ON SOME ELEMENTS OF THE BRAUER GROUP OF A CONIC

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Abstract. Let K be a field of characteristic not 2, X a nonsingular projective conic over K, n is a positive integer and $a_i \in K^*$ for $1 \le i \le n$. We investigate elements of ${}_2Br(X)$, which are sums of quaternion algebras (a_i, f_i) for some $f_i \in K(X)^*$. An application to a construction of indecomposable division algebras of exponent 2 is given.

The main purpose of this paper is to construct indecomposable central division algebras of an arbitrary 2-primary index more than 4 over fields with finite u-invariant (recall that the u-invariant of the field K is the maximal dimension of anisotropic quadratic forms over K). To do this we begin with a result concerning the Brauer group of a conic. Let K be a field of characteristic different from 2, and let X be a nonsingular projective conic over K. Recall that

$$ {}_2\mathrm{Br}(X) = \ker({}_2\mathrm{Br}\, K(X) \xrightarrow{\partial} \coprod_{x \in X} K(x)^*/K(x)^{*2}), $$

where $K(X)$ is the function field of X, x runs over all closed points of X, and $K(x)$ is the residue field at x. Moreover,

$$ \partial_x(f,g) = (-1)^{v_x(f)v_x(g)} \frac{f^{v_x(g)}}{g^{v_x(f)}} \in k(x)^*/k(x)^{*2}, $$

where v_x is the discrete valuation related to the point x, and the symbol (f,g) denotes a quaternion algebra. Notice also that

$$ \mathrm{res}_{K(X)/K}\ {}_2\mathrm{Br}\, K \subset {}_2\mathrm{Br}(X). $$

From now on we fix a field K of characteristic not 2, and elements $a_1, \dots a_n \in K^*$ such that $\overline{a_1}, \dots \overline{a_n} \in K^*/K^{*2}$ are linearly independent over $\mathbb{Z}/2\mathbb{Z}$. For any set $I \subset \{1, \dots, n\}$, put $a_I = \prod\limits_{i \in I} a_i$ (if $I = \emptyset$, then $a_I = 1$). For any polynomial $f \in K[t]$ denote by $l(f)$ the leading coefficient of f.

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Proposition 1. *Let X be a nonsingular projective conic over K, Q the corresponding quaternion algebra, $\alpha \in {}_2\mathrm{Br}(X)$. Suppose that the following conditions hold:*

1) $\alpha = \sum_{i=1}^{n}(a_i, f_i)$ *for some* $f_i \in K(X)^*$.

2) *There exist at most two sets* $I \subset \{1, \dots, n\}$ *such that* $Q_{K(\sqrt{a_I})} = 0$.

Then there exist $b_1, \dots b_n \in K^*$ *such that* $\alpha = \mathrm{res}_{K(X)/K} \sum_{i=1}^{n}(a_i, b_i)$.

Proof. If $Q = 0$, then $K(X) = K(t)$ is the rational function field. The exact sequence

$$0 \to {}_2\mathrm{Br}\, K \xrightarrow{\mathrm{res}} {}_2\mathrm{Br}\, K(t) \xrightarrow{\partial} \coprod_{x \in \mathbb{P}^1_K} K(x)^*/K(x)^{*2}$$

shows that ${}_2\mathrm{Br}(X) = \mathrm{res}_{K(t)/K}\, {}_2\mathrm{Br}\, K$. There is a splitting homomorphism

$$c:\ {}_2\mathrm{Br}\, K(t) \to {}_2\mathrm{Br}\, K$$

such that $c((f, g)) = (l(f), l(g))$ for any $f, g \in K[t]$. Applying the map c to the equality

$$\alpha = \sum_{i=1}^{n}(a_i, f_i),$$

we get what we need.

So assume that $Q = (a, b) \neq 0$. Let $z^2 = ax^2 + by^2$ be the equation determining the conic X, and let v_0 be the point of X determined by $y = 0$. Obviously, we have $K(v_0) = K(\sqrt{a})$. In particular, $\deg(v_0) = 2$. Assume first that $K(v_0) \neq K(\sqrt{a_I})$ for each $I \subset \{1, \dots, n\}$, i.e. $aa_I \notin K^{*2}$.

For any $f \in K(X)$ denote by $V(f)$ the finite set of all the points $v \neq v_0$ of X such that $v(f)$ is odd (we identify a point of X with the corresponding discrete K-valuation). Let $V = \bigcup_{i=1}^{n} V(f_i)$. For any $I \subset \{1, \dots, n\}$ put $V_I = \bigcap_{i \in I} V(f_i) \setminus \bigcup_{i \notin I} V(f_i)$. Clearly, $V = \bigcup_I V_I$, and, moreover, this union is disjoint and $V(f_i) = \bigcup_{i \in I} V_I$. For each $I \subset \{1, \dots, n\}$ consider the divisor

$$\mathfrak{a}_I = \sum_{v \in V_I} v - \frac{1}{2}\Big(\sum_{v \in V_I} \deg v\Big)v_0.$$

Obviously, $\deg \mathfrak{a}_I = 0$. Since any divisor on X of degree 0 is principal, we get that $\mathfrak{a}_I = \mathrm{div}(g_I) = \sum_{v \in X} v(g_I)v$ for some $g_I \in K(X)^*$. Notice that

$$v(g_I) = \begin{cases} 1, & \text{if } v \in V_I \\ 0, & \text{if } v \notin V_I, v \neq v_0 \end{cases},$$

and, therefore, $v(f_i(\prod_{i \in I} g_I)^{-1})$ is even for any i and $v \neq v_0$. Moreover, since $\deg(\mathrm{div}(f_i(\prod_{i \in I} g_I)^{-1})) = 0$, $\deg v_0 = 2$, and $2 | \deg v$ for any $v \in X$, we conclude that $v_0(f_i(\prod_{i \in I} g_I)^{-1})$ is also even, i.e. $v(f_i(\prod_{i \in I} g_I)^{-1})$ is even for any i and $v \in X$.

Therefore, we have $\operatorname{div}(f_i(\prod_{i\in I} g_I)^{-1}) = 2\operatorname{div}(h_i)$, i.e. $f_i(\prod_{i\in I} g_I)^{-1} = u_i h_i^2$ for some $u_i \in K^*$ and $h_i \in K(X)^*$. This implies

$$\alpha_{K(X)} = \sum_{i=1}^{n}(a_i, f_i) = \sum_{i=1}^{n}(a_i,\ u_i \prod_{i\in I} g_I h_i^2) = \sum_{i=1}^{n}(a_i, u_i) + \sum_{I}(a_I, g_I).$$

Now, let I_0 be a subset of $\{1,\dots,n\}$, $v \in V_{I_0}$. Then we have

$$1 = \partial_v(\alpha_{K(X)}) = \partial_v(\sum_{I}(a_I, g_I)) = \partial(a_{I_0}, g_{I_0}),$$

since $v(g_I) = 0$ if $I \neq I_0$.

On the other hand, if $v \notin V_{I_0}$ and $v \neq v_0$, then, obviously, $\partial_v(a_{I_0}, g_{I_0}) = 1$, since $v(g_{I_0}) = 0$. Thus, we conclude that $\partial_v(a_{I_0}, g_{I_0}) = 1$ for any $v \neq v_0$.

Our next step is to show that $\partial_{v_0}(a_{I_0}, g_{I_0}) = 1$ as well. Assume the converse. Then $v_0(g_{I_0})$ is odd. Since $\deg\operatorname{div}(g_{I_0}) = 0$, and $\deg v_0 = 2$, there exists $v \neq v_0$ such that $v(g_{I_0})$ is odd and $4 \not| \deg v$. We have

$$1 = \partial_v(a_{I_0}, g_{I_0}) = a_{I_0}(\operatorname{mod} K(v)^{*2}).$$

Hence we obtain that $K(\sqrt{a_{I_0}}) \subset K(v)$. Furthermore, since $\deg v$ is not divided by 4, we get that the degree $[K(v) : K(\sqrt{a_{I_0}})]$ is odd. Since $Q_{K(v)} = 0$, we conclude that $Q_{K(\sqrt{a_{I_0}})} = 0$. Therefore, by condition 2) of the proposition there are at most two sets I_0, which can satisfy the unequality $\partial_{v_0}(a_{I_0}, g_{I_0}) \neq 1$. Let, for instance, $\partial_{v_0}(a_{I_{01}}, g_{I_{01}}) \neq 1$ and $\partial_{v_0}(a_{I_{02}}, g_{I_{02}}) \neq 1$. So we have

$$\prod_{i\in I_{01}} a_i \prod_{i\in I_{02}} a_i = \partial_{v_0}((a_{I_{01}}, g_{I_{01}}) + (a_{I_{02}}, g_{I_{02}})) = \partial_{v_0}(\sum_{I\neq I_{01}, I_{02}}(a_I, g_I)) = 1.$$

Thus, $a_{I_{01}} = a_{I_{02}}(\operatorname{mod} K^{*2})$ (recall that $a \neq a_I(\operatorname{mod} K^{*2})$ for any I), which is impossible in view of linear independency of $\overline{a_1}, \dots, \overline{a_n} \in K^*/K^{*2}$.

The case where there is an only I_0 such that $\partial_{v_0}(a_{I_0}, g_{I_0}) \neq 1$ is treated similarly.

Therefore, we conclude that $\partial_v(a_I, g_I) = 1$ for any $I \subset \{1,\dots,n\}$ and $v \in X$.

Lemma 2. *Assume that* $\alpha \in {}_2\mathrm{Br}(X) \setminus \operatorname{res}_{K(X)/K}({}_2\mathrm{Br}\, K)$. *Then* $\operatorname{ind}(\alpha) \geq 4$.

Proof. The Merkurjev theorem on the norm residue homomorphism of degree 2 [4] claims that for any field F the natural map $K_2(F)/2K_2(F) \to {}_2\mathrm{Br}\, F$ is an isomorphism. In view of this it follows from [10] that, if ${}_2\mathrm{Br}(X) \neq \operatorname{res}_{K(X)/K}({}_2\mathrm{Br}\, K)$, then

$${}_2\mathrm{Br}(X)/\operatorname{res}_{K(X)/K}({}_2\mathrm{Br}\, K) \simeq \mathbb{Z}/2\mathbb{Z},\ Q = 2D \text{ for some } D \in \mathrm{Br}\, K \text{ and } D_{K(X)}$$

is the nonzero element in the factorgroup ${}_2\mathrm{Br}(X)/\operatorname{res}_{K(X)/K}({}_2\mathrm{Br}\, K)$.

Therefore, we get that $\alpha = (D + \beta)_{K(X)}$, where $\beta \in {}_2\mathrm{Br}\, K$. Obviously, we may assume that α is a division algebra. Let D_1 be the central division algebra over K similar to $D + \beta$. Since $\exp D_1 = 4$ we have $\operatorname{ind} D_1 \geq 4$. Suppose that $\operatorname{ind}(\alpha) = 2$.

Then by the Merkurjev index reduction theorem [5] $D_1 \simeq Q \otimes_K Q'$ for some central division K-algebra Q'. Obviously, ${Q'}_{K(X)} = {D_1}_{K(X)} = \alpha$. Since $\exp D_1 = 4$ and $\exp Q = 2$, we get that $\exp Q' = 4$. Applying the index reduction theorem again we obtain that $Q' \simeq Q \otimes_K Q''$ for some central division K-algebra Q'', which implies

$$D_1 \simeq Q'' \otimes_K Q \otimes_K Q \simeq Q'' \otimes_K M_4(K),$$

a contradiction, since D_1 is a division algebra. The lemma is proved. □

Now we are able to finish the proof of Proposition 1 in the case where $aa_I \notin {K^*}^2$ for each $I \subset \{1, \dots, n\}$. In view of Lemma 2 we get that $(a_I, g_I) = \operatorname{res}_{K(X)/K} \beta_I$ for some $\beta_I \in {}_2\operatorname{Br} K$. Since $(\beta_I)_{K(\sqrt{a_I})(X)} = 0$, we have $(\beta_I)_{K(\sqrt{a_I})}$ is either zero or $Q_{K(\sqrt{a_I})}$, which implies that β_I equals either (a_I, c_I) or $(a_I, c_I) + Q$ for some $c_I \in K^*$. Therefore,

$$\alpha = \operatorname{res}_{K(X)/K}(\sum_I (a_I, c_I)) = \operatorname{res}_{K(X)/K}(\sum_{i=1}^n (a_i, \prod_{i \in I} c_I)),$$

so we can put $b_i = \prod_{i \in I} c_I$.

It remains to consider the case where $aa_I \in {K^*}^2$ for some $I \in \{1, \dots, n\}$. Passing to the rational function field $K(t)$ we have

$$Q = (a, b) = (at^2 + b, -ab),$$

and, obviously $(at^2 + b)a_I \notin {K(t)^*}^2$. Hence $\alpha_{K(X)(t)} = \sum_{i=1}^n (a_i, p_i)$ for some polynomials $p_i \in K[t]$. This implies that $\alpha = \sum_{i=1}^n (a_i, l(p_i))$. The proposition is proved. □

Corollary 3. *In the notation of Proposition* 1 *assume that* $\alpha = \operatorname{res}_{K(X)/K} \gamma$ *for some* $\gamma \in \operatorname{Br} K$. *Then* γ *equals either* $\sum_{i=1}^n (a_i, b_i)$ *or* $Q + \sum_{i=1}^n (a_i, b_i)$ *for some* $b_i \in K^*$.

Proof. Since $\operatorname{res}_{K(X)/K}(\gamma - \sum_{i=1}^n (a_i, b_i)) = 0$, and $\ker(\operatorname{Br} K \to \operatorname{Br} K(X)) = \langle Q \rangle$ is the cyclic group generated by Q, the corollary follows. □

We can not drop condition 2) in Proposition 1 as the following two examples show.

Example A.

Let K be a field, $a_1, a_2, a_3 \in K^*$, $D \in {}_2\operatorname{Br} K$, $D_{K(\sqrt{a_1}, \sqrt{a_2}, \sqrt{a_3})} = 0$, and D is not a sum of 3 quaternion algebras (the existence of such D has been established in [6] and [1]). Hence by [3] we have

$$D_{K(\sqrt{a_3})} = (a_1, b_1 + c_1\sqrt{a_3}) + (a_2, b_2 + c_2\sqrt{a_3}) \tag{1}$$

for some $b_1, b_2, c_1, c_2 \in K$, and, moreover,

$$(a_1, b_1^2 - a_3c_1^2) + (a_2, b_2^2 - a_3c_2^2) = N_{K(\sqrt{a_3})/K}D = 0,$$

i.e. $(a_1, b_1^2 - a_3c_1^2) = (a_2, b_2^2 - a_3c_2^2)$. Put $Q = (a_1, b_1^2 - a_3c_1^2)$, and let X be the conic corresponding to Q. Then we get

$$N_{K(X)(\sqrt{a_3})/K(X)}(a_1, b_1 + c_1\sqrt{a_3}) = (a_1, b_1^2 - a_3c_1^2)_{K(X)} = 0,$$

$$N_{K(X)(\sqrt{a_3})/K(X)}(a_2, b_2 + c_2\sqrt{a_3}) = (a_2, b_2^2 - a_3c_2^2)_{K(X)} = 0.$$

Hence by [3]

(2) $$(a_1, b_1 + c_1\sqrt{a_3})_{K(X)(\sqrt{a_3})} = (a_1, f_1)_{K(X)(\sqrt{a_3})},$$

(3) $$(a_2, b_2 + c_2\sqrt{a_3})_{K(X)(\sqrt{a_3})} = (a_2, f_2)_{K(X)(\sqrt{a_3})}$$

for some $f_1, f_2 \in K(X)^*$. Combining (1), (2) and (3) we obtain that

$$(D - (a_1, f_1) - (a_2, f_2))_{K(X)(\sqrt{a_3})} = 0,$$

i.e.

$$D_{K(X)} = (a_1, f_1) + (a_2, f_2) + (a_3, f_3)$$

for some $f_1, f_2, f_3 \in K(X)^*$. On the other hand, if Corollary 3 of Proposition 1 were valid, then we would have

$$D = (a_1, e_1) + (a_2, e_2) + (a_3, e_3)$$

for some $e_1, e_2, e_3 \in K^*$, which would contradict to the hypothesis on D.

Notice that in this example

$$Q_{K(\sqrt{a_1})} = Q_{K(\sqrt{a_1a_3})} = Q_{K(\sqrt{a_2})} = Q_{K(\sqrt{a_2a_3})} = 0,$$

so there are at least four elements a_I such that $Q_{K(\sqrt{a_I})} = 0$.

Example B.

Let K be a field, $\sqrt{-1} \in K$, $Q = (a_1, a_2)$ a nontrivial quaternion algebra, X the corresponding conic. Put $\alpha = (a_1, y) + (a_2, x) \in {}_2\mathrm{Br}\, K(X)$, where $a_1x^2 + a_2y^2 = 1$ is the equation of an affine part of the conic. It is easy to see that $\alpha \in {}_2\mathrm{Br}(X)$ and $N_{K(X)/K(x)}\alpha = Q$. In particular, $\alpha \notin \mathrm{res}_{K(X)/K}({}_2\mathrm{Br}\, K)$. Notice that since $\sqrt{-1} \in K$, we have

$$Q = (a_1, a_2) = (a_1 + a_2, -a_1a_2) = (a_1 + a_2, a_1a_2),$$

hence

$$Q_{K(\sqrt{a_1})} = Q_{K(\sqrt{a_2})} = Q_{K(\sqrt{a_1a_2})} = 0,$$

so there are three elements a_I such that $Q_{K(\sqrt{a_I})} = 0$.

We keep the conditions of Proposition 1. Let L/K be any field extension. Consider the following abelian groups:

$$A(L/K) = \{\gamma \in {}_2\mathrm{Br}\, K \text{ such that } \mathrm{res}_{L/K}\, \gamma = \sum_{i=1}^{n} (a_i, f_i) \text{ for some } f_i \in L^*\},$$

$$B(L/K) = \{ \sum_{i=1}^{n} (a_i, b_i) \in {}_2\mathrm{Br}\, K \text{ where } b_i \in K^*\},$$

$$H(L/K) = A(L/K)/B(L/K).$$

Corollary 3 claims that if $L = K(X)$, where X is the nonsingular projective conic corresponding to the quaternion algebra Q, then $H(L/K) = \langle Q \rangle$ (of course, it may happen that the image of Q in $H(L/K)$ is zero). Now we will consider the case of a quadric of dimension more than 1. We will need the following

Lemma 4. 1) *If $F/L/K$ is a tower of field extensions, and $H(L/K) = H(F/L) = 0$, then $H(F/K) = 0$ as well. If $H(F/K) = 0$, then $H(L/K) = 0$.*

2) *If L/K is a purely transcendental extension, then $H(L/K) = 0$.*

3) *If $H(F/K) = 0$, and the extension L/K is such that LF/F is a purely transcendental extension, then $H(L/K) = 0$.*

Proof. 1) This follows immediately from the definition of the groups $H(F/K)$, $H(L/K)$ and $H(F/L)$.

2) In view of 1) it suffices to treat the case where $L = K(t)$. Assume that $u \in {}_2\mathrm{Br}\, K$ is such that $u_{K(t)} = \sum_{i=1}^{n} (a_i, f_i)$ for some $f_i \in K[t]$. Then $u = \sum_{i=1}^{n} (a_i, l(f_i))$, which proves the assertion.

3) This follows at once from 1) and 2). □

Proposition 5. *Let $X = X_\varphi$ be the projective quadric over K corresponding to the anisotropic quadratic form φ. Put $L = K(X)$. Then*

1) *If $\dim \varphi = 4$ and $\mathrm{disc}(\varphi) a_I \in K^{*2}$ for some $I \neq \emptyset$, then $H(L/K) = \langle (a,b) \rangle$, where (a,b) is an arbitrary quaternion K-algebra such that $C_0(\varphi) \simeq (a,b)_{K(\sqrt{\mathrm{disc}\,\varphi})}$.*

2) *If $\dim \varphi = 4$ and $\mathrm{disc}(\varphi) a_I \notin K^{*2}$ for every I, then $H(L/K) = 0$.*

3) *If $\dim \varphi \geq 5$, then $H(L/K) = 0$.*

Proof. 1), 2). Let $d = \mathrm{disc}(\varphi)$, and let $z^2 - ax^2 - by^2 + abd = 0$ be the equation of an affine part of X. We have

$$K(X) = K(x, y, \sqrt{ax^2 + by^2 - abd}) = K(x)(C),$$

where C is the conic over $K(x)$ corresponding to the quaternion $K(x)$-algebra $(b, ax^2 - abd)$.

Let $u \in A(L/K)$. By Corollary 3 we have

$$u_{K(x)} = \delta + \sum_{i=1}^{n} (a_i, f_i), \qquad (4)$$

where δ is either 0 or $(b, ax^2 - abd)$, and $f_i \in K[x]$. Furthermore,

$$u = c(u_{K(x)}) = c(\delta) + \sum_{i=1}^{n} (a_i, l(f_i)),$$

where $c(\delta)$ is either zero, or (a, b). This proves that $H(L/K) \subset \langle (a, b) \rangle$.

Assume first that $da_I \in K^{*2}$ for some I. Then, since

$$(ab)^2 d = a(by)^2 + b(ax)^2 - abz^2$$

and

$$(a, b)_{K(X)} = (a(by)^2 + b(ax)^2, -ab) = (a(by)^2 + b(ax)^2 - abz^2,\ ab(a(by)^2 + b(ax)^2),$$

we get that $(a, b) \in H(L/K)$.

Now assume that $da_I \notin K^{*2}$ for every I. Compare the residues at $(x^2 - bd)$ on the left-hand and the right-hand sides of (4). We have

$$\partial_{x^2-bd}(u_{K(x)}) = 0, \quad \partial_{x^2-bd}\Big(\sum_{i=1}^{n} (a_i, f_i)\Big) = a_I \text{ for some } I. \tag{5}$$

Suppose that $\delta \neq 0$, i.e. $\delta = (b, ax^2 - abd)$. The equality $\partial_{x^2-bd}(\delta) = b$ and (5) imply $ba_I \in K(\sqrt{bd})^{*2}$. Since $da_I \notin K^{*2}$, we obtain that $ba_I \in K^{*2}$. Thus, we have proved part 2) if $ba_I \notin K^{*2}$ for any I. On the other hand,

$$\varphi_{K(t)} \simeq \langle 1, -ab(at^2 + b), -(at^2 + b), abd \rangle,$$

and $(at^2 + b)a_I \notin K(t)^{*2}$. The above argument show that $H(LK(t)/K(t)) = 0$, and so by Lemma 4 we have $H(L/K) = 0$.

3) Obviously, we can choose a 4-dimensional subform ψ of the form $\varphi_{K(t)}$ such that $\operatorname{disc}(\psi)a_I \notin K(t)^{*2}$. Since the form $\varphi_{K(t)(X_\psi)}$ is isotropic, the extension $K(t)(X_\psi)(X_\varphi)/K(t)(X_\psi)$ is purely transcendental. Applying part 2) and Lemma 4 we get $H(K(X_\varphi)/K) = 0$. □

Now we are ready to strentghen the main results of [7] and [8]. Recall the corresponding notation.

Let F be an arbitrary field, $n \geq 2$, $a, b_1, \dots, b_n \in F^*$, $D \in {}_2\mathrm{Br}\, F$. We call the triple $(F, D, \{a, b_1, \dots, b_n\})$ admissible, if the following conditions hold:

a) The elements $\overline{a}, \overline{b_1}, \dots, \overline{b_n} \in F^*/F^{*2}$ are linearly independent.

b) $\operatorname{ind} D = 2$.

c) $D_{F(\sqrt{a}, \sqrt{b_1}, \dots, \sqrt{b_n})} = 0$.

d) For any tower $F \subset F_1 \subset F_2 \subset L = F(\sqrt{a}, \sqrt{b}_1, \dots, \sqrt{b}_n)$ such that

$$\sqrt{a} \notin F_2^*, \quad [L : F_1] = 8, \quad [F_2 : F_1] = 4$$

we have

$$D_{F_1} \notin {}_2\mathrm{Br}(F_1(\sqrt{a})/F_1) + {}_2\mathrm{Br}(F_2/F_1).$$

e) The field F has no proper extensions of odd degree.

Corollary 6. *Suppose that the triple $(F, D, \{a, b_1, \dots, b_n\})$ is admissible, φ is an anisotropic form over F, and $\dim \varphi \geq 5$. Put $X = X_\varphi$, and let $F(X)_{odd}$ be a maximal odd degree extension of $F(X)$. Then the triple $(F(X)_{odd}, D_{F(X)_{odd}}, \{a, b_1, \dots, b_n\})$ is admissible as well.*

Proof. Obviously, all the above conditions except maybe $d)$ hold. Let us check $d)$. First we will do this for the field $F(X)$. So, suppose that there is a tower of fields $F(X) \subset F_1(X) \subset F_2(X) \subset L(X) = F(\sqrt{a}, \sqrt{b}_1, \dots, \sqrt{b}_n)(X)$ such that

$$\sqrt{a} \notin F_2^*, \quad [L : F_1] = 8, \quad [F_2 : F_1] = 4$$

and

$$D_{F_1(X)} \in {}_2\mathrm{Br}(F_1(X)(\sqrt{a})/F_1(X)) + {}_2\mathrm{Br}(F_2(X)/F_1(X)).$$

Notice that $F_2 = F_1(\sqrt{d_1}, \sqrt{d_2})$, for some $d_1, d_2 \in F^*$, hence by [3] we have

$$D_{F_1(X)} = (a, f) + (d_1, f_1) + (d_2, f_2)$$

for some $f, f_1, f_2 \in F_1(X)^*$. By part 3) of Proposition 5 we conclude that

$$D_{F_1} = (a, u) + (d_1, u_1) + (d_2, u_2)$$

for some $u, u_1, u_2 \in F_1^*$, which contradicts to condition $d)$ for the admissible triple $(F, D, \{a, b_1, \dots, b_n\})$.

On the other hand, it has been shown in [8] that if the triple $(K, D, \{a, b_1, \dots, b_n\})$ is admissible, then the triple $K_{odd}, D_{K_{odd}}, \{a, b_1, \dots, b_n\})$ is admissible as well (see also Lemma 9 below). This finishes the proof of Corollary 6. □

We recall now some additional results from the papers [7] and [8].

If the triple $(F, D, \{a, b_1, \dots, b_n\})$ is admissible, and $E = F((t_0))((t_1))\dots((t_n))$ is the Laurent series field in variables $t_0, t_1, \dots, t_n$, then the division algebra A similar to the algebra

$$D \otimes_E (a, t_0) \otimes_E (b_1, t_1) \otimes \cdots \otimes_E (b_n, t_n)$$

does not decompose into a tensor product of two nontrivial central simple algebras over any odd degree extension of E, and $\operatorname{ind} A = 2^{n+1}$ [8]. Moreover, the multiquadratic extension $F(\sqrt{b_1}, \dots, \sqrt{b_n})/F$ is not 4-excellent, and if $D = (u, v)$, the form $\langle uv, -u, -v, a\rangle$ provides a corresponding counterexample [7]. Finally, if k is a field and elements $\overline{a}, \overline{b_1}, \dots, \overline{b_n} \in k^*/k^{*2}$ are linearly independent over $\mathbb{Z}/2\mathbb{Z}$, then there exists an extension F/k and a quaternion algebra D over F such that the triple $(F, D, \{a, b_1, \dots, b_n\})$ is admissible [8].

On the other hand, Corollary 6 and the argument in Lemma 9 below imply that if the triple $(F, D, \{a, b_1, \dots, b_n\})$ is admissible, then there exists an extension K/F such that the triple $(K, D_K, \{a, b_1, \dots, b_n\})$ is admissible as well, $u(K) = 4$, and K has no proper odd degree extension.

Summarizing all these results we obtain the following

Theorem 7. *Let k be a field, $n \geq 2$. Assume that elements $\overline{a}, \overline{b_1}, \dots, \overline{b_n} \in k^*/k^{*2}$ are linearly independent over $\mathbb{Z}/2\mathbb{Z}$. Then there exists an extension K/k such that*

1) $u(K) = 4$.

2) K has no proper extension of odd degree.

3) $K(\sqrt{b_1}, \dots, \sqrt{b_n})/K$ is not 4-excellent, and the counterexample is provided by some 4-dimensional form of discriminant a.

4) There exists a central division algebra A over $E = K((t_0))((t_1))\dots((t_n))$ such that $\operatorname{ind} A = 2^{n+1}$, and A_L does not decompose into a tensor product of two nontrivial central simple algebras over any odd degree extension L/E (notice that $u(E) = 2^{n+3}$ and $\operatorname{cd}_2(E) = n+3$).

Next we compute the group H for the extension determined by the function field of the Severi-Brauer variety related to a central simple algebra of exponent 2.

Corollary 8. *Let $D \in {}_2\mathrm{Br}\, K$, and either $\operatorname{ind} D \geq 4$, or $\operatorname{ind} D = 2$ and $D_{K(\sqrt{a_I})} \neq 0$ for every I. Denote by $K(D)$ the function field of the Severi-Brauer variety $SB(D)$ corresponding to D. Then $H(K(D)/K) = \langle D \rangle$.*

Proof. Let $\operatorname{ind} D = 2^n$, where $n \geq 2$. Let D_1 be the division algebra Brauer-equivalent to D, $F = SB(D_1)$, and $L = SB(D)$. Then, since $SB(D_F) \simeq \mathbb{P}_F^{\deg D - 1}$, the extension LF/F is purely transcendental. Hence by Lemma 4 it suffices to prove Corollary 8 in the case where D is a division algebra.

First consider the case where D is a tensor product of quaternion algebras. We will go on by induction on n. Let $\varphi \in I^2(K)$ be a quadratic form corresponding to D under the isomorphism $I^2(K)/I^3(K) \simeq {}_2\mathrm{Br}\, K$. It is easy to see that we can choose φ such that $\dim \varphi = 2n+2$, and, moreover, no form of dimension $< 2n+2$ corresponds to D under this isomorphism. Let $X = X_\varphi$ be the projective quadric determined by the form φ. Consider the following diagram:

$$\begin{array}{ccc} K & \longrightarrow & K(D) \\ \Big\downarrow & & \Big\downarrow \\ K(X) & \longrightarrow & K(X)(D) \end{array},$$

where all the maps are natural field embeddings.

If $n = 2$, then $\operatorname{ind} D_{K(X)} = 2$, and $D_{K(X)(\sqrt{a_I})} \neq 0$ for any I, since $D_{K(\sqrt{a_I})} \neq 0$, and $\ker({}_2\mathrm{Br}\, F \to {}_2\mathrm{Br}\, F(Y)) = 0$ for any field F and any quadric Y over F of dimension > 2.

If $n \geq 3$, then $4 \leq \operatorname{ind} D_{K(X)} < \operatorname{ind} D$. So in both cases ($n = 2$ or $n \geq 3$) we can apply the induction hypothesis to the extension $K(X)(D)/K(X)$. Let $u \in H(K(D)/K)$. By the induction hypothesis

$$u_{K(X)} \in H(K(X)(D)/K(X)) \subset \langle D_{K(X)} \rangle.$$

We conclude that either $u \in H(K(X)/K)$, or $u + D \in H(K(X)/K)$. Since by Proposition 5 we have $H(K(X)/K) = 0$, this completes the proof in the case where D is a tensor product of quaternion algebras.

Thus, to prove the proposition in the general case it suffices to construct an extension L/K such that D_L is a tensor product of quaternion algebras, $\operatorname{ind} D_L \geq 4$, and $H(L/K) = 0$. We need the following

Lemma 9.
1) *If L/K is a finite odd degree extension, then $H(L/K)=0$.*
2) *If π is a 3-fold Pfister form over K, then $H(K(X_\pi)/K)=0$.*
3) *Let F be a field, $D \in {}_2\mathrm{Br}\, F$ a division algebra. Assume that F has no proper extension of odd degree, and $I^3(F)=0$. Then D is a tensor product of quaternion algebras.*

Proof.
1) Assume that $u \in {}_2\mathrm{Br}\, K$ and $u_L = \sum_{i=1}^{n}(a_i, b_i)$, where $b_i \in L^*$. Then

$$u = N_{L/K}(u_L) = \sum_{i=1}^{n}(a_i, N_{L/K}b_i),$$

which proves 1).
2) This is a particular case of Proposition 5, since $\dim \pi = 8$.
3) This part is proved in [9] and [2]. □

We return to the proof of Corollary 8 in the general case. For an arbitrary field F denote by $\widehat{F}$ a maximal odd degree extension of F, and by $\widetilde{F}$ the composite of all extensions $F(X_\pi)$, where π runs over all anisotropic 3-fold Pfister forms over F. Consider the following infinite tower of fields $K = K_1 \subset K_2 \subset \dots$, where

$$K_{i+1} = \begin{cases} \widehat{K_i} & \text{if } i \text{ is odd}, \\ \widetilde{K_i} & \text{if } i \text{ is even} \end{cases}.$$

Set $L = \bigcup_{i=1}^{\infty} K_i$. It is easy to see that L has no proper odd degree extensions, and $I^3(L)=0$. Applying Lemma 9 we get that $H(L/K)=0$ and D_L is a tensor product of quaternion algebras. Moreover, by the index reduction formula for central simple algebras [5] we have $\mathrm{ind}\, D_L = \mathrm{ind}\, D$. These properties of the field L prove Corollary 8 in the general case. □

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