15)}$ are isotropic over $F(\langle 1 \rangle \perp 15 \times T_P)$, as the following argument demonstrates:

$16 \times T_{Q(15)}$ is clearly isotropic over $F(\langle 1 \rangle \perp 15 \times T_P)$. [EL, Theorem 1.4] implies that $i_W\left(16 \times T_{Q(15)}\right) \geq 16$. Since $\langle 1 \rangle \perp 12 \times T_{Q(15)}$ and $13 \times T_{Q(15)}$ are both subforms of $16 \times T_{Q(15)}$ of codimension < 16, they too are isotropic over $F\left(\langle 1\rangle \perp 15 \times T_P\right)$.

Hence, more refined quadratic form theoretic consequences of the sublevels of $Q(n)$ and $O(n)$ being bound above by $n-1$ are required, if we hope to prove Conjecture 3.2.

3.2 Constructions for the case where $n = l \cdot 2^k$

As outlined in Section 3.1.1, if we seek to construct quaternion and octonion algebras of level and sublevel n for all values of n, the most natural candidates to consider are $Q(n)$ and $O(n)$. However, if we restrict our attention to certain values of n, other constructions appear to be equally, if not more, appropriate.

As a first step in arriving at our constructions of $Q(n)$ and $O(n)$, we observed that the isotropy of the quadratic form $\langle 1\rangle \perp n \times T_P$ placed an upper bound on the level and sublevel of these algebras. Of course, it is possible to find other, though admittedly less natural, quadratic forms whose isotropy encodes such an upper bound on the level and sublevel of associated quaternion or octonion algebras.

As highlighted earlier, one such quadratic form is $(2^k+1) \times \langle 1\rangle \perp (2^k-1) \times T_P$, whose isotropy implies that $s(C) \leq 2^k$ ([Le, Theorem 2.2] for the Q case and [Pu, Proposition 2.12] for the O case). This result, and its method of proof, admit the following generalisation:

Theorem 3.17. *If $(2^h+1) \times \langle 1\rangle \perp (l \cdot 2^k - 1) \times T_P$ is isotropic, where h denotes the 2-adic order of $l \cdot 2^k$, then $s(C) \leq l \cdot 2^k$.*

Proof. Consider the case where C is a quaternion algebra. If $(2^h+1) \times \langle 1\rangle \perp (l \cdot 2^k - 1) \times T_P$ is isotropic, then there exists $-A \in D_F\left(\langle 1\rangle \perp (l \cdot 2^k - 1) \times T_P\right)$ for some nonzero $A \in D_F\left(2^h \times \langle 1\rangle\right)$. Hence for some $\alpha \in F$ and $B, C, D \in D_F\left((l \cdot 2^k - 1) \times \langle 1\rangle\right) \cup \{0\}$ we have $-A = \alpha^2 + aB + bC - abD$, that is $-1 = \frac{1}{A^2}\left(\alpha^2 A + aAB + bAC - abAD\right)$. Let $A = \sum_{\lambda=1}^{2^h} {x_\lambda}^2$. We show that there exist $y_\lambda \in F$ such that $\sum_{\lambda=1}^{l \cdot 2^k} {y_\lambda}^2 = AB$ and $\vec{x} \cdot \vec{y} = 0$, where $\vec{x} \cdot \vec{y}$ represents the scalar product of the two vectors, and we take $x_{2^h+1}, \ldots, x_{l \cdot 2^k} = 0$ if $2^h < l \cdot 2^k$. If $B = 0$, let each $y_\lambda = 0$. If $B \neq 0$, then $\langle A, AB\rangle \cong A \cdot \langle 1, B\rangle$ is a subform of $A \cdot (l \cdot 2^k) \times \langle 1\rangle \cong (l \cdot 2^k) \times \langle 1\rangle$, since $B \in D_F\left((l \cdot 2^k - 1) \times \langle 1\rangle\right)$ and $A \cdot 2^h \times \langle 1\rangle \cong 2^h \times \langle 1\rangle$. Therefore such a $\vec{y}$ exists. Similarly $\vec{z}, \vec{w}$ exist such that $\sum_{\lambda=1}^{l \cdot 2^k} {z_\lambda}^2 = AC$, $\sum_{\lambda=1}^{l \cdot 2^k} {w_\lambda}^2 = AD$ and $\vec{x} \cdot \vec{z} = \vec{x} \cdot \vec{w} = 0$. It follows that $\sum_{\lambda=1}^{l\cdot 2^k}\left(\frac{\alpha x_\lambda}{A}+\frac{y_\lambda}{A}i+\frac{z_\lambda}{A}j+\frac{w_\lambda}{A}k\right)^2 = \frac{1}{A^2}\left(\alpha^2 A+aAB+bAC-abAD\right) = -1$. Therefore $s(C)\leq l\cdot 2^k$.

The same method of proof applies in the octonionic case. ■

As the above form bounds the level of composition algebras, we may adapt our constructions of $Q(n)$ and $O(n)$ by hardwiring in the isotropy of this form as opposed to that of $\langle 1\rangle \perp n\times T_P$. We introduce the following notation to signify this alteration: let

$$Q' := \left(\frac{x,y}{F}\right)\otimes_F F\left((2^h+1)\times\langle 1\rangle \perp (l\cdot 2^k-1)\times T_P\right)$$

and

$$O' := \left(\frac{x,y,z}{F}\right)\otimes_F F\left((2^h+1)\times\langle 1\rangle \perp (l\cdot 2^k-1)\times T_P\right),$$

where h is the 2-adic order of $l\cdot 2^k$, T_P is respectively the pure trace form of $\left(\frac{x,y}{F}\right)$ and $\left(\frac{x,y,z}{F}\right)$ and F denotes $F_0(x,y)$ and $F_0(x,y,z)$ respectively.

By design, the level of these constructions is bounded above by $l\cdot 2^k$. There is no a priori reason why the level or sublevel of either Q' or O' might be strictly less than $l\cdot 2^k$ for any values of l or k. Moreover, given the general fashion of these constructions, we propose that Q' and O' are actually of level and sublevel $l\cdot 2^k$:

Conjecture 3.18. *$s(Q')=\underline{s}(Q')=s(O')=\underline{s}(O')=l\cdot 2^k$ for all l and k.*

In the case where l is a 2-power, Q' represents a construction of Laghribi and Mammone (see [LM]), with O' coinciding with one of Pumplün (see [Pu]), allowing us to conclude that $s(Q')=s(O')=2^k$. Moreover, we yield that $\underline{s}(Q')=\underline{s}(O')=2^k$ immediately for $k\geq 2$, by [Le, Theorem 2.5] for the Q case and [O'S, Theorem 3.11] for the O case, with an ad hoc argument proving the case where $k=1$.

Since the conjecture is true in the above case, we will restrict our attention to the situation where l is not a 2-power. Moreover, we will henceforth assume without loss of generality that l is odd, whereby k is the 2-adic order of $l\cdot 2^k$.

Having introduced Q' and O', it is reasonable to wonder why we might consider these algebras as opposed to $Q(l\cdot 2^k)$ and $O(l\cdot 2^k)$ in seeking quaternion and octonion algebras of level and sublevel $l\cdot 2^k$.

We begin our justification of this preference by noting that the dimension of $(2^k+1)\times\langle 1\rangle \perp (l\cdot 2^k-1)\times T_P$ is greater than that of $\langle 1\rangle \perp (l\cdot 2^k)\times T_P$ for $k\geq 1$ in the quaternionic case, and for $k \geq 2$ in the octonionic case. Clearly, if we can show that $i_1\left((2^k+1) \times \langle 1\rangle \perp (l \cdot 2^k - 1) \times T_P\right) = 1$, then $(2^k+1) \times \langle 1\rangle \perp (l \cdot 2^k - 1) \times T_P$ will be of greater essential dimension than $\langle 1\rangle \perp (l \cdot 2^k) \times T_P$ for k as above. Theorem 2.3 would then allow for the bounding of $s(Q')$ and $s(O')$ in a smaller interval than that in which we can currently place $s(Q(n))$ and $s(O(n))$.

The suggestion that $i_1\left((2^k+1) \times \langle 1\rangle \perp (l \cdot 2^k - 1) \times T_P\right) = 1$ is an entirely reasonable one. Consider the form $\varphi \simeq \pi \otimes \rho$, with π isometric to an n-fold Pfister form where $n \geq 1$. Invoking [EL, Theorem 1.4], we yield $i_1(\varphi) \geq \dim \pi$. Theorem 2.3 implies that for φ' a subform of φ whose codimension is strictly less than $\dim \pi$, we have $i_1(\varphi') > 1$. Hence, we know that the class of forms represented by φ' have first Witt index strictly greater than 1. We suspect that such forms might actually be the *only* forms whose first Witt index is strictly greater than 1.

The following theorem, which constitutes the main result of this section, shows that the first Witt index of $(2^k+1) \times \langle 1\rangle \perp (l \cdot 2^k - 1) \times T_P$ does indeed equal 1, as suspected:

Theorem 3.19. *For Q' and O' as above,* $i_1\left((2^k+1) \times \langle 1\rangle \perp (l \cdot 2^k - 1) \times T_P\right) = 1$.

Proof. For ease of notation, we let $\psi_{\langle x\rangle}$ and ψ_{T_P} denote $(2^k+1) \times \langle 1\rangle \perp (l \cdot 2^k - 1) \times \langle x\rangle$ and $(2^k+1) \times \langle 1\rangle \perp (l \cdot 2^k - 1) \times T_P$ respectively.

Let us first consider the quaternion algebra case. Suppose there exist k and l such that $i_1(\psi_{T_P}) > 1$. Hence $2^k \times \langle 1\rangle \perp (l \cdot 2^k - 1) \times T_P$ is isotropic over $F(\psi_{T_P})$, since it is a 1-codimensional subform of ψ_{T_P}. Since ψ_{T_P} is isotropic over $F\left(\psi_{\langle x\rangle}\right)$, there exists an F-place from $F(\psi_{T_P})$ to $F\left(\psi_{\langle x\rangle}\right)$ by [K, Theorem 3.3]. Thus $2^k \times \langle 1\rangle \perp (l \cdot 2^k - 1) \times T_P$ is isotropic over $F\left(\psi_{\langle x\rangle}\right)$ by [K, Proposition 3.1]. Springer's Theorem implies that either $2^k \times \langle 1\rangle \perp (l \cdot 2^k - 1) \times \langle x\rangle$ or $(l \cdot 2^k - 1) \times \langle 1, -x\rangle$ is isotropic over $F_0(x)\left(\psi_{\langle x\rangle}\right)$.

Suppose $(l \cdot 2^k - 1) \times \langle 1, -x\rangle$ becomes isotropic over $F_0(x)\left(\psi_{\langle x\rangle}\right)$. Hence $2^n \times \langle 1, -x\rangle$ becomes hyperbolic over $F_0(x)\left(\psi_{\langle x\rangle}\right)$, where $2^n > l \cdot 2^k - 1$. The Cassels-Pfister Subform Theorem implies that $\psi_{\langle x\rangle}$ is a subform of $2^n \times \langle 1, -x\rangle$, which is clearly false.

Thus $2^k \times \langle 1\rangle \perp (l \cdot 2^k - 1) \times \langle x\rangle$ must be isotropic over $F_0(x)\left(\psi_{\langle x\rangle}\right)$. Hence, $\dim_{es}\left(2^k \times \langle 1\rangle \perp (l \cdot 2^k - 1) \times \langle x\rangle\right) = \dim_{es}\left(\psi_{\langle x\rangle}\right)$ by Theorem 2.3.

Consider $2^k \times \langle 1\rangle \perp l \cdot 2^k \times \langle x\rangle \simeq 2^k \times \langle 1\rangle \otimes (\langle 1\rangle \perp l \times \langle x\rangle)$. Now $i_1\left(2^k \times \langle 1\rangle \perp l \cdot 2^k \times \langle x\rangle\right) \geq 2^k$ by [EL, Theorem 1.4]. In addition, choosing an ordering P of $F_0(x)$ such that x is negative implies that $i_1\left(2^k \times \langle 1\rangle \perp l \cdot 2^k \times \langle x\rangle\right) \leq 2^k$ by [ELW, Theorem 3.5 and Remark 3.6]. Thus $i_1\left(2^k \times \langle 1\rangle \perp l \cdot 2^k \times \langle x\rangle\right) = 2^k$, implying that $2^k \times \langle 1\rangle \perp (l \cdot 2^k - 1) \times \langle x\rangle$ is isotropic over $F_0\left(2^k \times \langle 1\rangle \perp l \cdot 2^k \times \langle x\rangle\right)$ and hence that $\dim_{es}\left(2^k \times \langle 1\rangle \perp l \cdot 2^k \times \langle x\rangle\right) = \dim_{es}\left(2^k \times \langle 1\rangle \perp (l \cdot 2^k - 1) \times \langle x\rangle\right)$ by Theorem 2.3.

We may therefore conclude that $i_1\left(\psi_{\langle x\rangle}\right) = 2^k$. Consequently $(2^k+1)\times\langle 1\rangle \perp ((l-1)\cdot 2^k)\times\langle x\rangle$ is isotropic over $F_0\left(\psi_{\langle x\rangle}\right)$, since it is a subform of $\psi_{\langle x\rangle}$ of codimension 2^k-1. Invoking Theorem 2.3 yields that $i_1\left((2^k+1)\times\langle 1\rangle \perp ((l-1)\cdot 2^k)\times\langle x\rangle\right) = 1$.

However l is odd, since 2^k is the highest 2-power dividing $l \cdot 2^k$, whereby 2^{k+1} divides $(l-1)\cdot 2^k$. Thus $2^{k+1} \times \langle 1\rangle \perp ((l-1)\cdot 2^k) \times \langle x\rangle \simeq 2^{k+1} \times \langle 1\rangle \otimes \left(\langle 1\rangle \perp \frac{l-1}{2} \times \langle x\rangle\right)$. Now $i_1\left(2^{k+1} \times \langle 1\rangle \perp ((l-1)\cdot 2^k) \times \langle x\rangle\right) = 2^{k+1}$ by [EL, Theorem 1.4] and [ELW, Theorem 3.5 and Remark 3.6]. Hence $(2^k+1) \times \langle 1\rangle \perp ((l-1)\cdot 2^k) \times \langle x\rangle$ becomes isotropic over $F_0\left(2^{k+1} \times \langle 1\rangle \perp ((l-1)\cdot 2^k) \times \langle x\rangle\right)$, whereby $\dim_{es}\left(2^{k+1} \times \langle 1\rangle \perp ((l-1)\cdot 2^k) \times \langle x\rangle\right) = \dim_{es}\left((2^k+1) \times \langle 1\rangle \perp ((l-1)\cdot 2^k) \times \langle x\rangle\right)$ by Theorem 2.3. Therefore we conclude that $i_1\left((2^k+1) \times \langle 1\rangle \perp ((l-1)\cdot 2^k) \times \langle x\rangle\right) = 2^k+1$, contradicting the supposition that $2^k \times \langle 1\rangle \perp (l \cdot 2^k - 1) \times \langle x\rangle$ is isotropic over $F_0(x)\left(\psi_{\langle x\rangle}\right)$.

Hence $2^k \times \langle 1\rangle \perp (l \cdot 2^k - 1) \times T_P$ is in fact anisotropic over $F\left(\psi_{T_P}\right)$, contradicting the supposition that there exist k and l such that $i_1\left(\psi_{T_P}\right) > 1$.

Next, we prove the result for O'. In this case, we let $\psi_{\langle x,y,-xy\rangle}$ denote $(2^k+1)\times\langle 1\rangle \perp (l \cdot 2^k - 1) \times \langle x, y, -xy\rangle$ and ψ_{T_P} be defined as above, with T_P denoting the pure trace form of O' here.

Again we suppose that there exist k and l such that $i_1\left(\psi_{T_P}\right) > 1$. Hence $2^k \times \langle 1\rangle \perp (l \cdot 2^k - 1) \times T_P$ is isotropic over $F\left(\psi_{T_P}\right)$, since it is a 1-codimensional subform of ψ_{T_P}. Since ψ_{T_P} is isotropic over $F\left(\psi_{\langle x,y,-xy\rangle}\right)$, there exists an F-place from $F\left(\psi_{T_P}\right)$ to $F\left(\psi_{\langle x,y,-xy\rangle}\right)$ by [K, Theorem 3.3]. Thus $2^k \times \langle 1\rangle \perp (l \cdot 2^k - 1) \times T_P$ is isotropic over $F\left(\psi_{\langle x,y,-xy\rangle}\right)$ by [K, Proposition 3.1]. Springer's Theorem implies that either $2^k \times \langle 1\rangle \perp (l \cdot 2^k - 1) \times \langle x, y, -xy\rangle$ or $(l \cdot 2^k - 1) \times \langle 1, -x, -y, xy\rangle$ is isotropic over $F_0(x,y)\left(\psi_{\langle x,y,-xy\rangle}\right)$.

Suppose $(l \cdot 2^k - 1) \times \langle 1, -x, -y, xy\rangle$ becomes isotropic over $F_0(x,y)\left(\psi_{\langle x,y,-xy\rangle}\right)$. Hence $2^n \times \langle 1, -x, -y, xy\rangle$ becomes hyperbolic over $F_0(x,y)\left(\psi_{\langle x,y,-xy\rangle}\right)$, where $2^n > l \cdot 2^k - 1$. The Cassels-Pfister Subform Theorem implies that $\psi_{\langle x,y,-xy\rangle}$ is a subform of $2^n \times \langle 1, -x, -y, xy\rangle$, which is clearly false.

Thus $2^k \times \langle 1\rangle \perp (l \cdot 2^k - 1) \times \langle x, y, -xy\rangle$ must be isotropic over $F_0(x,y)\left(\psi_{\langle x,y,-xy\rangle}\right)$.

However, we know that $i_1\left(\psi_{\langle x,y,-xy\rangle}\right) = 1$ (this is what we proved in the first half of the proof), whereby $\dim_{es}\left(\psi_{\langle x,y,-xy\rangle}\right) > \dim_{es}\left(2^k \times \langle 1\rangle \perp (l\cdot 2^k - 1)\times\langle x,y,-xy\rangle\right)$. Hence, $2^k \times \langle 1\rangle \perp (l\cdot 2^k-1)\times\langle x,y,-xy\rangle$ is anisotropic over $F_0(x,y)\left(\psi_{\langle x,y,-xy\rangle}\right)$ by Theorem 2.3, contradicting the supposition that there exist k and l such that $i_1\left(\psi_{T_P}\right) > 1$. ■

Since the above argument relies upon the fact that l is an odd number > 1, we cannot hope to apply it to determine the first Witt index of $(2^k+1)\times\langle 1\rangle \perp (2^k-1)\times T_P$. However, we recall that [LM, Proposition 3.4] and [Pu, Proposition 3.5] state that $2^k\times\langle 1\rangle \perp (2^k-1)\times T_P$ remains anisotropic over $F\left((2^k+1)\times\langle 1\rangle \perp (2^k-1)\times T_P\right)$, where T_P respectively represents the pure trace form of $\left(\frac{x,y}{F}\right)$ and $\left(\frac{x,y,z}{F}\right)$ and F respectively denotes $F_0(x,y)$ and $F_0(x,y,z)$. Since $2^k\times\langle 1\rangle \perp (2^k-1)\times T_P$ is a 1-codimensional subform of $(2^k+1)\times\langle 1\rangle \perp (2^k-1)\times T_P$, Theorem 2.3 allows us to conclude that $i_1\left((2^k+1)\times\langle 1\rangle \perp (2^k-1)\times T_P\right) = 1$, for all k. Hence, we can incorporate into Theorem 3.19 the case where l is a 2-power, whereby we obtain:

$$i_1\left((2^h+1)\times\langle 1\rangle \perp (l\cdot 2^k-1)\times T_P\right) = 1 \text{ for all values of } l.$$

As alluded to earlier, as an immediate consequence of Theorem 3.19 we may increase the lower bounds on the level of our prospective composition algebras of level $l\cdot 2^k$. As $\dim_{es}\left((2^k+1)\times\langle 1\rangle \perp (l\cdot 2^k-1)\times T_P\right) > \dim_{es}\left(\langle 1\rangle \perp (l\cdot 2^k)\times T_P\right)$ for $k\geq 1$ in the quaternionic case, and for $k\geq 2$ in the octonionic case, employing Theorem 2.3 allows us to obtain such a reduction.

For example, we know that $\underline{s}(Q(24))$ and $s\left(Q(24)\right)\in[18,24]$ by Theorem 3.9 and Theorem 3.11, whereas for $l=k=3$ we can say that $\underline{s}(Q')$ and $s(Q')\in[20,24]$:

Since $20\times T_{Q'} \simeq 4\times\langle 1\rangle\otimes(5\times T_{Q'})$, $i_1\left(20\times T_{Q'}\right)\geq 4$ by [EL, Theorem 1.4]. Hence, $\dim_{es}\left(20\times T_{Q'}\right)\leq 76$. Theorem 3.19 implies that $\dim_{es}\left(9\times\langle 1\rangle \perp 23\times T_P\right) = 77$. Hence, $20\times T_{Q'}$ remains anisotropic over $F\left(9\times\langle 1\rangle \perp 23\times T_P\right)$ by Theorem 2.3, whereby $\underline{s}(Q')\in[20,24]$.

Similarly, while we currently cannot make any stronger statement regarding $Q(48)$ than $\underline{s}\left(Q(48)\right)$ and $s\left(Q(48)\right)\in[36,48]$, we have that $\underline{s}\left(Q'\right)$ and $s(Q')\in[40,48]$ for $l=3$ and $k=4$.

As previously noted, Conjecture 3.18 is true in the case where l is a power of 2. As an additional application of Theorem 3.19, we conclude by resolving one further case of this conjecture:

Theorem 3.20. *For $l = 3$ and $k = 1$, $s(O') = \underline{s}(O') = l \cdot 2^k = 6$.*

Proof. $s(O') \leq l \cdot 2^k = 6$ by construction. Suppose $\underline{s}(O') \leq 5$. Then, as in the proof of Theorem 3.3, either $6 \times T_P$ or $6 \times (\langle 1 \rangle \perp \beta)$ is isotropic over $F(3 \times \langle 1 \rangle \perp 5 \times T_P)$, where β is some 5-dimensional subform of T_P. Hence, either $\dim_{es}(6 \times T_P)$ or $\dim_{es}(6 \times (\langle 1 \rangle \perp \beta)) \geq \dim_{es}(3 \times \langle 1 \rangle \perp 5 \times T_P)$ by Theorem 2.3. However, $\dim_{es}(6 \times T_P) = 32$ and $\dim_{es}(6 \times (\langle 1 \rangle \perp \beta)) \leq 35$, whereas $\dim_{es}(3 \times \langle 1 \rangle \perp 5 \times T_P) = 37$ by Theorem 3.19, yielding a contradiction. ∎

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