VALUE FUNCTIONS AND ASSOCIATED GRADED RINGS FOR SEMISIMPLE ALGEBRAS

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Introduction

Valuation theory is a time-honored subject, which has undergone a robust development for noncommutative division rings in the last two decades, spurred by its applications to the constructions of noncrossed products and of counterexamples to the Kneser–Tits conjecture: see [W_4] for a recent and fairly comprehensive survey. However, results that relate valuations with Brauer-group properties have been particularly difficult to establish; a major source of problems is that valuations are defined only on division algebras and not on central simple algebras with zero divisors. The purpose of this work is to introduce a more flexible tool, which we call *gauge*, inspired by the *normes carrées* of Bruhat and Tits [BT] (see Rem. 1.19). Gauges are valuation-like maps defined on finite-dimensional semisimple algebras over valued fields with arbitrary value group.

With any valuation there is an associated filtration of the ring, which yields an associated graded ring. Such filtrations and associated graded rings are actually defined not just for valuations, but also for more general value functions: the surmultiplicative value functions defined in (1.4) below, which are sometimes called *pseudo-valuations*. The gauges we consider here are the surmultiplicative value functions for which the associated graded algebra is semisimple, and which also satisfy a defectlessness condition, see Def. 1.4. It turns out that gauges exist in abundance and have good behavior with respect to tensor products, but that they still have sufficient uniqueness to reflect the structure of the algebras they are defined on.

Valuation theory typically derives information on fields or division algebras from properties of the residue field or algebra and of the ordered group of values. In a noncommutative setting, these structures interact since the value group acts naturally on the center of the residue algebra, see (1.13). It is therefore reasonable to consider the graded algebra associated with the valuation filtration, which encapsulates information on the residue algebra, the value group, and their interaction. This paper shows how fruitful it can be to work with the graded structures. Associated graded algebras have previously been studied for valuations on division algebras, as in [Bl_1], [Bl_2], and [HW_2]. But they have not been used in the earlier work with value functions on central simple algebras in [BT], nor with the value functions associated to Dubrovin valuation rings in [M_2]. (The relation between the value functions considered here and Morandi's value functions in [M_2] is described in Prop. 2.5.)

For a given semisimple algebra A over a field F, we fix a valuation v on F and consider gauges y on A which restrict to v on F, which we call v-gauges or (when v is understood) F-gauges. The associated graded ring $gr_y(A)$ is then a finite dimensional algebra over the graded field $gr_v(F)$. If A is central simple over F, there are typically many different v-gauges y on A; it turns out that $gr_y(A)$ is always a graded simple algebra (i.e., there are no nontrivial homogeneous ideals), and that the class of $gr_y(A)$ is uniquely determined in the graded Brauer group of its center, see Cor. 3.8.

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We get the strongest information when the valuation on F is Henselian. For any finite-dimensional division algebra D over F, it is well-known that the Henselian valuation v on F has a unique extension to a valuation w on D. For $A = End_D(M)$, where M is a finite dimensional right D-vector space, we prove in Th. 3.3 that for any v-gauge y on A there is a norm α (a kind of value function) on M such that up to isomorphism y is the gauge on $End_D(M)$ induced by α on M as described in §1.3. It follows that $gr_y(A)$ is isomorphic as a graded ring to $End_{gr_w(D)}(gr_\alpha(M))$; furthermore, the graded Brauer class of $gr_y(A)$ is the same as that of $gr_w(D)$, and $gr_y(A)$ has the same matrix size as A. In particular, if A is central simple over F and the gauge is *tame*, in the sense that the center of $gr_y(A)$ is $gr_v(F)$, then $gr_y(A)$ is a graded central simple algebra over $gr_v(F)$ with the same Schur index as A. We may then consider its Brauer class $[gr_y(A)]$ in the graded Brauer group $GBr(gr_v(F))$. The map $[A] \mapsto [gr_y(A)]$ defines an index-preserving group isomorphism Ψ from the tame Brauer group $TBr(F)$, which is the subgroup of $Br(F)$ split by the maximal tamely ramified extension of F, onto $GBr(gr_v(F))$. That Ψ is an isomorphism was proved earlier in [HW$_2$]; without the use of gauges the proof in [HW$_2$] that Ψ is a group homomorphism was particularly involved and arduous. The proof given here in Th. 3.9 is much easier and more natural, because we can work with central simple algebras, not just with division algebras, and because gauges work well with tensor products. The map Ψ should be compared with a similar map for Witt groups defined in [TW] to generalize Springer's theorem on quadratic forms over complete discretely valued fields.

When v is Henselian and A is assumed just to be semisimple, we show in Th. 3.1 that for any v-gauge on A the simple components of $gr_y(A)$ are the graded algebras for the restrictions of y to the simple components of A. Thus, the results described above apply component-by-component. Also, the information obtained in the Henselian case can be extrapolated to gauges with respect to non-Henselian valuations v. For, if the valuation v^h on field F^h is the Henselization of a valuation v on F, and y is any v-gauge on a semisimple F-algebra A, then there is a canonical extension of y to a v^h-gauge y^h on $A \otimes_F F^h$, and $gr_y(A)$ is graded isomorphic to $gr_{y^h}(A \otimes_F F^h)$. Thus, any v-gauge on A gives insight into what happens with A on passage to the Henselization of v.

In the last section, we apply gauges to obtain information on the division algebra Brauer-equivalent to a crossed product or to a tensor product of symbol algebras over valued fields. The idea is that, since we are now freed from the constraint to deal with division algebras, we may easily define gauges on these central simple algebras, and use the associated graded structure to derive properties of their Brauer-equivalent division algebras. We thus easily recover in a straightforward way several results that were previously obtained in [JW] and [W$_3$] by much more complicated arguments.

The organization of the paper is as follows: §1 gives the definition of gauges and describes various examples on division algebras, endomorphism algebras, and tensor products. In §2 we review some results on graded central simple algebras, complementing the discussion in [HW$_2$] with a result characterizing the graded group of the Brauer-equivalent graded division algebra. The main results quoted above, relating semisimple algebras with a gauge over a Henselian field to their associated graded algebras, are given in §3. This section also contains the definition of the map $\Psi\colon TBr(F) \to GBr(gr(F))$. The applications to crossed products and tensor products of symbols are in §4.

1. Value functions, norms, and gauges

Let D be a division ring finite-dimensional over its center. Let Γ be a divisible totally ordered abelian group. Let ∞ be an element of a set strictly containing Γ; extend the ordering on Γ to $\Gamma \cup \{\infty\}$ by requiring that $\gamma < \infty$ for each $\gamma \in \Gamma$. Further set $\gamma + \infty = \infty + \infty = \infty$ for all $\gamma \in \Gamma$. A valuation on D is a function $w\colon D \to \Gamma \cup \{\infty\}$ satisfying, for all c, $d \in D$,

$$w(d) = \infty \ \text{ iff } \ d = 0; \tag{1.1a}$$

$$w(cd) = w(c) + w(d); \tag{1.1b}$$

$$w(c + d) \geq \min\big(w(c), w(d)\big). \tag{1.1c}$$

(It follows that $w(1) = w(-1) = 0$ and if $w(c) \neq w(d)$ then $w(c + d) = w(c - d) = \min\big(w(c), w(d)\big)$.) Associated to the valuation on D, we have its value group $\Gamma_D = w(D^\times)$, where $D^\times$ is the group of units of D, i.e., $D^\times = D - \{0\}$; its valuation ring $V_D = \{\, d \in D \mid w(d) \geq 0\,\}$; the unique maximal left (and right) ideal M_D of V_D, $M_D = \{\, d \in D \mid w(d) > 0\,\}$; and the residue division ring $\overline{D} = V_D/M_D$. Another key structure is the associated graded ring: for $\gamma \in \Gamma$, set $D^{\geq\gamma} = \{\, d \in D \mid w(d) \geq \gamma\,\}$ and $D^{>\gamma} = \{\, d \in D \mid w(d) > \gamma\,\}$, which is a subgroup of $D^{\geq\gamma}$; let $D_\gamma = D^{\geq\gamma}/D^{>\gamma}$. The *associated graded ring* of D with respect to w is $gr_w(D) = \bigoplus_{\gamma\in\Gamma} D_\gamma$. For each γ, $\delta \in \Gamma$, the multiplication in D induces a well-defined multiplication $D_\gamma \times D_\delta \to D_{\gamma+\delta}$ given by $(c + D^{>\gamma}) \cdot (d + D^{>\delta}) = cd + D^{>\gamma+\delta}$. This multiplication is extended biadditively to all of $gr_w(D)$, making $gr_w(D)$ into a graded ring. When w is clear, we write $gr(D)$ for $gr_w(D)$. The grade group of $gr(D)$, denoted $\Gamma_{gr(D)}$, is $\{\, \gamma \in \Gamma \mid D_\gamma \neq 0\,\}$; note that $\Gamma_{gr(D)} = \Gamma_D$. Also, for the degree 0 component of $gr(D)$, we have $D_0 = D^{\geq 0}/D^{>0} = V_D/M_D = \overline{D}$. For $d \in D^\times$, we write d' for the image of d in $gr(D)$, i.e., $d' = d + D^{>w(d)} \in D_{w(d)}$. The homogeneous elements of $gr(D)$ are those in $\bigcup_{\gamma\in\Gamma} D_\gamma$. It follows from property (1.1b) that $gr(D)$ is a *graded division ring*, i.e., every nonzero homogeneous element of $gr(D)$ is a unit.

Now, let M be a right D-vector space, where D has a valuation w. A function $\alpha\colon M \to \Gamma \cup \{\infty\}$ is called a *D-value function* with respect to w (or a *w-value function*) if for all m, $n \in M$ and $d \in D$,

$$\alpha(m) = \infty \ \text{ iff } \ m = 0; \tag{1.2a}$$

$$\alpha(md) = \alpha(m) + w(d); \tag{1.2b}$$

$$\alpha(m + n) \geq \min\big(\alpha(m), \alpha(n)\big). \tag{1.2c}$$

Given such an α on M, we can form the associated graded module $gr_\alpha(M)$ just as before: for $\gamma \in \Gamma$, let $M^{\geq\gamma} = \{\, m \in M \mid \alpha(m) \geq \gamma\,\}$ and $M^{>\gamma} = \{\, m \in M \mid \alpha(m) > \gamma\,\}$; then set $M_\gamma = M^{\geq\gamma}/M^{>\gamma}$. Define $gr(M) = gr_\alpha(M) = \bigoplus_{\gamma\in\Gamma} M_\gamma$. For nonzero $m \in M$, let m' denote the image $m + M^{>\alpha(m)}$ of m in $gr(M)$; for $0 \in M$, let $0' = 0 \in gr(M)$. For γ, $\delta \in \Gamma$ there is a well-defined multiplication $M_\gamma \times D_\delta \to M_{\gamma+\delta}$ given by $(m + M^{>\gamma}) \cdot (d + D^{>\delta}) = (md) + M^{>\gamma+\delta}$. This is extended distributively to yield an operation $gr(M) \times gr(D) \to gr(M)$ which makes $gr(M)$ into a graded right $gr(D)$-module. It is well-known and easy to prove by a slight variation of the ungraded argument that every graded module over a graded division ring is a free module with a homogeneous base, and every two bases have the same cardinality. Thus, graded modules over graded division rings are called graded vector spaces; we write $dim_{gr(D)}(gr(M))$ for the cardinality of any $gr(D)$-module base of $gr(M)$. If $N = \bigoplus_{\gamma\in\Gamma} N_\gamma$ is another graded right $gr(D)$-vector space, we say that M and N are *graded isomorphic*, written $M \cong_g N$, if there is a $gr(D)$-vector space isomorphism $f\colon M \to N$ with $f(M_\gamma) = N_\gamma$ for each $\gamma \in \Gamma$.

Now, suppose M is finite-dimensional. A right D-vector space base $(m_i)_{1\leq i\leq k}$ of M is called a *splitting base* with respect to α if for all $d_1, \ldots, d_k \in D$,

$$\alpha\Big(\sum_{i=1}^{k} m_i d_i\Big) \;=\; \min_{1\leq i\leq k}\big(\alpha(m_i) + w(d_i)\big). \tag{1.3}$$

If there is a splitting base for the D-value function α, we say that α is a *D-norm* (or a *w-norm*) on M. Note that it is easy to construct D-norms on M: take any D-vector space base $(m_i)_{1\leq i\leq k}$ of M, and take any $\gamma_1, \ldots, \gamma_k \in \Gamma$. Define $\alpha(m_i) = \gamma_i$ for $1 \le i \le k$, and then define α on all of M by formula (1.3). It is straightforward to check that such an α is a D-norm on M and $(m_i)_{1\le i\le k}$ is a splitting base for α.

Recall the following from [RTW, Prop. 2.2, Cor. 2.3, Prop. 2.5]:

PROPOSITION 1.1. *Let α be a D-value function on M. Take any $m_1, \ldots, m_\ell \in M$.*

(i) *$m_1', \ldots, m_\ell'$ are $gr(D)$-linearly independent in $gr(M)$ iff $\alpha\big(\sum_{i=1}^{\ell} m_i d_i\big) = \min_{1\le i\le \ell}\big(\alpha(m_i)+w(d_i)\big)$ for all $d_1, \ldots, d_\ell \in D$. When this occurs, $m_1, \ldots, m_\ell$ are D-linearly independent in M.*
(ii) *$dim_{gr(D)}(gr(M)) \le dim_D(M)$. Equality holds iff α is a D-norm on M.*
(iii) *Suppose α is a D-norm on M. Then, for any D-subspace N of M, $\alpha|_N$ is a norm on N.*

We are interested here in value functions on algebras. Let F be a field with valuation $v\colon F \to \Gamma\cup\{\infty\}$, and let A be a finite-dimensional F-algebra. A function $y\colon A \to \Gamma \cup \{\infty\}$ is called a *surmultiplicative F-value function* on A if for any $a, b \in A$,

$$y(1) = 0, \quad \text{and} \quad y(a) = \infty \quad \text{iff} \quad a = 0; \tag{1.4a}$$

$$y(ca) = v(c) + y(a) \quad \text{for any } c \in F; \tag{1.4b}$$

$$y(a+b) \ge min\big(y(a), y(b)\big); \tag{1.4c}$$

$$y(ab) \ge y(a) + y(b). \tag{1.4d}$$

Note that for such a y, there is a corresponding "valuation ring" $V_A = A^{\ge 0} = \{\, a \in A \mid y(a) \ge 0 \,\}$. There is also an associated graded ring $gr(A) = gr_y(A) = \bigoplus_{\gamma\in\Gamma} A_\gamma$, where $A_\gamma = A^{\ge\gamma}/A^{>\gamma}$, as above, and the multiplication in $gr(A)$ is induced by that of A. Furthermore, $gr_y(A)$ is clearly a graded $gr_v(F)$-algebra. Also, $gr_v(F)$ is a graded field, i.e., a commutative graded ring in which every nonzero homogeneous element is a unit. Since axioms (1.4a) – (1.4c) show that y is an F-value function for A as an F-vector space, Prop. 1.1(ii) implies that $dim_{gr(F)}(gr(A)) \le dim_F(A)$, with equality iff y is an F-norm on A. The following lemma is convenient for verifying when an F-norm on A is surmultiplicative:

LEMMA 1.2. *Suppose $y\colon A \to \Gamma \cup \{\infty\}$ is an F-norm on A such that $y(1) = 0$. Let $(a_i)_{1\le i\le k}$ be a splitting base of A. If $y(a_i a_j) \ge y(a_i) + y(a_j)$ for all i, j, then y is a surmultiplicative F-value function on A.*

Proof. We need only to verify axiom (1.4d). For this, take any $b_1 = \sum c_i a_i$ and $b_2 = \sum d_i a_i$ in A with $c_i, d_i \in F$. Then,

$$\begin{aligned} y(b_1 b_2) &= y\Big(\sum_{i,j} c_i d_j a_i a_j\Big) \ \ge\ \min_{i,j}\big(y(c_i d_j a_i a_j)\big) \ \ge\ \min_{i,j}\big(v(c_i) + v(d_j) + y(a_i) + y(a_j)\big) \\ &\ge\ \min_i\big(v(c_i) + y(a_i)\big) \ +\ \min_j\big(v(d_j) + y(a_j)\big) \ =\ y(b_1) + y(b_2). \end{aligned}$$

□

If A has a surmultiplicative value function y, then for nonzero $a \in A$, we write a' for the image $a + A^{>y(a)}$ of a in $A_{y(a)}$. For 0 in A, we write $0' = 0 \in gr(A)$. The following immediate consequence of the definitions will be used repeatedly below: for nonzero $a, b \in A$,

$$a'b' \ =\ \begin{cases} (ab)', & \text{if } a'b' \ne 0', \text{ iff } y(ab) = y(a) + y(b); \\ 0', & \text{if } y(ab) > y(a) + y(b)\,. \end{cases} \tag{1.5}$$

With this, we can readily characterize the inverse image in A of the group of homogeneous units of $gr(A)$:

LEMMA 1.3. *Let y be a surmultiplicative F-value function on a finite-dimensional F-algebra A. For any nonzero $u \in A$, the following conditions are equivalent:*

(i) $u' \in gr(A)^\times$, *the group of units of* $gr(A)$*;*
(ii) $y(au) = y(a) + y(u)$ *for all* $a \in A$*;*
(ii′) $y(ua) = y(u) + y(a)$ *for all* $a \in A$*;*
(iii) $u \in A^\times$ *and* $y(u) + y(u^{-1}) = 0$.

Proof. (i) ⇒ (ii) Suppose $u' \in gr(A)^\times$. Then, for any nonzero $a \in A$, we have $a'u' \neq 0'$; hence, $y(au) = y(a) + y(u)$ by (1.5). (ii) ⇒ (i) By (1.5), (ii) implies that $a'u' \neq 0'$ for every nonzero $a \in A$. Therefore, as $gr(A)$ is a finite-dimensional graded algebra over the graded field $gr(F)$, $u' \in gr(A)^\times$. (i) ⇔ (ii′) is proved analogously. (ii) ⇒ (iii) Condition (ii) shows that u is not a zero divisor in the finite-dimensional algebra A. Therefore, $u \in A^\times$. The formula in (iii) follows by setting $a = u^{-1}$ in (ii). (iii) ⇒ (ii) For any $a \in A$, we have $y(a) = y(auu^{-1}) \geq y(au) + y(u^{-1})$. Therefore, (iii) yields

$$y(au) \;\leq\; y(a) - y(u^{-1}) \;=\; y(a) + y(u) \;\leq\; y(au)\,;$$

so equality holds throughout, proving (ii). □

It is easy to construct numerous surmultiplicative value functions y on A using Lemma 1.2. We next make further restrictions on y so as to be able to relate the structure of $gr(A)$ to that of A.

If K is a graded field, then a finite-dimensional graded K-algebra B is said to be *graded simple* if B has no homogeneous two-sided ideals except B and $\{0\}$. We say that B is a *graded semisimple* K-algebra if B is a direct product of finitely many graded simple K-algebras. By a variation of the ungraded argument, this is equivalent to: B has no nonzero nilpotent homogeneous ideals.

If B is an algebra (resp. graded algebra) over a field (resp. graded field) K, we write $[B{:}K]$ for $dim_K(B)$. Throughout the paper, all semisimple (resp. graded semisimple) algebras are tacitly assumed to be finite-dimensional.

Definition 1.4. Let F be a field with a valuation v. Let y be a surmultiplicative value function on a finite-dimensional F-algebra A. We say that y is an *F-gauge* (or a *v-gauge*) on A if y is an F-norm on A (i.e., $[gr(A){:}gr(F)] = [A{:}F]$) and $gr(A)$ is a graded semisimple $gr(F)$-algebra. Note that if A has an F-gauge then A must be semisimple. For, if A had a nonzero ideal N with $N^2 = \{0\}$, then $gr(N)$ would be a nonzero ideal of $gr(A)$ with $gr(N)^2 = \{0\}$.

For any ring R, let $Z(R)$ denote the center of R.

Definition 1.5. An F-gauge y on a finite-dimensional semisimple F-algebra A is called a *tame F-gauge* if $Z(gr(A)) = gr(Z(A))$ and $Z(gr(A))$ is separable over $gr(F)$. By [HW$_1$, Th. 3.11], the second condition holds if and only if $Z(gr(A))_0$ is separable over $gr(F)_0$ and $char(gr(F)_0) \nmid |\Gamma_{Z(A)}{:}\Gamma_F|$. Thus, the gauge is tame if and only if $Z(gr(A)) = gr(Z(A))$, $\overline{Z(A)}$ is separable over $\overline{F}$, and $char(\overline{F}) \nmid |\Gamma_{Z(A)}{:}\Gamma_F|$. It will be shown below (see Cor. 3.7) that whenever $char(\overline{F}) = 0$ every F-gauge is tame.

The notion of gauge generalizes that of defectless valuation on division algebras, and tame gauge generalizes tame valuation. We make this point clear in §1.2, and give fundamental examples of gauges on endomorphism algebras and on tensor products in §§1.3 and 1.4. We start our discussion of examples with commutative semisimple algebras.

1.1. **Gauges on commutative algebras.** For a commutative finite-dimensional algebra A over a field F, semisimplicity is equivalent to the absence of nonzero elements $x \in A$ such that $x^2 = 0$. A similar observation holds for graded algebras. Thus, if F has a valuation v and A has a surmultiplicative v-value function y, the following conditions are equivalent:

(a) $gr_y(A)$ is semisimple;
(b) $(x')^2 \neq 0$ for all nonzero $x \in A$;

(c) $(x')^n \neq 0$ for all nonzero $x \in A$ and for every positive integer n.

In view of (1.5), these conditions are also equivalent to:

(d) $y(x^2) = 2y(x)$ for all $x \in A$;
(e) $y(x^n) = ny(x)$ for all $x \in A$.

We first consider the case where A is a field.

PROPOSITION 1.6. *Let (F, v) be a valued field and let K/F be a finite-degree field extension. Suppose $y\colon K \to \Gamma \cup \{\infty\}$ is a surmultiplicative v-value function such that $gr_y(K)$ is semisimple. Then, there exist valuations $v_1, \ldots, v_n$ on K extending v such that*

$$y(x) \;=\; \min_{1\leq i\leq n}(v_i(x)) \qquad \text{for } x \in K. \tag{1.6}$$

Moreover, there is a natural graded isomorphism of graded $gr(F)$-algebras

$$gr_y(K) \;\cong_g\; gr_{v_1}(K) \times \ldots \times gr_{v_n}(K).$$

Proof. Let $\Gamma_K = y(K^\times) \subseteq \Gamma$ and $\Gamma_F = v(F^\times) = y(F^\times) \subseteq \Gamma_K$. If $x_1, \ldots, x_r \in K$ are such that $y(x_1), \ldots, y(x_r)$ belong to different cosets of Γ_F in Γ_K, then $x_1', \ldots, x_r' \in gr(K)$ are linearly independent over $gr(F)$, hence $x_1, \ldots, x_r$ are linearly independent over F, see Prop. 1.1(i). Since $[K{:}F]$ is finite, it follows that the index $|\Gamma_K{:}\Gamma_F|$ is finite, hence Γ_K/Γ_F is torsion.

Let V_F and M_F denote the valuation ring of F and its maximal ideal, and let

$$V_y \;=\; \{x \in K \mid y(x) \geq 0\} \qquad \text{and} \qquad M_y \;=\; \{x \in K \mid y(x) > 0\}.$$

Clearly, V_y is a subring of K containing V_F and M_y is an ideal of V_y containing M_F. Since $gr_y(K)$ is assumed to be semisimple, we have $y(x^n) = ny(x)$ for all $x \in K$ (see condition (e) above), hence the ideal M_y is radical. We may therefore find a set of prime ideals $P_\lambda \subseteq V_y$ (indexed by some set Λ) such that $M_y = \bigcap_{\lambda\in\Lambda} P_\lambda$. By Chevalley's Extension Theorem [EP, Th. 3.1.1], we may find for each $\lambda \in \Lambda$ a valuation ring V_λ of K with maximal ideal M_λ such that $V_y \subseteq V_\lambda$ and $P_\lambda = V_y \cap M_\lambda$.

Claim: The valuation v_λ corresponding to V_λ extends v. We have $V_F \subseteq V_y \subseteq V_\lambda$, hence $V_F \subseteq V_\lambda \cap F$. Similarly, $M_F \subseteq M_y \subseteq P_\lambda \subseteq M_\lambda$, so $M_F \subseteq M_\lambda \cap F$. Since $V_\lambda \cap F$ is a valuation ring of F with maximal ideal $M_\lambda \cap F$, the inclusions $V_F \subseteq V_\lambda \cap F$ and $M_F \subseteq M_\lambda \cap F$ imply $V_F = V_\lambda \cap F$, proving the claim. Hence, each value group Γ_{K,v_i} embeds canonically into the divisible group Γ.

Since there are only finitely many extensions of v to K, and since for $\lambda, \lambda' \in \Lambda$ the equality $V_\lambda = V_{\lambda'}$ implies $M_\lambda = M_{\lambda'}$, hence $P_\lambda = P_{\lambda'}$, it follows that Λ is a finite set. Let $\Lambda = \{1, \ldots, n\}$ and, for $x \in K$, let

$$w(x) \;=\; \min_{1\leq i\leq n}\big(v_i(x)\big).$$

Since $V_y \subseteq \bigcap_{i=1}^{n} V_i$, we have

$$y(x) \geq 0 \;\Rightarrow\; w(x) \geq 0 \qquad \text{for } x \in K.$$

Similarly, since $M_y = \bigcap_{i=1}^{n} P_i = V_y \cap \big(\bigcap_{i=1}^{n} M_i\big)$, we have

$$y(x) > 0 \iff (y(x) \geq 0 \text{ and } w(x) > 0).$$

It follows that

$$y(x) = 0 \;\Rightarrow\; w(x) = 0 \qquad \text{for } x \in K^\times. \tag{1.7}$$

Now, fix some $x \in K^\times$. Since Γ_K/Γ_F is a torsion group we may find an integer $m > 0$ and an element $u \in F^\times$ such that $my(x) = v(u)$, hence $y(x^m u^{-1}) = 0$. By (1.7), we then have $w(x^m u^{-1}) = 0$. Because each v_i extends v, we have $w(x^m u^{-1}) = w(x^m) - v(u)$. Hence,

$$mw(x) \;=\; w(x^m) \;=\; v(u) \;=\; my(x).$$

Since Γ has no torsion, it follows that $w(x) = y(x)$, which proves (1.6).

For $i = 1, \ldots, n$ we have $y(x) \le v_i(x)$ for all $x \in K$, hence the identity map on K induces a map $gr_y(K) \to gr_{v_i}(K)$. Combining these maps, we obtain a graded homomorphism of graded $gr(F)$-algebras

$$gr_y(K) \;\to\; gr_{v_1}(K) \times \ldots \times gr_{v_n}(K). \tag{1.8}$$

This map is injective since for every $x \in K^\times$ there is some index i such that $y(x) = v_i(x)$. For surjectivity, fix any k, $1 \le k \le n$ and any $b \in K^\times$.

Claim: There is $c \in K^\times$ with $v_k(c) = v_k(b)$ and $v_i(c) \ge v_k(b)$ for $i \ne k$. This will be proved below.

Now, since the valuations $v_1, \ldots, v_n$ are incomparable and $V_y = \bigcap_{i=1}^n V_i$ by (1.6), the map $V_y \to \prod_{i=1}^n V_i/M_i$ is surjective by [EP, Th. 3.2.7(3)]. Therefore, there is $d \in V_y$ with $d \in M_i$ for $i \ne k$ and $d - bc^{-1} \in M_k$. Let $a = cd$. Then, $v_i(a) > v_i(c) \ge v_k(b)$ for $i \ne k$, and $v_k(a) = v_k(b)$ with $a' = b'$ in $gr_{v_k}(K)$. Hence, $y(a) = v_k(b)$ and $a' \in gr_y(K)$ maps to $(0, \ldots, 0, b', 0 \ldots, 0) \in gr_{v_1}(K) \times \ldots \times gr_{v_n}(K)$ (b' in the k-th position). Since these n-tuples span $gr_{v_1}(K) \times \ldots \times gr_{v_n}(K)$, it follows that the natural map (1.8) is onto.

Proof of Claim. Assume for simplicity that Γ is the divisible hull of Γ_F ($= \Gamma_{F,v}$, the value group of v on F). For any valuation z on F which is coarser than v there is an associated convex subgroup $\Delta_F \subseteq \Gamma_F$, which is the kernel of the canonical epimorphism $\Gamma_F \to \Gamma_{F,z}$. Let Δ be the divisible hull of Δ_F in Γ, and let $\Lambda = \Gamma/\Delta$, which is a divisible group with ordering inherited from Γ. Since $\Gamma_F \cap \Delta = \Delta_F$, the order-preserving inclusion $\Gamma_F/\Delta_F \hookrightarrow \Lambda$ identifies $\Gamma_{F,z}$ canonically with a subgroup of the divisible group Λ. Likewise, the value group of every extension of z to K can be viewed as a subgroup of Λ.

For each pair of valuations v_i, v_j on K with $i \ne j$ there is a valuation v_{ij} on K which is the finest common coarsening of v_i and v_j. (v_{ij} is the valuation associated to the valuation ring V_iV_j.) Let $\Delta_{ij} \subseteq \Gamma$ be the divisible hull of the convex subgroup of Γ_F associated to the restriction of v_{ij} to F. Let $\Gamma_i = \Gamma_{K,v_i}$ for $1 \le i \le n$. We say that an n-tuple $(\beta_1, \ldots, \beta_n) \in \Gamma_1 \times \ldots \times \Gamma_n$ is *compatible* if $\beta_i - \beta_j \in \Delta_{ij}$ for all $i \ne j$. (This is equivalent to the definition of compatibility in [R, p. 127], though stated a little differently.) For our fixed $b \in K^\times$, let $\gamma_i = v_i(b) \in \Gamma_i$, and note that $(\gamma_1, \ldots, \gamma_n)$ is compatible since $v_i(b)$ and $v_j(b)$ have the same image $v_{ij}(b)$ in Γ/Δ_{ij}. For our fixed k and for each i, let $\epsilon_i = \gamma_k - \gamma_i$. So, for all $i \ne j$, $\epsilon_i - \epsilon_j = \gamma_j - \gamma_i \in \Delta_{ij}$. Since $0 \le \big||\epsilon_i| - |\epsilon_j|\big| \le |\epsilon_i - \epsilon_j|$ and Δ_{ij} is convex, we have $|\epsilon_i| - |\epsilon_j| \in \Delta_{ij}$. Because each Γ/Γ_i is a torsion group, there is a positive integer m such that $m|\epsilon_i| \in \Gamma_i$ for each i. Let $\delta_i = \gamma_i + m|\epsilon_i| \in \Gamma_i$. Then $\delta_k = \gamma_k$, and for each i we have

$$\delta_i \;=\; \gamma_i + m|\epsilon_i| \;\ge\; \gamma_i + \epsilon_i \;=\; \gamma_k\,.$$

Note that $(\delta_1, \ldots, \delta_n)$ is compatible, since $(\gamma_1, \ldots, \gamma_n)$ and $(m|\epsilon_1|, \ldots, m|\epsilon_n|)$ are compatible. Since the valuations $v_1, \ldots, v_n$ are incomparable, by [R, Th. 1, p. 135] there is $c \in K^\times$ with $v_i(c) = \delta_i$ for each i. This c has the properties of the claim. □

Let $v_1, \ldots, v_r$ be all the extensions of v to K. For $i = 1, \ldots, r$, let $e_i = e(v_i/v)$ be the ramification index and $f_i = f(v_i/v)$ be the residue degree. We say that *the fundamental equality holds for K/F* if $[K{:}F] = \sum_{i=1}^r e_i f_i$.

Corollary 1.7. *There is an F-gauge on K if and only if the fundamental equality holds for K/F. When that condition holds, the F-gauge on K is unique and is defined by*

$$y(x) \;=\; \min_{1\le i\le r}\big(v_i(x)\big) \qquad \text{for } x\in K, \tag{1.9}$$

where $v_1, \ldots, v_r$ are all the extensions of v to K.

Proof. Suppose y is an F-gauge on K. By Prop. 1.6, we may find some extensions $v_1, \ldots, v_n$ of v to K such that $gr_y(K) \cong_g gr_{v_1}(K) \times \ldots \times gr_{v_n}(K)$. Now, $[gr_{v_i}(K){:}gr(F)] = e_i f_i$ by [Bl$_1$, Cor. 2], and $[gr_y(K){:}gr(F)] = [K{:}F]$ since y is an F-norm, so

$$[K{:}F] \;=\; \sum_{i=1}^{n} e_i f_i.$$

This implies $v_1, \ldots, v_n$ is the set of *all* extensions of v to K by [EP, Th. 3.3.4], and the fundamental equality holds. Conversely, if the fundamental equality holds, then formula (1.9) defines an F-gauge on K. □

The following special case will be used in §1.2:

Corollary 1.8. *Let K/F be a finite-degree field extension and let v be a valuation on F. Suppose $char(\overline{F}) \nmid [K{:}F]$. If v extends uniquely to K, this extension is an F-gauge on K.*

Proof. In view of Cor. 1.7, it suffices to show that

$$[K{:}F] \;=\; [\overline{K}{:}\overline{F}]\,|\Gamma_K{:}\Gamma_F|, \tag{1.10}$$

which may be regarded as a (weak) version of Ostrowski's theorem. We include a proof for lack of a convenient reference. When the equality in (1.10) holds, we say that K/F is defectless.

Let N be a Galois closure of K/F, let z be the extension of v to K, and let w be an extension of z to N. Denote by D the decomposition group of w/v, by R the ramification group, and by N^D and N^R the corresponding fixed subfields of N. Since

$$\begin{aligned}[N^D{:}F] \;&=\; \text{number of extensions of } v \text{ to } N\\ &=\; \text{number of extensions of } z \text{ to } N \;=\; [KN^D{:}K]\,,\end{aligned}$$

we have $[K{:}F] \;=\; [KN^D{:}N^D]$. Also, N^D/F and KN^D/K are immediate extensions by [EP, Cor. 5.3.8(0)]. Therefore, by substituting N^D for F and KN^D for K, we may assume that $N^D = F$. Now, N^R/F is Galois and $[N{:}N^R]$ is a power of $char(\overline{F})$ ($N = N^R$ if $char(\overline{F}) = 0$), by [EP, Th. 5.3.3]. Since $char(\overline{F}) \nmid [K{:}F]$, and $[KN^R{:}N^R] = [K{:}K \cap N^R]$ by Galois theory, we must have $K \subseteq N^R$. By [EP, Cor. 5.3.8], N^R/F is defectless, hence K/F is defectless. □

We now turn to the general type of commutative semisimple algebras.

Proposition 1.9. *Let $K_1, \ldots, K_m$ be finite-degree field extensions of a field F and $A = K_1 \times \ldots \times K_m$. Let v be a valuation on F, let y be a surmultiplicative v-value function on A, and let $y_i = y|_{K_i}$ for $i = 1, \ldots, m$. If $gr_y(A)$ is graded semisimple, then each y_i has the form* (1.6) *and, for $a = (a_1, \ldots, a_n) \in A$,*

$$y(a) \;=\; \min_{1\le i\le m}\big(y_i(a_i)\big). \tag{1.11}$$

Moreover, there is a canonical graded isomorphism of graded $gr(F)$-algebras

$$gr_y(A) \;\cong_g\; gr_{y_1}(K_1) \times \ldots \times gr_{y_m}(K_m). \tag{1.12}$$

There is a v-gauge on A if and only if the fundamental equality holds for each K_i/F. When that condition holds, there is a unique v-gauge y on A, defined by (1.11) *where each y_i is the unique v-gauge on K_i as in Cor. 1.7.*

Proof. Let $e_1, \ldots, e_m$ be the primitive idempotents of A, such that $e_iA = K_i$. Since $gr_y(A)$ is graded semisimple, we have $y(e_i^2) = 2y(e_i)$, hence $y(e_i) = 0$ for $i = 1, \ldots, m$. For $a \in A$, we have $a = ae_1 + \ldots + ae_m$, hence

$$y(a) \ \geq \ \min_{1\leq i\leq m} \big(y(ae_i)\big).$$

On the other hand, the surmutliplicativity of y yields

$$y(ae_i) \ \geq \ y(a) + y(e_i) \ = \ y(a) \qquad \text{for } i = 1, \ldots, m,$$

hence

$$y(a) \ = \ \min_{1\leq i\leq m} \big(y(ae_i)\big),$$

proving (1.11). Since $y(a^2) = 2y(a)$ for $a \in A$, we also have $y_i(a^2) = 2y(a)$ for $a \in K_i$, hence $gr_{y_i}(K_i)$ is semisimple. Therefore, Prop. 1.6 shows that y_i has the form (1.6) for $i = 1, \ldots, m$. The isomorphism (1.12) is clear; it implies

$$[gr_y(A){:}gr(F)] \ = \ \sum_{i=1}^{m} [gr_{y_i}(K_i){:}gr(F)],$$

hence y is a v-gauge if and only if each y_i is a v-gauge. The last assertions then follow from Cor. 1.7. □

1.2. **Gauges on division algebras.** Consider a finite-dimensional (not necessarily central) division algebra D over F. Suppose w is a valuation on D which extends the valuation v on F, and consider the canonical homomorphism

$$\theta_D\colon\ \Gamma_D/\Gamma_F \to Aut\big(Z(\overline{D})\big) \tag{1.13}$$

which for $d \in D^\times$ maps $w(d) + \Gamma_F$ to the automorphism $\overline{z} \mapsto \overline{dzd^{-1}}$ (see [JW, (1.6)]).

PROPOSITION 1.10. *With the notation above, the valuation w is an F-gauge on D if and only if*

$$[D{:}F] \ = \ [\overline{D}{:}\overline{F}]\,|\Gamma_D{:}\Gamma_F|. \tag{1.14}$$

When this condition holds, the gauge w is tame if and only if $Z(\overline{D})$ is separable over $\overline{F}$ and $char(\overline{F}) \nmid |ker(\theta_D)|$.

If $char(\overline{F}) \nmid [D{:}F]$, the following conditions are equivalent:

(i) *w is an F-gauge;*
(ii) *w is a tame F-gauge;*
(iii) *v extends uniquely to $Z(D)$.*

Proof. Since $gr(D)$ is a graded division algebra, it is graded semisimple. Therefore, w is an F-gauge if and only if $[gr(D){:}gr(F)] = [D{:}F]$. By an easy calculation, cf. [HW$_2$, (1.7)] or [Bl$_2$, p. 4278], we have

$$[gr(D){:}gr(F)] \ = \ [\overline{D}{:}\overline{F}]\,|\Gamma_D{:}\Gamma_F|. \tag{1.15}$$

The first statement follows.

To prove the second statement, assume w is a gauge, hence $[gr(D){:}gr(F)] = [D{:}F]$. Since

$$[gr(D){:}gr(F)] \ = \ [gr(D){:}gr(Z(D))]\,[gr(Z(D)){:}gr(F)] \quad \text{and} \quad [D{:}F] \ = \ [D{:}Z(D)]\,[Z(D){:}F],$$

and since, by Prop. 1.1,

$$[D{:}Z(D)] \ \geq \ [gr(D){:}gr(Z(D))] \quad \text{and} \quad [Z(D){:}F] \ \geq \ [gr(Z(D)){:}gr(F)],$$

it follows that

$$[D{:}Z(D)] \ = \ [gr(D){:}gr(Z(D))]. \tag{1.16}$$

From the definition of θ_D, it is clear that $\Gamma_{Z(D)}/\Gamma_F \subseteq ker(\theta_D)$, hence there is an induced map

$$\overline{\theta}_D\colon\ \Gamma_D/\Gamma_{Z(D)} \to Aut(Z(\overline{D})).$$

Clearly, $|ker(\theta_D)| = |ker(\overline{\theta}_D)|\,|\Gamma_{Z(D)}{:}\Gamma_F|$. When (1.16) holds, Boulagouaz proved in [Bl_2, Cor. 4.4] that $Z(gr(D)) = gr(Z(D))$ if and only if $Z(\overline{D})/\overline{Z(D)}$ is separable and $char(\overline{F}) \nmid |ker(\overline{\theta}_D)|$. Therefore, the following statements are equivalent:

(a) $Z(gr(D)) = gr(Z(D))$, $\overline{Z(D)}/\overline{F}$ is separable and $char(\overline{F}) \nmid |\Gamma_{Z(D)}{:}\Gamma_F|$ (i.e., the gauge w is tame);
(b) $Z(\overline{D})/\overline{F}$ is separable and $char(\overline{F}) \nmid |ker(\theta_D)|$.

The second statement is thus proved.

If v does not extend uniquely to $Z(D)$, then the fundamental inequality for extensions of valuations [B, VI.8.2, Th. 1] yields

$$[Z(D){:}F] \;>\; [\overline{Z(D)}{:}\overline{F}]\,|\Gamma_{Z(D)}{:}\Gamma_F|,$$

hence (1.14) does not hold, and w is not a gauge. This proves (i) ⇒ (iii) (without hypothesis on $char(\overline{F})$). For the rest of the proof, assume $char(\overline{F}) \nmid [D{:}F]$. Then $Z(\overline{D})/\overline{F}$ is separable and $char(\overline{F}) \nmid |ker(\theta_D)|$, hence (i) ⟺ (ii). Finally, assume (iii). By Cor. 1.8, we have

$$[Z(D){:}F] \;=\; [\overline{Z(D)}{:}\overline{F}]\,|\Gamma_{Z(D)}{:}\Gamma_F|.$$

On the other hand, a noncommutative version of Ostrowski's theorem [M_1, Th. 3] yields

$$[D{:}Z(D)] \;=\; [\overline{D}{:}\overline{Z(D)}]\,|\Gamma_D{:}\Gamma_{Z(D)}|.$$

Therefore, (1.14) holds, and (i) follows. □

In the case where D is central and v is Henselian, various other characterizations of tame F-gauges are given in the following proposition:

PROPOSITION 1.11. *With the same notation as above, suppose $F = Z(D)$ and v is Henselian. The following conditions are equivalent:*

(i) *the valuation w is a tame F-gauge;*
(ii) *D is split by the maximal tamely ramified extension of F;*
(iii) *either $char(\overline{F}) = 0$ or the $char(\overline{F})$-primary component of D is split by the maximal unramified extension of F;*
(iv) *D has a maximal subfield which is tamely ramified over F.*

Proof. By [HW_2, Prop. 4.3], conditions (ii)–(iv) above are equivalent to: $[D{:}F] = [gr(D){:}gr(F)]$ and $Z(gr(D)) = gr(F)$, hence also to (i). □

DEFINITION 1.12. As in [HW_2], a central division algebra D over a Henselian valued field F is called *tame* if the equivalent conditions of Prop. 1.11 hold. Note that by Cor. 3.4 below, w is the unique F-gauge on D (if any exists).

EXAMPLE 1.13. *A non-tame gauge.* Let $F = \mathbb{Q}(x, y)$, the field of rational fractions in two indeterminates over the rationals. The 2-adic valuation of $\mathbb{Q}$ extends to a valuation v on F with residue field $\mathbb{F}_2(x, y)$, see [EP, Cor. 2.2.2]. This valuation further extends to a valuation w on the quaternion algebra $D = (x, y)_F$, with residue division algebra $\overline{D} = \mathbb{F}_2(x, y)(\sqrt{x}, \sqrt{y})$. The valuation w is an F-gauge on D which is not tame.

EXAMPLE 1.14. *Gauges that are not valuations.* Let D be the quaternion division algebra $(-1, -1)_{\mathbb{Q}}$ over the field of rational numbers. This algebra is split by the field $\mathbb{Q}_3$ of 3-adic numbers, hence the 3-adic valuation v on $\mathbb{Q}$ does not extend to a valuation on D, by [C, Th. 1] or [M_1, proof of Th. 2]. Let $(1, i, j, k)$ be the quaternion base of D with $i^2 = j^2 = -1$ and $k = ij = -ji$, and define a $\mathbb{Q}$-norm y on D by

$$y(a_0 + a_1 i + a_2 j + a_3 k) \;=\; min\big(v(a_0), v(a_1), v(a_2), v(a_3)\big).$$

(This y is in fact the armature gauge on D with respect to v on $\mathbb{Q}$ and the abelian subgroup of $D^\times/\mathbb{Q}^\times$ generated by the images of i and j, as described in §4.2 below. But we will not use the §4 results here.) Lemma 1.2 shows that y is a surmultiplicative value function on D. We have $gr(\mathbb{Q}) = \mathbb{F}_3[t, t^{-1}]$, where the indeterminate t is the image of 3. With primes denoting images in $gr(D)$, we have $i'^2 = -1'$, $j'^2 = -1'$, and $i'j' = k' = -j'i'$. Thus, $1', i', j', k'$ span a copy of the 4-dimensional graded simple graded quaternion algebra $(-1,-1)_{gr(\mathbb{Q})}$. Since $[gr(D){:}gr(\mathbb{Q})] = [D{:}\mathbb{Q}] = 4$, we have

$$gr(D) \;=\; (-1,-1)_{gr(\mathbb{Q})} \;\cong\; (-1,-1)_{\mathbb{F}_3} \otimes_{\mathbb{F}_3} gr(\mathbb{Q}) \;\cong\; M_2(\mathbb{F}_3)[t, t^{-1}]\,.$$

Thus, y is a tame gauge on D with $D_0 \cong M_2(\mathbb{F}_3)$ and $\Gamma_D = \Gamma_{\mathbb{Q}} = \mathbb{Z}$. One can obtain other gauges on D by conjugation, by using Prop. 1.15 below.

The residue ring D_0 in the preceding example is a simple ring, though not a division ring. Here is an example where the residue is not simple: Let k be any field of chacteristic not 2, let $F = k(x,z)$, where x and z are algebraically independent over k. Let Q be the quaternion division algebra $(1+x, z)_F$. Let v be the valuation on F obtained by restriction from the standard Henselian valuation on $k((x))((z))$. So, $\Gamma_F = \mathbb{Z}\times\mathbb{Z}$ with right-to-left lexicographic ordering, with $v(x) = (1,0)$ and $v(z) = (0,1)$. Again let $(1, i, j, k)$ denote the quaternion base of Q with $i^2 = 1 + x$, $j^2 = y$, and $k = ij = -ji$. Define an F-norm y on Q by, for $a_0, \ldots, a_3 \in F$,

$$y(a_0 + a_1 i + a_2 j + a_3 k) \;=\; min\big(v(a_0), v(a_1), v(a_2) + (0, \tfrac{1}{2}), v(a_3) + (0, \tfrac{1}{2})\big)\ .$$

This y is the armature gauge of Q with respect to v and the abelian subgroup of $Q^\times/F^\times$ generated by the images of i and j. By using results in §4.2 or by easy direct calculations, one sees that y is a tame gauge on Q with $gr(Q)$ graded isomorphic to the graded quaternion algebra $(1, y')_{gr(F)}$, with $\Gamma_Q = \mathbb{Z}\times\frac{1}{2}\mathbb{Z}$, and $Q_0 \cong k \times k$, which is clearly not simple. let is

A gauge y on a division ring D is a valuation iff $gr_y(D)$ is a graded division ring. When this occurs, the gauge is invariant under conjugation. But for a gauge which is not a valuation, the associated graded ring is not a graded division ring, so it has nonzero homogenous elements which are not units. Then, conjugation yields different gauges, on D, as the next proposition shows:

Proposition 1.15. *Let (F, v) be a valued field and let y be a v-gauge on a central simple F-algebra A. For any unit $u \in A^\times$, the following conditions are equivalent:*

(i) *u' is a unit of $gr(A)$;*
(ii) *$y(uxu^{-1}) = y(x)$ for all $x \in A$.*

Proof. (i) ⇒ (ii) If u' is a unit of $gr(A)$, then u' is not a zero divisor. So, $u'(u^{-1})' = (uu^{-1})' = 1'$ in $gr(A)$ by (1.5). Hence, $(u^{-1})' = (u')^{-1}$, which is a unit of $gr(A)$. It follows by Lemma 1.3 that $y(u) + y(u^{-1}) = 0$ and for any $x \in A$,

$$y(uxu^{-1}) \;=\; y(ux) + y(u^{-1}) \;=\; y(u) + y(x) + y(u^{-1}) \;=\; y(x)\,.$$

(ii) ⇒ (i) The surmultiplicativity of y yields

$$y(u) + y(u^{-1}) \;\le\; y(1) \;=\; 0.$$

If equality holds here, then (i) follows by Lemma 1.3. Therefore, if (ii) holds and (i) does not, then for all $x \in A$

$$y(uxu^{-1}) \;=\; y(x) \;>\; y(u) + y(x) + y(u^{-1}).$$

By (1.5), it follows that $u'x'(u^{-1})' = 0$ for all $x \in A$, hence $u'gr(A)(u^{-1})' = \{0\}$ since $gr(A)$ is spanned by its homogeneous elements. This equation shows that $gr(A)u'gr(A) \neq gr(A)$ since $(u^{-1})' \neq 0$. On the other hand, $gr(A)u'gr(A) \neq \{0\}$ since $u' \neq 0$, hence the 2-sided homogeneous ideal $gr(A)u'gr(A)$ is not trivial, and $gr(A)$ is not graded simple. This is a contradiction to Cor. 3.8 below. (Observe that Prop. 1.15 is not used in the sequel; thus, the argument is not circular.) □

1.3. **Gauges on endomorphism algebras.** Let D be a finite-dimensional division algebra over a field F, let M be a finite-dimensional right D-vector space, and let $A = End_D(M)$. Suppose $w\colon D \to \Gamma \cup \{\infty\}$ is a valuation on D, and let $v = w|_F$. Let α be a D-norm on M.

LEMMA 1.16. *Let $(m_i)_{1\le i\le k}$ be a splitting base of M for α. For every $f \in A$ and nonzero $m \in M$,*

$$\alpha(f(m)) - \alpha(m) \;\ge\; \min_{1\le i\le k}\big(\alpha(f(m_i)) - \alpha(m_i)\big).$$

Proof. Let $m = \sum_{i=1}^{k} m_i d_i$ with $d_i \in D$. Then

$$\alpha(f(m)) \;=\; \alpha\Big(\sum_{i=1}^{k} f(m_i)d_i\Big) \;\ge\; \min_{1\le i\le k}\big(\alpha(f(m_i)) + w(d_i)\big). \tag{1.17}$$

Writing $\alpha(f(m_i)) + w(d_i) = \alpha(f(m_i)) - \alpha(m_i) + \alpha(m_i) + w(d_i)$, we obtain

$$\min_{1\le i\le k}\big(\alpha(f(m_i)) + w(d_i)\big) \;\ge\; \min_{1\le i\le k}\big(\alpha(f(m_i)) - \alpha(m_i)\big) \;+\; \min_{1\le i\le k}\big(\alpha(m_i) + w(d_i)\big)\,. \tag{1.18}$$

Since $(m_i)_{1\le i\le k}$ is a splitting base of M, the second term on the right side is $\alpha(m)$. The lemma then follows from (1.17) and (1.18). □

In view of the lemma, we may define a function $y_\alpha\colon A \to \Gamma \cup \{\infty\}$ as follows: for $f \in A$,

$$y_\alpha(f) \;=\; \min_{m\in M,\ m\ne 0}\big(\alpha(f(m)) - \alpha(m)\big)\,. \tag{1.19}$$

Indeed, the lemma shows that $y_\alpha(f) = \min_{1\le i\le k}\big(\alpha(f(m_i)) - \alpha(m_i)\big)$ for any splitting base $(m_i)_{1\le i\le k}$ of M. If $f(m_j) = \sum_{i=1}^{k} m_i d_{ij}$ with $d_{ij} \in D$, we have,

$$y_\alpha(f) \;=\; \min_{1\le i,j\le k}\big(\alpha(m_i) + w(d_{ij}) - \alpha(m_j)\big)\,. \tag{1.20}$$

Now, let $E = End_{gr(D)}(gr(M))$. Recall that E is graded as follows: for $\gamma \in \Gamma$,

$$E_\gamma \;=\; \{f \in E \mid f(M_\delta) \subseteq M_{\delta+\gamma} \text{ for all } \delta \in \Gamma\}.$$

PROPOSITION 1.17. *The map y_α of (1.19) is a surmultiplicative F-value function on A, and there is a canonical $gr(F)$-algebra isomorphism $gr(A) \cong_g E$. Moreover, y_α is an F-gauge (resp. a tame F-gauge) on A if and only if w is an F-gauge (resp. a tame F-gauge) on D.*

Proof. We omit the easy proof that y_α is a surmultiplicative F-value function on A. To define the canonical isomorphism $gr(A) \to E$, take any $f \in A$ with $f \ne 0$, and let $\gamma = y_\alpha(f) \in \Gamma$. The definition of y_α says that $\alpha(f(m)) \ge \alpha(m) + \gamma$ for all $m \in M$, and equality holds for some nonzero m. For any $\delta \in \Gamma$, this shows f maps $M^{\ge\delta}$ into $M^{\ge\delta+\gamma}$ and $M^{>\delta}$ into $M^{>\delta+\gamma}$; so, f induces a well-defined additive group homomorphism

$$\widetilde{f}_\delta\colon M_\delta \to M_{\delta+\gamma}, \qquad m' \mapsto f(m) + M^{>\delta+\gamma} \quad \text{for all } m \in M \text{ with } \alpha(m) = \delta.$$

Define $\widetilde{f} = \bigoplus_{\delta\in\Gamma} \widetilde{f}_\delta\colon gr(M) \to gr(M)$, an additive group homomorphism which shifts graded components by γ. For any $d \in D^\times$ and nonzero $m \in M$, we have

$$\widetilde{f}_{\alpha(m)+w(d)}(m' \cdot d') \;=\; \widetilde{f}_\alpha(m') \cdot d' \in M_{\alpha(m)+w(d)+\gamma}.$$

Thus, $\widetilde{f} \in E_\gamma$, and the definition of y_α assures that $\widetilde{f} \neq 0$. Since for any $g \in A^{>\gamma}$ we have $\widetilde{f+g} = \widetilde{f}$, there is a well-defined injective map $A_\gamma \to E_\gamma$ given by $f + A^{>\gamma} \mapsto \widetilde{f}$. The direct sum of these maps is a graded (i.e., grade-preserving) $gr(F)$-algebra homomorphism $\rho\colon gr(A) \to E$; this map is injective, since it is injective on each homogeneous component of $gr(A)$. To prove ρ is onto, let $(m_i)_{1\le i\le k}$ be a splitting base of M with respect to α; so, $(m_i')_{1\le i\le k}$ is a homogeneous $gr(D)$-base of $gr(M)$. If we fix any i, j with $1 \le i, j \le k$ and any $d \in D^\times$, and define $g \in A$ by $g(m_j) = m_i d$ and $g(m_\ell) = 0$ for $\ell \neq j$, then $y_\alpha(g) = w(d) + \alpha(m_i) - \alpha(m_j)$; so, $\widetilde{g}(m_j') = (m_i d) + A^{>w(d)+\alpha(m_i)-\alpha(m_j)+\alpha(m_j)} = m_i' \cdot d' \in M_{w(d)+\alpha(m_i)}$ and $\widetilde{g}(m_\ell') = 0$ for $\ell \neq j$. Since such maps generate E as an additive group, ρ is onto, hence an isomorphism, as desired.

It follows from the isomorphism $gr(A) \cong_g E$ that $gr(A)$ is a graded simple $gr(F)$-algebra. Thus, y_α is an F-gauge on A iff y_α is an F-norm on A, iff, by Prop. 1.1(ii), $[gr(A){:}gr(F)] = [A{:}F]$. We have

$$[A{:}F] \;=\; k^2[D{:}F] \qquad \text{and} \qquad [gr(A){:}gr(F)] \;=\; [E{:}gr(F)] \;=\; k^2[gr(D){:}gr(F)].$$

Thus, y_α is an F-norm on A iff $[gr(D){:}gr(F)] = [D{:}F]$ iff, by (1.15) and Prop. 1.10, w is an F-gauge on D.

Since $Z(A) = Z(D)$ (up to canonical isomorphism) with $y_\alpha|_{Z(A)} = w|_{Z(D)}$ and $Z(gr(A)) \cong_g Z(gr(D))$ (a $gr(F)$-algebra isomorphism), y_α is a tame F-gauge on A iff w is a tame F-gauge on D. □

We now compare the "gauge ring" $V_A = \{f \in A \mid y_\alpha(f) \ge 0\}$ with the valuation rings V_D and V_F.

LEMMA 1.18. *The following conditions are equivalent:*

(i) *V_A is integral over V_F;*
(ii) *V_D is integral over V_F;*
(iii) *the valuation v on F has a unique extension to $Z(D)$.*

In particular, these conditions all hold if v is Henselian.

Proof. (i) $\Longleftrightarrow$ (ii) Let $(m_i)_{1\le i\le k}$ be a splitting base of M with respect to α, and let $f \in A$. If $f(m_j) = \sum_{i=1}^{k} m_i d_{ij}$, eq. (1.20) shows that $f \in V_A$ iff $w(d_{ij}) \ge \alpha(m_i) - \alpha(m_j)$ for all i, j. Let $e_1, \ldots, e_k$ be the orthogonal idempotents of A for the splitting base $(m_i)_{1\le i\le k}$ of M, i.e., $e_i(m_j) = \delta_{ij} m_i$ (Kronecker delta). We have $y_\alpha(e_i) = 0$, so each $e_i \in V_A$. For f with matrix (d_{ij}) as above, $e_i f e_i$ has matrix with $i\,i$-entry d_{ii} and all other entries 0. Hence, $e_i V_A e_i \cong V_D$, a V_F-algebra isomorphism. If V_A is integral over V_F, then $V_D \cong e_1 V_A e_1$ must also be integral over V_F. Conversely, suppose V_D is integral over V_F. Then, each $e_i V_A e_i$ is integral over V_F, since it is isomorphic to V_D. For $j \neq i$, each element of $e_i V_A e_j$ has square 0, so is integral over V_F. Since $\sum_{i=1}^{k} e_i = 1$, we have $V_A = \sum_{1\le i,j\le k} e_i V_A e_j$, with the elements of each summand integral over V_F. Because A is a p.i.-algebra over F, the theorem [AS, Th. 2.3] of Amitsur and Small shows that V_A is integral over V_F.

(ii) $\Longleftrightarrow$ (iii) Let $Z = Z(D)$, the center of D, and let V_Z be the valuation ring of $w|_Z$. Then V_D is always integral over V_Z, by [W_1, Cor.], and V_Z is integral over V_F iff $w|_Z$ is the unique extension of the valuation v to Z, see [B, Ch. VI, §8.3, Remark] or [EP, Cor. 3.1.4]. □

REMARK 1.19. The value function y_α on $End_D(M)$ associated to a norm α on M is defined by Bruhat and Tits in [BT, 1.11, 1.13], where it is denoted by *End* α and called a *norme carrée*. In [BT, 2.13, Cor.] (see also the Appendix of [BT]), Bruhat and Tits establish a bijection between the set of *normes carrées* on $End_D(M)$ and the building of $GL(M)$, when the rank of the ordered group Γ is 1.

The following result is in [BT, 1.13]:

Proposition 1.20. *For D-norms α, β on M, the following conditions are equivalent:*

(i) *$\alpha - \beta$ is constant, i.e. there exists $\gamma \in \Gamma$ such that $\alpha(m) = \beta(m) + \gamma$ for all $m \in M$;*
(ii) *$y_\alpha = y_\beta$.*

Proof. (i) ⇒ (ii) is clear from the definition (1.19). To prove (ii) ⇒ (i), choose any m, $n \in M$ with $m \neq 0$ and let $A(m,n) = \{f \in A \mid f(m) = n\}$. By definition of y_α, we have

$$y_\alpha(f) \le \alpha(n) - \alpha(m) \qquad \text{for } f \in A(m,n).$$

On the other hand, we may choose a splitting base $(m_i)_{1\le i\le k}$ of M with $m_1 = m$ (see [RTW, Prop. 2.5]) and define $g \in A(m,n)$ by $g(m_1) = n$, $g(m_i) = 0$ for $i > 1$; then $y_\alpha(g) = \alpha(n) - \alpha(m)$. Therefore,

$$\alpha(n) - \alpha(m) = \max_{g\in A(m,n)} \big(y_\alpha(g)\big). \tag{1.21}$$

If (ii) holds, it follows from (1.21) that

$$\alpha(n) - \alpha(m) = \beta(n) - \beta(m) \qquad \text{for all } m,\ n \in M \text{ with } m \neq 0.$$

Condition (i) readily follows. □

1.4. **Gauges on tensor products.** If P and Q are two graded vector spaces over a graded field K, then the grading on $P \otimes_K Q$ is given by $(P \otimes_K Q)_\gamma = \sum_{\delta\in\Gamma} P_\delta \otimes_{K_0} Q_{\gamma-\delta}$.

Proposition 1.21. *Let (F, v) be a valued field, and let M and N be F-vector spaces such that M has an F-norm α and N has an F-value function β. There is a unique F-value function t on $M \otimes_F N$ such that there is a graded isomorphism of $gr(F)$-vector spaces*

$$gr_t(M \otimes_F N) \cong_g gr_\alpha(M) \otimes_{gr(F)} gr_\beta(N)$$

satisfying $(m\otimes n)' \mapsto m' \otimes n'$ for all $m \in M$ and $n \in N$. So, $t(m\otimes n) = \alpha(m) + \beta(n)$. If β is an F-norm, then t is also an F-norm.

The unique value function t on $M \otimes_F N$ satisfying the condition in the proposition will be denoted $\alpha \otimes \beta$, and the canonical isomorphism of graded vector spaces will be viewed as an identification.

The following lemma will be used in the proof of Prop. 1.21.

Lemma 1.22. *Let D be a division ring with a valuation w, and let P be a right D-vector space. Let u and t be D-value functions $P \to \Gamma \cup \{\infty\}$ such that $t(p) \le u(p)$ for all $p \in P$. Then, there is a canonical induced $gr(D)$-vector space homomorphism $\chi_{t,u}\colon gr_t(P) \to gr_u(P)$ which is injective iff $t = u$.*

Proof. Clear. □

Proof of Prop. 1.21. Let $(m_i)_{1\le i\le k}$ be a splitting base of M with respect to α. Define $t\colon M \otimes_F N \to \Gamma \cup \{\infty\}$ by

$$t\Big(\sum_{i=1}^k m_i \otimes n_i\Big) \;=\; \min_{1\le i\le k} \big(\alpha(m_i) + \beta(n_i)\big) \qquad \text{for any } n_i \in N. \tag{1.22}$$

Clearly, t is an F-value function on $M \otimes_F N$, and

$$(M \otimes_F N)^{\ge\gamma} \;=\; \bigoplus_{i=1}^k m_i \otimes N^{\ge\gamma-\alpha(m_i)} \qquad \text{for } \gamma \in \Gamma.$$

Also, since $(m_i')_{1\le i\le k}$ is a homogeneous $gr(F)$-base of $gr(M)$, we have

$$\big(gr(M) \otimes_{gr(F)} gr(N)\big)_\gamma \;=\; \bigoplus_{i=1}^k m_i' \otimes N_{\gamma-\alpha(m_i)}.$$

For $\gamma \in \Gamma$, let $\pi_\gamma\colon N^{\geq\gamma} \to N_\gamma$ be the canonical map. Define a surjective map $\psi_\gamma\colon (M \otimes_F N)^{\geq\gamma} \to \big(gr(M) \otimes_{gr(F)} gr(N)\big)_\gamma$ as follows:

$$\psi_\gamma\Big(\sum_{i=1}^{k} m_i \otimes n_i\Big) \ = \ \sum_{i=1}^{k} m_i' \otimes \pi_{\gamma-\alpha(m_i)}(n_i) \qquad \text{for } n_i \in N^{\geq\gamma-\alpha(m_i)}.$$

This ψ_γ is clearly an additive group homomorphism. Moreover, $\psi_\gamma\big(\sum m_i \otimes n_i\big) = 0$ iff each $n_i \in N^{>\gamma-\alpha(m_i)}$, iff $\sum m_i \otimes n_i \in (M \otimes_F N)^{>\gamma}$. Thus, ψ_γ induces a group isomorphism $\varphi_\gamma\colon (M \otimes_F N)_\gamma \to \big(gr(M) \otimes_{gr(F)} gr(N)\big)_\gamma$ and $\varphi = \bigoplus_{\gamma\in\Gamma} \varphi_\gamma$ is a graded $gr(F)$-vector space isomorphism $gr_t(M \otimes_F N) \to gr_\alpha(M) \otimes_{gr_v(F)} gr_\beta(N)$.

To see that $\varphi\big((m \otimes n)'\big) = m' \otimes n'$ for $m \in M$ and $n \in N$, let $m = \sum_{i=1}^{k} m_i r_i$ with $r_i \in F$. Then $m \otimes n = \sum_{i=1}^{k} m_i \otimes r_i n$ and $\alpha(m) = \min\{\alpha(m_i) + v(r_i)\}$, so

$$t(m \otimes n) \ = \ \min_{1\leq i\leq k}\big(\alpha(m_i) + \beta(n) + v(r_i)\big) \ = \ \alpha(m) + \beta(n)$$

and

$$\varphi((m \otimes n)') \ = \ \psi_{\alpha(m)+\beta(n)}\Big(\sum_{i=1}^{k} m_i \otimes r_i n\Big) \ = \ \sum_{i=1}^{k} m_i' \otimes \pi_{\alpha(m)-\alpha(m_i)+\beta(n)}(r_i n).$$

Note that

$$\pi_{\alpha(m)-\alpha(m_i)+\beta(n)}(r_i n) \ = \ \begin{cases} r_i' n' & \text{if } \alpha(m_i) + v(r_i) = \alpha(m), \\ 0 & \text{if } \alpha(m_i) + v(r_i) > \alpha(m). \end{cases}$$

Changing the indexing if necessary, we may assume $\alpha(m_i) + v(r_i) = \alpha(m)$ for $i = 1, \ldots, \ell$ and $\alpha(m_i) + v(r_i) > \alpha(m)$ for $i > \ell$. Then $m' = \sum_{i=1}^{\ell} m_i' r_i'$ and

$$\varphi((m \otimes n)') \ = \ \sum_{i=1}^{\ell} m_i' \otimes r_i' n' \ = \ m' \otimes n' \,,$$

as desired.

Now, suppose $u\colon M \otimes_F N \to \Gamma \cup \{\infty\}$ is an F-value function with the same property. For $m \in M$ and $n \in N$ we have $u(m \otimes n) = \alpha(m) + \beta(n)$, since the degree of $(m \otimes n)'$ in $gr_u(M \otimes N)$ is the same as the degree of $m' \otimes n'$ in $gr_\alpha(M) \otimes gr_\beta(N)$. Therefore, for $n_i \in N$ we have

$$u\big(\sum_{i=1}^{k} m_i \otimes n_i\big) \ \geq \ \min_{1\leq i\leq k}\big(u(m_i \otimes n_i)\big) \ = \ \min_{1\leq i\leq k}\big(\alpha(m_i) + \beta(n_i)\big) \ = \ t\big(\sum_{i=1}^{k} m_i \otimes n_i\big)\,. \tag{1.23}$$

We have graded $gr(F)$-vector space isomorphisms $\varphi_t\colon gr_t(M \otimes_F N) \to gr_\alpha(M) \otimes_{gr(F)} gr_\beta(N)$ and $\varphi_u\colon gr_u(M \otimes_F N) \to gr_\alpha(M) \otimes_{gr(F)} gr_\beta(N)$. Because of the inequality in (1.23), Lemma 1.22 yields a canonical $gr(F)$-vector space homomorphism $\chi_{t,u}\colon gr_t(M \otimes_F N) \to gr_u(M \otimes_F N)$. Our hypotheses on φ_t and φ_u imply that $\varphi_u \circ \chi_{t,u}$ and φ_t agree on $(m \otimes n)'$ for all $m \in M$ and $n \in N$. Since such $(m \otimes n)'$ form a generating set for $gr_t(M \otimes_F N)$, we have $\varphi_u \circ \chi_{t,u} = \varphi_t$. Then, $\chi_{t,u}$ is an isomorphism, since φ_t and φ_u are each isomorphisms. Lemma 1.22 then shows that $u = t$, proving the desired uniqueness of t.

If β is an F-norm on N, say with splitting base $(n_i)_{1\leq i\leq \ell}$, then it follows easily from (1.22) that $(m_i \otimes n_j)_{\substack{1\leq i\leq k\\ 1\leq j\leq \ell}}$ is a splitting base for t on $M \otimes_F N$, so t is an F-norm for $M \otimes_F N$. ☐

REMARK 1.23. A basis-free description of $\alpha \otimes \beta$ is stated in [BT, p. 269]: for $s \in M \otimes N$,

$$\alpha \otimes \beta(s) \ = \ sup\Big\{ \min_{1\leq j\leq \ell}\big(\alpha(p_j) + \beta(q_j)\big) \ \Big|\ s = \sum_{j=1}^{\ell} p_j \otimes q_j \Big\}\,.$$

To see this equality, for $s \in M \otimes_F N$, let $u(s) = sup\{ \min\limits_{1 \le j \le \ell} \big(\alpha(p_j) + \beta(q_j)\big) \mid s = \sum\limits_{j=1}^{\ell} p_j \otimes q_j \}$. Take any representation for s as $\sum\limits_{j=1}^{\ell} p_j \otimes q_j$. Write each $p_j = \sum\limits_{i=1}^{k} m_i r_{ij}$ with $r_{ij} \in F$. Then, $s = \sum\limits_{i=1}^{k} m_i \otimes \big(\sum\limits_{j=1}^{\ell} r_{ij} q_j \big)$, and

$$\begin{aligned}
\min_{1\le j\le \ell}\big(\alpha(p_j)+\beta(q_j)\big) \;&=\; \min_{1\le j\le \ell}\big(\alpha(\sum_{i=1}^{k} m_i r_{ij})+\beta(q_j)\big) \;=\; \min_{1\le j\le \ell}\big(\big(\min_{1\le i\le k}\alpha(m_i)+v(r_{ij})\big) \,+\, \beta(q_j)\big)\\
&=\; \min_{i,\,j}\big(\alpha(m_i)+v(r_{ij})+\beta(q_j)\big) \;=\; \min_{1\le i\le k}\big(\alpha(m_i) \,+\, \min_{1\le j\le \ell}\big(v(r_{ij})+\beta(q_j)\big)\\
&\le\; \min_{1\le i\le k}\big(\alpha(m_i) \,+\, \beta\big(\sum_{j=1}^{\ell} r_{ij}q_j\big)\big) \;=\; \alpha\otimes\beta\big(\sum_{i=1}^{k} m_i\otimes\big(\sum_{j=1}^{\ell} r_{ij}q_j\big)\big) \;=\alpha\otimes\beta(s)\,.
\end{aligned}$$

So, $\alpha \otimes \beta(s)$ is an upper bound for the quantities in the description of $u(s)$. But, by using the representation $s = \sum\limits_{i=1}^{k} m_i \otimes \big(\sum\limits_{j=1}^{\ell} r_{ij} q_j \big)$, we see that $\alpha \otimes \beta(s)$ is one of those quantities. Hence, the *sup* exists, and $u(s) = \alpha \otimes \beta(s)$.

Corollary 1.24. *Suppose (F, v) is a valued field and A is a semisimple F-algebra with an F-gauge y. Let (L, w) be any valued field extending (F, v). Then, $y \otimes w$ is a surmultiplicative L-value function on $A \otimes_F L$ and the canonical isomorphism*

$$gr(A \otimes_F L) \cong_g gr(A) \otimes_{gr(F)} gr(L)$$

is an isomorphism of $gr(L)$-algebras. Moreover, $y \otimes w$ is an L-gauge iff $gr(A) \otimes_{gr(F)} gr(L)$ is semisimple, iff $Z(gr(A)) \otimes_{gr(F)} gr(L)$ is a direct sum of graded fields. Furthermore, $y \otimes w$ is a tame L-gauge iff y is a tame F-gauge.

Proof. Let $z = y \otimes w$. Prop. 1.21 shows that z is a well-defined F-value function on $A \otimes_F L$ with $z(a \otimes \ell) = y(a) + w(\ell)$ for all $a \in A$, $\ell \in L$. This equation shows that z is actually an L-value function. Moreover, if $(a_i)_{1 \le i \le k}$ is an F-splitting base of A with respect to y, then formula (1.22) shows that $(a_i \otimes 1)_{1 \le i \le k}$ is an L-splitting base of $A \otimes_F L$ with respect to z. Since

$$z\big((a_i \otimes 1)(a_j \otimes 1)\big) \;=\; z(a_i a_j \otimes 1) \;=\; y(a_i a_j) \;\ge\; y(a_i) + y(a_j) \;=\; z(a_i \otimes 1) + z(a_j \otimes 1)\,,$$

Lemma 1.2 shows that z is surmultiplicative. The value-function compatible F-algebra homomorphism $A \to A \otimes_F L$ induces a graded $gr(F)$-algebra homomorphism $gr(A) \to gr(A \otimes_F L)$, and hence a graded $gr(L)$-algebra homomorphism $\varphi\colon gr(A) \otimes_{gr(F)} gr(L) \to gr(A \otimes_F L)$. This φ is bijective, since it coincides with the $gr(F)$-vector space isomorphism of Prop. 1.21. Since z satisfies all the other conditions for an L-gauge, z is an L-gauge iff $gr(A \otimes_F L)$ is a graded semisimple $gr(L)$-algebra, iff $gr(A) \otimes_{gr(F)} gr(L)$ is a graded semisimple $gr(L)$-algebra. We have $gr(A) \otimes_{gr(F)} gr(L) \cong_g gr(A) \otimes_{Z(gr(A))} \big(Z(gr(A)) \otimes_{gr(F)} gr(L)\big)$. Since $gr(A)$ is semisimple, $gr(A) \otimes_{gr(F)} gr(L)$ is semisimple iff $Z(gr(A)) \otimes_{gr(F)} gr(L)$ is a direct sum of graded fields. This is justified just as in the ungraded analogue by reducing to the simple case and using the fact [HW_2, Prop. 1.1] that if B is a graded central simple algebra over a graded field K and M is any graded field extension of K, then $B \otimes_K M$ is a graded central simple algebra over M.

Applying the first part of the corollary to $Z(A)$ with the gauge $y|_{Z(A)}$, we obtain

$$gr\big(Z(A \otimes_F L)\big) = gr(Z(A) \otimes_F L) \cong_g gr\big(Z(A)\big) \otimes_{gr(F)} gr(L).$$

On the other hand,

$$Z\big(gr(A \otimes_F L)\big) \cong_g Z\big(gr(A) \otimes_{gr(F)} gr(L)\big) = Z\big(gr(A)\big) \otimes_{gr(F)} gr(L), \tag{1.24}$$

hence

$$[gr\big(Z(A\otimes L)\big):gr(L)] = [gr\big(Z(A)\big):gr(F)] \quad \text{and} \quad [Z\big(gr(A\otimes L)\big):gr(L)] = [Z\big(gr(A)\big):gr(F)]. \qquad (1.25)$$

Since $gr\big(Z(A\otimes L)\big) \subseteq Z\big(gr(A\otimes L)\big)$, these finite-dimensional $gr(L)$-algebras coincide iff they have the same dimension. Similarly, $gr\big(Z(A)\big) = Z\big(gr(A)\big)$ iff $[gr\big(Z(A)\big):gr(F)] = [Z\big(gr(A)\big):gr(F)]$. Therefore, (1.25) shows that

$$gr\big(Z(A\otimes L)\big) = Z\big(gr(A\otimes L)\big) \quad \text{iff} \quad gr\big(Z(A)\big) = Z\big(gr(A)\big).$$

Moreover, since $gr(L)$ is a free $gr(F)$-module, it follows from (1.24) that $Z\big(gr(A\otimes L)\big)$ is separable over $gr(L)$ iff $Z\big(gr(A)\big)$ is separable over $gr(F)$, by [KO, Ch. III, Prop. 2.1 and 22]. Therefore, $y\otimes w$ is a tame L-gauge iff y is a tame F-gauge. □

Remark 1.25. In the context of Cor. 1.24, since $gr(A)$ is semisimple, $Z(gr(A))$ is a direct sum of graded fields finite-dimensional over $gr(F)$. If each of these graded fields is separable over $gr(F)$, or if $gr(L)$ is separable over $gr(F)$, then $Z(gr(A)) \otimes_{gr(F)} gr(L)$ is a direct sum of graded fields. Recall from [HW_1, Th. 3.11, Def. 3.4] that if $[gr(L):gr(F)] < \infty$, then $gr(L)$ is separable over $gr(F)$ iff $\overline{L}$ is separable over $\overline{F}$ and $char(\overline{F}) \nmid \big|\Gamma_L:\Gamma_F\big|$.

Corollary 1.26. *Suppose (F, v) is a valued field and A and B are semisimple F-algebras with respective F-gauges y and z. If $gr(A)\otimes_{gr(F)} gr(B)$ is graded semisimple, then $y\otimes z$ is an F-gauge on $A\otimes_F B$, and the canonical isomorphism*

$$gr(A\otimes_F B) \cong_g gr(A)\otimes_{gr(F)} gr(B)$$

is an isomorphism of graded $gr(F)$-algebras. Moreover $gr(A)\otimes_{gr(F)} gr(B)$ is graded semisimple iff $Z(gr(A))\otimes_{gr(F)} Z(gr(B))$ is a direct sum of graded fields. If A and B are central simple and y, z are tame gauges, then $y\otimes z$ is a tame gauge.

Proof. Since y is an F-norm on A, say with splitting base $(a_i)_{1\le i\le k}$, and z is an F-norm on B, say with splitting base $(b_j)_{1\le j\le \ell}$, we saw in the proof of Prop. 1.21 that $(a_i\otimes b_j)_{\substack{1\le i\le k\\ 1\le j\le \ell}}$ is a splitting base for the F-norm $y\otimes z$ on $A\otimes_F B$. For any a_i, a_p, b_j, b_q, we have, by Prop. 1.21,

$$\begin{aligned}(y\otimes z)\big((a_i\otimes b_j)\cdot(a_p\otimes b_q)\big) \;&=\; (y\otimes z)(a_ia_p\otimes b_jb_q) \;=\; y(a_ia_p) \,+\, z(b_jb_q)\\ &\ge\; y(a_i)+y(a_p)+z(b_j)+z(b_q) \;=\; (y\otimes z)(a_i\otimes b_j) \,+\, (y\otimes z)(a_p\otimes b_q)\,.\end{aligned}$$

Therefore, Lemma 1.2 shows that $y\otimes z$ is surmultiplicative. We have $gr(A\otimes_F B)\cong_g gr(A)\otimes_{gr(F)} gr(B)$ by Prop. 1.21. Thus, $y\otimes z$ is an F-gauge on $A\otimes_F B$ iff $gr(A)\otimes_{gr(F)} gr(B)$ is graded semisimple. Since $gr(A)\otimes_{gr(F)} gr(B) \cong_g gr(A)\otimes_{Z(gr(A))} [Z(gr(A))\otimes_{gr(F)} Z(gr(B))]\otimes_{Z(gr(B))} gr(B)$, the desired graded semisimplicity holds iff $Z(gr(A))\otimes_{gr(F)} Z(gr(B))$ is a direct sum of graded fields. This follows just as in the ungraded case, using [HW_1, Prop. 1.1].

The last statement is immediate since a graded tensor product of graded central simple $gr(F)$-algebras is graded central simple over $gr(F)$, by [HW_2, Prop. 1.1]. □

Note that the constructions in §§1.3 and 1.4 could be done in more generality. For instance, the valuation w in §1.3 could be replaced by a gauge, and in §1.4 the tensor products could be taken over division algebras instead of fields. Moreover, given a norm α on a right D-vector space M one can define a dual norm α^* on the left D-vector space $M^* = Hom_D(M, D)$ and check that the tensor product $\alpha\otimes\alpha^*$ corresponds to the gauge y_α under the canonical isomorphism $M\otimes_D M^* \cong End_D(M)$. (See [BT, §1].)

2. Graded central simple algebras

Let K be a graded field, and let B be a (finite-dimensional) graded central simple K-algebra. By the graded version of Wedderburn's theorem, see, e.g., [HW$_2$, Prop. 1.3], there is a (finite-dimensional) graded central K-division algebra E and a finite-dimensional graded right E-vector space N such that $B \cong_g End_E(N)$. We identify B with $End_E(N)$. The grading on B is given as follows: for any $\epsilon \in \Gamma$,

$$B_\epsilon \;=\; \{f \in B \mid f(N_\delta) \subseteq N_{\delta+\epsilon} \text{ for all } \delta \in \Gamma\}.$$

Since E is a graded division ring, E_0 is a division ring, and for each $\gamma \in \Gamma_E$, E_γ is a 1-dimensional left and right E_0-vector space. The grade set of N, $\Gamma_N = \{\,\gamma \in \Gamma \mid N_\gamma \neq \{0\}\,\}$, need not be a group, but it is clearly a union $\Gamma_N = \Gamma_1 \cup \ldots \cup \Gamma_k$ where each Γ_i is a (non-empty) coset of the group Γ_E. Then, N has a canonical direct sum decomposition into graded E-vector subspaces

$$N \;=\; \bigoplus_{i=1}^{k} N_{\Gamma_i} \qquad \text{where} \quad N_{\Gamma_i} \;=\; \bigoplus_{\gamma\in\Gamma_i} N_\gamma. \tag{2.1}$$

For each coset Γ_i, choose and fix a representative $\gamma_i \in \Gamma_i$.

Proposition 2.1. *The grade set of B is*

$$\Gamma_B \;=\; \bigcup_{i,j=1}^{k} (\gamma_i - \gamma_j) + \Gamma_E \tag{2.2}$$

and there is a canonical isomorphism of K_0-algebras

$$B_0 \;\cong\; \prod_{i=1}^{k} End_{E_0}(N_{\gamma_i}). \tag{2.3}$$

Moreover,

$$dim_E(N) \;=\; \sum_{i=1}^{k} dim_{E_0}(N_{\gamma_i}). \tag{2.4}$$

Proof. Eq. (2.2) readily follows from the description of the grading on B, and (2.4) from (2.1), since $N_{\Gamma_i} = N_{\gamma_i} \otimes_{E_0} E$. To prove (2.3), observe that every element in B_0 maps each N_{Γ_i} to itself, so

$$B_0 \;=\; \prod_{i=1}^{k} End_E(N_{\Gamma_i})_0.$$

Since $N_{\Gamma_i} = N_{\gamma_i} \otimes_{E_0} E$, restriction to N_{γ_i} defines a canonical isomorphism of K_0-algebras $End_E(N_{\Gamma_i})_0 \cong End_{E_0}(N_{\gamma_i})$. The proof is thus complete. □

This proposition shows that B_0 is in general semisimple but not simple; however all its simple components are equivalent to E_0 in the Brauer group $Br(Z(E_0))$. Also, the grade set Γ_B is in general not a group. We next show how Γ_E can be detected within Γ_B.

Let $\mathcal{H}_B$ be the multiplicative group of homogeneous units of B and let $\Delta_B \subseteq \Gamma_B$ be the image of $\mathcal{H}_B$ under the grade homomorphism mapping each nonempty $\mathcal{H}_B \cap B_\gamma$ to γ. The group action of $\mathcal{H}_B$ by conjugation on B preserves the grading, so sends B_0, hence also $Z(B_0)$, to itself. If b, $c \in \mathcal{H}_B \cap B_\gamma$, then $b^{-1}c \in B_0^\times$ centralizes $Z(B_0)$, hence b and c have the same action on $Z(B_0)$. Therefore, there is a well-defined homomorphism

$$\theta_B \colon \Delta_B/\Gamma_K \;\to\; Aut(Z(B_0)/K_0), \tag{2.5}$$

which maps $\gamma + \Gamma_K$ to $z \mapsto bzb^{-1}$ for $z \in Z(B_0)$ and $b \in \mathcal{H}_B \cap B_\gamma$. Of course, if $B = E$ then $\Delta_B = \Gamma_E$. The homomorphism θ_B then coincides with the homomorphism θ_E defined in [HW$_2$, (2.2)].

Proposition 2.2. *Let e be any primitive idempotent of $Z(B_0)$. Then*

$$\Gamma_E \;=\; \{\gamma \in \Delta_B \mid \theta_B(\gamma + \Gamma_K)(e) = e\}.$$

Moreover, the following diagram is commutative:

$$\begin{array}{ccc} \Gamma_E/\Gamma_K & \xrightarrow{\;i\;} & \Delta_B/\Gamma_K \\ {\scriptstyle\theta_E}\big\downarrow & & \big\downarrow{\scriptstyle\theta_B} \\ Aut(Z(E_0)) & \xrightarrow{\;d\;} & Aut(Z(B_0)) \end{array} \tag{2.6}$$

where i is induced by the inclusion $\Gamma_E \subseteq \Delta_B$ and d is the diagonal map, letting an automorphism of $Z(E_0)$ act on each component of $Z(B_0) \cong Z(E_0) \times \ldots \times Z(E_0)$.

Proof. From the description of B_0 in Prop. 2.1, it follows that the primitive idempotents of $Z(B_0)$ are the maps $e_1, \ldots, e_k$ such that $e_i|_{N_{\Gamma_i}} = id$ and $e_i|_{N_{\Gamma_j}} = 0$ for $i \neq j$. Suppose $\gamma \in \Delta_B$ is such that $\theta_B(\gamma + \Gamma_K)$ fixes some e_i, and let $h \in \mathcal{H}_B \cap B_\gamma$; then $he_i = e_ih$, hence $h(N_{\Gamma_i}) = N_{\Gamma_i}$ and therefore $\gamma \in \Gamma_E$.

Conversely, suppose $\gamma \in \Gamma_E$. Take any nonzero $c \in E_\gamma$ and any homogeneous E-vector space base of N built from bases of the N_{Γ_i}; let $f \in B$ be defined by mapping each base vector n to nc. Then $fe_i = e_if$ for all i and $f \in \mathcal{H}_B \cap B_\gamma$, so $\theta_B(\gamma + \Gamma_K)$ fixes each $e_1, \ldots, e_k$. Moreover, $\theta_B(\gamma + \Gamma_K)$ induces on each component $e_iZ(B_0) \cong Z(E_0)$ of $Z(B_0)$ the automorphism of conjugation by c, which is $\theta_E(\gamma + \Gamma_K)$. Therefore, diagram (2.6) commutes. □

Corollary 2.3. *If B_0 is simple, then $\Gamma_B = \Delta_B = \Gamma_E$ and $\theta_B = \theta_E$. Moreover, B and B_0 have the same matrix size, and $[B{:}K] = [B_0{:}K_0]|\Gamma_B{:}\Gamma_K|$.*

Proof. If B_0 is simple, Prop. 2.1 yields $\Gamma_B = \Gamma_E$. It also yields $B_0 \cong End_{E_0}(N_{\gamma_1})$ where $dim_{E_0}(N_{\gamma_1}) = dim_E(N)$, hence B and B_0 have the same matrix size. The equalities for Γ_B and θ_B follow from Prop. 2.2. Since $B \cong_g End_E(N)$, we have

$$[B{:}K] \;=\; (dim_E(N))^2\,[E{:}K] \;=\; (dim_{E_0}(N_{\gamma_1}))^2\,[E_0{:}K_0]\,|\Gamma_E{:}\Gamma_K| \;=\; [B_0{:}K_0]\,|\Gamma_B{:}\Gamma_K|\,.$$

□

Example 2.4. With the notation of Prop. 2.1, if the dimensions $dim_{E_0}(N_{\gamma_1}), \ldots, dim_{E_0}(N_{\gamma_k})$ are all different then every invertible homogeneous element in B has grade in Γ_E; therefore, $\Delta_B = \Gamma_E$ and $\theta_B = \theta_E$.

We can now see how the gauges considered here are related to Morandi value functions. The main earlier approach to value functions for central simple algebras is that of P. Morandi in [M$_2$] and [MW]. Let A be a central simple algebra over a field F with a valuation v. Let $y\colon A \to \Gamma \cup \{\infty\}$ be a surmultiplicative v-value function, and let

$$st(y) \;=\; \{\, a \in A^\times \mid y(a^{-1}) = -y(a)\,\}\,,$$

a subgroup of $A^\times$, cf. Lemma 1.3. Let $V_y = A^{\geq 0}$, the "valuation ring" of y, and let $A_0 = A^{\geq 0}/A^{>0}$, the degree 0 part of $gr_y(A)$. Then, y is a *Morandi value function* if

(i) A_0 is a simple ring;
(ii) $y(st(y)) = \Gamma_A$.

When this occurs, it is known that V_y is a Dubrovin valuation ring integral over its center, which is V_v, and y is completely determined by V_y. Conversely, to every Dubrovin valuation ring B of A with B integral over its center, there is a canonically associated Morandi value function y_B with $V_{y_B} = B$. (For the theory of Dubrovin valuation rings, see [MMU], [W$_2$], or [G], and the references given there. In particular, it is known that for every central simple algebra A over a field F and every valuation v on F there is a Dubrovin valuation ring B of A with $B \cap F = V_v$, and such a B is unique up to conjugacy, so unique up to isomorphism. See [W$_2$, Th. F] for characterizations of when B is integral over V_v.) For a Morandi value function y on A, the *defect* $\delta(y)$ (an integer) is defined by

$$\delta(y) \;=\; [A{:}F] \,/\, [A_0{:}F_0]\, |\Gamma_A{:}\Gamma_F|\,.$$

If (F^h, v^h) is a Henselization of (F, v), and D^h is the division algebra Brauer-equivalent to $A \otimes_F F^h$, it is known (see. [W$_2$, Th. C]) that $\delta(y)$ coincides with the defect of the valuation on D^h extending v^h on F^h. Hence, $\delta(y) = 1$ if $char(\overline{F}) = 0$ and $\delta(y)$ is a power of $char(\overline{F})$ otherwise. The requirement of integrality of V_y over V_v has been a significant limitation in applying the machinery of Dubrovin valuation rings in connection with value functions on central simple algebras.

Proposition 2.5. *Let A be a central simple algebra over a field F with a valuation v, and let y be a surmultiplicative v-value function on A. Then, y is a Morandi value function on A with $\delta(y) = 1$ if and only if y is a gauge on A with A_0 simple.*

Proof. $\Rightarrow$ Suppose y is a Morandi value function on A with $\delta(y) = 1$. For any $\gamma \in \Gamma_A = y(st(y))$, there is $u_\gamma \in st(y)$ with $y(u_\gamma) = \gamma$. Then, u'_γ is a homogeneous unit of $gr(A)$, by Lemma 1.3. Take any nonzero homogeneous ideal I of $gr(A)$. Then, there is a $\gamma \in \Gamma_A$ with $I \cap A_\gamma \neq \{0\}$. So, for the ideal $I \cap A_0$ of A_0 we have $I \cap A_0 \supseteq u'^{-1}_\gamma(I \cap A_\gamma) \neq \{0\}$. Because A_0 is a simple ring, we must have $I \cap A_0 = A_0$, so $1 \in I$, so $I = gr(A)$. Thus, $gr(A)$ is a simple ring, which is finite-dimensional over $gr(F)$ by Prop. 1.1(ii). Furthermore, as $\delta(y) = 1$, Cor. 2.3 yields

$$[A{:}F] \;=\; [A_0{:}F_0]\, |\Gamma_A{:}\Gamma_F| \;=\; [gr(A){:}gr(F)]\,.$$

Hence, y is a gauge on A. By hypothesis, A_0 is simple.

$\Leftarrow$ Suppose y is a gauge on A with A_0 simple. Since $gr(A)$ is graded semisimple, it is a graded direct product of graded simple rings. But, if $gr(A)$ has nontrivial graded direct product decomposition $gr(A) = C \times D$, then $A_0 = C_0 \times D_0$ with C_0 and D_0 nontrivial. This cannot occur as A_0 is simple. Hence, $gr(A)$ is graded simple. (This also follows from the simplicity of A. See Cor. 3.8 below.) Since A_0 is simple, Cor. 2.3 applies with $B = gr(A)$. The equality $\Gamma_{gr(A)} = \Delta_{gr(A)}$ from Cor. 2.3 shows that for each $\gamma \in \Gamma_A = \Gamma_{gr(A)}$, there is a homogeneous unit u in A_γ. Pick any $a \in A$ with $y(a) = \gamma$ and $a' = u$. Then, $a \in st(y)$ by Lemma 1.3. Hence, $y(st(y)) = \Gamma_A$, proving that y is a Morandi value function. Furthermore, as y is a norm on A, by Cor. 2.3 and Prop. 1.1(ii) we have

$$\delta(y) \;=\; [A{:}F] \,/\, [A_0{:}F_0]\, |\Gamma_A{:}\Gamma_F| \;=\; [A{:}F] \,/\, [gr(A){:}gr(F)] \;=\; 1\,.$$

□

3. Gauges over Henselian fields

We write $ms(B)$ for the matrix size of a simple (or graded simple) algebra B.

Theorem 3.1. *Let F be a field with a Henselian valuation v, and let A be a semisimple F-algebra with an F-gauge y. Let $A_1, \ldots, A_n$ be the simple components of A,*

$$A \;=\; A_1 \times \ldots \times A_n.$$

For $i = 1, \ldots, n$, the restriction $y|_{A_i}$ is a gauge on A_i, the graded algebra $gr_{y|_{A_i}}(A_i)$ is graded simple with $ms(A_i) = ms(gr(A_i))$, and $gr(A_1), \ldots, gr(A_n)$ are the graded simple components of $gr(A)$,

$$gr(A) \;=\; gr(A_1) \times \ldots \times gr(A_n).$$

For $a = (a_1, \ldots, a_n) \in A_1 \times \ldots \times A_n$,

$$y(a) = \min_{1 \le i \le n} \big(y|_{A_i}(a_i)\big).$$

Moreover, the gauge y *is tame if and only if each* $y|_{A_i}$ *is tame.*

A major tool in the proof is the following lemma, which allows us to lift idempotents from $gr(A)$:

LEMMA 3.2. *With the same hypotheses as in Th. 3.1, let* $\widetilde{e}_1, \ldots, \widetilde{e}_k$ *be a family of nonzero homogeneous orthogonal idempotents in* $\mathrm{gr}(A)$ *such that* $\widetilde{e}_1 + \ldots + \widetilde{e}_k = 1$*. There is a family of orthogonal idempotents* $e_1, \ldots, e_k$ *in* A *such that* $e_i' = \widetilde{e}_i$ *for* $i = 1, \ldots, k$, $e_1 + \ldots + e_k = 1$, *and*

$$\mathrm{gr}(e_iAe_j) \;=\; e_i'\mathrm{gr}(A)e_j', \qquad \mathrm{gr}(e_iA) \;=\; e_i'\mathrm{gr}(A), \qquad \mathrm{gr}(Ae_j) \;=\; \mathrm{gr}(A)e_j' \quad \textit{for } i, j = 1, \ldots, k.$$

Proof. Let $C = End_F(A)$, and let $z = z_y \colon C \to \Gamma \cup \{\infty\}$ be the gauge on C arising from the norm y on A, as in §1.3. Take any $a \in V_A$. For each $b \in A$, we have

$$y(ab) - y(b) \;\ge\; y(a) + y(b) - y(b) \;\ge\; 0.$$

This shows that for the left multiplication map $\lambda_a \in C$ given by $b \mapsto ab$, we have $z(\lambda_a) \ge 0$. That is, $\lambda_a \in V_C$. Lemma 1.18 shows that V_C is integral over V_F hence λ_a is integral over V_F, and so is a. This shows that V_A is integral over V_F. Because V_F is Henselian, the integrality implies that we can lift families of orthogonal idempotents to V_A from any homomorphic image of it, by [MMU, Th. A.18, p. 180]. The homogeneous idempotents $\widetilde{e}_i$ are necessarily of grade 0; they lie in $A_0 = V_A/A^{>0}$. We may thus find orthogonal idempotents $e_1, \ldots, e_{k-1}$ of V_A with each $e_i' = \widetilde{e}_i$ in A_0; let $e_k = 1-(e_1+\ldots+e_{k-1})$. Then, $e_k^2 = e_k$, $e_ke_i = e_ie_k = 0$ for $i < k$; also, $e_k' = 1' - (e_1+\ldots+e_{k-1})' = 1' - (\widetilde{e}_1+\ldots+\widetilde{e}_{k-1}) = \widetilde{e}_k$. So, $e_1, \ldots, e_k$ are pairwise orthogonal idempotents with each $y(e_i) = 0$ and $e_i' = \widetilde{e}_i$, and $e_1 + \ldots + e_k = 1$.

To simplify notation, for any F-subspace N of A, we let $N' = gr(N)$, the associated graded $gr(F)$-vector space determined by $y|_N$; we view N' as an $F' = gr(F)$-subspace of $A' = gr(A)$. Since y is a norm on A, we have $dim_{F'}(N') = dim_F(N)$, by Prop. 1.1(iii) and (ii). For $1 \le i, j \le k$, we claim that

$$(e_iAe_j)' \;=\; e_i'A'e_j' \quad \text{and} \quad dim_F(e_iAe_j) \;=\; dim_{F'}(e_i'A'e_j')\,. \tag{3.1}$$

For, we have $e_i'A'e_j' \subseteq (e_iAe_j)'$ (cf. (1.5)). Since $A = \bigoplus_{1 \le i,j \le k} e_iAe_j$ and $A' = \bigoplus_{1 \le i,j \le k} e_i'A'e_j'$, we have

$$dim_F(A) \;=\; dim_{F'}(A') \;=\; \sum_{i,j} dim_{F'}(e_i'A'e_j') \;\le\; \sum_{i,j} dim_{F'}((e_iAe_j)') \;=\; \sum_{i,j} dim_F(e_iAe_j) \;=\; dim_F(A)\,.$$

Thus, equality must hold throughout, so that $dim_{F'}(e_i'A'e_j') = dim_{F'}((e_iAe_j)') = dim_F(e_iAe_j)$, for all i, j; this shows that $e_i'A'e_j' = (e_iAe_j)'$, proving (3.1). A similar calculation shows that for all i,

$$(e_iA)' \;=\; e_i'A' \quad \text{and} \quad (Ae_i)' \;=\; A'e_i'\,.$$

□

We now consider the special case of Th. 3.1 where $gr(A)$ is graded simple, and show that all gauges in this case are obtained from the construction in §1.3.

THEOREM 3.3. *Let* F *be a field with a Henselian valuation* v*. Let* A *be a semisimple* F*-algebra with an* F*-gauge* y *such that* $\mathrm{gr}(A)$ *is graded simple. Then,* A *is simple. Let* D *be the division algebra Brauer-equivalent to* A*, and let* w *be the valuation on* D *extending* v *on* F*. Then* w *is defectless. Furthermore, there is a finite-dimensional right* D*-vector space* M *with a* w*-norm* α *such that*

$$A \;\cong\; End_D(M), \qquad \mathrm{gr}(A) \;\cong_g\; End_{\mathrm{gr}(D)}(\mathrm{gr}(M)),$$

and y *on* A *corresponds to the* F*-gauge* y_α *on* $End_D(M)$ *induced by* α *as in §1.3. In particular,* $ms(A) = ms(gr(A))$*, and the gauge* y *is tame if and only* w *is a tame* F*-gauge.*

Proof. To simplify notation, let $A' = gr(A)$. Let $\widetilde{e}_1, \ldots, \widetilde{e}_n$ be a family of orthogonal primitive idempotents of the semisimple ring A_0 with $\widetilde{e}_1 + \ldots + \widetilde{e}_n = 1$. The $\widetilde{e}_i$ are primitive homogeneous idempotents of $gr(A)$, since all homogeneous idempotents in $gr(A)$ have grade 0. Lift these idempotents to a family of orthogonal primitive idempotents $e_1, \ldots, e_n$ of V_A as in Lemma 3.2. Let $e = e_1$ and let $D = eAe$. Then $gr(D) = e'A'e'$. Because $e' = \widetilde{e}_1$ is a primitive homogeneous idempotent of A', we have $e'A'e'$ is a graded division ring. Therefore,

$$y(cd) \;=\; y(c) + y(d) \quad \text{for all } c,\, d \in D^\times. \tag{3.2}$$

It follows that D has no zero divisors; since D is finite-dimensional over F, it must be a division ring. Furthermore, (3.2) shows that $y|_D$ is a valuation on D which restricts to v on F (when we identify F with $eF \subseteq D$). Thus, $y|_D = w$, since there is only one extension of v to D. Note that since y is a gauge, its restriction to any subspace of A is a norm, by Prop. 1.1(iii). Hence $[gr(D){:}gr(F)] = [D{:}F]$, which means the valuation w is also an F-gauge on D, so D is defectless over F.

Let $M = Ae$, which is a right D-vector space. By Lemma 3.2, we have $gr(M) = A'e'$. This is a graded right $gr(D)$-vector space, hence by (1.5),

$$y(md) \;=\; y(m) + y(d) \qquad \text{for all } m \in M \text{ and } d \in D.$$

Thus, $y|_M$ is a D-value function on M with respect to w. Furthermore,

$$dim_{gr(D)}(gr(M)) \;=\; dim_{gr(F)}(gr(M))/dim_{gr(F)}(gr(D)) \;=\; dim_F(M)/dim_F(D) \;=\; dim_D(M).$$

Hence, $y|_M$ is actually a D-norm on M. We let $\alpha = y|_M$.

We have an F-homomorphism $\beta\colon A \to End_D(M)$ given by $\beta(a)(m) = am$. Let $y_\alpha\colon End_D(M) \to \Gamma \cup \{\infty\}$ be the F-gauge on $End_D(M)$ induced by the D-norm α on M, as in §1.3. That is, for $f \in End_D(M)$, we have $y_\alpha(f) = \min\limits_{m\in M,\, m\neq 0}\big(\alpha(f(m)) - \alpha(m)\big)$. We claim that for every nonzero $a \in A$,

$$y_\alpha(\beta(a)) \;=\; y(a)\,. \tag{3.3}$$

For, we have $y_\alpha(\beta(a)) = \min\limits_{m\in M,\, m\neq 0}\big(y(am) - y(m)\big) \geq y(a)$, since $y(am) \geq y(a) + y(m)$ for all m. Suppose $y_\alpha(\beta(a)) > y(a)$. Then, $y(am) - y(m) > y(a)$, for all m. Hence, $a' \cdot m' = 0$ in $gr(M)$ for all $m \in M$, showing that $a'gr(M) = 0$, since $gr(M)$ is generated by its homogeneous elements. But, $gr(M) = A'e'$, by Lemma 3.2. Thus, $0 = a'gr(M) = a'A'e'$. Because A' is graded simple, we have $A'e'A' = A'$, and hence $0 = a'A'e'A' = a'A'$, which shows that $a' = 0$. This contradicts $a \neq 0$, proving (3.3), and showing that β is injective. We have

$$[A{:}F] \;=\; [A'{:}gr(F)] \;=\; (dim_{gr(D)}gr(M))^2[gr(D){:}gr(F)] \;=\; (dim_D M)^2[D{:}F] \;=\; [End_D(M){:}F].$$

Therefore, the injective F-algebra map β is an isomorphism. Furthermore, from (3.3) and Prop. 1.17 we have

$$gr_y(A) \;\cong_g\; gr_{y_\alpha}(End_D(M)) \;\cong_g\; End_{gr_w(D)}(gr_\alpha(M)),$$

as desired. Note that $ms(A) = dim_D(M) = dim_{gr(D)}(gr(M)) = ms(gr(A))$. Moreover, Prop. 1.17 shows that the gauge y_α is tame if and only if w is a tame F-gauge. This completes the proof of Th. 3.3. □

Proof of Th. 3.1. Since y is a gauge, the graded algebra $gr(A)$ is graded semisimple. Consider its decomposition into graded simple components:

$$gr(A) \;=\; \widetilde{A}_1 \times \ldots \times \widetilde{A}_\ell.$$

Let $\widetilde{e}_1, \ldots, \widetilde{e}_\ell$ be the homogeneous central idempotents of $gr(A)$ which are the unity elements of $\widetilde{A}_1, \ldots, \widetilde{A}_\ell$. By Lemma 3.2, we may lift these idempotents to idempotents $e_1, \ldots, e_\ell$ of A such that $e_i' = \widetilde{e}_i$ for each i, and $e_1 + \ldots + e_\ell = 1$. Moreover, for $i \neq j$ we have

$$gr(e_iAe_j) \;=\; \widetilde{e}_i gr(A)\widetilde{e}_j \;=\; \{0\}\,,$$

hence $e_iAe_j = \{0\}$. For $a \in A$, we have

$$a = ae_1 + \ldots + ae_\ell = e_1a + \ldots + e_\ell a.$$

Since $e_iae_j = 0$ for $i \neq j$, it follows by multiplying on the left or on the right by e_i that

$$e_ia = e_iae_i = ae_i \qquad \text{for } i = 1, \ldots, \ell.$$

Therefore, $e_1, \ldots, e_\ell$ are central idempotents in A. Let $A_i = e_iA$ for $i = 1, \ldots, \ell$. Then,

$$A = A_1 \times \ldots \times A_\ell.$$

Moreover, by Lemma 3.2 we have

$$gr_{y|_{A_i}}(A_i) = \widetilde{e}_i gr(A) = \widetilde{A}_i \qquad \text{for } i = 1, \ldots, \ell.$$

Since $\widetilde{A}_i$ is graded simple, $y|_{A_i}$ is an F-gauge on A_i, and it follows from Th. 3.3 that A_i is simple, with $ms(A_i) = ms(\widetilde{A}_i)$. Therefore, $A_1, \ldots, A_\ell$ are the simple components of A.

For $a = (a_1, \ldots, a_n) \in A_1 \times \ldots \times A_n$ we have $a_i = ae_i$ hence by the surmultiplicativity of y

$$y|_{A_i}(a_i) = y(ae_i) \geq y(a) + y(e_i) = y(a).$$

On the other hand, since $a = ae_1 + \ldots + ae_n$ we have $y(a) \geq \min_{1\leq i\leq n} \big(y(ae_i)\big)$, hence

$$y(a) = \min_{1\leq i\leq n} \big(y(a_i)\big).$$

To complete the proof, it remains to show that y is a tame gauge if and only if each $y|_{A_i}$ is tame. We have

$$gr\big(Z(A)\big) = gr(Z(A)e_1 \oplus \ldots \oplus Z(A)e_\ell) = gr\big(Z(A_1)\big) \times \ldots \times gr\big(Z(A_\ell)\big)$$

and

$$Z\big(gr(A)\big) = Z\big(gr(A_1) \times \ldots \times gr(A_\ell)\big) = Z\big(gr(A_1)\big) \times \ldots \times Z\big(gr(A_\ell)\big).$$

Therefore, $gr\big(Z(A)\big) = Z\big(gr(A)\big)$ if and only if $gr\big(Z(A_i)\big) = Z\big(gr(A_i)\big)$ for all i, and $Z\big(gr(A)\big)$ is separable over $gr(F)$ if and only if each $Z\big(gr(A_i)\big)$ is separable over $gr(F)$. □

Corollary 3.4. *Let F be a field with a Henselian valuation v, let $D_1, \ldots, D_n$ be (finite-dimensional) division F-algebras and let*

$$A = D_1 \times \ldots \times D_n.$$

For $i = 1, \ldots, n$, let w_i be the unique valuation on D_i extending v. The F-algebra A carries a v-gauge if and only if each D_i is defectless (i.e. (1.14) holds). When this condition holds, there is a unique v-gauge on A, defined by

$$y(d_1, \ldots, y_n) = \min_{1\leq i\leq n} \big(w_i(d_i)\big) \qquad \text{for } d_i \in D_i. \tag{3.4}$$

Proof. If D_i is defectless for $i = 1, \ldots, n$, then w_i is a v-gauge on D_i by Prop. 1.10 and it is straightforward to check that formula (3.4) defines a gauge on A. For the converse, suppose y is a gauge on A, and let $y_i = y|_{D_i}$ for $i = 1, \ldots, n$. By Th. 3.1, each y_i is a v-gauge on D_i and $y(d_1, \ldots, d_n) = \min_{1\leq i\leq n} \big(y_i(d_i)\big)$. Moreover, $ms\big(gr_{y_i}(D_i)\big) = ms(D_i)$ hence $gr_{y_i}(D_i)$ is a graded division ring. Therefore, for all $a, b \in D_i^\times$ we have $(ab)' = a'b'$, hence $y_i(ab) = y_i(a) + y_i(b)$. This shows that y_i is a valuation, hence $y_i = w_i$. However, w_i is a gauge only when D_i is defectless, by Prop. 1.10. The proof is thus complete. □

If the valuation v is not Henselian, we may still apply Th. 3.1 after scalar extension to a Henselization (F^h, v^h) of (F, v). If y is a v-gauge on the semisimple F-algebra A, then Cor. 1.24 shows that $y \otimes v^h$ is a v^h-gauge on $A \otimes_F F^h$ with $gr(A \otimes_F F^h) = gr(A)$, since $gr(F^h) = gr(F)$. Let $B_1, \ldots, B_n$ be the simple components of $A \otimes_F F^h$,

$$A \otimes_F F^h = B_1 \times \ldots \times B_n.$$

For $i = 1, \ldots, n$, let D_i be the division algebra Brauer-equivalent to B_i.

PROPOSITION 3.5. *With the notation above, let ℓ be the number of simple components of A and ℓ' be the number of graded simple components of $gr(A)$. Then $\ell \leq \ell' = n$. Moreover, the following conditions are equivalent:*

(i) *y is a tame v-gauge;*
(ii) *$y \otimes v^h$ is a tame v^h-gauge;*
(iii) *for $i = 1, \ldots, n$, the unique valuation w_i on D_i extending v^h is a tame v^h-gauge.*

Proof. The equality $\ell' = n$ follows from Th. 3.1 and the equality $\ell \leq n$ is clear. The equivalence (i) $\Longleftrightarrow$ (ii) readily follows from Cor. 1.24. For $i = 1, \ldots, n$, let y_i be the restriction of $y \otimes v^h$ to B_i. Th. 3.1 shows that $y \otimes v^h$ is a tame v^h-gauge if and only if each y_i is a tame v^h-gauge. By Th. 3.3, this condition is equivalent to (iii). □

COROLLARY 3.6. *Suppose A is a simple algebra finite-dimensional over a field F with valuation v. If y is a v-gauge on A, then the number of simple components of $gr(A)$ equals the number of extensions of v from F to $Z(A)$.*

Proof. Let (F^h, v^h) be a Henselization of (F, v), and let $K = Z(A)$. Since F^h is separable over F, we can write $K \otimes_F F^h = L_1 \times \ldots \times L_n$, where each L_i is a field. Then, $A \otimes_F F^h \cong A \otimes_K (K \otimes_F F^h) \cong B_1 \times \ldots \times B_n$, where each $B_i = A \otimes_K L_i$, a central simple L_i-algebra. With respect to the F^h-gauge $y \otimes v^h$ on $A \otimes_F F^h$, Th. 3.1 shows that

$$gr(A \otimes_F F^h) \;\cong_g\; gr(B_1) \times \ldots \times gr(B_n) \tag{3.5}$$

with each $gr(B_i)$ simple. Now, $y|_K$ is a surmultiplicative norm on K by Prop. 1.1(iii), and $gr(K)$ is semisimple since it is a central subalgebra of the semisimple $gr(F)$-algebra $gr(A)$; so, $y|_K$ is a gauge. Hence, by Prop. 1.6 and Cor. 1.7, the number of simple summands of $gr(K)$ equals the number of extensions of v to K. We have $y|_K \otimes v^h$ is a gauge on $K \otimes_F F^h$, and also,

$$gr(K \otimes_F F^h) \;\cong_g\; gr(L_1) \times \ldots \times gr(L_n) \tag{3.6}$$

by Prop. 1.9. Furthermore, each $gr(L_i)$ is simple by Cor. 1.7 and Prop. 1.6 because the Henselian valuation v^h has a unique extension from F^h to L_i. Since $gr(A) \cong_g gr(A \otimes_F F^h)$ and $gr(K) \cong_g gr(K \otimes_F F^h)$ by Cor. 1.24, equations (3.5) and (3.6) show that

$$\begin{aligned} n &= \text{number of simple components of } gr(A) \\ &= \text{number of simple components of } gr(K) \\ &= \text{number of extensions of } v \text{ to } K\,. \end{aligned}$$

□

COROLLARY 3.7. *If $char(\overline{F}) = 0$, then every F-gauge on a semisimple F-algebra is tame.*

Proof. Condition (iii) of Prop. 3.5 holds if $char(\overline{F}) = 0$, by Prop. 1.11. □

In the rest of this section, we consider the case where the semisimple F-algebra A is central simple. Recall from [HW$_2$] that the graded Brauer group $GBr(E)$ of a graded field E can be defined on the same model as the classical Brauer group of fields. The elements of $GBr(E)$ are graded isomorphism classes of graded division algebras with center E.

COROLLARY 3.8. *Let (F, v) be a valued field and let (F^h, v^h) be a Henselization of (F, v). Let A be a central simple F-algebra, let D be the division algebra Brauer-equivalent to $A \otimes_F F^h$, and let w be the valuation on D extending v^h. Let y be any v-gauge on A. Then, the $gr_v(F)$-algebra $gr_y(A)$ is graded* *simple and Brauer-equivalent to* $gr_w(D)$. *Moreover, if* y *is tame then* D *is tame (see Def. 1.12) and* $gr_y(A)$ *is central simple over* $gr_v(F)$.

Proof. Let $A^h = A \otimes_F F^h$ and let $y^h = y \otimes v^h$. Th. 3.1 shows that $gr_{y^h}(A^h)$ is graded simple, and Th. 3.3 yields $gr_{y^h}(A^h) \cong_g End_{gr(D)}\big(gr(M)\big)$ for some finite-dimensional right D-vector space M, hence $gr_{y^h}(A^h)$ is Brauer-equivalent to $gr(D)$. These properties carry over to $gr_y(A)$ because $gr_y(A) = gr_{y^h}(A^h)$ by Cor. 1.24, since $gr(F^h) = gr(F)$. When y is tame, it follows by definition that $Z(gr(A)) = gr(F)$, and from Prop. 3.5 that D is tame. □

For any valued field (F, v), define the *tame part* of the Brauer group $Br(F)$ (with respect to v) to be

$$TBr(F) \;=\; \{\,[A] \mid A \text{ is a central simple } F\text{-algebra with a tame } F\text{-gauge}\,\}\,.$$

Cor. 3.8 shows that if y is a tame v-gauge on a central simple F-algebra A, then the Brauer class $[gr_y(A)]$ in $GBr(gr(F))$ does not depend on the choice of the tame v-gauge but only on the Brauer class of A, since it coincides with the Brauer class $[gr(D)]$ where D is the division algebra Brauer-equivalent to $A \otimes_F F^h$. Therefore, there is a well-defined map

$$\Psi\colon TBr(F) \to GBr(gr(F)), \quad [A] \mapsto [gr_y(A)] \text{ for any } v\text{-gauge } y \text{ on } A.$$

By Cor. 1.26, $TBr(F)$ is a subgroup of $Br(F)$ and the map Ψ is a group homomorphism.

Theorem 3.9. *The kernel of* Ψ *consists of the elements in* $TBr(F)$ *which are split by any Henselization* F^h *of* F *with respect to* v. *Moreover, for any valued field* (L, w) *extending* (F, v), *there is a commutative diagram*

$$\begin{array}{ccc} TBr(F) & \xrightarrow{\Psi_F} & GBr(gr(F)) \\ \big\downarrow & & \big\downarrow \\ TBr(L) & \xrightarrow{\Psi_L} & GBr(gr(L)). \end{array} \tag{3.7}$$

If v *is Henselian, then* Ψ *is an index-preserving group isomorphism.*

Proof. Cor. 1.24 shows that the scalar extension map $-\otimes_F L$ sends $TBr(F)$ to $TBr(L)$, and that diagram (3.7) is commutative. When v is Henselian, Th. 3.3 shows that $ms(gr_y(A)) = ms(A)$ for any central simple F-algebra, hence Ψ is index-preserving and injective. In this Henselian case, Ψ is also surjective, by [HW$_2$, Th. 5.3]. Another more direct proof of the surjectivity is possible, by showing that we can construct algebra classes of unramified algebras and inertially split cyclic algebras and tame totally ramified symbol algebras over F which map onto generators of $GBr(F)$. No longer assuming that v is Henselian, take $L = F^h$ in commutative diagram (3.7); since Ψ_{F^h} is bijective and $gr(F^h) = gr(F)$, the kernel of Ψ_F is the kernel of the scalar extension map $TBr(F) \to TBr(F^h)$. □

This theorem generalizes [HW$_2$, Th. 5.3], which showed that Ψ_F is a group isomorphism when v is Henselian. The proof given here is vastly simpler than the one in [HW$_2$].

4. Applications

The utility of Theorems 3.1 and 3.3 depends on being able to construct gauges on algebras over valued fields. We give several examples where this can be done, obtaining as a result considerably simplified and more natural proofs of some earlier theorems. In each case, we use the following result, which can be viewed as a detection theorem: it allows one to use a gauge to determine whether the division algebra D Brauer-equivalent to a given central simple algebra A has a valuation extending a given valuation on the center, without first determining D.

Theorem 4.1. *Let (F, v) be a valued field, and let A be a central simple F-algebra, and let D be the division algebra Brauer-equivalent to A. Suppose A has an F-gauge y. Then, $gr(A)$ is simple, and $ms(gr(A)) \geq ms(A)$. Moreover, the following conditions are equivalent:*

(i) *v extends to a valuation on D;*
(ii) *$ms(gr(A)) = ms(A)$;*
(iii) *$[D{:}F] = [E{:}gr(F)]$, where E is the graded division algebra Brauer-equivalent to $gr(A)$.*

When these conditions hold, $gr(D) \cong_g E$ and $gr(A) \cong_g End_{gr(D)}(N)$ for some graded right $gr(D)$-vector space N. Hence, $\overline{D}$ is Brauer equivalent to any simple component of A_0, and Γ_D and θ_D are determinable from $gr(A)$, as described in Prop. 2.2.

Proof. Let (F^h, v^h) be a Henselization of (F, v). Let $A^h = A \otimes_F F^h$ and $D^h = D \otimes_F F^h$. So, $A^h \cong M_n(D^h)$, where $A \cong M_n(D)$. As noted in the proof of Cor. 3.8, $y \otimes v^h$ is an F^h-gauge on A^h with $gr(A^h) \cong_g gr(A)$. Since A^h is simple, Th. 3.1 shows that $gr(A^h)$ (so also $gr(A)$) is graded simple, and that $ms(gr(A)) = ms(A^h)$. Of course, $ms(A^h) \geq ms(A)$. Thus, $ms(gr(A)) = ms(A)$ iff $ms(A^h) = ms(A)$, iff D^h is a division ring, iff v extends to a valuation on D, by Morandi's theorem [M_1, Th. 2].

Let $\widetilde{D}$ be the division algebra Brauer-equivalent to A^h. By Th. 3.3, we have $A^h \cong End_{\widetilde{D}}(N)$ for some right $\widetilde{D}$-vector space N with a $\widetilde{D}$-norm, and $gr(A^h) \cong_g End_{gr(\widetilde{D})}(gr(M))$. Hence, $E \cong_g gr(\widetilde{D})$, by the uniqueness part of the graded Wedderburn theorem. Since Th. 3.3 also says that the valuation on $\widetilde{D}$ extending v^h is an F^h-gauge, we have $[E{:}gr(F)] = [gr(\widetilde{D}){:}gr(F^h)] = [\widetilde{D}{:}F^h]$. But, $[D{:}F] = [D^h{:}F^h]$. Hence, $[E{:}gr(F)] = [D{:}F]$ iff $[\widetilde{D}{:}F^h] = [D^h{:}F^h]$ iff D^h is a division ring, iff (as above) v extends to D. When this occurs, since the valuation on D^h extends the one on D, we have an inclusion $\iota\colon gr(D) \hookrightarrow gr(D^h)$; but ι is actually an isomorphism, since $\overline{D} \cong \overline{D^h}$ and $\Gamma_D = \Gamma_{D^h}$ by [M_1, Th. 2], hence $D_0 \cong D^h_0$ and $\Gamma_{gr(D)} = \Gamma_{gr(D^h)}$. Thus, $E \cong_g gr(\widetilde{D}) \cong_g gr(D^h) \cong_g gr(D)$, so that $gr(A) \cong_g End_{gr(D)}(N)$. Hence the Brauer class of $\overline{D} = D_0$ coincides with that of any simple component of A_0, and θ_D is determinable from $gr(A)$ as described in Prop. 2.2. □

4.1. **Crossed products.** We now show how to construct tame gauges on crossed product algebras when the Galois extension is indecomposed and defectless with respect to the valuation.

Let K/F be a finite Galois extension of fields, and let G be the Galois group $\mathcal{G}(K/F)$. Let A be a crossed product algebra $(K/F, G, f)$, where f is a 2-cocycle in $Z^2(G, K^\times)$, and assume for convenience that f is normalized. Explicitly, write

$$A \;=\; \bigoplus_{\sigma\in G} Kx_\sigma,$$

where

$$x_{id} \;=\; 1, \quad x_\sigma c x_\sigma^{-1} \;=\; \sigma(c) \text{ for all } c \in K \quad \text{and} \quad x_\sigma x_\tau \;=\; f(\sigma,\tau)x_{\sigma\tau} \text{ for all } \sigma,\, \tau \in G.$$

Assume v is a valuation on F which has a unique extension to a valuation w of K, and that w is defectless over v. Thus, w is a v-norm on K as an F-vector space, and every automorphism $\sigma \in \mathcal{G}(K/F)$ induces an automorphism σ' of $gr(K)$. Assume further that $\overline{K}$ is separable over $\overline{F}$ and that $char(\overline{F}) \nmid |\Gamma_K{:}\Gamma_F|$. Then $gr(K)$ is Galois over $gr(F)$ and the canonical map $\mathcal{G}(K/F) \to \mathcal{G}(gr(K)/gr(F))$ given by $\sigma \mapsto \sigma'$ is an isomorphism. Let $G' = \mathcal{G}(gr(K)/gr(F))$ and define $f'\colon G' \to gr(K)^\times$ by $f'(\sigma', \tau') = f(\sigma,\tau)'$; then $f' \in Z^2(G', gr(K)^\times)$ and we may consider the crossed product algebra $(gr(K)/gr(F), G', f')$. This is a graded simple $gr(F)$-algebra, see [HW_2, Lemma 3.1].

Toward defining a v-gauge on A, we set for $\sigma \in G$

$$y(x_\sigma) \;=\; \tfrac{1}{|G|} \sum_{\rho\in G} w(f(\sigma,\rho))\,. \tag{4.1}$$

We extend y to a value function on A by letting

$$y\big(\sum_{\sigma\in G} c_\sigma x_\sigma\big) \;=\; \min_{\sigma\in G}\big(w(c_\sigma)+y(x_\sigma)\big)\,. \tag{4.2}$$

Proposition 4.2. *The value function y is a v-gauge on A, and there is a canonical isomorphism*

$$gr(A) \;\cong_g\; (gr(K)/gr(F), G', f').$$

Proof. Let $(a_i)_{1\le i\le n}$ be any splitting base of w on K as an F-norm. Then, it follows from the definition in (4.2) that $(a_i x_\sigma \mid 1\le i\le n,\ \sigma\in G)$ is a splitting base for y, so that y is an F-norm on A. Note that w is invariant under the action of G on K, since w is the unique extension of v to K. Thus, when we apply w to the basic cocycle equation $f(\sigma,\tau)f(\sigma\tau,\rho)=\sigma(f(\tau,\rho))f(\sigma,\tau\rho)$ and sum over all $\rho\in G$, we obtain in Γ,

$$|G|w(f(\sigma,\tau)) \;+\; |G|y(x_{\sigma\tau}) \;=\; |G|y(x_\tau) \;+\; |G|y(x_\sigma)\,.$$

Since Γ is torsion-free this yields

$$y(x_\sigma) \;+\; y(x_\tau) \;=\; w(f(\sigma,\tau)) \;+\; y(x_{\sigma\tau})\,, \tag{4.3}$$

for all σ, $\tau\in G$. Therefore, for any i, j, σ, τ we have

$$\begin{aligned} y\big((a_i x_\sigma)(a_j x_\tau)\big) \;&=\; y\big(a_i\sigma(a_j)f(\sigma,\tau)x_{\sigma\tau}\big) \;=\; w(a_i)+w(a_j)+w(f(\sigma,\tau))+y(x_{\sigma\tau})\\ &=\; w(a_i)+w(a_j)+y(x_\sigma)+y(x_\tau) \;=\; y(a_i x_\sigma)+y(a_j x_\tau)\,. \end{aligned}$$

By Lemma 1.2 it follows that y is surmultiplicative.

Now consider $gr(A)$. Since $(a_i')_{1\le i\le n}$ is a homogeneous $gr(F)$-base for $gr(K)$ and $\big((a_i x_\sigma)' \mid 1\le i\le n,\ \sigma\in G\big)$ is a homogeneous $gr(F)$-base of $gr(A)$, and since $(a_i x_\sigma)' = a_i' x_\sigma'$ by (4.2), we have $gr(A)=\bigoplus_{\sigma\in G} gr(K)x_\sigma'$. For any $c\in K^\times$ and $\sigma\in G$, we have $y(x_\sigma c)=y(\sigma(c)x_\sigma)=w(c)+y(x_\sigma)$; hence, in $gr(A)$,

$$x_\sigma' c' \;=\; (x_\sigma c)' \;=\; (\sigma(c)x_\sigma)' \;=\; \sigma(c)'x_\sigma' \;=\; \sigma'(c')x_\sigma'\,.$$

Moreover, formula (4.3) shows that $x_\sigma' x_\tau' = f(\sigma,\tau)'x_{\sigma\tau}'$ for all σ, $\tau\in G$. Therefore, $gr(A)\cong_g (gr(K)/gr(F), G', f')$. It follows that $gr(A)$ is a graded central simple $gr(F)$-algebra, hence y is a tame F-gauge on A. □

We can describe A_0 to some extent for this A. The value function y yields a map

$$\lambda\colon G\to\Gamma/\Gamma_K \quad\text{given by}\quad \sigma\mapsto y(x_\sigma)+\Gamma_K, \tag{4.4}$$

and (4.3) shows that λ is a group homomorphism. Let $H=ker(\lambda)$. Write $A=\bigoplus_{\sigma\in G} Kz_\sigma$, where

$$z_\sigma \;=\; d_\sigma x_\sigma,\ \text{with the } d_\sigma\in K^\times \text{ chosen so that } w(d_\sigma)=-y(x_\sigma) \text{ if } \sigma\in H\,. \tag{4.5}$$

Thus, $d_\sigma d_\tau = g(\sigma,\tau)d_{\sigma\tau}$, where $g(\sigma,\tau)=d_\sigma\sigma(d_\tau)d_{\sigma\tau}^{-1}f(\sigma,\tau)$, so g is a 2-cocycle cohomologous to f. We have $y(z_\rho)=0$ for $\rho\in H$, and the analogue to (4.3) for g shows that $w(g(\rho,\tau))=0$ for all ρ, $\tau\in H$. For $\sigma\notin H$, because $y(z_\sigma)\notin\Gamma_K$ the summand $gr(K)z_\sigma'$ makes no contribution to A_0. Thus,

$$A_0 \;=\; \bigoplus_{\rho\in H} K_0 z_\rho'\,.$$

This A_0 is semisimple, as $gr(A)$ is simple, see (2.3), but it need not be a crossed product algebra, nor even simple, depending on how H acts on K_0. Recall that $K_0=\overline{K}$. Because the extension of v to K is indecomposed, each $\sigma\in G$ induces an automorphism $\widetilde{\sigma}$ of $\overline{K}$ which coincides with the restriction of σ' to K_0.

Proposition 4.3. *In the situation just described where* $A = (K/F, G, f) = \bigoplus_{\sigma\in G} Kx_\sigma$ *and* $H = \ker(\lambda)$ *for* λ *as in* (4.4), *suppose that each* $\rho \in H$ *induces a different automorphism of* $\overline{K}$. *Then,* $Z(A_0) = K_0^H$, *the subfield of* K_0 *fixed by* H, *and* $A_0 = \bigoplus_{\rho\in H} K_0 z'_\rho$ *is a crossed product algebra over* $Z(A_0)$. *Let* E *be the graded division algebra Brauer-equivalent to* $gr(A)$. *Then,* $\Gamma_E = \Gamma_{gr(A)} = \Gamma_K + \langle y(x_\sigma) \mid \sigma \in G\rangle$ *and* E_0 *is the division algebra Brauer-equivalent to* A_0. *The map* $\theta_{gr(A)} \colon \Gamma_E/\Gamma_F \to \mathcal{G}(Z(A_0)/F_0)$ *induces* $\widetilde{\theta} \colon \Gamma_{gr(A)}/\Gamma_K \to \mathcal{G}(K_0^H/F_0)$, *and we have a commutative diagram,*

$$\begin{array}{ccc} G & \xrightarrow{\ \lambda\ } & \Gamma_{gr(A)}/\Gamma_K \\ \big\downarrow & & \big\downarrow \widetilde{\theta} \\ \mathcal{G}(\overline{K}/\overline{F}) & \longrightarrow & \mathcal{G}(K_0^H/F_0)\,. \end{array} \tag{4.6}$$

Proof. Since each $\rho \in H$ induces a different automorphism of $K_0 = \overline{K}$, it is clear that $A_0 = \bigoplus_{\rho\in H} K_0 z'_\rho$ is a crossed product algebra over its center $K_0^H = \overline{K}^H$. Clearly, $\Gamma_{gr(A)} = \Gamma_K + \langle y(x_\sigma) \mid \sigma \in G\rangle$. Since A_0 is simple, we have $\Gamma_E = \Gamma_{gr(A)}$ and $\theta_E = \theta_{gr(A)}$, by Cor. 2.3. Furthermore, since A_0 has only one simple component, for any $\gamma \in \Gamma_E$, $\theta_E(\gamma + \Gamma_F)$ is the automorphism of $Z(A_0)$ induced by conjugation by any $a \in A_\gamma \cap A^\times$. If $\gamma \in \Gamma_K$, then a can be chosen in $gr(K)$, and the conjugation is trivial, as $Z(A_0) \subseteq K_0$. Hence, $\theta_{gr(A)}$ induces $\widetilde{\theta}$. For each $\gamma \in \Gamma_{gr(A)}$, there is $\sigma \in G$ with $\lambda(z_\sigma) \equiv \gamma \ (mod\, \Gamma_K)$. Then $\widetilde{\theta}(\gamma+\Gamma_K)$ is given by conjugation by z'_σ on K_0^H, which coincides with $\widetilde{\sigma}|_{\overline{K}^H}$. Therefore, diagram (4.6) is commutative, where the left map is $\sigma \mapsto \widetilde{\sigma}$ and the bottom map is restriction of the automorphism from $\overline{K} = K_0$ to $\overline{K}^H = K_0^H$. □

Now consider the unramified case of the preceding discussion. Suppose the field K is Galois over F, and suppose F has a valuation v which has a unique and unramified (and defectless) extension to a valuation w of K. So, $\overline{K}$ is Galois over $\overline{F}$ and $\mathcal{G}(\overline{K}/\overline{F}) \cong \mathcal{G}(K/F)$. Let $G = \mathcal{G}(K/F)$. The short exact sequence of trivial G-modules

$$0 \to \Gamma_F \to \Gamma \to \Gamma/\Gamma_F \to 0$$

yields a connecting homomorphism $\partial \colon H^1(G, \Gamma/\Gamma_F) \to H^2(G, \Gamma_F)$. In fact, ∂ is an isomorphism, since for the divisible torsion-free group Γ we have $H^1(G,\Gamma) = H^2(G,\Gamma) = \{0\}$. Thus, we have a succession of maps

$$H^2(G, K^\times) \to H^2(G, \Gamma_F) \xrightarrow{\partial^{-1}} H^1(G, \Gamma/\Gamma_F) = Hom(G, \Gamma/\Gamma_F)\,, \tag{4.7}$$

where the left map is induced by the G-module homomorphism $w \colon K^\times \to \Gamma_K = \Gamma_F$.

Corollary 4.4. *Let* K *be an unramified and defectless Galois extension field of* F *with respect to the valuation* v *on* F. *Let* $G = \mathcal{G}(K/F)$, *and take any* $f \in Z^2(G, K^\times)$. *Let* $A = (K/F, G, f) = \bigoplus_{\sigma\in G} Kx_\sigma$, *and let* D *be the division algebra Brauer-equivalent to* A. *Let* y *be the tame* F-*gauge on* A *defined in* (4.2) *above, and let* λ *be as defined in* (4.4), *let* $H = \ker(\lambda)$, *and the* z_σ *as defined in* (4.5). *So, for the graded simple ring* $gr_y(A)$, *we have* $A_0 = \bigoplus_{\rho\in H} K_0 z'_\rho$, *as above, and* A_0 *is a crossed product algebra over* $\overline{K}^H$. *Furthermore,*

(i) *The map* λ *of* (4.4) *is the image of* $[f] \in H^2(G, K^\times)$ *under the maps of* (4.7).
(ii) v *extends to a valuation on* D *iff* $ms(A_0) = ms(A)$.
(iii) *Suppose* v *extends to* D. *Then,* $\Gamma_D/\Gamma_F = im(\lambda)$; $Z(\overline{D}) = \overline{K}^H$; $\overline{D}$ *is the division algebra Brauer-equivalent to* A_0; *and* $\theta_D \colon \Gamma_D/\Gamma_F \to \mathcal{G}(Z(\overline{D})/\overline{F})$ *is the isomorphism which is the inverse to the composite map* $\mathcal{G}(Z(\overline{D})/\overline{F}) \xrightarrow{\cong} G/H \xrightarrow{\cong} \Gamma_D/\Gamma_F$ *induced by* λ.

Proof. Note that Prop. 4.3 applies here, since the map $\mathcal{G}(K/F) \to \mathcal{G}(\overline{K}/\overline{F})$ is injective. Hence, A_0 is a crossed product with center $\overline{K}^H$.

(i) follows from (4.3) above and the definition of the connecting homomorphism in group cohomology.

For (ii), we have $ms(gr(A)) = ms(A_0)$, as A_0 is simple, by Cor. 2.3. But, by Th. 4.1, v extends to D iff $ms(A) = ms(gr(A))$. This proves (ii).

(iii) Suppose v extends to D. Let $E = gr(D)$. By Th. 4.1, we have $gr(A) \cong_g End_E(N)$ for some graded right E-vector space N. Then, using Cor. 2.3 and Prop. 4.3, we have $\Gamma_D/\Gamma_F = \Gamma_E/\Gamma_F = \Gamma_{gr(A)}/\Gamma_F = im(\lambda)$. Also, $\overline{D} = E_0$, which is the division algebra Brauer-equivalent to A_0, so $Z(\overline{D}) \cong Z(A_0) = K_0^H = \overline{K}^H$. Finally, again using Cor. 2.3, we have $\theta_D = \theta_E = \theta_{gr(A)}$. In commutative diagram (4.6), $\widetilde{\theta} = \theta_{gr(A)}$, as $\Gamma_K = \Gamma_F$. The diagram shows that the surjective map $\widetilde{\theta}$, is also injective, as $ker(\lambda) = H = ker(\widetilde{\theta}\circ\lambda)$. So, $\widetilde{\theta}$ is an isomorphism, and the diagram shows that $\widetilde{\theta}$ is the inverse of the isomorphism induced by λ. □

In the context of Cor. 4.4, if the valuation v on F is Henselian, then v always extends to D, so Cor. 4.4 applies. It yields a new proof of [JW, Th. 5.6(b)] for inertially split division algebras over Henselian fields, which is significantly simpler and more direct than previous proofs. It does not use generalized crossed products, as in the proof in [JW], nor Dubrovin valuation rings, as in the proof in [MW, Cor. 3.7].

4.2. **Tensor products of symbol algebras.** Let A be a finite-dimensional algebra over an arbitrary field F. Recall from [TW, §2] (see also [T]) that an *armature* of A is an abelian subgroup $\mathcal{A} \subset A^\times/F^\times$ such that $|\mathcal{A}| = [A{:}F]$ and $\{a \in A \mid aF^\times \in \mathcal{A}\}$ spans A as an F-vector space.

For example, suppose F contains a primitive n-th root of unity ω for some $n \geq 2$, and A is a *symbol algebra* of degree n, i.e. an F-algebra generated by two elements i, j subject to the relations $i^n \in F^\times$, $j^n \in F^\times$, and $ij = \omega ji$. The images in $A^\times/F^\times$ of the standard generators i, j generate an armature of A. More generally, in a tensor product of symbol algebras the images of the products of standard generators generate an armature. Tensor products of symbol algebras can actually be characterized by the existence of armatures of a certain type, see [TW, Prop. 2.7].

Although tensor products of symbol algebras are the main case of interest to us, we first consider commutative algebras. Let $\mathcal{Z}$ be an armature of a commutative F-algebra Z. Suppose F contains a primitive s-th root of unity for some multiple s of the exponent $exp(\mathcal{Z})$, and let $\mu_s \subseteq F$ denote the group of s-th roots of unity. Since $char(F) \nmid s$, we have $|\mathcal{Z}| \neq 0$ in F. Let $\pi\colon Z^\times \to Z^\times/F^\times$ be the canonical map and let

$$X = \pi^{-1}(\mathcal{Z}) \subseteq Z^\times.$$

Since $exp(\mathcal{Z})$ divides s, we have $x^s \in F^\times$ for all $x \in X$. Therefore, there is a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & F^\times & \longrightarrow & X & \xrightarrow{\pi} & \mathcal{Z} & \longrightarrow & 1 \\ & & \big\downarrow{\scriptstyle s} & & \big\downarrow{\scriptstyle s} & & \big\downarrow{\scriptstyle \rho} & & \\ 1 & \longrightarrow & F^{\times s} & \longrightarrow & F^\times & \longrightarrow & F^\times/F^{\times s} & \longrightarrow & 1. \end{array} \tag{4.8}$$

Let $\mathcal{K} = ker(\rho)$ and $\mathcal{L} = im(\rho)$, and let $L = F(\{\sqrt[s]{c} \mid cF^{\times s} \in \mathcal{L}\})$ be the s-Kummer extension field of F associated with $\mathcal{L}$. Let also $G = Hom(\mathcal{Z}, \mu_s)$, the character group of $\mathcal{Z}$, and let $H \subseteq G$ be the subgroup orthogonal to $\mathcal{K}$,

$$H = \{\chi \in G \mid \chi(k) = 1 \text{ for all } k \in \mathcal{K}\}.$$

Let also $r = |\mathcal{K}| = |G{:}H|$, and let $K \subseteq Z$ be the subalgebra spanned by $\pi^{-1}(\mathcal{K})$. The following proposition extends [TW, Lemma 2.9]:

Proposition 4.5. *The F-algebra Z is G-Galois, and contains r primitive idempotents $e_1, \ldots, e_r$, which form an F-base of K and are conjugate under the G-action. The isotropy subgroup of any e_i is H, and $e_iZ \cong L$ is a Galois extension of F with Galois group isomorphic to H. In particular, $Z \cong L^r$, a direct product of r copies of L.*

Proof. For each $z \in \mathcal{Z}$, choose $x_z \in X$ such that $\pi(x_z) = z$. By definition of an armature, $(x_z)_{z\in\mathcal{Z}}$ is an F-base of Z. Let ${}_sX = \{x \in X \mid x^s = 1\}$. Applying the snake lemma to (4.8), we get the exact sequence

$$1 \to \mu_s \to {}_sX \xrightarrow{\pi} \mathcal{K} \to 1.$$

Since μ_s is a cyclic group of order s in the finite abelian group ${}_sX$ of exponent s, this exact sequence splits. Therefore, we may assume that the elements x_z satisfy

$$x_k x_{k'} = x_{kk'} \qquad \text{for } k,\, k' \in \mathcal{K}. \tag{4.9}$$

In particular, $x_1 = 1$. In the base $(x_z)_{z\in\mathcal{Z}}$, the matrix of multiplication by x_z is monomial, and the corresponding permutation is multiplication by z in $\mathcal{Z}$. This permutation has no fixed point if $z \neq 1$, hence the trace map $T_{Z/F}\colon Z \to F$ satisfies

$$T_{Z/F}(x_1) = |\mathcal{Z}| \neq 0 \qquad \text{and} \qquad T_{Z/F}(x_z) = 0 \quad \text{for } z \neq 1.$$

It is then straightforward to check that the bilinear trace form on Z is not degenerate, hence Z is étale. An action of G on Z is defined by

$$\chi * x_z = \chi(z)x_z \qquad \text{for } z \in \mathcal{Z}.$$

If $z \neq 1$, there exists $\chi \in G$ with $\chi(z) \neq 1$, hence $F \subseteq Z$ is the set of fixed points under the G-action. Since $|G| = |\mathcal{Z}| = [Z{:}F]$, it follows that Z is a G-Galois F-algebra, see [KMRT, Sec. 18B].

Now, consider $e = \frac{1}{r}\sum_{k\in\mathcal{K}} x_k$. In view of (4.9), we have $ex_k = e$ for all $k \in \mathcal{K}$, hence $e^2 = e$. Let $z_1, \ldots, z_m \in \mathcal{Z}$ be representatives of the cosets modulo $\mathcal{K}$. Since $x_{z_i}x_k \in x_{z_ik}F^\times$, the products $x_{z_i}x_k$ for $k \in \mathcal{K}$ and $i = 1, \ldots, m$ form a base of Z. For $i = 1, \ldots, m$ the product ex_{z_i} is in the F-span of $(x_{z_i}x_k)_{k\in\mathcal{K}}$, hence $ex_{z_1}, \ldots, ex_{z_m}$ are linearly independent. These elements span eZ since $ex_{z_i}x_k = ex_{z_i}$ for $k \in \mathcal{K}$, hence they form a base of eZ. Let

$$eX = \{ex \mid x \in X\} = \bigcup_{i=1}^{m} ex_{z_i}F^\times \subseteq (eZ)^\times.$$

Mapping ex to $\rho\pi(x) \in \mathcal{L}$ defines a surjective map $eX \to \mathcal{L}$ with kernel $eF^\times$, hence $\mathcal{L}$ may be identified with an armature of eZ. By [TW, Lemma 2.9], it follows that $eZ \cong L$. Since L is a field, e is a primitive idempotent in Z. From the definition of e, it is clear that $H \subseteq G$ is the subgroup of elements that leave e fixed, hence the orbit of e has r elements, which span K. The structure theorem of Galois algebras (see [KMRT, (18.18)]) shows that the primitive idempotents of Z are the conjugates of e, and that eZ is H-Galois. The proof is thus complete. □

Remark 4.6. The G-structure of Z can be made explicit by [KMRT, Prop. (18.18)]: it is an induced algebra $Z = \mathrm{Ind}_H^G(eZ)$.

For an armature $\mathcal{A}$ of an arbitrary finite-dimensional F-algebra A, there is an associated *armature pairing*

$$\beta_{\mathcal{A}}\colon \mathcal{A} \times \mathcal{A} \to \mu(F) \qquad \text{given by} \qquad (aF^\times, bF^\times) \mapsto aba^{-1}b^{-1},$$

where $\mu(F)$ denotes the group of roots of unity in F. It is shown in [TW, §2] that $\beta_{\mathcal{A}}$ is a well-defined symplectic bimultiplicative pairing, and if $\beta_{\mathcal{A}}$ is nondegenerate, then A is isomorphic to a tensor product of symbol algebras. Conversely, in any tensor product of symbol algebras the standard generators generate an armature whose associated pairing is nondegenerate. For any subgroup $\mathcal{B} \subseteq \mathcal{A}$, we let

$$\mathcal{B}^{\perp} \;=\; \{a \in \mathcal{A} \mid \beta_{\mathcal{A}}(a,b) = 1 \text{ for all } b \in \mathcal{B}\},$$

which is a subgroup of $\mathcal{A}$. Note that when $\beta_{\mathcal{A}}$ is nondegenerate, i.e., $\mathcal{A} \cap \mathcal{A}^{\perp} = \{1\}$, we have $|\mathcal{B}|\,|\mathcal{B}^{\perp}| = |\mathcal{A}|$.

We now fix the setting we will consider for the rest of the paper. Let A be an F-algebra with an armature $\mathcal{A}$ such that $\beta_{\mathcal{A}}$ is nondegenerate. Let $s = exp(\mathcal{A})$. The nondegeneracy of $\beta_{\mathcal{A}}$ implies that $\mu_s \subseteq F$. We denote by $\pi\colon A^{\times} \to A^{\times}/F^{\times}$ the canonical map. Let $v\colon F \to \Gamma \cup \{\infty\}$ be a valuation on F. Assume that $char(\overline{F}) \nmid s$. Hence, the group $\mu_s = \mu_s(F)$ of s-th roots of unity in F maps bijectively to $\mu_s(\overline{F})$. We build a tame F-gauge on A using the armature $\mathcal{A}$. For this, define functions

$$w\colon \pi^{-1}(\mathcal{A}) \to \Gamma \;\text{ given by }\; w(x) \;=\; \tfrac{1}{s}v(x^s), \quad\text{and}\quad \widetilde{w}\colon \mathcal{A} \to \Gamma/\Gamma_F, \;\;\text{the map induced by } w\,.$$

Note that w and $\widetilde{w}$ are group homomorphisms, since the commutators of elements of $\pi^{-1}(\mathcal{A})$ are roots of unity, hence elements in $F^{\times}$ of value 0. Clearly, $w|_{F^{\times}} = v$. For each $a \in \mathcal{A}$, pick $x_a \in A^{\times}$ such that $\pi(x_a) = a$. Then $(x_a)_{a\in\mathcal{A}}$ is an F-base of A. Define an F-norm $y\colon A \to \Gamma \cup \{\infty\}$ by

$$y\big(\sum_{i=1}^{n} \lambda_a x_a\big) \;=\; \min_{a\in\mathcal{A}}\big(v(\lambda_a) + w(x_a)\big)\,.$$

The definition of y depends on $\mathcal{A}$, but is independent of the choice of the x_a.

Proposition 4.7. *The F-norm y is a tame F-gauge on A and Γ_A is determined by*

$$\Gamma_A/\Gamma_F \;=\; \widetilde{w}(\mathcal{A})\,.$$

The graded algebra $gr(A)$ has an armature isometric to $(\mathcal{A}, \beta_{\mathcal{A}})$. Moreover, every homogeneous component of $gr(A)$ contains an invertible element, hence the subgroup $\Delta_{gr(A)} \subseteq \Gamma_{gr(A)} = \Gamma_A$ defined in Sec. 2 coincides with Γ_A, and the map $\theta_{gr(A)}$ of (2.5) is a homomorphism

$$\theta_{gr(A)}\colon \Gamma_A/\Gamma_F \to Aut(Z(A_0)).$$

Proof. Note that $y|_{\pi^{-1}(\mathcal{A})} = w$. Hence, for all a, $b \in \mathcal{A}$, we have $y(x_ax_b) = y(x_a) + y(x_b)$; so for the image x'_a of x_a in $gr(A)$, $(x_ax_b)' = x'_ax'_b$. It follows by Lemma 1.2 that y is surmultiplicative, and that $\pi^{-1}(\mathcal{A})$ maps to a subgroup of $gr(A)^{\times}$. Furthermore, $(x'_a)_{a\in\mathcal{A}}$ is a $gr(F)$-base of $gr(A)$ by Prop. 1.1(i), since $(x_a)_{a\in\mathcal{A}}$ is an F-splitting base of A. Thus, the image $\mathcal{A}'$ of $\{x'_a \mid a \in \mathcal{A}\}$ in $gr(A)^{\times}/gr(F)^{\times}$ could be called a graded armature for $gr(A)$. The map $\mathcal{A} \to \mathcal{A}'$ given by $x_aF^{\times} \mapsto x'_a gr(F)^{\times}$ is clearly a group isomorphism and also an isometry between the armature pairings $\beta_{\mathcal{A}}$ and $\beta_{\mathcal{A}'}$ when we identify $\mu_s(F)$ with $\mu_s(\overline{F})$. The pairing $\beta_{\mathcal{A}'}$ is therefore nondegenerate, so an argument analogous to the ungraded one in [TW, Prop. 2.7] shows that $gr(A)$ is isomorphic to a graded tensor product of graded symbol algebras over $gr(F)$. Since it is easy to see that graded symbol algebras are graded central simple $gr(F)$-algebras (by a slight variation of the ungraded argument), it follows that $gr(A)$ is graded central simple over $gr(F)$. Thus, y is a tame F-gauge on A. □

Our next goal is to describe the degree 0 component $A_0 \subseteq gr(A)$, which is a semisimple algebra over $F_0 = \overline{F}$. For this, we consider

$$\mathcal{B} \;=\; ker(\widetilde{w}) \;\subseteq\; \mathcal{A}, \qquad \mathcal{Z} \;=\; \mathcal{B} \cap \mathcal{B}^{\perp},$$

and denote by $Z \subseteq A$ the subalgebra spanned by $\pi^{-1}(\mathcal{Z})$. Since $\beta_{\mathcal{A}}$ is trivial on $\mathcal{Z}$, the F-algebra Z is commutative.

Proposition 4.8. *The F_0-algebra A_0 has an armature $\mathcal{B}_0$ canonically isomorphic to $\mathcal{B}$ with armature pairing $\beta_{\mathcal{B}_0}$ isometric to the restriction of $\beta_{\mathcal{A}}$ to $\mathcal{B}$. Its center $Z(A_0)$ is the degree 0 component of Z,* *i.e.* $Z(A_0) = Z_0$; *it is an* $(\mathcal{A}/\mathcal{Z}^\perp)$*-Galois* F_0*-algebra. For the map* $\psi\colon \mathcal{A}/\mathcal{Z}^\perp \hookrightarrow Aut_{F_0}(Z(A_0))$ *given by the Galois action, the following diagram is commutative:*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\ \widetilde{w}\ } & \Gamma_A/\Gamma_F \\ \big\downarrow & & \big\downarrow \theta_{gr(A)} \\ \mathcal{A}/\mathcal{Z}^\perp & \xrightarrow{\ \psi\ } & Aut_{F_0}(Z(A_0)). \end{array} \tag{4.10}$$

Proof. We first fix a convenient choice of $x_a \in A^\times$ such that $\pi(x_a) = a \in \mathcal{A}$: for $b \in \mathcal{B}$, we choose x_b such that $w(x_b) = 0$. As observed in the proof of Prop. 4.7, $(x'_a)_{a\in\mathcal{A}}$ is a homogeneous $gr(F)$-base of $gr(A)$. We have $y(x_a) = 0$ if and only if $a \in \mathcal{B}$, hence $(x'_b)_{b\in\mathcal{B}}$ is an F_0-base of A_0. We have

$$x'_b x'_c {x'_b}^{-1} {x'_c}^{-1} \;=\; \overline{\beta_{\mathcal{A}}(b,c)} \;\in\; \mu_s(F_0),$$

hence $\mathcal{B}_0 = \{x'_b F_0^\times \mid b \in \mathcal{B}\}$ is an armature of the F_0-algebra A_0, with armature pairing isometric to the restriction of $\beta_{\mathcal{A}}$ to $\mathcal{B}$. It follows that $Z(A_0)$ is spanned by $(x'_z)_{z\in\mathcal{Z}}$, hence $Z(A_0) = Z_0$. As in Prop. 4.5, Z_0 is $Hom(\mathcal{Z}, \mu_s)$-Galois, for the action defined by

$$\chi * x'_z \;=\; \chi(z) x'_z \qquad \text{for } \chi \in Hom(\mathcal{Z}, \mu_s) \text{ and } z \in \mathcal{Z}.$$

Since $\beta_{\mathcal{A}}$ is nondegenerate, the map $\mathcal{A} \to Hom(\mathcal{Z}, \mu_s)$ that carries $a \in \mathcal{A}$ to the character χ defined by

$$\chi(z) \;=\; \beta_{\mathcal{A}}(a, z) \qquad \text{for } z \in \mathcal{Z} \tag{4.11}$$

is surjective, and its kernel is $\mathcal{Z}^\perp$. Therefore, $\mathcal{A}/\mathcal{Z}^\perp \cong Hom(\mathcal{Z}, \mu_s)$, and Z is $(\mathcal{A}/\mathcal{Z}^\perp)$-Galois. For $z \in \mathcal{Z}$ and $a \in \mathcal{A}$, (4.11) yields

$$\chi * x_z \;=\; \beta_{\mathcal{A}}(a,z)\, x_z \;=\; x_a x_z x_a^{-1},$$

hence the action of χ on Z is conjugation by x_a; the induced action on Z_0 is conjugation by a', so diagram (4.10) commutes. □

The arguments above also show that $\{\, x'_z F_0^\times \mid z \in \mathcal{Z} \,\} \subseteq Z_0^\times / F_0^\times$ is an armature of Z_0 which is isomorphic to $\mathcal{Z}$. We may use this armature to determine the primitive idempotents of Z_0 as in Prop. 4.5: consider the map

$$\rho_0\colon \mathcal{Z} \to F_0^\times / F_0^{\times s} \qquad \text{given by } z \mapsto {x'_z}^{s} F_0^{\times s}.$$

Let $\mathcal{K}_0 = ker(\rho_0)$, $\mathcal{L}_0 = im(\rho_0)$, and $r_0 = |\mathcal{K}_0|$. Let also E be the graded division $gr(F)$-algebra Brauer-equivalent to $gr(A)$.

Proposition 4.9. *The* F_0*-algebra* $Z(A_0)$ *contains* r_0 *primitive idempotents, which are conjugate in* $gr(A)$*. Letting* t *denote the index of any simple component of* A_0*, we have*

$$ms(gr(A)) \;=\; r_0 t^{-1} \sqrt{|\mathcal{B}{:}\mathcal{Z}|}\,. \tag{4.12}$$

Moreover, $Z(E_0)$ *is the* s*-Kummer extension of* F_0 *associated with* $\mathcal{L}_0$*, and* $\Gamma_E/\Gamma_F = \widetilde{w}(\mathcal{K}_0^\perp)$*.*

Proof. Prop. 4.5 shows that Z_0 contains r_0 primitive idempotents, which are conjugate under the $Hom(\mathcal{Z}, \mu_s)$-Galois action, and whose isotropy subgroup is the orthogonal of $\mathcal{K}_0$ in $Hom(\mathcal{Z}, \mu_s)$. On the other hand, Prop. 4.8 shows that the $Hom(\mathcal{Z}, \mu_s)$-Galois action is also realized by inner automorphisms of $gr(A)$, and yields an isomorphism $Hom(\mathcal{Z}, \mu_s) \cong \mathcal{A}/\mathcal{Z}^\perp$ (see (4.11)) carrying the orthogonal of $\mathcal{K}_0$ in the character group to $\mathcal{K}_0^\perp/\mathcal{Z}^\perp$. Therefore, the primitive idempotents of Z_0 are conjugate in $gr(A)$. Prop. 2.2 and 4.8 show that the inverse image of Γ_E/Γ_F in $\mathcal{A}$ is $\mathcal{K}_0^\perp$; hence, $\Gamma_E/\Gamma_F = \widetilde{w}(\mathcal{K}_0^\perp)$.

The center $Z(E_0)$ is isomorphic to the simple components of $Z_0 = Z(A_0)$ (see Prop. 2.2), and hence also to the s-Kummer extension of F_0 associated with $\mathcal{L}_0$, by Prop. 4.5.

Finally, we compute the matrix size of $gr(A)$. First, note that $\widetilde{w}\colon \mathcal{K}_0^\perp \to \Gamma_E/\Gamma_F$ is surjective with kernel $\mathcal{B}$, hence $|\Gamma_E{:}\Gamma_F| = |\mathcal{K}_0^\perp|\,|\mathcal{B}|^{-1}$. Since the pairing $\beta_{\mathcal{A}}$ is nondegenerate, we have $|\mathcal{K}_0^\perp|\,|\mathcal{K}_0| = |\mathcal{A}|$, hence

$$|\Gamma_E{:}\Gamma_F| \;=\; \frac{|\mathcal{A}|}{r_0|\mathcal{B}|} \;=\; \frac{[gr(A){:}gr(F)]}{r_0|\mathcal{B}|}.$$

On the other hand,

$$[E_0{:}F_0] \;=\; t^2[Z(E_0){:}F_0] \;=\; t^2|\mathcal{L}_0| \;=\; t^2|\mathcal{Z}|r_0^{-1}.$$

Since $[E{:}gr(F)] = [E_0{:}F_0]\,|\Gamma_E{:}\Gamma_F|$, it follows that

$$[E{:}gr(F)] \;=\; \frac{t^2|\mathcal{Z}|\,[gr(A){:}gr(F)]}{r_0^2|\mathcal{B}|} \;=\; \frac{t^2[gr(A){:}gr(F)]}{r_0^2|\mathcal{B}{:}\mathcal{Z}|}. \tag{4.13}$$

Since $ms(gr(A)) = \sqrt{[gr(A){:}gr(F)]\,[E{:}gr(F)]^{-1}}$, formula (4.12) follows. □

Let D be the division F-algebra Brauer-equivalent to A. By combining Th. 4.1 and Prop. 4.9, we readily obtain a criterion for the extension of the valuation v on F to D:

Corollary 4.10. *The valuation v on F extends to a valuation on D if and only if $ms(A) = r_0t^{-1}\sqrt{|\mathcal{B}{:}\mathcal{Z}|}$. When this occurs, $\overline{D} = E_0$ and $\Gamma_D = \Gamma_E$.*

Note that when it exists the valuation on D is necessarily tame since $char(\overline{F}) \nmid s = exp(\mathcal{A})$ while $deg(D) \mid deg(A) = \sqrt{|\mathcal{A}|}$.

Finally, we consider the case where v extends to a valuation v_D on D that is totally ramified over F. Recall from [TW, Sec. 3] that in this case there is a canonical pairing $C_D\colon \Gamma_D/\Gamma_F \times \Gamma_D/\Gamma_F \to \mu(\overline{F})$ defined by $C_D(\gamma_1 + \Gamma_F, \gamma_2 + \Gamma_F) = \overline{x_1x_2x_1^{-1}x_2^{-1}}$ for any x_1, $x_2 \in D^\times$ with $v_D(x_i) = \gamma_i$ for $i = 1, 2$.

Proposition 4.11. *The valuation v on F extends to a valuation on D that is tamely and totally ramified over F if and only if $deg(D) = \sqrt{|\mathcal{B}^\perp{:}\mathcal{Z}|}$. When this occurs, we have $\Gamma_D/\Gamma_F = \widetilde{w}(\mathcal{B}^\perp)$ and $\widetilde{w}$ defines an isometry from $\mathcal{B}^\perp/\mathcal{Z}$ with the nondegenerate pairing induced by $\beta_{\mathcal{A}}$ onto Γ_D/Γ_F with the pairing C_D.*

Proof. Since $[gr(A){:}gr(F)] = |\mathcal{A}| = |\mathcal{B}|\,|\mathcal{B}^\perp|$, equation (4.13) yields

$$[E{:}gr(F)] \;=\; t^2|\mathcal{B}^\perp|\,|\mathcal{Z}|r_0^{-2} \;=\; t^2|\mathcal{B}^\perp{:}\mathcal{Z}|\,|\mathcal{Z}|^2\,r_0^{-2} \;=\; t^2|\mathcal{L}_0|^2\,|\mathcal{B}^\perp{:}\mathcal{Z}|.$$

On the other hand, Th. 4.1 yields $[D{:}F] \geq [E{:}gr(F)]$. Therefore, if $[D{:}F] = |\mathcal{B}^\perp{:}\mathcal{Z}|$, then we must have $[D{:}F] = [E{:}gr(F)]$ and $t = |\mathcal{L}_0| = 1$, hence v extends to valuation on D that is totally ramified over F.

For the converse, we apply Cor. 4.10 with $t = 1$ and $r_0 = |\mathcal{Z}|$, and obtain $ms(A) = \sqrt{|\mathcal{B}|\,|\mathcal{Z}|}$. Since $[A{:}F] = [D{:}F]ms(A)^2$ and $[A{:}F] = |\mathcal{A}| = |\mathcal{B}|\,|\mathcal{B}^\perp|$, it follows that $[D{:}F] = |\mathcal{B}^\perp{:}\mathcal{Z}|$.

For the rest of the proof, assume v extends to a valuation on D that is tamely and totally ramified over F. Then $gr(D) \cong_g E$, and $r_0 = |\mathcal{Z}|$, hence $\mathcal{K}_0 = \mathcal{Z}$ and $\Gamma_D/\Gamma_F = \Gamma_E/\Gamma_F = \widetilde{w}(\mathcal{Z}^\perp)$, by Prop. 4.9. Since $\mathcal{Z} = \mathcal{B} \cap \mathcal{B}^\perp$, we have $\mathcal{Z}^\perp = \mathcal{B} + \mathcal{B}^\perp$, hence $\widetilde{w}(\mathcal{Z}^\perp) = \widetilde{w}(\mathcal{B}^\perp)$ since $\mathcal{B} = ker(\widetilde{w})$. We may identify the canonical pairing C_D with the pairing C on Γ_E/Γ_F given by

$$C(\gamma + \Gamma_F, \delta + \Gamma_F) = \xi\eta\xi^{-1}\eta^{-1} \qquad \text{for any nonzero } \xi \in E_\gamma,\ \eta \in E_\delta.$$

In order to relate C to $\beta_{\mathcal{A}}$, we identify a copy of E in $gr(A)$. First, we choose for each $a \in \mathcal{A}$ an element $x_a \in A^\times$ such that $\pi(x_a) = a$. As in the proof of Prop. 4.8, we choose x_b such that $w(x_b) = 0$ for $b \in \mathcal{B}$. Note that $\mathcal{Z}_0 = \{\, x'_z F_0^\times \mid z \in \mathcal{Z} \,\}$ is an armature of Z_0 which is isomorphic to $\mathcal{Z}$. Since $\mathcal{Z} = ker(\rho_0)$, $\mathcal{Z}_0$ is the kernel of the s-power map $\mathcal{Z}_0 \to F_0^\times/F_0^{\times s}$. Therefore, the proof of Prop. 4.5 shows that after scaling x_z for $z \in \mathcal{Z}$ by suitable units in $F^\times$ we may assume $x'_{z_1}x'_{z_2} = x'_{z_1z_2}$ for z_1, $z_2 \in \mathcal{Z}$.

As in the proof of Prop. 4.5, we consider $e = \frac{1}{|\mathcal{Z}|}\sum_{z\in\mathcal{Z}} x'_z$, which is a primitive idempotent in Z_0 such that $ex'_z = e$ for $z \in \mathcal{Z}$. For $a \in \mathcal{A}$, we have

$$x'_a e x'^{\,-1}_a \;=\; \tfrac{1}{|\mathcal{Z}|}\sum_{z\in\mathcal{Z}} \overline{\beta_{\mathcal{A}}(a,z)}\, x'_z,$$

which is e if $a \in \mathcal{Z}^\perp$, and is another primitive idempotent of Z_0 if $a \notin \mathcal{Z}^\perp$. Thus, $ex'_a e = ex'_a = x'_a e$ if $a \in \mathcal{Z}^\perp$, and $e(x'_a e x'^{\,-1}_a) = 0$, hence $ex'_a e = 0$, if $a \notin \mathcal{Z}^\perp$. Therefore, $e gr(A) e$ is spanned by $(ex'_a)_{a\in\mathcal{Z}^\perp}$. If $a_1, \ldots, a_r \in \mathcal{Z}^\perp$ are in different cosets modulo $\mathcal{Z}$, then $ex'_{a_1}, \ldots, ex'_{a_r}$ are linearly independent since each ex'_{a_i} lies in the span of $(x'_{za_i})_{z\in\mathcal{Z}}$. Let $n = |\mathcal{B}{:}\mathcal{Z}|$, $m = |\mathcal{B}^\perp{:}\mathcal{Z}|$, and let $b_1, \ldots, b_n \in \mathcal{B}$ (resp. $c_1, \ldots, c_m \in \mathcal{B}^\perp$) be representatives of the various cosets of $\mathcal{B}$ (resp. $\mathcal{B}^\perp$) modulo $\mathcal{Z}$. Since $\mathcal{Z}^\perp = \mathcal{B} + \mathcal{B}^\perp$ and $\mathcal{Z} = \mathcal{B} \cap \mathcal{B}^\perp$, we have $\mathcal{Z}^\perp/\mathcal{Z} = (\mathcal{B}/\mathcal{Z}) \oplus (\mathcal{B}^\perp/\mathcal{Z})$, hence $\{b_i c_j \mid 1 \le i \le n,\ 1 \le j \le m\}$ is a set of representatives of the various cosets of $\mathcal{Z}^\perp$ modulo $\mathcal{Z}$. For $i = 1, \ldots, n$ and $j = 1, \ldots, m$, let

$$\xi_i \;=\; ex'_{b_i} \;=\; x'_{b_i} e \in eA_0 \;\subseteq\; e gr(A) e \qquad \text{and} \qquad \eta_j \;=\; ex'_{c_j} \;=\; x'_{c_j} e \;\in\; e gr(A) e.$$

Then $(\xi_i\eta_j \mid 1 \le i \le n,\ 1 \le j \le m)$ is a $gr(F)$-base of $e gr(A) e$. Moreover, $\xi_i\eta_j = \eta_j\xi_i$ since $\beta_{\mathcal{A}}(b_i, c_j) = 1$. Therefore, the graded subalgebras B, $B' \subseteq e gr(A) e$ spanned respectively by $\xi_1, \ldots, \xi_n$ and by $\eta_1, \ldots, \eta_m$ centralize each other, and

$$e gr(A) e \;\cong_g\; B \otimes_{gr(F)} B'.$$

The degree of each ξ_i is 0 since $b_i \in \mathcal{B} = ker(\widetilde{w})$, hence $B = B_0 \otimes_{F_0} gr(F)$. On the other hand, the degree of η_j is 0 if and only if $\eta_j \in eF_0$ since $\mathcal{B}^\perp \cap \mathcal{B} = \mathcal{Z}$. Therefore, $eA_0 = B_0$. This algebra is split by hypothesis. Therefore, B is split and $gr(A)$ is Brauer-equivalent to B'. Since $[B'{:}gr(F)] = |\mathcal{B}^\perp{:}\mathcal{Z}| = [E{:}gr(F)]$, we may identify B' with E. Clearly, under this identification the canonical pairing on Γ_E/Γ_F coincides with the pairing on $\mathcal{B}^\perp/\mathcal{Z}$ induced by $\beta_{\mathcal{A}}$. □

REMARKS 4.12. (i) The description of $\overline{D}$ and Γ_D in Cor. 4.10 (with additional information from Prop. 4.8 and 4.9) were given in [W₃, Th. 1], and proved using Morandi value functions. The proof given here is easier and more direct. By Prop. 2.5 the tame gauge y defined here is a Morandi value function (so the associated valuation ring $A^{\ge 0}$ is a Dubrovin valuation ring) if and only if $|\mathcal{B}_0| = 1$, i.e., if and only if $A^{\ge 0}$ has a unique maximal two-sided ideal.

(ii) Suppose the valuation v on F is strictly Henselian, i.e., v is Henselian and $\overline{F}$ is separably closed. Then, in the setting of Prop. 4.11 with $char(\overline{F}) \nmid exp(\mathcal{A})$, v necessarily extends to a valuation on the division algebra D Brauer-equivalent to A, and D is totally and tamely ramified over F. In that situation, the description of the canonical pairing on D (which then determines D up to isomorphism by [TW, Prop. 4.2]) was given in [TW, Th. 4.3], with a more difficult proof.

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