ON THE FADDEEV INDEX OF AN ALGEBRA OVER THE FUNCTION FIELD OF A CURVE

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Abstract. We give an upper bound for the Faddeev index of a central simple algebra of prime exponent p over the rational function field in the case where the ramification sequence of the algebra consists of rational points. This bound depends only on the number of ramification points and in many cases turns out to be strict.

Let k be a field containing a primitive p-th root of unity, X a smooth complete curve over k. We show that there exist algebras of exponent p over $k(X)$ with arbitrarily large Faddeev index, provided that there are algebras of exponent p and arbitrarily large index over k.

Let k be a field, p a fixed prime, $\operatorname{char} k \neq p$, ξ_p a primitive p-th root of unity, $\xi_p \in k$. For any $a, b \in k^*$ the cyclic algebra $(a,b)_{\xi_p}$ is denoted just by (a,b). For an abelian group G we denote by $G\{p\}$ its p-primary part, and by ${}_nG$ its n-torsion. Let further X be a smooth curve over k. Consider the exact sequence

$$0 \to \operatorname{Br}(X)\{p\} \to \operatorname{Br} k(X)\{p\} \to \coprod_{x\in X} H^1(k(x), \lim_{m\geq 1} p^{-m}\mathbb{Z}/\mathbb{Z}).$$

For any $A \in \operatorname{Br} k(X)\{p\}$ the set of $x \in X$ such that $\partial_x(A) \neq 0$ is called the ramification sequence of A. Any x from the ramification sequence is called a ramification point of A.

If $A_1, A_2 \in \operatorname{Br} k(X)\{p\}$ and $A_1 - A_2 \in \operatorname{Br}(X)\{p\}$, then we say that A_1 and A_2 are Faddeev equivalent and denote this as $A_1 \sim A_2$. This means that the algebras A_1 and A_2 have the same residues at all points of X.

For any $A \in \operatorname{Br}(X)\{p\}$ define the Faddeev index of A as

$$F(A) = \operatorname{Min} \operatorname{ind} B, \text{ where } A \sim B.$$

Put also

$$F_m(A) = \operatorname{Min} \operatorname{ind} B, \text{ where } B \in_{p^m} \operatorname{Br}\, k(X),\ (A-B) \in \operatorname{Br}(X)\{p\},$$

$$F_m(k(X)) = \operatorname{Max} F_m(A), \text{ where } A \in_p \operatorname{Br}\, k(X),$$

$$F(k(X)) = \operatorname{Max} F(A), \text{ where } A \in_p \operatorname{Br}\, k(X),$$

$$I(k) = \operatorname{Max} \operatorname{ind}(A), \text{ where } A \in_p \operatorname{Br}\, k.$$

Clearly,

$$\operatorname{Min}(F_m(A),\ p^{m+1}) \leq F(A) \leq F_m(A).$$

1. Case of the affine line

Computation of the Faddeev index of an algebra seems to be very hard problem in general even when X is the affine line. However, in some particular cases this problem can be solved. We start with recalling the notion of specialization at a rational point of the affine line. Let $x \in \mathbb{A}^1_k$. The specialization map s_x with respect to the point x is defined as follows:

$$s_x : \ \mathrm{Br}\, k(t)\{p\} \xrightarrow{\text{res}} \mathrm{Br} k((t-x))\{p\} = \mathrm{Br}\, k\{p\} \oplus \chi(k)\{p\} \xrightarrow{\text{p}} \mathrm{Br}\, k\{p\},$$

where the map p is the projection, and $\chi(k)$ is the character group of k. Notice that if $f, g \in k(t)$ and $f(x), g(x)$ are nonzero elements of k, then $s_x(f,g) = (f(x), g(x))$. The following lemma is well known.

Lemma 1. *Let $x \in \mathbb{A}^1_k$ be a rational point, $B \in \mathrm{Br}\, k(t)\{p\}$. Then*

$$\text{ind } \ s_x(B) \leq \text{ind } \ B.$$

Proof. There exists a field extension l/k such that $l((t-x))$ is a maximal subfield of $B_{k((t-x))}$ [S]. In particular, $B_{l((t-x))} = 0$. Therefore, we get

$$0 = p(B_{l((t-x))}) = p(B_{k((t-x))})_l = s_x(B)_l,$$

hence

$$\text{ind } \ s_x(B) \leq [l:k] = \text{ind}(\text{res}\, B) \leq \text{ind}\, B.$$

□

Now we consider the case where the ramification sequence of an algebra of exponent p consists of rational points. Then it is possible to give a good upper bound for the Faddeev index of the algebra depending only on the number of the ramification points. Moreover, this bound is strict if the ground field is good enough.

Proposition 2. *Suppose $X = \mathbb{A}^1_k$, $A \in_p \mathrm{Br}\ k(t)$ and the ramification sequence of A consists of n rational points. Then*

1) $F(A) \leq p^{[\frac{n+1}{2}]}$

2) *Suppose that $a_i, b_i \in k^*$ $(1 \leq i \leq n)$ are such that* $\text{ind}(\sum\limits_{i=1}^{n}(a_i, b_i)) = p^n$. *Put* $A = \sum\limits_{i=1}^{n}((1-a_i)t + a_i, b_i)$. *Then $F(A) = p^{[\frac{n+1}{2}]}$. In particular, the bound in part 1) is strict in this case.*

3) *If $I(k) = \infty$, then $F(k(\mathbb{A}^1_k)) = \infty$.*

Proof. 1), 2). Consider the case of even n. If $n = 2$ and f_1, f_2 are the monic linear polynomials corresponding to the ramification points of A, one can choose some $c_1, c_2 \in k^*$ such that $A \sim (c_1 f_1, c_2 f_2)$. Let now $n = 2m$ and $\{x_1, \dots x_n\}$ be the ramification sequence of A. Let further B_i be an algebra with ramification sequence $\{x_{2i-1}, x_{2i}\}$ $(1 \leq i \leq m)$. Then

$$F(A) \leq \prod_{i=1}^{m} F(B_i) = p^m,$$

which proves part 1). As for part 2), let s_0, s_1 be the specialization maps with respect to the points $0, 1 \in \mathbb{A}^1_k$. If $A = C + B$, where $C \in \operatorname{Br} k\{p\}$, $B \in \operatorname{Br} k(t)\{p\}$ and $F(A) = \operatorname{ind} B$, then

$$\sum_{i=1}^{n} (a_i, b_i) = s_0(A) - s_1(A) = s_0(B) - s_1(B),$$

hence

$$p^n = \operatorname{ind} \sum_{i=1}^{n} (a_i, b_i) \leq \operatorname{ind} s_0(B) \operatorname{ind} s_1(B) \leq (\operatorname{ind} B)^2,$$

so $F(A) = \operatorname{ind} B \geq p^{\frac{n}{2}}$. The case of an odd n is treated quite similarly.

3) In view of the Merkurjev-Suslin theorem [MS], any element of ${}_p\mathrm{Br}\ (k)$ is a sum of cyclic algebras (a, b). The same argument as in the proof of part 2) shows that if $\operatorname{ind}(\sum_{i=1}^{r} (a_i, b_i)) = p^n$, and $A = \sum_{i=1}^{r} ((1 - a_i)t + a_i, b_i)$, then $F(A) \geq p^{[\frac{n+1}{2}]}$. The proposition is proved. □

Now consider the case of $p = 2$ and suppose the ramification sequence of an algebra A consists of three rational points, say a_1, a_2, a_3. It turns out that in this case one can always compute $F(A)$. Namely, let $\partial_{a_i}(A) = b_i$. This means that

$$A \sim (t - a_1,\ b_1) + (t - a_2,\ b_2) + (t - a_3,\ b_3).$$

Moreover, by the reciprocity law $\partial_\infty(A) = b_1 b_2 b_3$. Put

$$\pi = (b_1 b_2 (a_3 - a_1)(a_3 - a_2),\ b_1 b_3 (a_2 - a_1)(a_2 - a_3)).$$

Proposition 3. 1) *If* $\partial_\infty(A) = 1$, *then* $F(A) = 2$.

2) *If* $\partial_\infty(A) \neq 1$, *then the following conditions are equivalent:*

a) $F(A) = 2$.

b) $\pi = 0$.

c) There exist $\alpha, \beta \in k$ *such that* $\alpha a_i + \beta \equiv b_i (k^{*2})$.

Proof. 1) Computing the residues one check immediately that

$$A \sim (\frac{b_2(t - a_1)(t - a_3)}{(a_2 - a_1)(a_2 - a_3)},\ \frac{b_1(t - a_2)(t - a_3)}{(a_1 - a_2)(a_1 - a_3)}),$$

hence $F(A) = 2$.

2) *b)* $\Longleftrightarrow$ *c)*.

Obviously, $\pi = 0$ iff the form $\langle b_1(a_2 - a_3), b_2(a_3 - a_1), b_3(a_1 - a_2)\rangle$ is isotropic iff there exists $x_1, x_2, x_3 \in k^*$ such that

$$b_1(a_2 - a_3)x_1^2 + b_2(a_3 - a_1)x_2^2 + b_3(a_1 - a_2)x_3^2 = 0$$

iff there exist $x_1, x_2, x_3 \in k^*$ such that

$$\frac{b_1 x_1^2 - b_2 x_2^2}{a_1 - a_2} = \frac{b_2 x_2^2 - b_3 x_3^2}{a_2 - a_3}. \qquad (1)$$

If equality (1) holds put $\alpha=\frac{b_1x_1^2-b_2x_2^2}{a_1-a_2}$, $\beta=b_1x_1^2-\alpha a_1$. Then for each i we have $\alpha a_i+\beta=b_ix_i^2$. Conversely, if $\alpha a_i+\beta=b_ix_i^2$, then equality (1) holds.

c) $\Longrightarrow$ *a*). Obviously one can choose $\gamma\in k^*$ such that

$$A\sim(\gamma(t-a_1)(t-a_2)(t-a_3),\ \alpha t+\beta),$$

so $F(A)=2$.

a) $\Longrightarrow$ *c*). For any polynomial $f\in k[t]$ denote by $d(f)$ its degree. Suppose that $A\sim(f_1,f_2)$, where $f_1,f_2\in k[t]$ and $d(f_1)+d(f_2)$ is minimal. Obviously, the polynomials $t-a_i$ divide f_1 or f_2, and f_1,f_2 are squarefree. There are two possible cases:

a) $(t-a_1)(t-a_2)(t-a_3)$ divides f_1 or f_2.

b) The case opposite to *a*).

Taking into account that A is ramified only at the points a_i, it is easy to check that in the case *a*), where

$$A=(p(t-a_1)(t-a_2)(t-a_3),q) \qquad (2)$$

and $p,q\in k[t]$, the polynomial $p(t-a_1)(t-a_2)(t-a_3)$ is a square in $k[t]/q$, and the polynomial q is a square in $k[t]/p$. This means that there exist $X,Y,p',q'\in k[t]$ such that

$$X^2-p(t-a_1)(t-a_2)(t-a_3)=qq'$$

$$Y^2-q=pp'.$$

Consider the first of the last two equalities. We may assume that $d(X)\le d(q)-1$, and, moreover, $d(q)\le d(q')$ (otherwise, we could change q for q' in presentation (2) of A, which would contradict minimality of $d(f_1)+d(f_2)$). From this we get

$$d(p)+3\ge 2d(q).$$

Similarly, from the second equality we get

$$d(q)\ge 2d(p).$$

Taking these two unequalities together we easily conclude that either $d(p)=0$ and $d(q)\le 1$, or $d(p)=1$ and $d(q)=2$. But the last case is impossible, because then $\partial_\infty(A)=1$, which contradicts the condition of the proposition. Therefore, case *a*) is done.

In the case *b*) we have, say, $A=(p(t-a_2)(t-a_3),q(t-a_1))$, where $t-a_1\not| p,q$, hence there exist $X,Y,p',q'\in k[t]$ such that

$$X^2-p(t-a_2)(t-a_3)=qq',$$

$$Y^2-q(t-a_1)=pp'.$$

Quite similarly to case *a*) we get

$$d(p)+2\ge 2d(q),$$

$$d(q) + 1 \geq 2d(p),$$

which, since $\partial_\infty(A) \neq 1$, implies that either $d(p) = d(q) = 0$, or $d(p) = d(q) = 1$.

Consider the case $d(p) = d(q) = 1$. We have for some $\alpha,\ \beta,\ c,\ d \in k$ the equality

$$A = (\alpha(t-c)(t-a_2)(t-a_3), \beta(t-d)(t-a_1),$$

where $c, d \neq a_1$. Assume first that $d \neq a_2, a_3$. Then, computing the residues we get the following system:

$$\alpha(a_1 - c)(a_1 - a_2)(a_1 - a_3) \equiv b_1$$

$$\beta(a_2 - d)(a_2 - a_1) \equiv b_2$$

$$\beta(a_3 - d)(a_3 - a_1) \equiv b_3$$

$$\alpha(d - c)(d - a_2)(d - a_3) \equiv 1$$

$$\beta(c - d)(c - a_1) \equiv 1.$$

From the last two equalities we obtain

$$\alpha\beta \equiv (d - a_2)(d - a_3)(a_1 - c).$$

Hence

$$b_1 b_2 (a_3 - a_1)(a_3 - a_2) \equiv (a_3 - d)(a_3 - a_2),$$

$$b_1 b_3 (a_2 - a_1)(a_2 - a_3) \equiv (a_2 - d)(a_2 - a_3),$$

which, since

$$(a_3 - d)(a_3 - a_2) + (a_2 - d)(a_2 - a_3) = (a_3 - a_2)^2,$$

implies

$$\pi = ((a_3 - d)(a_3 - a_2),\ (a_2 - d)(a_2 - a_3)) = 0.$$

This finishes the proof of this case. The remaining cases are simpler and treated in a similar way. The proposition is proved. □

Remark 1. The Pfister form π is related to the notion of cross ratio. Recall that for any pairwise distinct $a_i = (\lambda_i : \mu_i) \in \mathbb{P}^1_k$ $(1 \leq i \leq 4)$ the cross ratio of these four points is defined as

$$[a_1, a_2, a_3, a_4] = \frac{(\lambda_3\mu_1 - \lambda_1\mu_3)(\lambda_4\mu_2 - \lambda_2\mu_4)}{(\lambda_3\mu_2 - \lambda_2\mu_3)(\lambda_4\mu_1 - \lambda_1\mu_4)} \in k^*,$$

and this element does not change under a linear change of the homogeneous coordinates of $\mathbb{P}^1_k$. Moreover, $[a_1, a_2, a_3, a_4] = [b_1, b_2, b_3, b_4]$ if and only if there exists a k- automorphism $f : \mathbb{P}^1_k \to \mathbb{P}^1_k$ such that $f(a_i) = b_i$ for each i.

Let an algebra $A \in_2 \mathrm{Br}\, k(t)$ have four ramification points in $\mathbb{P}^1_k$, say a_1, a_2, a_3, a_4, and suppose all of them are rational. Let us make an automorphism of $\mathbb{P}^1_k$ taking one of these points to infinity, say by means of the sustitution $t - a_l = \frac{1}{u}$ for some

$1 \le l \le 4$. With respect to the variable u the ramification sequence of A consists of the points $\frac{1}{a_i - a_l}$, so we can apply Proposition 2. It is easy to check that

$$\pi = (b_i b_j [a_k, a_l, a_i, a_j],\ b_i b_k [a_j, a_l, a_i, a_k]).$$

Remark 2. The method used in Proposition 2 can be applied to treat a more general case, namely the one where the ramification sequence in $\mathbb{A}^1_k$ consists of points p_i whose sum of degrees equals 3. This means that the ramification sequence consists either of three rational points (this is just the case considered in Proposition 3), or of one rational point and one point of degree 2, or of an only point of degree 3. Clearly, there exists a polynomial of degree ≤ 2, say $at^2 + bt + c$, which provides the needed residues at all the ramification points. On the other hand, by the same argument as in Proposition 3 one can prove that the condition $F(A) = 2$ holds if and only if either $\partial_\infty(A) = 1$, or there exists a polynomial of degree ≤ 1 providing the needed residues at all the points. The last condition means that for some $x_0, x_1, x_2 \in k$ the residue of the polynomial

$$(at^2 + bt + c)(x_2 t^2 + x_1 t + x_0)^2 (\mathrm{mod} \prod p_i)$$

is of degree ≤ 1. The coefficient at t^2 of this residue is some quadratic form q in variables x_0, x_1, x_2. Thus we conclude that if $\partial_\infty(A) \ne 1$, then $F(A) = 2$ if and only if the form q is isotropic. It can be easily checked that in the case of three rational points the 2-fold Pfister form π associates with the form q, i.e. q is similar to the pure subform of π.

So far we have considered algebras defined over the rational function field in one variable. However, our investigation naturally gives rise to some algebra defined over the rational function field in two variables.

For any field k and elements $u, v, a_1, a_2, a_3, b_1, b_2, b_3 \in k$ consider the algebra

$$D = ((u - a_1)(v - a_1),\ b_1) + ((u - a_2)(v - a_2),\ b_2) + ((u - a_3)(v - a_3),\ b_3)$$

(if all the entries are nonzero). Now assume that $F(A) = 2$, where

$$A = (t - a_1,\ b_1) + (t - a_2,\ b_2) + (t - a_3,\ b_3).$$

Let $A = C + B$, where $C \in \mathrm{Br}\, k\{2\}$, $B \in \mathrm{Br}\, k(t)\{2\}$, and $\mathrm{ind}\, B = 2$. We have

$$D = s_u(A) - s_v(A) = s_u(B) - s_v(B).$$

Therefore, $\mathrm{ind}\, D \le 4$.

Consider a particular case, where $k = k_0(u, v, b_1, b_2, b_3)$, elements $a_1, a_2, a_3 \in k_0$ are pairwise distinct, and u, v, b_1, b_2, b_3 are indeterminates. Then, obviously, $\mathrm{ind}\, D = 8$. On the other hand, by the argument above we have $\mathrm{ind}\, D_{k(\pi)} = 4$. Hence by the index reduction theorem we get $D \simeq \pi \otimes D_1$, where D_1 is a biquaternion algebra. By specialization argument we get the same in the general case.

One can obtain the explicit form of the algebra D_1. To do this, for any elements $A_1, A_2, A_3, B \in k^*$ consider the field $k[x_1, x_2, x_3]/(A_1x_1 + A_2x_2 + A_3x_3 - B)$. Computing the residues it is easy to see that

$$(b_1, x_1) + (b_2, x_2) + (b_3, x_3) = R + (-A_1A_2A_3Bx_1x_2x_3,\ b_1A_2Bx_2 + b_1A_3Bx_3) + (b_1b_3A_2Bx_2,\ b_1b_2A_3Bx_3), \qquad (3)$$

where

$$R = (b_1, A_1B) + (b_2, A_2B) + (b_3, A_3B) + (b_1, b_2) + (b_2, b_3) + (b_1, b_3).$$

By specialization argument equality (3) is universal. Now apply the equality

$$(a_3-a_2)(u-a_1)(v-a_1)+(a_1-a_3)(u-a_2)(v-a_2)+(a_2-a_1)(u-a_3)(v-a_3) = (a_3-a_1)(a_3-a_2)(a_2-a_1).$$

Applying equality (3), we easily get

$$D = \pi + A + B,$$

where

$$A = (x_1x_2x_3,\ ((a_2-a_1)^2(a_3-a_1)(a_3-a_2)x_3 + (a_3-a_1)^2(a_2-a_1)(a_2-a_3)x_2)b_1)$$

and

$$B = (b_1b_2(a_3-a_1)(a_3-a_2)x_3,\ b_1b_3(a_2-a_1)(a_2-a_3)x_2).$$

It is rather interesting to compute ind D in the case where u and v are indeterminates. Unfortunately we are able to give only a partial answer to this question.

Proposition 4. *Let u, v be indeterminates, $k = k_0(u, v)$, $a_i, b_i \in k_0$ $(1 \le i \le 3)$. Assume that $\overline{b_i} \in {k_0}^*/{k_0}^{*2}$ are linearly independent over $\mathbb{Z}/2\mathbb{Z}$, and $\pi_{k_0(\sqrt{b_i})} \ne 0$ for some $1 \le i \le 3$, or $\pi_{k_0(\sqrt{b_1b_2b_3})} \ne 0$. Then* $\operatorname{ind} D = 8$.

Proof. Assume for instance that $\pi_{k_0(\sqrt{b_3})} \ne 0$. We will prove then a stronger statement, namely that $\operatorname{ind} D_{k_0(v)((u-a_3))} = 8$. Indeed, the last statement is equivalent to that

$$\operatorname{ind}[((a_3-a_1)(v-a_1),\ b_1) + ((a_3-a_2)(v-a_2),\ b_2)]_{k_0(\sqrt{b_3})(v)} = 4.$$

Denote the last biquaternion algebra over $k_0(\sqrt{b_3})(v)$ by R. It is easy to check that

$$R = \pi + (x,\ b_1b_2((a_2-a_1)^2b_2 - x)),$$

where $x = b_2(a_2-a_1)(v-a_1)$. Now the assertion follows from [RST], Corollary 5.3.

Assume now that $\pi_{k_0(\sqrt{b_1b_2b_3})} \ne 0$. Put $t = \frac{1}{u}$. Then over $k_0(v)((t))$ we have

$$D = (v-a_1, b_1) + (v-a_2, b_2) + (v-a_3, b_3) + (b_1b_2b_3, t),$$

hence $\operatorname{ind} D_{k_0(v)((t))} = 8$ if and only if

$$\operatorname{ind}\ [(v-a_1, b_1) + (v-a_2, b_2) + (v-a_3, b_3)]_{k_0(\sqrt{b_1b_2b_3})(v)} = 4.$$

On the other hand, computing the residues, it is easy to check that

$$[(v-a_1, b_1) + (v-a_2, b_2) + (v-a_3, b_3)]_{k_0(\sqrt{b_1b_2b_3})(v)} = \pi + (y,\ b_1b_2(b_2 - y)),$$

where $y = \frac{b_2(a_2-a_3)(v-a_1)}{(a_2-a_1)(v-a_3)}$. Applying again Corollary 5.3 from [RST], we are done. □

Open question. Does Proposition 4 remain true under the weaker condition, namely, $\pi \ne 0$?

2. Case of an arbitrary smooth complete curve

The case of an arbitrary complete curve is more difficult because in this case the Brauer group of the curve is usually bigger than the Brauer group of the ground field. However, even in this situation it is possible to produce an algebra with the prescribed p-primary Faddeev index, provided that the ground field k is good enough, more precisely that $I(k) = \infty$. Unfortunately, it is hardly possible to describe such an algebra explicitly. We have to apply a bit of the etale cohomology theory. By $H^n(X, *)$ we denote the nth etale cohomology group with coefficients in $*$, and by μ_l the sheaf of lth roots of unity in the etale topology.

Lemma 5. *Let X be a complete geometrically irreducible smooth curve over a field k having a rational point, m a positive integer. There are a number C and elements $\alpha_i \in_{p^m} \mathrm{Br}(X)$ $(1 \le i \le p^m)$ such that for any $D \in_{p^m} \mathrm{Br}(X)$ there exists a field extension l/k with the following properties:*

1) $[l : k] \le C$.
2) $(D_{l(X)} - \alpha_i) \in \mathrm{Im}({}_{p^m}\mathrm{Br}\ l \to_{p^m} \mathrm{Br}(X_l))$ *for some* i.

Proof. Argument quite similar to that in [AEJ], Prop. 3.3 shows that if X has a rational point over k, then for any r the following holds:

1) There is a natural surjective homomorphism $f : H^r(X, \mu_{p^m}) \to H^{r-2}(k, \mathbb{Z}/p^m\mathbb{Z})$.
2) The kernel $\underline{H}^r(X, \mu_{p^m})$ of f contains the group $H^r(k, \mu_{p^m})$.
3) The quotient $\underline{H}^r(X, \mu_{p^m})/H^r(k, \mu_{p^m})$ is naturally isomorphic to the group $H^{r-1}(k, {}_{p^m}\mathrm{Pic}(X_s))$.

We will apply the last statement for $r = 2$. The exact sequence

$$H^2(X, \mu_{p^m}) \to H^2(k(X), \mu_{p^m}) \to \coprod_{x \in X^1} H^1(k(x), \mathbb{Z}/p^m\mathbb{Z})$$

shows that the map $g : H^2(X, \mu_{p^m}) \to_{p^m} \mathrm{Br}(X)$ is onto. Let $\widetilde{D}$ be a preimage of D under the map g, and let $\widetilde{\alpha}_i \in H^2(X, \mu_{p^m})$ $(1 \le i \le p^m)$ be some preimages of all the elements of $H^0(k, \mathbb{Z}/p^m\mathbb{Z}) = \mathbb{Z}/p^m\mathbb{Z}$ under the map f. Put $\alpha_i = g(\widetilde{\alpha}_i)$. Assume that $\widetilde{D} - \widetilde{\alpha}_1 \in \underline{H}^2(X, \mu_{p^m})$. Let l_0 be such a finite extension of k that the group ${}_{p^m}\mathrm{Pic}(X_s)$ is defined over l_0. There is a group homomorphism

$${}_{p^m}\mathrm{Pic}(X_s) \simeq (\mathbb{Z}/p^m\mathbb{Z})^{2g},$$

where g is the genus of X, hence

$$H^1(l_0,\ {}_{p^m}\mathrm{Pic}(X_s)) \simeq H^1(l_0,\ \mathbb{Z}/p^m\mathbb{Z})^{2g}.$$

The commutative diagram

$$\begin{array}{ccc}
\underline{H}^2(X_{l_0}, \mu_{p^m}) & \longrightarrow & H^1(l_0,\ {}_{p^m}\mathrm{Pic}(X_s)) \\
{\scriptstyle res}\downarrow & & {\scriptstyle res}\downarrow \\
\underline{H}^2(X_l, \mu_{p^m}) & \longrightarrow & H^1(l,\ {}_{p^m}\mathrm{Pic}(X_s))
\end{array}$$

shows that there is an extension l/l_0 of degree $\leq p^{2gm}$ such that the image of $(\widetilde{D}-\widetilde{\alpha_1})_l$ in $H^1(l,\ _{p^m}\operatorname{Pic}(X_s))$ is zero. This means that

$$(\widetilde{D}-\widetilde{\alpha_1})_l \in \operatorname{Im}\ (H^2(l,\mu_{p^m}) \to H^2(l(X),\mu_{p^m}).$$

Now the commutative diagramm

$$\begin{array}{ccc} H^2(l,\mu_{p^m}) & \longrightarrow & H^2(X_l,\mu_{p^m}) \\ {\scriptstyle id}\big\downarrow & & {\scriptstyle g_l}\big\downarrow \\ H^2(l,\mu_{p^m}) & \longrightarrow & _{p^m}\operatorname{Br}(X_l) \end{array}$$

implies that $(D-\alpha_1)_{l(X)} \in \operatorname{Im}\ ({_{p^m}\operatorname{Br}}\ l \to_{p^m} \operatorname{Br}(X_l))$, which proves the lemma. □

Proposition 6. *Let X be a smooth complete curve over a field k such that $I(k) = \infty$, and let m be a positive integer. Then $F_m(k(X)) = F(k(X)) = \infty$.*

Proof. Suppose first that k is perfect and X is geometrically irreducible. The field $k(X)$ is a finite extension of the field $k(t)$. Let $A = \sum_{i=1}^{r}((1-a_i)t+a_i, b_i)$ be an algebra constructed in part 3) of Proposition 2. We are going to prove that $F(A_{k(X)})$ is big enough, provided that so is ind $(\sum_{i=1}^{r}(a_i,b_i))$. Obviously, it suffices to treat only the case, where X has a rational point and the field k coincides with its extension l from Lemma 5. Assume then that $A_{k(X)} = D + B$, where $D \in_{p^m} \operatorname{Br}\ (X)$. Using Lemma 5 with its notation, and changing D to $D-\alpha_i$ and B to $B+\alpha_i$ we may assume that $D = D_{1k(X)}$, for some $D_1 \in_{p^m} \operatorname{Br}\ k$. Since $\operatorname{ind}(A-D_1)_{k(t)} \geq F(A)$ is big enough and $[k(X):k(t)]$ is fixed, we get that $\operatorname{ind} B_{k(X)} = \operatorname{ind}(A-D)_{k(X)}$ is also big enough, hence $F_m(k(X)) = \infty$. Since

$$\operatorname{Min}(F_m(A),\ p^{m+1}) \leq F(A),$$

and m is arbitrary, we conclude that $F(k(X)) = \infty$.

To drop the condition that the ground field k is perfect and the curve X is geometrically irreducible it suffices to pass to the pure inseparable closure of some finite extension of k. □

Corollary 7. *Under the condition of Proposition* 6 *for any positive integer n there exists $A \in_p \operatorname{Br}\ k(X)$ such that $F(A) = p^n$.*

Proof. Choose $A = \sum_{i=1}^{r}(a_i,b_i) \in_p \operatorname{Br}\ k(X)$ such that $F(A) \geq p^n$, and r is minimal. Suppose that $F(A) > p^n$. Then we have $F(\sum_{i=1}^{r-1}(a_i,b_i)) \geq p^n$, which contradicts minimality of r. □

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