CROSSED PRODUCT CONDITIONS FOR CENTRAL SIMPLE ALGEBRAS IN TERMS OF IRREDUCIBLE SUBGROUPS

T. KESHAVARZIPOUR & M. MAHDAVI-HEZAVEHI

ABSTRACT. Let $M_m(D)$ be a finite dimensional F-central simple algebra. It is shown that $M_m(D)$ is a crossed product over a maximal subfield if and only if $GL_m(D)$ has an irreducible subgroup G containing a normal abelian subgroup A such that $C_G(A) = A$ and $F[A]$ contains no zero divisor. Various other crossed product conditions on subgroups of D^* are also investigated. In particular, it is shown that if D^* contains either an irreducible finite subgroup or an irreducible soluble-by-finite subgroup that contains no element of order dividing $deg(D)^2$, then D is a crossed product over a maximal subfield.

1. INTRODUCTION

Let D be an F-central division algebra of index n. Assume that the algebra $M_m(D)$ is generated by a subgroup G of $GL_m(D)$ over a subring S which is normalized by G (written $M_m(D) = S[G]$). For $H = G \cap S \trianglelefteq G$, we say that $M_m(D)$ is a *crossed product* of S by G/H if $M_m(D) = \oplus_{t \in T} tS$, for some transversal T of H in G, and denote it by $M_m(D) = S * G/H$. The classical crossed product simple algebra corresponds to the case where S is a maximal subfield of $M_m(D)$ which is Galois over F. Also, we call a subgroup G of $GL_m(D)$ *irreducible* if $F[G] = M_m(D)$. In [4] and [10], the authors ask whether a division algebra generated by a soluble-by-finite irreducible subgroup G is necessarily a crossed product over a maximal subfield. For some special cases where D is of a prime power degree and G is soluble or finite, the answer to the above mentioned question is shown to be positive. In [8], this problem for arbitrary degree is investigated, and it is proved that when G is irreducible soluble-by-finite, then D is a crossed product over a (not necessarily maximal) subfield K. But for the case where G is torsion free, or metabelian, or the degree of D has the property

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that every finite group of order $deg(D)^2$ is nilpotent, then K may be shown to be a maximal subfield. In this paper, we present a necessary and sufficient condition for which a central simple algebra is a crossed product over a maximal subfield in terms of its irreducible subgroups that contain a self-centralizing normal abelian subgroup. Various other crossed product conditions on subgroups of D^* are also investigated. To be more precise, it is proved that D is a crossed product over a maximal subfield if D is spanned by an irreducible subgroup G such that G is either finite or soluble-by-finite containing no element of order dividing $deg(D)^2$, or locally nilpotent. Other special cases such as G being abelian-by-supersoluble or metabelian are also reviewed and it is shown that in these cases the irreducible subgroups involved all contain a self-centralizing normal abelian subgroup.

2. Notations and Conventions

Throughout, D is an F-central division algebra of index n and G is a subgroup of $GL_m(D)$, the group of units of $M_m(D)$. The F-linear hull of G, i.e., the F-subalgebra generated by elements of G over F in $M_m(D)$, is denoted by $F[G]$. Given a subgroup H of G, $N_G(H)$ means the normalizer of H in G and $C_G(H)$ the centralizer of H in G. By $Z(G)$ or $Z(M_m(D))$, we mean the center of the group G or the center of the central simple algebra $M_m(D)$, respectively.

3. Crossed products in terms of irreducible subgroups

We begin this section by a lemma which gives us a useful tool to realize maximal Galois subfields of a central simple algebra in terms of irreducible subgroups containing a self-centralizing normal abelian subgroup.

Lemma 1. *Let D be an F-central division algebra and G an irreducible subgroup of $GL_m(D)$. If A is any abelian normal subgroup of G such that $F[A]$ contains no zero divisor, then $M_m(D)$ is isomorphic to the crossed product*

$$M_m(D) = F[C_G(A)] * G/C_G(A),$$

where $F[C_G(A)]$ is a simple subalgebra of $M_m(D)$. Moreover, we have $[G : C_G(A)] = \dim(Z(F[C_G(A)]) : F)$

Proof. Consider $F[A]$, the F-linear hull of A in $M_m(D)$. By assumption, $F[A]$ is a commutative integral domain. Since D is of finite dimension over F, $F[A]$ is algebraic over F. Thus, $F[A]$ is a subfield of

$M_m(D)$. Now, applying Lemma 2.4 of [8], we obtain $C_{M_m(D)}(F[A]) = F[C_G(A)]$ and that $M_m(D)$ is a crossed product of $C_{M_m(D)}(F[A])$ by $G/C_G(A)$. Now, by Centralizer Theorem ([2], p. 42), it is easily seen that $C_{M_m(D)}(F[A])$ is a simple subalgebra of $M_m(D)$. Hence, its center is a subfield containing F. Also, the group $G/C_G(A)$ acts as a group of automorphisms on the center of $C_{M_m(D)}(F[A])$, and F is the fixed field of this action. Therefore, we obtain

$$[G : C_G(A)] = \dim(Z(F[C_G(A)]) : F),$$

as required. □

Now, we are prepared to prove the main theorem of this section as follows:

Theorem 1. *Let D be a finite dimensional F-central division algebra. Then, $M_m(D)$ is a crossed product over a maximal subfield if and only if there exists an irreducible subgroup G of $M_m(D)$ and a normal abelian subgroup A of G such that $C_G(A) = A$, and $F[A]$ contains no zero divisor.*

Proof. Assume that $M_m(D)$ is a crossed product over a maximal subfield K. Then, K/F is Galois and by a theorem of ([2], p.92), we can write $M_m(D) = \oplus_{\sigma \in Gal(K/F)} K e_\sigma$, where $e_\sigma \in GL_m(D)$ and for each $x \in K$ and $\sigma \in Gal(K/F)$ there exists $\sigma(x) \in K$ such that $e_\sigma x = \sigma(x) e_\sigma$. Therefore, the elements e_σ's as well as the group K^* are contained in $N_{GL_m(D)}(K^*)$. This implies that $N_{GL_m(D)}(K^*)$ is an irreducible subgroup of $M_m(D)$. Now, using the Skolem-Noether Theorem ([2], p.39) and the fact that $C_{M_m(D)}(K) = K$, we obtain the isomorphism $N_{GL_m(D)}(K^*)/K^* \simeq Gal(K/F)$. Hence, taking $G := N_{GL_m(D)}(K^*)$ and $A := K^*$, one side of the proof is done.
On the other hand, let G be an irreducible subgroup of $M_m(D)$, and A a normal abelian subgroup of G such that $C_G(A) = A$, and $F[A]$ contains no zero divisor. By Lemma 1, we conclude that $M_m(D)$ is a crossed product of $F[A]$ by G/A. Furthermore, $F[A]$ is a subfield of $M_m(D)$ such that $[G : A] = \dim(Z(F[A]) : F) = \dim(F[A] : F)$. Therefore, $F[A]$ is in fact a maximal Galois subfield of $M_m(D)$ with Galois group isomorphic to G/A. Hence, the result follows. □

We observe that in the special case $m = 1$, the above result reduces to the following:

Theorem 2. *Let D be a finite dimensional F-central division algebra. Then D is a crossed product over a maximal subfield if and only if there* *exists an irreducible subgroup $G \subseteq D^*$ and a normal abelian subgroup A of G such that $C_G(A) = A$.*

4. Irreducible subgroups which imply that D is a crossed product over a maximal subfield

In this section we deal with the special case where our central simple algebra is a non-commutative finite dimensional F-central division algebra D. In fact, we investigate some classes of irreducible subgroups G whose existence in D^* give rise to a crossed product division algebra D over a maximal subfield. To do this, we consider the following cases and apply mainly Theorem 2 of the last section. In all cases below G will be an irreducible subgroup of D^*. We also remark that given an irreducible subgroup G of an F-central division algebra D, if $\bar{F}$ is the algebraic closure of F, then $D \otimes_F \bar{F} = M_n(\bar{F})$ and we have $\bar{F}[G] = M_n(\bar{F})$. So, G is also an irreducible subgroup in the linear group sense.

Case 1. Abelian-by-supersoluble groups

Let G be an abelian-by-supersoluble subgroup of D^*. We shall prove that D is a crossed product over a maximal subfield. By Theorem 2, it is enough to find an abelian normal subgroup in G such that $C_G(A) = A$. To do this, take A maximal abelian normal in G such that G/A is supersoluble. If $C_G(A)/A \neq 1$, then there exists a normal subgroup H of $C_G(A)$ such that H/A is a nontrivial normal cyclic subgroup of G/A. Now, take H/A to be the smallest nontrivial intersection of $C_G(A)/A$ by the terms of the normal cyclic series of group G/A. It is easily seen that H is an abelian normal subgroup of G properly containing A. This contradicts the choice of A. Hence, we have $C_G(A) = A$.

Case 2. Metabelian groups

Let G be a metabelian subgroup of D^*. Take A maximal abelian normal in G such that G/A is also abelian. If $C_G(A)/A \neq 1$, then for every element $x \in C_G(A) \setminus A$ the group $H =< A, x >$ is easily seen to be an abelian normal subgroup properly containing A. This contradicts the choice of A. So, we must have $C_G(A) = A$.

Case 3. Locally nilpotent groups

Assume that G is a locally nilpotent subgroup of D^*. Then, by a theorem in [3], G is hypercentral. This reduces to the next item.

Case 4. Hypercentral groups

If G is hypercentral, then by an exercise of ([7],p.354), we conclude that every maximal abelian normal subgroup of G is self centralizing. Hence, we get the result.

Case 5. Soluble-by-finite maximal subgroups of D^*

If M is a maximal subgroup of D^* that is soluble-by-finite, then M is not D^*. Otherwise, D^* satisfies a group identity which is not possible [1]. We now claim that M is an irreducible subgroup of D. Take $F[M]$, the division subring of D generated by M over F. Since M is maximal two cases are possible, either $F[M] = D$ or $F[M]^* = M$. But the second case is not possible due to the fact that the multiplicative group of a division algebra does not satisfy a group identity. Therefore, $D = F[M]$, i.e., M is an irreducible subgroup of D. Now, by Mal'cev Theorem ([11], p.44), M contains a normal abelian subgroup of finite index. Take A maximal. First, we prove that $C_{D^*}(A) \subseteq M$. If not, then $D^* =< M, C_{D^*}(A) >\subseteq N_{D^*}(A)$. This means that $D = N_D(F[A])$, which is a contradiction, by Cartan-Brauer-Hua Theorem [5]. Hence, $C_{D^*}(A) \subseteq M$. Now, by Centralizer Theorem ([2], p.42), $C_D(A)$ is a division subring of D with center $F[A]$ whose multiplicative group is contained in M. Since M is a soluble by finite group, so is $C_{D^*}(A)$. It implies that $C_D(A) = F[A]$. Since $F[A]^*$ is a normal abelian subgroup of M containing A, we must have $A = F[A]^*$. Therefore, $C_D(A) = A$, as desired.

We observe that the above result actually generalizes Corollary 4 of [6].

Case 6. Finite groups

First we prove the problem for the case where G is a finite soluble subgroup with a normal subgroup N such that G/N is supersolube and N has all its Sylow subgroups abelian. For such a group G, take A maximal abelian normal in G. If $A \neq C_G(A)$, then $C_G(A)/A$ contains a nontrivial normal abelian subgroup L/A of G/A. Since G/A is soluble, we may take L/A to be the smallest nontrivial intersection of $C_G(A)/A$ with the terms of the derived series of G/A. Clearly, $A \subseteq Z(L)$, and because A is the maximal abelian normal subgroup of G, we necessarily have $A = Z(L)$. Also, $[L, L] = L' \subseteq A \subseteq Z(L)$, So L is nilpotent of class two. Hence, we may write $L \cap N \simeq \prod L_p$, where L_p's are the p-Sylow subgroups of $L \cap N$. By assumption , all p-Sylow subgroups of N are abelian. Thus, we conclude that $L \cap N$ is an abelian normal subgroup of G. We claim that $L \cap N \subseteq A$. If not, since $L \cap N \subseteq C_G(A)$, the

group $(L \cap N)A$ is an abelian normal subgroup of G properly containing A. This contradicts the choice of A, so we must have $L \cap N \subseteq A$. We also claim that $AN \neq LN$. Otherwise, if $LN = AN$, then for all $x \in L$, we have $x = an$ for some $a \in A$ and $n \in N$. It follows that $xa^{-1} = n \in L \cap N \subseteq A$, and so $x \in A$, a contradiction. Hence, we necessarily have $AN/N \trianglelefteq LN/N \trianglelefteq G/N$. Now, G/N is supersoluble and consequently G/AN is a supersoluble group. Therefore, the group LN/AN must contain a nontrivial normal cyclic subgroup in G/AN denoted by $< x > AN/AN$, where $x \in L$. Now, if we take $A_0 =< A, x >$, then A_0 is an abelian normal subgroup of G properly containing A, which is a contradiction to the choice of A. Therefore, we have $C_G(A) = A$ and we obtain the result.
Now, we consider the general case. By Amitsur's Theorem ([9], p.46), if G is a finite subgroup of the multiplicative group of a division ring, then G is isomorphic to one of the following list of groups, and only the last one is insoluble.

(a) A group that all of whose Sylow subgroups are cyclic.
(b) $C_m]Q$, where Q is a quaternion of degree 2^t, and C_m a cyclic group of order m. By $C_m]Q$, we mean the split extension of C_m by Q in which Q acts on C_m.
(c) $Q \times M$, where Q is quaternion of degree 8, and M a group of type (a).
(d) The binary octahedral group of order 48.
(e) $SL(2, 3) \times M$, where M is a group of type (a).
(f) $SL(2, 5)$.

The groups of (a), (b) and (c) are special cases of the soluble groups that are dealt with previously. Now, we examine the case (d). The binary octahedral group G of order 48, which has a normal quaternion subgroup Q of order 8. If $D = F[G]$ is a division algebra generated by the octahedral subgroup G over the center F, then the division subring $F[Q]$ has degree 2 since Q is a nonabelian group with a center containing two elements. It implies that $Z(F[Q]) = F$. Now, by Theorem 2.6 of [8], we can write D as $F[X_1] \otimes_F F[X_2]$, where $F[X_1] = F[Q]$ and $F[X_2] = C_D(F[Q])$. We also have $F^* \subseteq X_2$, and $[G : C_G(X_2)] = [X_2 : F^*]$ in which $C_G(X_2) = G \cap F[X_1]$. This means that $[X_2 : F^*]$ is at most 6, and then it follows that X_2 is abelian-by-supersoluble. Hence, both $F[X_1]$ and $F[X_2]$ are crossed product algebras over their maximal subfields by part 1, and so is D as desired.
To deal with the case (e), assume that $D = F[G]$, where G is as in (e) and let Q be the normal quaternion subgroup of $SL(2, 3)$ of order 8. We first claim that $F[Q] = F[SL(2, 3)]$. To do this, consider $SL(2, 3)$ as a

semidirect product of Q by C_3, written by $Q]C_3$, where C_3 is the cyclic group of order 3 with generator c and $Q =< x, y|x^2 = y^2, y^4 = 1, x^y = x^{-1} >$. c acts on Q by cyclically permuting y, x, xy. But conjugation by the element $t = -(1 + x + y + xy)/2 \in F[Q]$ also has the same effect. Thus ct^{-1} centralizes $F[Q]$, and hence so does t. Therefore, c commutes with t. But t and c are both cube roots of unity in the field $F[t, c]$. It follows that $c \in < t >$. Thus, $F[Q] = F[SL(2, 3)]$ and hence $Z(F[Q]) = Z(F[SL(2, 3)]) = F$. Now, again by Theorem 2.6 of [8], we may write $D = F[SL(2, 3)] \otimes_F F[X_2]$. Repeating the same argument as used in the proof of Theorem 2.6 of [8], we may take $X_2 = M$. So, one sees that both factors of the decomposition of D are crossed product division algebras over their maximal subfields. Hence, D is also a crossed product over a maximal subfield, as required.
To prove the final case (f), let D be the division algebra generated by $G = SL(2, 5)$. It is known from Theorem 2.1.11 of [[9], p. 51] that D can be generated by the quaternion subgroup of order 8. Hence, D is a crossed product over a maximal subfield.

We remark that the last result generalizes the case where the degree of D is a prime power, which is dealt with in [4].

Case 7. Soluble subgroups containing no element of order dividing $deg(D)^2$

Let G be an irreducible soluble subgroup of D^*. By Mal'cev Theorem ([11], p.44), G contains an abelian normal subgroup A of finite index . As before, take A maximal abelian normal in G. If $C_G(A)/A \neq 1$, then due to solublity of G/A, the group $C_G(A)/A$ contains a nontrivial abelian subgroup L/A which is normal in G/A. We have also $Z(L) = A$. For every $a, b, c \in L$, it is easily seen that for $(a, b) = aba^{-1}b^{-1} \in A$ we have $(a, b)(a, c) = (a, bc)$. Now, for every $a \in L$, define the group homomorphism $\phi_a : L \to A$ by $\phi_a(x) = (a, x)$, where $x \in L$. It is clear that $A \subseteq \ker \phi_a$. Therefore, the image of ϕ_a is a finite subgroup of A whose elements are of order dividing $[L : A]$. We now claim that $[L : A]|(D : F)$. To do this, take a transversal $l_1, ..., l_t$ of A in L, and consider $F[L]$, the F-linear hull of L in D. We want to prove that l_i's are a linearly independent generating set of $F[L]$ over $F[A]$. That $F[L]$ is generated by l_i's over $F[A]$ is evident. But for the latter, take the relation $\sum_{i=1}^{s} a_i l_i = 0$, where $a_i \in F[A]$, and s is chosen minimal. For each l_j, we have $\sum_{i=1}^{s} a_i l_j l_i l_j^{-1} = 0$, and by subtracting two relations we obtain $\sum_{i=1}^{s} a_i(l_j l_i l_j^{-1} l_i^{-1} - 1)l_i = 0$. Since s is minimal for each i we obtain $l_j l_i l_j^{-1} l_i^{-1} = 1$. In other words, L

is an abelian normal subgroup of G properly containing A. This is not possible by choice of A, hence l_i's are linearly independent over $F[A]$. Therefore, $[L : A] = \dim(F[L] : F[A])$ divides $deg(D)^2$. Because each element of G is not of order dividing $deg(D)^2$, we conclude that $\phi_a(L) = \{1\}$. Since a is chosen arbitrary in L, we conclude that L is an abelain normal subgroup of G properly containing A. This is again a contradiction. Therefore, we have $C_G(A) = A$, as desired.

Case 8. G is soluble-by-finite satisfying the condition of part 7, and G does not contain any normal subgroup isomorphic to SL(2,5).

Take A maximal abelian normal in G of finite index. If $C_G(A)/A \neq 1$, then consider S, the maximal soluble normal subgroup of $C_G(A)$ which clearly contains A in itself. We have $A \neq S$ for otherwise from Lemma 5.5 in [8] we obtain $A = C_G(A)$, a contradiction. Hence, S/A is a nontrivial normal soluble subgroup of G/A which is contained in $C_G(A)/A$. Therefore, we can choose a nontrivial abelian normal subgroup L/A of G/A that is contained in $C_G(A)/A$. Now, apply a similar argument as in the previous case to obtain an abelian normal subgroup in G which is self-centralizing.

We observe that Cases 7 and 8 in fact generalize the last theorem of [8] in which the torsion free case is investigated. Finally, we remark that the counterexample given in the last section of [8] provides us with an irreducible subgroup G whose maximal abelian normal subgroup A does not satisfy the condition $C_G(A) = A$. Therefore, in general it is not true that every maximal abelian normal subgroup of an irreducible subgroup G of D^* is a self-centralizing abelian normal subgroup.

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Department of Mathematical Sciences, Sharif University of Technology, P. O. Box 11365-9415. Tehran, Iran
E-mail address: keshavarzipour@math.sharif.edu &, mahdavih@sharif.edu