NONEXCELLENCE OF CERTAIN FIELD EXTENSIONS

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Abstract. Consider towers of fields $F_1 \subset F_2 \subset F_3$, where F_3/F_2 is a quadratic extension and F_2/F_1 is an extension, which is either quadratic, or of odd degree, or purely transcendental of degree 1. We construct numerous examples of the above types such that the extension F_3/F_1 is not 4-excellent. Also we show that if k is a field, *char* $k \neq 2$ and l/k is an arbitrary field extension of degree 4, then there exists a field extension F/k such that the extension lF/F is not 4-excellent.

1. Notation and terminology

The purpose of this paper is to construct nonexcellent field extensions of certain types. Recall that a field extension L/F is called excellent (resp. n-excellent) if for any quadratic form φ over F (resp. for any quadratic form φ over F of dimension at most n) the anisotropic part of the form φ_L is defined over F. It is well known that any quadratic extension is excellent (see for example [4]). Obviously, the same is true for odd degree and purely transcendental extensions, since they can not make an anisotropic form isotropic. We consider below towers of fields $F_1 \subset F_2 \subset F_3$, where the extension F_3/F_2 is quadratic and the extension F_2/F_1 is either quadratic or of odd degree or purely transcendental of degree 1. We give numerous examples of extensions F_3/F_1 of such types, which are not 4-excellent.

The main source of reference concerning quadratic forms over fields is the Scharlau book [4]. We keep the standard notation. All the fields considered in the sequel are of characteristic different from 2. By a form we mean a nonsingular quadratic form. If F is a field, then $W(F)$ is the Witt ring of F and $I(F)$ is the ideal of evendimensional forms in $W(F)$. The anisotropic part of a form φ is denoted by φ_{an}. Slightly abusing notation we write $\varphi = 0$ if φ is hyperbolic, i.e. its anisotropic part is zero. By $D(\varphi)$ we denote the set of nonzero values of φ. The function field of the projective quadric related to the form φ is denoted by $F(\varphi)$. If L/F is a field extension, then φ_L is the extension of φ to L, i.e. the form $\varphi \otimes_F L$. For $a, b \in F^*$ the symbol $\langle\langle a, b\rangle\rangle$ means the 2-fold Pfisterform $\langle 1, -a, -b, ab\rangle$. If k is a field and $p \in k[x]$ is an irreducible polynomial, by $k(p)$ we denote the residue field $k[x]/p$. If

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p is fixed, $f \in k[x]$ and p does not divide f, then unless specified otherwise, $\overline{f}$ is the image of f in $k(p)^*/k(p)^{*2}$.

The main tool in constructing of the examples in question is the notion of the second residue homomorphism $\partial_p : I^2/I^3(k(x)) \to k(p)^*/k(p)^{*2}$ for an arbitrary irreducible polynomial p in x over a field k ([4]). Recall that the residue of the 2-fold Pfisterform $\langle\langle f, g\rangle\rangle$ at the polynomial p is computed as follows:

$$\partial_p(\langle\langle f, g\rangle\rangle) = \begin{cases} 0 \text{ if } v_p(f) = v_p(g) = 0, \\ \overline{f} \text{ if } v_p(f) = 0,\ v_p(g) = 1 \end{cases}.$$

(v_p is the corresponding discrete valuation on $k(x)$). In fact these equations determine the map ∂_p on the whole group $I^2/I^3(k(x))$.

2. Nonexcellence of field extensions of degree 4

We begin with the following

Theorem 1. *Let k be a field, $a, b, c \in k$, $a \in k^* \setminus k^{*2}$, $b^2 - ac^2 \in k^* \setminus k^{*2}$. Put $l = k(\sqrt{b + c\sqrt{a}})$ and $F = k(t, x, y)$, where t, x, y are indeterminates. Then there exists a 4-dimensional quadratic form φ over F such that $\dim\ (\varphi_{lF})_{an} = 2$ and the form $(\varphi_{lF})_{an}$ is not defined over F. (This means that the field extension lF/F is not 4-excellent).*

Proof. Let $f = ((t^2 - a)x^2 - (b^2 - ac^2))((t^2 - a)y^2 - 1)$,

$$\varphi_1 \simeq (t + \sqrt{a})\langle 1,\ -(b + c\sqrt{a})\rangle,$$

$$\varphi_2 \simeq (t + \sqrt{a})\langle 1,\ -f(b + c\sqrt{a})\rangle.$$

Let us find $q \in F^*$ such that the forms $(t + \sqrt{a})\langle\langle q\rangle\rangle$ and $q(t + \sqrt{a})(b + c\sqrt{a})\langle\langle qf\rangle\rangle$ are defined over F. Provided q satisfies this condition we get that the form

$$\varphi \simeq \varphi_2 \perp -q\varphi_1 \simeq (t + \sqrt{a})\langle\langle q\rangle\rangle \perp q(t + \sqrt{a})(b + c\sqrt{a})\langle\langle qf\rangle\rangle$$

is defined over F as well.

The following lemma is well known, but in view of the absence of a convenient reference and for the sake of completeness we give the proof.

Lemma 2. *Suppose that $q \notin F^{*2} \cup aF^{*2}$ and the elements $\alpha, \beta \in F$ are such that $\langle\langle q, \alpha^2 - a\beta^2\rangle\rangle = 0$. Then the form $(\alpha + \beta\sqrt{a})\langle\langle q\rangle\rangle$ is defined over F. Moreover, if elements $X, Y \in F$ are such that $X^2 - qY^2 = \alpha^2 - a\beta^2$, then $(\alpha + \beta\sqrt{a})\langle\langle q\rangle\rangle \simeq (2X + 2\alpha)\langle\langle q\rangle\rangle$.*

Proof. Since $(X + Y\sqrt{q})(X - Y\sqrt{q}) = (\alpha + \beta\sqrt{a})(\alpha - \beta\sqrt{a})$, we have

$$N_{F(\sqrt{a},\sqrt{q})/F(\sqrt{a})}(\alpha + \beta\sqrt{a} + X + Y\sqrt{q}) =$$
$$= (\alpha + \beta\sqrt{a} + X + Y\sqrt{q})(\alpha + \beta\sqrt{a} + X - Y\sqrt{q}) = (\alpha + \beta\sqrt{a})(2X + 2\alpha).$$

Therefore, since obviously, $\langle\langle q, N_{F(\sqrt{a},\sqrt{q})/F(\sqrt{a})}(\alpha + \beta\sqrt{a} + X + Y\sqrt{q})\rangle\rangle = 0$, we get

$$\langle\langle q, \alpha + \beta\sqrt{a}\rangle\rangle \simeq \langle\langle q, (\alpha + \beta\sqrt{a})N_{F(\sqrt{a},\sqrt{q})/F(\sqrt{a})}(\alpha + \beta\sqrt{a} + X + Y\sqrt{q})\rangle\rangle \simeq \langle\langle q, 2X + 2\alpha\rangle\rangle,$$

or, equivalently $(\alpha+\beta\sqrt{a})\langle\langle q\rangle\rangle \simeq (2X+2\alpha)\langle\langle q\rangle\rangle$. The lemma is proved. □

Since

$$((t^2-a)y)^2 - \frac{t^2-a}{(t^2-a)y^2-1}((t^2-a)y^2-1)^2 = t^2-a,$$

and

$$((t^2-a)x)^2 - \frac{t^2-a}{(t^2-a)y^2-1}f = (t^2-a)(b^2-ac^2) = N_{F(\sqrt{a})/F}(b+c\sqrt{a})(t+\sqrt{a}),$$

we obtain that

$$\langle\langle \frac{t^2-a}{(t^2-a)y^2-1},\ t^2-a\rangle\rangle = \langle\langle \frac{t^2-a}{(t^2-a)y^2-1}f,\ (t^2-a)(b^2-ac^2)\rangle\rangle = 0.$$

Therefore, constructing the form φ we can put $q = \frac{t^2-a}{(t^2-a)y^2-1}$. By Lemma 2 we get

$$(t+\sqrt{a})\langle\langle q\rangle\rangle \simeq (2(t^2-a)y+2t)\langle\langle q\rangle\rangle,$$

$$q(t+\sqrt{a})(b+c\sqrt{a})\langle\langle qf\rangle\rangle \simeq (2(t^2-a)x+2(bt+ac))\langle q,-f\rangle,$$

hence

$$\varphi \simeq (2(t^2-a)y+2t)\langle 1,-q\rangle \perp (2(t^2-a)x+2(bt+ac))\langle q,-f\rangle.$$

We are going to prove that the form φ provides a counterexample to 4-excellency of the field F. However, we don't need the explicit form of the coefficients of φ, which look somewhat complicated.

It is obvious that

$$(\varphi_{lF})_{an} \simeq (\varphi_2)_{lF} \simeq (t+\sqrt{a})\langle 1,\ -f\rangle_{lF}.$$

In particular, $\dim(\varphi_{lF})_{an} = 2$. If the form $(\varphi_{lF})_{an}$ is defined over F, then there is $p\in F^*$ such that

$$(t+\sqrt{a})\langle 1,\ -f\rangle \simeq p\langle 1,\ -f\rangle,$$

i.e. $\langle\langle f,\ (t+\sqrt{a})p\rangle\rangle_{lF} = 0$, which is equivalent to

$$\langle\langle f,\ (t+\sqrt{a})p\rangle\rangle \simeq \langle\langle b+c\sqrt{a}, u+v\sqrt{a}\rangle\rangle$$

over $F(\sqrt{a})$ for some $u,v\in F$. We will show that in fact this is impossible. More precisely we will prove the following a bit more general

Lemma 3. *For any $u,v,p\in F$, $w\in k(\sqrt{a})$ one has*

$$\langle\langle f,\ (t+\sqrt{a})wp\rangle\rangle \not\simeq \langle\langle b+c\sqrt{a},\ u+v\sqrt{a}\rangle\rangle. \tag{1}$$

Proof. Put $e=b^2-ac^2$. If $e\notin ak^{*2}$, i.e. l/k is not a Galois extension, then the residues at either $t+\sqrt{a}$ or $t-\sqrt{a}$ of the left hand and on the right hand parts of (1) are different, which proves the lemma in this case.

If $e \in ak^{*2}$, the argument is more complicated. Assume that

$$\langle\langle f,\ (t+\sqrt{a})wp\rangle\rangle \simeq \langle\langle b+c\sqrt{a},\ u+v\sqrt{a}\rangle\rangle \tag{1'}$$

for some $u, v, p \in k(t,y)[x]$, and that under this condition $m = \deg p$ is minimal (in this section by deg we mean the degree with respect to the variable x). In particular, the polynomial p is squarefree. Moreover, we may assume that $\deg(u+v\sqrt{a})$ is minimal provided that $\deg p = m$. It follows that $(t^2-a)x^2-e$ does not divide p, for otherwise we would change p for $-p\frac{(t^2-a)y^2-1}{(t^2-a)x^2-e}$, a contradiction to minimality of $\deg p$. Also $(t^2-a)x^2-e$ does not divide $u+v\sqrt{a}$, for in the opposite case the isomorphism

$$\langle\langle f,\ (t+\sqrt{a})w(b+c\sqrt{a})p\rangle\rangle \simeq \langle\langle b+c\sqrt{a},\ \frac{u+v\sqrt{a}}{(t^2-a)x^2-e}((t^2-a)y^2-1)\rangle\rangle,$$

obtained from $(1')$ by adding $\langle\langle b+c\sqrt{a}, f\rangle\rangle$ to its both parts, would imply a contradiction to minimality of $\deg(u+v\sqrt{a})$.

Let $p = \prod p_i$, where $p_i \in k(t,y)[x]$ are irreducible polynomials, $\deg p_i \geq 1$. We need the following

Lemma 4. *For any p_i the elements f and af are not squares in $k(t,y)[x]/p_i$.*

Proof. Suppose for instance that af is a square in $k(t,y)[x]/p_1$. Consider two cases: a) $\deg p_1 \geq 2$, b) $\deg p_1 = 1$.

Case a).

Since af is a square in $k(t,y)[x]/p_1$ we get $P^2 - af = p_1 s$, for some $P, s \in k(t,y)[x]$, $\deg P \leq deg\ p_1 - 1$. Since $\deg p_1 \geq 2$ and $\deg f = 2$ it follows that $\deg(p_1 s) \leq 2(\deg p_1 - 1)$, hence $\deg s \leq \deg p_1 - 2$. Over the field $k(\sqrt{a}, t, y, x)$ we have

$$\langle\langle f, p_1 s\rangle\rangle \simeq \langle\langle f,\ P^2 - af\rangle\rangle \simeq \langle\langle af,\ P^2-af\rangle\rangle = 0,$$

so

$$\langle\langle b+c\sqrt{a},\ u+v\sqrt{a}\rangle\rangle \simeq \langle\langle f,\ (t+\sqrt{a})wp\rangle\rangle \simeq \langle\langle af,\ (t+\sqrt{a})wp\rangle\rangle \simeq \langle\langle af,\ (t+\sqrt{a})w\frac{p}{p_1}s\rangle\rangle \simeq \langle\langle f,\ (t+\sqrt{a})w\frac{p}{p_1}s\rangle\rangle,$$

a contradiction to minimality of $\deg p$, since $\deg \frac{p}{p_1}s \leq \deg p - 2$.

Case b).

Let $p_1 = Qx + P$. We may assume that $Q, P \in k(t)[y]$. Since af is a square in $k(t,y)[x]/p_1$, we conclude that $a((t^2-a)\frac{P^2}{Q^2} - e)((t^2-a)y^2-1) \in k(t,y)^{*2}$. This implies that $(t^2-a)y^2-1$ is a divisor of $(t^2-a)P^2 - eQ^2$ as a polynomial in y. It follows that there are nonzero polynomials $P_1, Q_1 \in k(t)[y]$ and a rationl function $g \in k(t)^*$ such that $\deg_y P_1$, $\deg_y Q_1 \leq 1$ and $(t^2-a)P_1^2 - eQ_1^2 = g((t^2-a)y^2 - 1)$ (here by $\deg_y$ we mean the degree with respect to y). It is easy to see that this implies $g\langle(t^2-a), -1\rangle \simeq \langle(t^2-a), -e\rangle$ over $k(t)$, a contradiction, since the discriminants on the left-hand and on the right-hand sides of the last isomorphism are different. The case where f is a would-be square in $k(t,y)[x]/p_1$ is treated quite similar. Lemma 4 is proved. $\square$

We return to the proof of Lemma 3. By Lemma 4 we get that every p_i is a divisor of the polynomial $u+v\sqrt{a} \in k(\sqrt{a},t,y)[x]$, since otherwise the residues at p_i (more precisely at some irreducible divisor of p_i over $k(\sqrt{a},t,y)[x]$) on the left-hand and the right- hand sides of the equality $\langle\langle f,\ (t+\sqrt{a})wp\rangle\rangle \simeq \langle\langle b+c\sqrt{a},\ u+v\sqrt{a}\rangle\rangle$ would be different. So we obtain that $u+v\sqrt{a} = p(u_1+v_1\sqrt{a})$, hence

$$\langle\langle f,\ (t+\sqrt{a})wp\rangle\rangle \simeq \langle\langle b+c\sqrt{a}\ ,p(u_1+v_1\sqrt{a})\rangle\rangle \tag{2}$$

for some $u_1, v_1 \in k(t,y)[x]$, and, since p is squarefree, the polynomials $u_1+v_1\sqrt{a}$ and p are relatively prime. Multiplying by some square in $k(t)^*$ we may assume that $v_{t^2-a}(p)$ equals either 0 or 1. Suppose that $v_{t^2-a}(p) = 1$. Then applying the automorphism $W(F(\sqrt{a})) \to W(F(\sqrt{a}))$ determined by $t \to -t$ to the equality (2) written as

$$\langle\langle f,\ (t-\sqrt{a})\frac{p}{t^2-a}w\rangle\rangle \simeq \langle\langle b+c\sqrt{a},\ \frac{p}{t^2-a}(u_1+v_1\sqrt{a})(t^2-a)\rangle\rangle$$

and changing $p(t,x,y)$ for $p(-t,x,y)(t^2-a)^{-1}$, we reduce the problem to the case $v_{t^2-a}(p) = 0$.

We claim that $\deg(u_1+v_1\sqrt{a}) = 0$. Indeed, suppose $s \in k(\sqrt{a},t,y)[x]$ is a prime divisor of $u_1+v_1\sqrt{a}$. Since s does not divide $((t^2-a)x^2-e)p$, we get that

$$\partial_s(\langle\langle b+c\sqrt{a},\ p(u_1+v_1\sqrt{a})\rangle\rangle) = \partial_s(\langle\langle f,\ (t+\sqrt{a})wp\rangle\rangle) = 1.$$

By the same argument as in Lemma 4 we can diminish $\deg(u_1+v_1\sqrt{a})$ in (2), which contradicts minimality of $\deg(u+v\sqrt{a})$ in (1′). Thus, we conclude that there are no prime divisors of $u_1+v_1\sqrt{a}$ at all, i.e. $u_1, v_1 \in k(t,y)$. The equality (2) implies

$$\langle\langle (b+c\sqrt{a})f,\ (t+\sqrt{a})wp\rangle\rangle \simeq \langle\langle b+c\sqrt{a},\ (t+\sqrt{a})(u_1+v_1\sqrt{a})w\rangle\rangle,$$

and the right- hand side of the last isomorphism is defined over $k(\sqrt{a},t,y)$. Denote it by π. Since π splits by the square root of

$$(b+c\sqrt{a})((t^2-a)y^2-1)((t^2-a)x^2-e),$$

we conclude that $\pi_{k(\sqrt{a},t,y)(\tau)} = 0$, where

$$\tau \simeq \langle\langle ((t^2-a)y^2-1)(b+c\sqrt{a})(t^2-a), -((t^2-a)y^2-1)(b+c\sqrt{a})e\rangle\rangle.$$

Therefore, either $\pi = 0$, or

$$\pi \simeq \tau \simeq \langle\langle ((t^2-a)y^2-1)(b+c\sqrt{a})(t^2-a),\ -((t^2-a)y^2-1)(b+c\sqrt{a})e\rangle\rangle. \tag{3}$$

On the one hand, since $v_{t^2-a}(p) = 0$, we have

$$\partial_{(t+\sqrt{a})}(\pi) = \partial_{(t+\sqrt{a})}(\langle\langle f(b+c\sqrt{a}),\ (t+\sqrt{a})wp\rangle\rangle) = b-c\sqrt{a},$$

so $\pi \neq 0$, hence (3) holds. On the other hand,

$$\partial_{(t-\sqrt{a})}(\pi) = \partial_{(t-\sqrt{a})}(\langle\langle f(b+c\sqrt{a}),\ (t+\sqrt{a})wp\rangle\rangle) = 1,$$

and

$$\partial_{(t-\sqrt{a})}(\langle\langle ((t^2-a)y^2-1)(b+c\sqrt{a})(t^2-a),\ -((t^2-a)y^2-1)(b+c\sqrt{a})e\rangle\rangle) = \\ (b+c\sqrt{a})e = b-c\sqrt{a},$$

a contradiction in view of (3). This proves Lemma 3, hence also Theorem 1. □

□

Corollary 5. *Let k be a field, let l/k be an arbitrary field extension of forth degree. Then there is a field extension F/k such that the extension lF/F is not 4-excellent.*

Proof. There is a field extension k_1/k, which is either cubic or trivial, such that lk_1/k_1 is the tower of quadratic extensions. If this tower is not the composite of two quadratic extensions of k_1 we can apply Theorem 1. The opposite case is considered in [5]. $\square$

If $k(\sqrt{b+c\sqrt{a}})/k$ is not a Galois extension i.e. $e \notin k^{*2} \cup ak^{*2}$, then Theorem 1 can be strengthened and the proof becomes much simpler. Namely, we have the following

Proposition 6. *Let k be a field, $a, b, c \in k$, $a \in k^* \setminus k^{*2}$, $e = b^2 - ac^2 \in k^* \setminus (k^{*2} \cup ak^{*2})$, $l = k(\sqrt{b+c\sqrt{a}})$. Let t be an indeterminate, $F = k(t)$. Then there exists a 4-dimensional form φ over F such that the following holds:*

For any field $\widetilde{l} \supset l$ such that $b - c\sqrt{a}$ is not a square in $\widetilde{l}$ and t is transcendental over $\widetilde{l}$ the form $(\varphi_{\widetilde{l}F})_{an}$ is not defined over F. In particular, the field l itself satisfies these conditions.

Proof. Let

$$\varphi \simeq \langle 2t,\ 2t(t^2-a),\ -2(bt+ac),\ -2e(bt+ac)(t^2-a)\rangle.$$

It is easy to see that

$$\varphi_{F(\sqrt{a})} \simeq (t-\sqrt{a})\langle 1,\ -(b-c\sqrt{a})\rangle \perp (t+\sqrt{a})\langle 1,\ -(b+c\sqrt{a})\rangle,$$

hence $\dim(\varphi_{\widetilde{l}F})_{an} = 2$. Assume that the form $(\varphi_{\widetilde{l}F})_{an}$ is define over F. Then there is $z \in F^*$ such that

$$(t-\sqrt{a})\langle 1,\ -(b-c\sqrt{a})\rangle \simeq z\langle 1,\ -(b-c\sqrt{a})\rangle,$$

i.e. $\langle\langle b-c\sqrt{a},\ (t-\sqrt{a})z\rangle\rangle_{\widetilde{l}F} = 0$. On the other hand, since $b-c\sqrt{a}$ is not a square in $\widetilde{l}$ and $z \in F^*$, the residue of this 2-fold Pfisterform at either $t-\sqrt{a}$ or $t+\sqrt{a}$ is nontrivial, a contradiction. $\square$

3. Nonexcellence of $2n$-degree extensions for an odd n

In this section we consider towers of two field extensions one of which is a Galois extension of odd degree and the other is quadratic.

Proposition 7. *Let k be a field, $l = k(\alpha, \beta)$ is a finite Galois field extension of odd degree $n > 1$. Assume that $\alpha \notin k$ and $\beta \notin k(\alpha)$. Let further t, x, y be indeterminates, $a, b, d \in k^*$ such that $d, -ab, -abd \notin k^{*2}$. Set $\varphi \simeq \langle a, b, x, abdx\rangle$, $K = k(t, x, y)$, $L = lK$, $E = L(\sqrt{-(xt^2 + a\alpha^2)(by^2 + abdx\beta^2)})$. Then*

1) $\dim(\varphi_E)_{an} = 2$.
2) *The form $(\varphi_E)_{an}$ is not defined over K.*

Proof. The argument is somewhat similar to the one in Theorem 1.

1) Obviously,

$$\varphi_L \simeq \langle xt^2 + a\alpha^2, ax(xt^2 + a\alpha^2), by^2 + abdx\beta^2, adx(by^2 + abdx\beta^2)\rangle,$$

hence φ_E is isotropic and $(\varphi_E)_{an} \simeq ax(xt^2 + a\alpha^2)\langle 1, -d\rangle$.

2) Assume that the form $(\varphi_E)_{an}$ is defined over K. Then there is $p \in K^*$ such that

$$ax(xt^2 + a\alpha^2)\langle 1, -d\rangle_E \simeq p\langle 1, -d\rangle_E,$$

i.e. $\langle\langle d, ax(xt^2 + a\alpha^2)p\rangle\rangle_E = 0$. This means that

$$\langle\langle d, ax(xt^2 + a\alpha^2)p\rangle\rangle_L \simeq \langle\langle -(xt^2 + a\alpha^2)(by^2 + abdx\beta^2), q\rangle\rangle_L \tag{4}$$

for some $q \in L^*$.

We may assume that $p \in k(t,x)[y]$ and $m = \deg p$ is minimal (in this section we mean by deg the degree with respect to the variable y). Notice also that $by^2 + abdx\beta^2 \not| p$, for otherwise $by^2 + abdx(\sigma\beta)^2 | p$ for any $\sigma \in G(l/k)$ and comparing the residues of both parts of (4) at $by^2 + abdx(\sigma\beta)^2$, with $\sigma\beta \neq \beta$ would give us a contradiction.

Now let $p = \prod p_i$, where $p_i \in k(t,x)[y]$ are irreducible polynomials in y.

Lemma 8. *d is not a square in $k(t,x)[y]/p_i$ for any p_i.*

Proof. Suppose for instance that d is a square in $k(t,x)[y]/p_1$. We get $P^2 - d = p_1 s$ for some $P, s \in k(t,x)[y]$, $\deg P \leq \deg p_1 - 1$. Hence

$$\deg s \leq 2(\deg p_1 - 1) - \deg p_1 = \deg p_1 - 2.$$

Since $\langle\langle d,\ p_1 s\rangle\rangle \simeq \langle\langle d, P^2 - d\rangle\rangle = 0$, we can change the divisor p_1 of p for s in (4), a contradiction to minimality of $\deg p$. The lemma is proved. □

Let $p_i = \prod p_{ij}$, where $p_{ij} \in l(t,x)[y]$ are irreducible polynomials. Since l/k is an odd degree Galois extension, we have $l(t,x)[y]/p_{ij} \simeq l(k(t,x)[y]/p_i)$ and, moreover, $\deg[l(k(t,x)[y]/p_i) : k(t,x)[y]/p_i]$ is odd. Therefore, by Lemma 8 d is not a square in $l(k(t,x)[y]/p_i)$, hence in $l(t,x)[y]/p_{ij}$. This means that the form (4) has nonzero residues at each p_{ij}. This implies that $q = pf$, where $f \in l(t,x)[y]$ and the polynomials f and p are relatively prime, so we can represent (4) as

$$\langle\langle d, ax(xt^2 + a\alpha^2)p\rangle\rangle_L \simeq \langle\langle -(xt^2 + a\alpha^2)(by^2 + abdx\beta^2), pf\rangle\rangle_L. \tag{5}$$

Just as in Lemma 2 we may assume that in equality (5), where $\deg p = m$ is fixed, $\deg f$ is minimal. Denote the Pfisterform in (5) by π. Notice that $by^2 + abdx\beta^2$ does not divide f, since otherwise we could change f for $\frac{f}{(xt^2+a\alpha^2)(by^2+abdx\beta^2)}$, a contradiction to minimality of $\deg f$.

Lemma 9. $\deg f = 0$.

Proof. Let $f = \prod f_i$, where $f_i \in l(t,x)[y]$ are irreducible polynomials. Since f_i do not appear on the left- hand side of (5), we have $\partial_{f_i}(\pi) = 1$, for any f_i. Assume first that, say $\deg f_1 \geq 2$. Since $\partial_{f_1}(\pi) = 1$, we get that

$$P^2 + (xt^2 + a\alpha^2)(by^2 + abdx\beta^2) = f_1 s$$

for some $P, s \in l(t,x)[y]$ such that $\deg P \le \deg f_1 - 1$. Since $\deg f_1 \ge 2$ we get that

$$\deg(P^2 + (xt^2 + a\alpha^2)(by^2 + abdx\beta^2)) \le 2(\deg f_1 - 1),$$

hence $\deg s \le \deg f_1 - 2$, and we are done just as in Lemma 8. Now suppose that $\deg f_1 = 1$, i.e. $f_1 = R(y - \frac{P}{Q})$, where $P, Q \in l(x)[t]$, $R \in l(x,t)$. Since $\partial_{f_1}(\pi) = 1$, we get that $-(xt^2 + a\alpha^2)(bP^2 + abdx\beta^2Q^2)$ is a square in $l(x)[t]$. In particular,

$$xt^2 + a\alpha^2 | bP^2 + abdx\beta^2Q^2.$$

Therefore, there are some nonzero polynomials $P_1, Q_1 \in l(x)[t]$ such that $\deg_t P_1, \deg_t Q_1 \le 1$ and $xt^2 + a\alpha^2 | b{P_1}^2 + abdx\beta^2Q_1^2$, i.e.

$$b{P_1}^2 + abdx\beta^2Q_1^2 = g(xt^2 + a\alpha^2)$$

for some $g \in l(x)$. It is easy to see that this implies $\langle b, abdx\rangle_{l(x)} \simeq \langle gx,\ ga\rangle$, a contradiction, since the discriminants on the left-hand and on the right-hand sides of the last isomorphism are different. The lemma is proved. □

Adding to the both parts of (5) the Pfisterform $\langle\langle d,\ pf\rangle\rangle$ we can represent (5) as

$$\langle\langle d,\ ax(xt^2 + a\alpha^2)f\rangle\rangle_L \simeq \langle\langle -d(xt^2 + a\alpha^2)(by^2 + abdx\beta^2), pf\rangle\rangle_L. \tag{6}$$

By Lemma 8 the left-hand side of the last isomorphism is defined over $l(t,x)$. Moreover, the right-hand side splits over $l(t,x)(\sqrt{-d(xt^2 + a\alpha^2)(by^2 + abdx\beta^2)})$. So, just as for the form π from section 2, we have that either

a) $\langle\langle d,\ ax(xt^2 + a\alpha^2)f\rangle\rangle_{l(t,x)} = 0$, or

b) $\langle\langle d,\ ax(xt^2 + a\alpha^2)f\rangle\rangle_{l(t,x)} \simeq \langle\langle -bd(xt^2 + a\alpha^2), -abx(xt^2 + a\alpha^2)\rangle\rangle_{l(t,x)}$.

Assume that case *a*) holds. We may suppose that $f \in l(x)[t]$, $p \in k(x,y)[t]$ and that f and p are squarefree as polynomials in t. Obviously, $xt^2 + a\alpha^2 | f$, for otherwise we would have

$$d = \partial_{xt^2 + a\alpha^2}(\langle\langle d,\ ax(xt^2 + a\alpha^2)f\rangle\rangle) = \partial_{xt^2+a\alpha^2}(0) = 1,$$

a contradiction. By the similar reason if $\sigma\alpha \ne \alpha$ (such σ exists, since $\beta \notin k(\alpha)$), then $xt^2 + a(\sigma\alpha)^2 \not| f$. Hence from (6) it follows that $xt^2 + a(\sigma\alpha)^2 \not| p$, if $\sigma\alpha \ne \alpha$, or equivalently $xt^2 + a\alpha^2 \not| p$.

Let $f = f_1(xt^2 + a\alpha^2)$. Then

$$\begin{gathered} 1 = \partial_{xt^2+a\alpha^2}(\langle\langle -d(xt^2 + a\alpha^2)(by^2 + abdx\beta^2),\ pf\rangle\rangle) = \\ \overline{df_1(by^2 + abdx\beta^2)p \mod (xt^2 + a\alpha^2)} \in l(t)(u)^*/l(t)(u)^{*2}. \end{gathered} \tag{7}$$

Since $\beta \notin k(\alpha)$ and $[k(\alpha,\beta) : k]$ is odd we get that $\beta^2 \notin k(\alpha)$, and so there is $\sigma \in G(l/k)$ such that $\sigma\alpha = \alpha$ and $\sigma(\beta^2) \ne \beta^2$. Applying such σ to (7) we get

$$\overline{d\sigma f_1(by^2 + abdx(\sigma\beta)^2)p \mod (xt^2 + a\alpha^2)} = 1. \tag{8}$$

Combining (7) and (8) we get that

$$\overline{f_1\sigma f_1(by^2+abdx\beta^2)(by^2+abdx(\sigma\beta)^2) \mod(xt^2+a\alpha^2)} = 1.$$

Since $\overline{f_1\sigma f_1 \mod(xt^2+a\alpha^2)} \in l(t,x,\sqrt{-ax})$ does not depend on y, and $\sigma(\beta^2)\neq\beta^2$, we come to a contradiction. This shows that case $a)$ is impossible.

Now assume that case $b)$ holds, i.e.

$$\langle\langle d,\ ax(xt^2+a\alpha^2)f\rangle\rangle_{l(t,x)} \simeq \langle\langle -bd(xt^2+a\alpha^2),\ -abx(xt^2+a\alpha^2)\rangle\rangle_{l(t,x)}.$$

Comparing residues at x on the left-hand and on the right-hand sides of the last isomorphism we obtain that either $-ab\in k^{*2}$, or $-dab\in k^{*2}$, which is not so by the hypothesis. This shows that case $b)$ is also impossible, which finishes the proof of Proposition 7. □

Corollary 10. *Let $n\geq 2$ be an integer, which is not an odd prime. Then there exists a finite 4-nonexcellent field extension of degree $2n$.*

Proof. If n is even, then $2n=2^km$, where $k\geq 2$ and m is odd. Let L/F be a 4-nonexcellent multiquadratic extension of degree 2^k (the existence of such extensions has been proven in [5]). Let further φ be a 4-dimensional form providing a corresponding counterexample. Consider the extension $L(t^{\frac{1}{m}})/F(t)$ of degree $2n$. Applying the specialization $t=0$ (the first residue map) it is easy to see that the extension $L(t)/F(t)$ is 4-nonexcellent with the form $\varphi_{F(t)}$ as a counterexample. Since the extension $L(t^{\frac{1}{m}})/L(t)$ is of odd degree, we conclude that $\varphi_{F(t)}$ is a counterexample for the extension $L(t^{\frac{1}{m}})/F(t)$ as well.

Now assume that a number n is odd, a prime $p\neq n$ divides n, and l/k is a Galois extension of degree n. Let $k\subset k_1\subset l$, where the Galois group $G(l/k_1)$ is cyclic of order p, $l=k(\beta)$ and $\alpha\in k_1\setminus k$. Then the extension l/k and the elements α and β satisfy the hypothesis of Proposition 7, which implies an existence of a 4-nonexcellent extension of degree $2n$. □

4. Nonexcellence of extensions determined by hyperelliptic curves

In the last section of the present work we give examples of 4-nonexcellent extensions L/F, where L is the function field of a hyperelliptic curve over F. In this case the extension L/F is a tower of a purely trancendental extension of degree 1 and a quadratic extension. This question is of some interest, since the function field of a conic always determines an excellent extension ([1], [3]). More precisely the following statement holds.

Proposition 11. *Let k be a field, $a,b\in k^*$ and the Pfisterform $\langle\langle a,b\rangle\rangle$ is anisotropic. Suppose $n>m\geq 1$ are positive integers. Let further $F=k((t))$, t being an indeterminate, and let X be a hyperelliptic curve over F defined by the equation $y^2=ax^{2n}+bx^{2m}-t$. Then the form $(\langle 1,-a,-b,t\rangle_{F(X)})_{an}$ is not defined over F.*

Proof. Since the hyperbolic plane $\langle 1,\ -(ax^{2n}+bx^{2m}-t)\rangle_{F(X)}$ is a subform of $\langle 1,-a,-b,t\rangle_{F(X)}$, we get that the form $\langle 1,-a,-b,t\rangle_{F(X)}$ is isotropic, hence

$$\dim\left(\langle 1,-a,-b,t\rangle_{F(X)}\right)_{an} = 2.$$

Assume for a moment that there are $d, e \in F^*$ such that

$$(\langle 1, -a, -b, t\rangle_{F(X)})_{an} \simeq \langle d, e\rangle_{F(X)},$$

or, equivalently that the form $\varphi \simeq \langle 1, -a, -b, t, -d, -e\rangle$ becomes hyperbolic over $F(X)$. Then we have

$$(\varphi_{an})_{F(x)} \simeq \langle\langle ax^{2n} + bx^{2m} - t\rangle\rangle \otimes \tau \tag{8}$$

for some form τ. Assume that $\dim \tau$ is odd. Then $\mathrm{disc}(\varphi_{an}) = ax^{2n} + bx^{2m} - t$. On the other hand,

$$\mathrm{disc}(\varphi_{an}) = -abdet \neq ax^{2n} + bx^{2m} - t \ (\mathrm{mod}\, F(x)^{*2}),$$

a contradiction. Hence $\dim \tau$ is even, and so $\dim \varphi_{an} = 4$. Applying the first residue map (in fact the specialization $x = 0$) to (8) we conclude that φ_{an} is similar to the form $\langle\langle -t, -c\rangle\rangle$ for some $c \in F^*$. Since $\langle\langle -t, t\rangle\rangle = 0$, we can assume that $c \in k^*$ and $-c \notin k^{*2}$. Furthermore, since $\varphi_{F(X)} = 0$, we have

$$ax^{2n} + bx^{2m} - t \in D(\langle -c, -t, -ct\rangle_{F(x)}).$$

Let r be a minimal nonnegative integer such that

$$t^{2r}(ax^{2n} + bx^{2m} - t) = -(cp_1^2 + tp_2^2 + ctp_3^2)$$

for some $p_1, p_2, p_3 \in k[[t]][x]$ (such r exists by the Cassels-Pfister theorem, (see, for example [4])). Suppose $r \geq 1$. Then $p_1 = tq_1$, $q_1 \in k[[t]][x]$, and so

$$t^{2r-1}(ax^{2n} + bx^{2m} - t) = -(p_2^2 + cp_3^2 + ctq_1^2).$$

Hence $\overline{p_2}^2 + c\overline{p_3}^2 = 0$, where $\overline{p_2}, \overline{p_3} \in k[x]$ are the reductions of p_2, p_3 modulo the ideal $(t) \subset k[[t]][x]$. Since $-c \notin k^{*2}$, we have $\overline{p_2} = \overline{p_3} = 0$ and so

$$p_2 = tq_2,\ p_3 = tq_3,\ q_2,\ q_3 \in k[[t]][x].$$

Therefore,

$$t^{2r-2}(ax^{2n} + bx^{2m} - t) = -(cq_1^2 + tq_2^2 + ctq_3^2),$$

a contradiction to minimality of r. Thus, we conclude that $r = 0$, hence

$$ax^{2n} + bx^{2m} - t = -(cp_1^2 + tp_2^2 + ctp_3^2),$$

which implies $ax^{2n} + bx^{2m} = -c\overline{p_1^2}$. On the other hand, obviously, $\frac{-(ax^{2n}+bx^{2m})}{c} \notin k[x]^2$, a contradiction, which proves Proposition 9. □

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