Nondegenerate semiramified valued and graded division algebras[*†]

Karim Mounirh

Département des Mathématiques
Faculté des Sciences
Université Catholique de Louvain
Belgium
e.mail: akamounirh@hotmail.com

Abstract

In this paper, we define what we call (non)degenerate valued and graded division algebras [Definition 3.1] and use them to give examples of division p-algebras that are not tensor product of cyclic algebras [Corollary 3.17] and examples of indecomposable division algebras of prime exponent [Theorem 5.2, Corollary 5.3 and Remark 5.5]. We give also, many results concerning subfields of these division algebras.

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Preliminaries

Let F be an associative ring with a unit and Γ a totally ordered abelian group. We say that F is a graded ring of type Γ if there are subgroups F_γ ($\gamma \in \Gamma$) of F such that $F = \oplus_{\gamma\in\Gamma} F_\gamma$ and $F_\gamma F_\delta \subseteq F_{\gamma+\delta}$, for all γ, $\delta \in \Gamma$. In this case, the set $\Gamma_F = \{\gamma \in \Gamma \mid F_\gamma \neq 0\}$ is called the support of F.

If F is a graded ring of type Γ and $x \in F_\gamma$ for some $\gamma \in \Gamma_F$, we say that x is a homogeneous element of F. In particular, if x is a nonzero element of F_γ, we say that x has grade γ and we write $gr(x) = \gamma$. We denote by F^h [resp. F^*] the set of homogeneous [resp. nonzero homogeneous] elements of F. A graded ring F which is commutative and for which all nonzero homogeneous elements are invertible is called a graded field. If in addition F has support Γ and there is a ring isomorphism $\sigma : F \mapsto F_0[\Gamma]$ such that $\sigma(F_0) = F_0$ and $\sigma(F_\gamma) = F_0\gamma$ for any $\gamma \in \Gamma - \{0\}$, then F is called a graded field of group-ring type.

Let F be a commutative graded ring of type Γ. A (left) algebra A over F is called a graded algebra over F (of type Γ) if A is a graded ring of type Γ and $F_\gamma \subseteq A_\gamma$, for all $\gamma \in \Gamma$. In particular, if F is a graded field, then graded algebras over F [resp. commutative graded algebras over F] for which homogeneous elements are invertible are called graded division algebras (GDA) over F [resp. graded field extensions of F]. If F is the center of a graded division algebra A, then A is called a graded central division algebra (GCDA) over F.

We recall that a graded field extension K of a graded field F is totally ramified over F if $[K : F] = (\Gamma_K : \Gamma_F)$. It is inertial over F if $[K : F] = [K_0 : F_0]$ and K_0 is separable over F_0. K is called tame over F if K_0 is separable over F_0 and Γ_K/Γ_F has no p-torsion, where $p = char(F)$. Finally, K is purely wild over F if K_0 is purely inseparable over F_0 and Γ_K/Γ_F is a p-group. By [HW(1)99, Lemma 3.6], K/F is purely wild if and only if $Frac(E)/Frac(F)$ is purely inseparable.

Let F be a graded field and $Frac(F)_{alg}$ an algebraic closure of $Frac(F)$. We recall that for any element λ of the divisible hull $\Delta_F(= \Gamma_F \otimes_{\mathbb{Z}} \mathbb{Q})$ of Γ_F, there is a unique grading on the polynomial ring $F[X]$ extending the grading of F and for which X is a homogeneous element with $gr(X) = \lambda$. We denote this grading by $F[X]^{(\lambda)}$. A polynomial $f \in F[X]$ is called homogenizable if there is $\lambda \in \Delta_F$ such that $f \in (F[X]^{(\lambda)})^h$. Let $x \in Frac(F)_{alg}$ and denote by $f_{x,Frac(F)}$ its minimal polynomial over $Frac(F)$. We say that x is algebraic over F if $f_{x,Frac(F)}$ is a homogenizable polynomial of $F[X]$. If K is a graded field extension of F, we say that K is algebraic over F if every homogeneous element of K is algebraic over F.
Let $F_{alg} = F[\{x \in Frac(F)_{alg} \mid x \text{ is algebraic over } F\}]$, then by [HW(1)99, Corollary 2.7(c)] F_{alg} is an algebraic graded field extension of F and all subrings of $Frac(F)_{alg}$ that are algebraic graded field extensions of F are graded subfields of F_{alg}. We call F_{alg} 'the' algebraic closure of F.

If F is a graded field and A is a graded division algebra over F, we denote by $Cq(A)$ the algebra of central quotients of A. Obviously, to the graded structure of A corresponds a canonical valuation on $Cq(A)$, defined by $v(a) = gr(a)$ for any $a \in A^*$ (See [B98, §4] or [HW(2)99, §4]). We denote by $HCq(A)$ the algebra $Cq(A) \otimes_{Frac(F)} HFrac(F)$, where $HFrac(F)$ is the Henselization of $Frac(F)$ with respect to its canonical valuation (see [E72, §16]).

Conversely, let E be a field and v a valuation of E. Then, the filtration of E induced by v yields a canonical graded field GE. Namely, let $E^\gamma = \{x \in E \mid v(x) \geq \gamma\}$, $E^{>\gamma} = \{x \in E \mid v(x) > \gamma\}$. Obviously, $E^{>\gamma}$ is a subgroup of the additive group E^γ.

So, we can define the quotient group $GE_\gamma = E^\gamma/E^{>\gamma}$. For $x \in E\backslash\{0\}$, we denote by $\tilde{x}$ the element $x + E^{>v(x)}$ of $GE_{v(x)}$. One can easily see that the additive group $GE = \oplus_{\gamma\in\Gamma} GE_\gamma$ with the multiplication law defined by $\tilde{x}\tilde{y} = \widetilde{xy}$ is a graded field.

In the same way, if D is a valued division algebra over a field E (we refer to [JW90] for non-commutative valuative definitions used in this paper), then the filtration of D by the principal fractional ideals yields a graded division algebra GD (see [B98, §4] or [HW(2)99, §4]). We recall that if F is a graded field and A is a graded central division algebra over F, then A is graded isomorphic to $GHCq(A)$ by means of the mapping $x \mapsto \tilde{x}$, where x is an arbitrary homogeneous element of A. Indeed, we have $A_0 = (GHCq(A))_0$ and $\Gamma_A = \Gamma_{GHCq(A)}$. We write $A \cong_g GHCq(A)$.

A valued central division algebra D over a field E is called defectless (over E) if $[D : E] = [\bar{D} : \bar{E}](\Gamma_D : \Gamma_E)$, where $[\bar{D} : \bar{E}]$ [resp. $(\Gamma_D : \Gamma_E)$] is the residue degree [resp. ramification index] of D over E. If in addition, $\bar{D}$ is a field and $[\bar{D} : \bar{E}] = (\Gamma_D : \Gamma_E)$, we say that D is semiramified. If D has an inertial and a totally ramified maximal subfields, we say that D is nicely semiramified (see [M05, Theorem 4]).

Let E be a Henselian valued field and D a defectless central division algebra over E. We know that the map

$$\theta_D : \Gamma_D/\Gamma_E \to Gal(Z(\bar{D})/\bar{E})$$

$$\gamma + \Gamma_E \mapsto \theta_D(\gamma + \Gamma_E) : \bar{a} \mapsto \overline{dad^{-1}}$$

(d being an arbitrary element of D such that $v(d) = \gamma$) is a surjective group homomorphism [JW90, Proposition 1.7]. We say that D is tame (over E) if it is defectless over E, the center $Z(\bar{D})$ of $\bar{D}$ is separable over $\bar{E}$ and the characteristic of $\bar{E}$ does not divide the cardinality of the kernel of θ_D.

We recall that if F is a graded field and D is a graded central division algebra over F, then $HCq(D)$ is a tame central division algebra over $HFrac(F)$ (See [HW(2)99, Theorem 5.1]. It suffices also to apply [B95, Corollary 4.4]). Analogously, to the valued case, we say that D is semiramified [resp., nicely semiramified] if D_0 is a field and $[D_0 : E_0] = (\Gamma_D : \Gamma_E)$ [resp., if D has an inertial and a totally ramified maximal graded subfields].

It is well known that graded central division algebras over a graded field F play the same role as central division algebras over a Henselian valued field. Indeed, their equivalent classes form a Graded Brauer group $GBr(F)$ and there is a group isomorphism $GBr(F) \to TBr(HFrac(F))$, where $TBr(HFrac(F))$ is the tame part of $Br(HFrac(F))$ [HW(2)99, Theorem 5.1]. Conversely, for any Henselian valued field E, there is a canonical group isomorphism $TBr(E) \to GBr(GE)$ [HW(2)99, Theorem 5.3].

Now, let M/E be a finite-dimensional abelian field extension with Galois group G; $S = (\sigma_i)_{1 \leq i \leq r}$ be a basis of G (i.e. such that $G = \langle \sigma_1 \rangle \oplus ... \oplus \langle \sigma_r \rangle$); $q_i = ord(\sigma_i)$ for $1 \leq i \leq r$; M_i (resp. M_{ij}) be the subfield of M fixed by σ_i (resp. σ_i and σ_j); N_i (resp. N_{ij}) be the norm of M/M_i (resp. M/M_{ij}); and $I = \{(m_1, ..., m_r) \in I\!N^r \mid 0 \leq m_i < q_i \text{ for } 1 \leq i \leq r\}$. We recall that if M is a maximal subfield of a central simple E-algebra A, then there exist invertible elements $x_1, ..., x_r$ of A such that the following conditions hold :

(0.1) $A = \oplus_{\bar{m} \in I} M x^{\bar{m}}$, where $x^{\bar{m}} = x_1^{m_1} ... x_r^{m_r}$;

(0.2) $x_i a = \sigma_i(a) x_i$ for $a \in M$ and $1 \leq i \leq r$.

Let $u_{ij} = x_i x_j x_i^{-1} x_j^{-1}$ and $b_i = x_i^{q_i}$ for $1 \leq i, j \leq r$, then the matrices $U = (u_{ij})_{1 \leq i,j \leq r}$ and $B = (b_i)_{1 \leq i \leq r}$ satisfy the following conditions (for $1 \leq i, j, k \leq r$) :

(0.3) u_{ij} and b_i are non-zero elements of M ;

(0.4) $u_{ji} = u_{ij}^{-1}$;

(0.5) $\sigma_i(u_{jk})\sigma_j(u_{ki})\sigma_k(u_{ij}) = u_{jk}u_{ki}u_{ij}$;

(0.6) $N_{ij}(u_{ij}) = 1$;

(0.7) $\sigma_j(b_i)b_i^{-1} = N_i(u_{ji})$.

Conversely, for any pair of matrices (U, B) satisfying conditions 0.3 to 0.7, we can construct a central simple E-algebra $(M, G, U, B) = \oplus_{\bar{m}\in I} M z^{\bar{m}}$, where z_i are independent indeterminates that satisfy the following conditions : $z_i a = \sigma_i(a) z_i$, $z_i z_j = u_{ij} z_j z_i$ and $z_i^{q_i} = b_i$ (See [AS78, section 1]).

J.P. Tignol showed that conditions 0.3 to 0.7 mean that (U, B) is a 2-cocycle with respect to a G-complex described in [T81, §1, p. 421]. Hence he denoted $(U, B) \in Z_S^2(G, M^*)$. He wrote also $U_S(G, M^*)$ to denote the set of matrices U (with entries in M^*) satisfying conditions (0.3) to (0.6). A matrix $U \in U_S(G, M^*)$ is said to be *degenerate* (with respect to the field extension M/E) if there exist $\bar{m}, \bar{n} \in I$ and $a, b \in M^*$ such that the subgroup $\langle \sigma^{\bar{m}}, \sigma^{\bar{n}} \rangle$ of G, generated by $\sigma^{\bar{m}}$ and $\sigma^{\bar{n}}$ is non-cyclic, and $u_{\bar{m},\bar{n}} = \sigma^{\bar{m}}(a)a^{-1}\sigma^{\bar{n}}(b)b^{-1}$, where $u_{\bar{m},\bar{n}} = x^{\bar{m}}x^{\bar{n}}(x^{\bar{m}})^{-1}(x^{\bar{n}})^{-1}$ [AS78, Definition, p. 81].

Now, for a matrix $U = (u_{ij})_{1\le i,j\le r}$ of $U_S(G, M^*)$, consider the amalgamated sum $M.E\langle z_1, ..., z_r\rangle$ over E of M and the free E-algebra on the independent indeterminates z_i ($1 \le i \le r$); then let I be the ideal of $M.E\langle z_1, ..., z_r\rangle$, generated by the elements $z_i z_j - u_{ij} z_j z_i$ and $z_i a - \sigma_i(a) z_i$, for $1 \le i, j \le r$ and $a \in M$. We recall that the generic abelian crossed product (M, G, U) is the algebra of central quotients of $M.E\langle z_1, ..., z_r\rangle/I$. Let $B = (b_i)_{1\le i\le r}$ be a matrix of M such that $(U, B) \in Z_S^2(G, M^*)$ and let $Z_i = b_i^{-1} z_i^{q_i}$, then the center of (M, G, U) is $E(Z_1, ..., Z_r)$ [T81, §2, p. 428].

We have proved in [BM00, Theorem 1.1] that there is a graded field F and a semiramified graded division algebra D over F such that $E(Z_1, ..., Z_r) = Frac(F)$ and $(M, G, U) = Cq(D)$. The valuation of $E(Z_1, ..., Z_r)$ defined by the graded structure of F will be called the canonical valuation of $E(Z_1, ..., Z_r)$.

Let M/E be as above an abelian finite-dimensional field extension, F a graded field with $F_0 = E$ and with support Γ_F and MF the graded subfield of F_{alg} generated by M and F. Assume there is a totally ordered abelian group Δ containing Γ_F with ordering extending that of Γ_F and suppose $\epsilon : \Delta \to G$ is a group homomorphism with kernel Γ_F. Let $\delta_i \in \Delta$ such that $\epsilon(\delta_i) = \sigma_i$, and consider a family $(x_{q_i\delta_i})_{1\leq i\leq r}$ of elements of F^* with $x_0 = 1$ and $gr(x_{q_i\delta_i}) = q_i\delta_i$. For a cocycle $(U, B) \in Z^2_S(G, M^*)$, consider the iterated twisted polynomial ring $A := MF[X_1, ..., X_r; \sigma_1, ..., \sigma_r]$ defined by $X_iX_j = u_{ij}X_jX_i$, $X_ia = \sigma_i(a)X_i$ for all $1 \leq i, j \leq r$ and $a \in MF$ (see [AS, p. 78]), and let I be the left ideal of A, generated by the elements $X_i^{q_i} - b_ix_{q_i\delta_i}$. Then, I is a two-sided ideal of A. Indeed, for any $1 \leq i, j \leq r$, we have $(X_i^{q_i} - b_ix_{q_i\delta_i})X_j = N_i(u_{ij})X_jX_i^{q_i} - X_j\sigma_j^{-1}(b_i)x_{q_i\delta_i} = X_j(\sigma_j^{-1}(N_i(u_{ij}))X_i^{q_i} - \sigma_j^{-1}(b_i)x_{q_i\delta_i}) = X_j\sigma_j^{-1}(N_i(u_{ij}))(X_i^{q_i} - b_ix_{q_i\delta_i})$. So, we can consider the quotient ring $(MF, \epsilon, G, U, B) := A/I$. One can easily see that (MF, ϵ, G, U, B) is a semiramified graded division algebra over F.

In the first section of the present paper we give some new results in Galois theory of graded fields. In particular, we show that if E is a field and M is a defectless normal finite-dimensional valued field extension of E, then GM is a normal graded field extension of GE (and that we can drop the condition 'defectless' when E is Henselian) [Theorem 1.4]. We prove also that if E is a Henselian valued field, M is a finite-dimensional field extension of E, $g = \sum_{i=0}^{n} \tilde{a}_iX^i$ is a homogenizable irreducible polynomial of $GE[X]$ ($a_i \in E$) with a simple root α in GM and $f = \sum_{i=0}^{n} a_iX^i$, then

f is irreducible and has a simple root a in M such that $\tilde{a} = \alpha$ [Proposition 1.5]. We show also that if E is a Henselian valued field with residue characteristic $p > 0$ and L is a purely wild [resp., a defectless simple purely wild] finite-dimensional graded field extension of GE, then there is a defectless field [simple field] extension K of E such that $GK = L$. Furthermore, K can be chosen to be a purely inseparable field extension of E when $char(E) = p$ [Proposition 1.7]. Then, we use Proposition 1.5 to give a new proof of [HW(1)99, Theorem 5.2] which for an arbitrary Henselian valued field E establish a one-to-one correspondence between the set of isomorphism classes of finite-dimensional tame field extensions of E and the set of isomorphism classes of finite-dimensional tame graded field extensions of GE [Corollary 1.8]. Finally, in the last part of this section, we prove some results concerning cyclic and Kummer graded field extensions.

The second section deals with the structure of nicely semiramified valued division algebras. We recall that these algebras were defined in [JW90, Definition p.149] to be defectless finite-dimensional valued division algebras with inertial and totally ramified of radical type maximal subfields. As indicated in [JW90, p.128], nicely semiramified division algebras appeared in [Pl76] as examples of division algebras with nonzero SK_1. They appeared also in the similarity decomposition of any tame division algebra over a Henselian valued field [JW90, Lemma 5.14 and Lemma 6.2]. As seen above, we have proved in [M05, Theorem 4] that any finite-dimensional central division algebra over a Henselian valued field E with an inertial maximal subfield and a totally ramified maximal subfield (not necessarily of radical type) [resp. split by an inertial and a totally ramified field extensions of E] is nicely semiramified. We use in this section this new characterization of nicely semiramified division algebras to give many new properties describing them. We are precisely interested here in determining relations between cyclic division p-algebras and nicely semiramified division algebras over Henselian valued fields (see [Proposition 2.5 and Proposition 2.9]).

In the third section, we define what we call degenerate (valued and graded) division algebras [Definition 3.1] and we determine when a tame semiramified division algebra is degenerate [Proposition 3.6, Proposition 3.9, Proposition 3.11 and Proposition 3.12]. We prove also many equivalent statements for a semiramified graded division algebra D over a graded field F to be degenerate [Proposition 3.16]. In particular, we show that D is degenerate if and only if $D_0 Frac(F)$ is a degenerate maximal subfield of $Cq(D)$ (see [BM00, Definition 0.12]). We use then nondegenerate division algebras to give examples of division p-algebras that are not tensor products of cyclic algebras [Corollary 3.17] (see also Remark 3.19(1)). As a consequence of Corollary 3.17, we get a new proof of [AS78, Theorem 3.2] (see [Corollary 3.18]).

We show in the forth section many results concerning subfields of nondegenerate tame semiramified division algebras of prime power degree over Henselian valued fields. In particular, we prove that a nondegenerate tame semiramified division algebra D over a Henselian valued field E with Γ_D/Γ_E non-cyclic cannot have a totally ramified (non-trivial) subfield [Proposition 4.2]. This enables us to prove that if E has residue characteristic $p > 0$, D is a nondegenerate tame semiramified division algebra of degree a power of p over E with Γ_D/Γ_E non-cyclic, and K is a normal subfield of D, then K is an inertial Galois subfield of D. Hence, there is a subgroup H of $Gal(\bar{D}/\bar{E})$ such that $Gal(K/E) \cong Gal(\bar{D}/\bar{E})/H$. In particular, if K is a Galois maximal subfield of D, then $Gal(K/E) \cong Gal(\bar{D}/\bar{E})$ [Proposition 4.3]. This Proposition generalizes [S78, Theorem 3.1] and [McK05, Theorem 2.3]. We prove also that if $char(\bar{E})$ is arbitrary, $deg(D)$ is a power of a prime integer p and $rk(\Gamma_D/\Gamma_E) \geq 3$, then all abelian subfield of D are inertial over E [Proposition 4.9]. Then, we show that if $rk(\Gamma_D/\Gamma_E)$ is arbitrary but $exp(\Gamma_D/\Gamma_E) = p$, then any non-maximal subfield of D is inertial over E. Also, if K is a non-quaternion normal maximal subfield of D, then K is either cyclic with dimension $\leq p^2$ or inertial over E [Theorem 4.11]. More results for the case where $char(\bar{E})$ does not divide $deg(D)$ are proved at the end of this section.

Finally, in the last section we prove that nondegenerate tame semiramified division algebras of prime power degree over Henselian valued fields are indecomposable [Theorem 5.2]. This can be used to construct indecomposable division algebras of prime exponent (see [Remark 5.4]).

We will give in a next paper many results concerning prime to p-extensions of nondegenerate inertially split division algebras of degree a power of p over a Henselian valued field (where p is a prime positive integer).

1 Graded and valued field extensions

Let F be a graded field and E an algebraic graded field extension of F. We recall that E is called normal over F if every homogenizable irreducible polynomial of $F[X]$ that has a root in E splits completely (i.e., into degree one factors) in $E[X]$. Let g be a homogenizable irreducible polynomial of $F[X]$. Assume g has a root $x \in E\backslash\{0\}$ and let $f_{x,Frac(F)}$ be the minimal polynomial of x over $Frac(F)$. By Claim A.1 comments below, we have $g = af_{x,Frac(F)}$ for some $a \in F^*$. So, by [HW(1)99, Proposition 2.2 and corollary 2.5] $x \in E^*$. Therefore, E is normal over F if and only if for any $x \in E^*$, $f_{x,Frac(F)}$ splits into one degree polynomials in $E[X]$.

Lemma 1. 1 *Let E/F be an algebraic graded field extension. Then, E is normal over F if and only if $Frac(E)$ is a normal field extension of $Frac(F)$.*

Proof. Suppose that E is normal over F and consider a $Frac(F)$-isomorphism σ from $Frac(E)$ into $Frac(E)_{alg}$. Let $x \in E^*$ and let $f_{x,Frac(F)}$ be its minimal polynomial over $Frac(F)$. Obviously, we have $f_{x,Frac(F)}(\sigma(x)) = 0$. So, $\sigma(x) \in E^*$. See that we have $Frac(E) = EFrac(F)$, hence $\sigma(Frac(E)) = Frac(E)$. Therefore, $Frac(E)$ is a normal field extension of $Frac(F)$.
Conversely, suppose that $Frac(E)$ is a normal field extension of $Frac(F)$ and let $x \in E^*$. Then, $f_{x,Frac(F)}$ splits into one degree polynomials in $Frac(E)[X]$. Let y be a root of $f_{x,Frac(F)}$ and let σ be a $Frac(F)$-automorphism of $Frac(E)$ such that $y = \sigma(x)$. By [HW(1)99, Corollary 2.5(d)] the restriction of σ to E is a graded F-automorphism of E. So, $y \in E^*$.

Proposition 1. 2 *Let E/F be a finite-dimensional graded field extension. Then the following statements are equivalent :*
(1) E/F is tame and normal.
(2) E is a Galois graded field extension of F.

Proof. By [HW(1)99, Theorem 3.11(a), (b)] and Lemma 1.1.

Let E/F be a normal finite-dimensional graded field extension. As in the ungraded case, we call the group G consisting of graded F-isomorphisms of E into E_{alg} the Galois group of E/F.

Proposition 1. 3 *Let E/F be a finite-dimensional normal graded field extension of*

Galois group G and let $Fix_G(E)$ be the graded subfield of E, elementwise invariant under the action of G. Then, $Fix_G(E)$ is purely wild over F and E is Galois over $Fix_G(E)$. Moreover, if T is the tame closure of F in E, then $E = TFix_G(E) \cong_g T \otimes_F Fix_G(E)$.

Proof. Since E is normal over F, then by Lemma 1.1 $Frac(E)/Frac(F)$ is a normal field extension. Moreover, we have $Frac(E) \cong E \otimes_F Frac(F)$, so $Gal(E/F) = Gal(Frac(E)/Frac(F))$ -up to a group isomorphism- (See that by [HW(1)99, Corollary 2.5(d)] the restriction to E of any $\sigma \in Gal(Frac(E)/Frac(F))$ is in $Gal(E/F)$). By identification of $Frac(E)$ and $E \otimes_F Frac(F)$, for any element $x \in Frac(E)$ there are $\alpha \in E$ and $\beta \in Frac(F)$ such that $x = \alpha \otimes \beta$ (indeed, we can write $x = \sum a_i \otimes b_i$ where $a_i \in E$ and $b_i \in Frac(F)$. Let $0 \neq b \in F$ such that $h_i := b_i b \in F$ for all i, then we have $x = (\sum a_i h_i) \otimes b^{-1}$). So, for any $\sigma \in Gal(Frac(E)/Frac(F))$, $\sigma(x) = x$ if and only if $\sigma(\alpha) = \alpha$. Accordingly, $Fix_G(Frac(E)) = Fix_G(E) \otimes_F Frac(F) = Frac(Fix_G(E))$. Therefore, by [Karp89, Proposition 7.7, p.283] and [HW(1)99, Theorem 3.11(b) and Lemma 3.6], $E/Fix_G(E)$ is Galois and $Fix_G(E)/F$ is purely wild. By [HW(1)99, (3.8)], $Frac(T)$ is the separable closure of $Frac(F)$ in $Frac(E)$. Furthermore, we have $TFix_G(E) \otimes_F Frac(F) = Frac(T)Fix_G(Frac(E))$. So, by [Karp89, Proposition 7.7, p.283] $TFix_G(E) \otimes_F Frac(F) = Frac(E)$. Therefore, $TFix_G(E) = E$. Remark that $Frac(T)$ is separable over $Frac(F)$ and $Frac(Fix_G(E))$ is purely inseparable over $Frac(F)$. Therefore, T and $Fix_G(E)$ are linearly disjoint over F. Hence, $E \cong_g T \otimes_F Fix_G(E)$.

Let E be a Henselian valued field, M a finite-dimensional field extension of E, $(x_i)_{i=1}^n$ a family of elements of M such that $v(x_i) = v(x_j)$ for all $1 \leq i, j \leq n$ and such that $f := \prod_{i=1}^{n}(X - x_i)$ is an element of $E[X]$. We aim to show here that $\prod_{i=1}^{n}(X - \tilde{x}_i)$ is an element of $GE[X]$. Let for $\bar{m} = (m_1, ..., m_n) \in I\!N^n$, $supp(\bar{m}) = \{i \mid \text{such that } m_i \neq 0\}$, then let $I_k^n = \{\bar{m} = (m_1, ..., m_n) \in I\!N^n \mid card(supp(\bar{m})) = k \text{ and } m_i = 0 \text{ or } 1 \text{ for all } 1 \leq i \leq n\}$. For any $\bar{m} = (m_1, ..., m_n) \in I_k^n$, denote $x^{\bar{m}} = \prod_{i \in supp(\bar{m})} x_i$. Then, $\prod_{i=1}^{n}(X - x_i) = \sum_{i=0}^{n} d_k X^{n-k}$ [resp. $\prod_{i=1}^{n}(X - \tilde{x}_i) = \sum_{i=0}^{n} d'_k X^{n-k}$] , where $d_0 = 1$ [resp. $d'_0 = 1$] and $d_k = \sum_{\bar{m} \in I_k^n} (-1)^k x^{\bar{m}}$ [resp. $d'_k = \sum_{\bar{m} \in I_k^n} (-1)^k \tilde{x}^{\bar{m}}$] (for $1 \leq k \leq n$). See that $v(x^{\bar{m}}) = kv(x)$ for any $\bar{m} \in I_k^n$. So, in $GE_{kv(x)} (= E^{kv(x)}/E^{>kv(x)})$ we have $\bar{d}_k = \overline{\sum_{\bar{m} \in I_k^n} (-1)^k x^{\bar{m}}} = \sum_{\bar{m} \in I_k^n} (-1)^k \tilde{x}^{\bar{m}} = d'_k$. So, $\prod_{i=1}^{n}(X - \tilde{x}_i) \in GE[X]$.

Theorem 1. 4 *Let E be a field and M a defectless finite-dimensional valued field extension of E. If M is normal over E, then GM is normal over GE. If E is Henselian, then this is true even if M is not defectless over E.*

Proof. Let v be the valuation of M and let HM be the henselization of M with respect to v [E72, §16]. Since $\bar{M} \otimes_{\bar{E}} \overline{HE} = \bar{M}$ and $\Gamma_M \cap \Gamma_{HE} = \Gamma_E$, then by [Mor89, Theorem 1] $M \otimes_E HE$ is a field. So, $HM = MHE = M \otimes_E HE$. Moreover, because $M \subseteq HM$, $(GM)_0 = \bar{M} = (GHM)_0$ and $\Gamma_{GM} = \Gamma_{GHM}$, then $GM = GHM$. Similarly, $GE = GHE$. Therefore, we can assume that E is Henselian.
Now, let x be an element of M and let $f_{x,E}$ be the minimal polynomial of x over E. M being normal over E, then we can write $f_{x,E} = \prod_{i=1}^{n}(X - x_i)$, where $x_i = \sigma_i(x)$ for some $\sigma_i \in Gal(M/E)$. Since E is Henselian, then $v(x_i) = v(x)$. Therefore, by the above the polynomial $\prod_{i=1}^{n}(X - \tilde{x}_i)$ is in $GE[X]$. Hence, the minimal polynomial $f_{\tilde{x},Frac(GE)}$ of $\tilde{x}$ over $Frac(GE)$ splits into one degree polynomials in $GM[X]$.

Proposition 1. 5 *Let E be a Henselian valued field, M a finite-dimensional field extension of E, $g = \sum_{i=0}^{n} \tilde{a}_i X^i$ a homogenizable irreducible polynomial of $GE[X]$, with*

$a_i \in E$, and $f = \sum_{i=0}^{n} a_i X^i$. If g has a simple root α in GM, then f is irreducible and has a simple root a in M such that $\tilde{a} = \alpha$.

Proof. Since $X - \alpha$ divides g and g is homogenizable, then $\alpha \in GM^h$ and $gr(\alpha) = gr(X)$. Therefore, $\alpha = \tilde{e}$ for some $e \in M$. Let $f_{\alpha,Frac(GE)}$ be the minimal polynomial of α over $Frac(GE)$. Then, by Claim A.1 comments below, $g = \beta f_{\alpha,Frac(GE)}$ for some $\beta \in GE^h$. Let Γ_M be the value group of M (for the extension of the valuation of E to M), $\Delta_M = \Gamma_M \otimes_{\mathbb{Z}} \mathbb{Q}$ (the divisible hull of Γ_M) and $\mu = gr(g)(\in \Delta_M)$. Then, $gr(\tilde{a_i}) + igr(\tilde{e}) = \mu$ for all $1 \leq i \leq n$ when $a_i \neq 0$. See that $\tilde{a}_0 \neq 0$ (because g is irreducible). Hence $e \neq 0$.
We distinguish the following two cases:
i) If $\mu = 0$, then setting $Y = \tilde{e}^{-1}X$, we get $g = \sum_{i=0}^{n} \overline{a_i e^i} Y^i$, with $Y = 1$ as a simple root in $\bar{M}$. Therefore, by Hensel Lemma [E72, Corollary 16.6(iv)], there is a $b \in V_M$ (where V_M is the valuation ring of the extension of the valuation of E to M) such that $\bar{b} = 1$ and $\sum_{i=0}^{n} a_i (eb)^i = 0$. So, eb is a root of f and $\widetilde{eb} = \tilde{e} = \alpha$.
ii) If $\mu \neq 0$, then setting $c = a_k e^k$, where a_k is a nonzero coefficient of f, $Y = \tilde{e}^{-1}X$ and $h = \tilde{c}^{-1}g$, we get $h = \sum_{i=0}^{n} \overline{c^{-1}a_i e^i} Y^i$. So again by Hensel Lemma, there is $b \in V_M$ such that $c^{-1}\sum_{i=0}^{n} a_i (eb)^i = 0$ and $\bar{b} = 1$. Hence, eb is a root of f with $\widetilde{eb} = \alpha$.
Now to prove that f is irreducible, remark that because $g = \beta f_{\alpha,Frac(GE)}$, then $[GE[\tilde{a}] : GE] = deg(g) = deg(f)$. Moreover, since $f(a) = 0$, then the minimal polynomial $f_{a,E}$ of a over E divides f, hence $[E[a] : E] \leq deg(f)$. On the other hand, we have $[E[a] : E] \geq [G(E[a]) : GE] \geq [GE[\tilde{a}] : GE]$. Therefore, $[E[a] : E] = [G(E[a]) : GE] = [GE[\tilde{a}] : GE] = deg(g) = deg(f)$. So, f is irreducible. (The reader can see that $E[a]$ is necessaraly defectless over E).
α being a simple root of g, then g has $deg(g)$ distinct roots. Let N be a normal field

extension of E which contains M. By Theorem 1.4, GN is a normal graded field extension of GE. Hence all the roots of g are in GN. Moreover, by the previous part of this proof, for any root δ of g there is a root d of f in N such that $\tilde{d} = \delta$. Therefore, f has $deg(f)$ distinct roots. This shows that a is a simple root of f.

Remark 1.6 Let E be a Henselian valued field, $g = \sum_{i=0}^{n} \tilde{a}_i X^i$ a homogenizable irreducible polynomial of $GE[X]$, with $a_i \in E$, and $f = \sum_{i=0}^{n} a_i X^i$. If f has a root a such that $\tilde{a}$ is a root of g, then using the same arguments as in Proposition 1.5, f is irreducible in $E[X]$ and $G(E[a]) = GE[\tilde{a}]$.

Proposition 1. 7 *Let E be a Henselian valued field with residue characteristic $p > 0$ and L a purely wild [resp., simple purely wild] finite-dimensional graded field extension of GE, then there is a defectless field extension [resp., simple field extension] K of E such that $GK = L$. If $char(E) = p$, then K can be chosen to be a purely inseparable field extension of E.*

Proof. Let N be a normal field extension of E such that $L \subseteq GN$. Assume first that $L = GE[\tilde{a}]$ (for some $a \in N$) and let $p^n = [L : GE]$. Since $Frac(L)$ is purely inseparable over $Frac(GE)$, then the minimal polynomial of $\tilde{a}$ over $Frac(GE)$ is $g := X^{p^n} - \tilde{a}^{p^n}$. We have $L \cap Frac(GE) \subseteq GN \cap Frac(GE) = GE$. So, there is $b \in E$ such that $\tilde{a}^{p^n} = \tilde{b}$. Let $f = X^{p^n} - b$ and let x be a root of f in N. Clearly, $\tilde{x}$ is a root of g. So, $\tilde{x} = \tilde{a}$ (because $\tilde{x}^{p^n} = \tilde{a}^{p^n}$ in GN). By remark 1.6, we have $G(E[x]) = GE[\tilde{x}] = GE[\tilde{a}] = L$.

Now, let L be an arbitrary finite-dimensional purely wild graded field extension of GE. Then, we can write $L = GE[\tilde{a}_1, ..., \tilde{a}_r]$. Therefore, one can easily proceed by induction on r to end the proof.

As a consequence of Proposition 1.5, we have the following Corollary that gives explicitly the correspondence between (finite-dimensional) tame valued field extensions over a Henselian valued field and tame graded field extensions.

Corollary 1. 8 [*HW(1)99, Theorem 5.2*] *Let E be a Henselian valued field. Then, the map $K \mapsto GK$ gives a one-to-one correspondence between the set of isomorphism classes of finite-dimensional tame field extensions of E and the set of isomorphism classes of finite-dimensional tame graded field extensions of GE. Moreover, K is a Galois tame (finite-dimensional) field extension of E if and only if GK is a Galois (finite-dimensional) graded field extension of GE and in such a case $Gal(K/E)$ is isomorphic to $Gal(GK/GE)$.*

Proof. If K is a tame field extension of E, then obviously GK is a tame graded field extension of GE. Let K' be a tame field extension of E such that $K' \cong K$. Since E is Henselian, then clearly $GK \cong_g GK'$. Conversely, if L is a tame finite-dimensional graded field extension of GE, then we can write $L = GE[\tilde{x}_1, ..., \tilde{x}_r]$, where $x_i \in E_{alg}$ with $\tilde{x}_i$ separable over $Frac(GE)$. Assume first that $r = 1$ and let $M = E[x_1]$. By Proposition 1.5, there is $a_1 \in M$ such that $\tilde{a}_1 = \tilde{x}_1$ and $[E[a_1] : E] = [L : GE]$. Hence, $G(E[a_1]) = L$. By induction, assume there are $a_1, ..., a_{r-1} \in E[x_1, ..., x_{r-1}]$ such that $G(E[a_1, ..., a_{r-1}]) = GE[\tilde{x}_1, ..., \tilde{x}_{r-1}]$. Then, again by Proposition 1.5, there is $a_r \in E[x_1, ..., x_r]$ such that $G(E[a_1, ..., a_r]) = L$ (this is obtained by applying Proposition 1.5 on the field extension M/E', where $E' = E[a_1, ..., a_{r-1}]$). Clearly, $K := E[a_1, ..., a_r]$ is tame over E. If K' is an other tame field extension of E such that $GK' \cong_g L$, then by Proposition 1.5, there is $a'_1 \in K'$ such that $G(E[a'_1]) \cong_g G(E[a_1])$ and both a'_1 and a_1 have the same minimal polynomial over E. So, $E[a'_1] \cong E[a_1]$. By induction, we prove that $K' \cong K$.

Now, let K be a Galois tame finite-dimensional field extension of E. By Proposition 1.2 and Theorem 1.4, GK is a Galois graded field extension of GE. Since v is Henselian, then for any $\sigma \in Gal(K/E)$, we can define a graded GE-automorphism $\tilde{\sigma} : GK \to GK$ by extending the mapping $\tilde{x} \mapsto \widetilde{\sigma(x)}$. Let $\phi : G \to Gal(GK/GE)$ be the group homomorphism defined by $\phi(\sigma) = \tilde{\sigma}$, B be the valuation ring of the extension of v to K and let G^v the ramification group of B over E. See that $\phi(\sigma) = id_{GK}$ if and only if $v(\frac{\sigma(x)}{x} - 1) > 0$, for any $x \in K^*$. Otherwise said, $\phi(\sigma) = id_{GK}$ if and only if $\sigma \in G^v$. But $G^v = \{id_K\}$ (since K is tame over E). Therefore, ϕ is injective. Remark that $Gal(K/E)$ and $Gal(GK/GE)$ have the same cardinality (since $Gal(GK/GE) \cong Gal(Frac(GE)/Frac(GE))$). So, ϕ is a group isomorphism.
Let M be a tame finite-dimensional field extension of E such that GM is a Galois graded field extension of GE and consider a Galois tame finite-dimensional field extension N of E containing M. By the above GN is a Galois graded field extension of GE [resp., of GM] and $Gal(N/E) \cong Gal(GN/GE)$ [resp., $Gal(N/M) \cong Gal(GN/GM)$]. Since GM is a Galois graded field extension of GE, then $Gal(GN/GM)$ is a normal subgroup of $Gal(GN/GE)$, therefore $Gal(N/M)$ is a normal subgroup of $Gal(N/E)$. Hence, M is a Galois field extension of E.

Let F be a graded field and K a finite-dimensional graded field extension of F. For an arbitrary abelian group A -namely for $A = \Gamma_K/\Gamma_F$- and a family a_1, a_2..., a_r of elements of A, we say that a_1, a_2..., a_r are independent if the subgroup $\langle a_1, a_2, .., a_r\rangle$ of A, generated by a_1, a_2..., a_r, equals $\oplus_{i=1}^r \langle a_i\rangle$. We recall that K is called totally ramified of radical type ($TRRT$) over F if there are homogeneous elements $t_1, ..., t_r \in F^*$ and nonnegative integers $n_1, ..., n_r$ such that the following conditions are satisfied:
(1) $K = F[t_1^{1/n_1}, ..., t_r^{1/n_r}]$ and

(2) $gr(t_i^{1/n_i}) + \Gamma_F$ $(1 \leq i \leq r)$ are independent elements of Γ_K/Γ_F, with order n_i, respectively.

One can see that in the same way as for $TRRT$ valued field extensions (See [JW90, Lemma 4.1]), a totally ramified finite-dimensional graded field extension K of F is $TRRT$ over F if and only if there is a subgroup G of K^*/F^* such that the mapping $G \to \Gamma_K/\Gamma_F$, defined by $xF^* \mapsto gr(x) + \Gamma_F$, is a group isomorphism.

Lemma 1. 9 *Let F be a graded field and K a totally ramified finite-dimensional graded field extension of F. Then, K is $TRRT$ over F.*

Proof. [M05, Lemma 1].

Corollary 1. 10 [*Sch50, p.64, Theorem 3*] *Let E be a Henselian valued field and K a tame totally ramified finite-dimensional field extension of E, then K is totally ramified of radical type.*

Proof. By Lemma 1.9 GK is totally ramified of radical type. Write $GK = GE[\alpha_1] \otimes_{GE} ... \otimes_{GE} GE[\alpha_r]$ for some $\alpha_1, ..., \alpha_r \in GK^*$ where $gr(\alpha_i) + \Gamma_{GE}$ are independent elements of Γ_{GK}/Γ_{GE}. Assume first that $r = 1$. By Proposition 1.5 there is $x_1 \in K$ such that $\tilde{x}_1 = \alpha_1$ and $[E[x_1] : E] = [GE[\alpha_1] : GE] = [GK : GE]$. Hence, $E[x_1] = K$ and $v(E[x_1]) = \langle gr(\alpha_1) + \Gamma_{GE} \rangle = \langle v(x_1) + \Gamma_E \rangle$. More generally, for an arbitrary positive integer r, by the above there are $x_i \in K$ such that $[E[x_i] : E] = [GE[\alpha_i] : GE]$ and $v(x_i) + \Gamma_E$ generates $\Gamma_{E[x_i]}/\Gamma_E$. Hence, by [Mor89, Theorem 1] $K = E[x_1] \otimes_E ... \otimes_E E[x_r]$.

Let E/F be a Galois finite-dimensional graded field extension. In the same way as for ungraded fields, one may define the norm $N_{E/F}$ of E/F (i.e. $N_{E/F} = \prod_{\sigma \in Gal(E/F)} \sigma(x)$ for all $x \in E$). Then, the following lemma is a direct consequence of [L84, Theorem 6.1, p323].

Lemma 1. 11 *Let E/F be a finite-dimensional cyclic graded field extension with dimension n and with Galois group generated by σ and $x \in E^*$. Then, $N_{E/F}(x) = 1$ if and only if there exists $y \in E^*$ such that $x = y\sigma(y)^{-1}$*

Proof. Assume that $N_{E/F}(x) = 1$. Then, $N_{Frac(E)/Frac(F)}(x)(= N_{E/F}(x)) = 1$. So, by [L84, Theorem 6.1, p. 323] there is $z \in Frac(E)^*$ such that $x = z\sigma(z)^{-1}$. We may assume $z \in E^*$. Write $z = z_1 + ... + z_r$, where z_i $(1 \leq i \leq r)$ are nonzero homogeneous elements of E with $gr(z_i) < gr(z_{i+1})$, for all $1 \leq i < r$. Since $\sigma(z)x = z$ and x is homogeneous, then for all $1 \leq i \leq r$, $\sigma(z_i)x = z_i$. The converse is clear.

Remark that an alternative cohomological proof of Lemma 1.11 can be obtained by considering the G-module E^*.

Proposition 1. 12 *Let F be a graded field with characteristic not dividing a positive integer n, and assume F_0 contains a primitive n^{th} root of unity ζ. Then, the following statements hold :*

(1) If E is a cyclic graded field extension of F with dimension n, then there is $x \in E^*$ such that $E = F[x]$, $x^n \in F^*$ and $Gal(E/F)$ is generated by the graded F-isomorphism σ defined by $\sigma(x) = \zeta x$.

(2) Conversely, if $a \in F^*$ and x is a root of the polynomial $X^n - a$ in F_{alg}, then $F[x]$ is a cyclic graded field extension of F with $[F[x] : F] = m$ dividing n and $x^m \in F$.

Proof. (1) Let σ a generator of $Gal(E/F)$. We have $N_{E/F}(\zeta^{-1}) = 1$, so by Lemma 1.11 there is $x \in E^*$ such that $\sigma(x) = \zeta x$. Accordingly, $\sigma(x^n) = \sigma(x)^n = (\zeta x)^n = x^n$. Hence, $x^n \in F^*$. Since $\sigma^i(x)(= \zeta^i x)$ $(1 \leq i \leq n)$ are pairwise distinct, then the minimal polynomial of x over $Frac(F)$ is $X^n - x^n$. So, $F[x] = E$.
(2) By [L84, Theorem 6.2, p. 324] $Frac(F)(x)/Frac(F)$ is cyclic with dimension m dividing n and with $x^m \in Frac(F)^*$. Hence, $F[x]/F$ is cyclic of dimension m and by [HW(1)99, Corollary 2.5(b)] $x^m \in F[x]^* \cap Frac(F)^* = F^*$.

Let F be a graded field with characteristic $p > 0$. Then, Galois graded p-extensions of F are inertial over F, so they are exactly graded fields of the form KFF, where K runs over Galois p-extensions of F_0. This because Galois graded p-extensions are necessarily inertial. Therefore, a graded field extension E/F of dimension a power of p is cyclic if $E = F(x_1, ..., x_n)$, where $x = (x_1, ..., x_n) \in W_n(E_0)$ and $(x_1^p, ..., x_n^p) - (x_1, ..., x_n) \in W_n(F_0)$ (here $W_n(E_0)$ is the ring of Witt vectors associated to the field E_0). In particular, cyclic extensions of degree p of F are $F[x]$, where x is a root of a polynomial $X^p - X - a$ for some $a \in F_0$ with $x \notin F_0$.

Let F be a graded field and K a finite-dimensional abelian graded field extension of F. We say that K is a Kummer graded field extension of F if F_0 contains a primitive m^{th} root of unity, where m is the exponent of $Gal(K/F)$. In such a case, we denote $KUM(K/F) = \{x \in K^* \mid x^m \in F\}$ and $kum(K/F) = KUM(K/F)/F^*$ (i.e., the quotient group). As for Kummer field extensions, one can see that $kum(K/F)$ is isomorphic to $Gal(K/F)$.

Let F be a graded field of characteristic not dividing a positive integer n. Readily, F_0 contains a primitive n^{th} root of unity if and only if $Frac(F)$ does (indeed, if ξ is a primitive n^{th} root of unity in $Frac(F)$, then clearly $F[\xi]$ is a graded field extension of F. So, $[F[\xi] : F] = [Frac(F[\xi]) : Frac(F)] = 1$. Hence $\xi \in F_0$). Therefore, for any finite-dimensional abelian graded field extension K of F, K is a Kummer graded field extension of F if and only if $Frac(K)$ [resp. $HFrac(K)$] is a Kummer field extension of $Frac(F)$ [resp. $HFrac(F)$]. Moreover, if K is a Kummer graded field extension of F, then $kum(K/F)$ embeds canonically in $kum(Frac(K)/Frac(F))$ [resp. in $kum(HFrac(K)/HFrac(F))$]. Since both groups have the same cardinality, then -up to a group isomorphism- $kum(Frac(K)/Frac(F)) = kum(K/F)$ [resp. $kum(HFrac(K)/HFrac(F)) = kum(K/F)$]. Obviously, this isomorphism can be deduced easily from the fact that $Gal(K/F) \cong_g Gal(Frac(K)/Frac(F))$ [resp., $Gal(K/F) \cong_g Gal(HFrac(K)/HFrac(F))$], but the above shows that every class of $kum(Frac(K)/Frac(F))$ [resp. $kum(HFrac(K)/HFrac(F))$] can be represented by an element of $KUM(K/F)$.

Proposition 1. 13 *Let F be a graded field and K a totally ramified graded field extension of F. Let m be the exponent of Γ_K/Γ_F. Then, the following statements are equivalent :*
(1) K is a Galois graded field extension of F.
(2) K is a Kummer graded field extension of F.
(3) F_0 contains a primitive m^{th} root of unity.
If K satisfies these conditions, then $Gal(K/F) \cong kum(K/F) \cong \Gamma_K/\Gamma_F$.

Proof. By [HW(1)99, Proposition 3.3].

2 Nicely semiramified division algebras over Henselian valued fields

Let F be a graded field and D a graded central division algebra over F. We recall that in the same way as for valued central division algebras, D is called nicely semiramified (NSR) (over F) if it has an inertial and a totally ramified (of radical type) maximal graded subfields. We recall also that D is NSR (over F) if and only if $Cq(D)$ is NSR (over $Frac(F)$) ([B98, Proposition 6.4]). Moreover, $Cq(D)$ is NSR (over $Frac(F)$) if and only if $HCq(D)$ is NSR (over $HFrac(F)$). Indeed, assume that $Cq(D)$ is NSR and let K [resp. L] be an inertial [resp. a $TRRT$] maximal subfield of $Cq(D)$. Then, by [Mor89, Theorem 1], $HK(= K \otimes_{Frac(F)} HFrac(F))$ [resp. $HL(= L \otimes_{Frac(F)} HFrac(F))$] is an inertial [resp. a $TRRT$] maximal subfield of $HCq(D)$. Conversely, if $HCq(D)$ is NSR, then it has an inertial [resp. a $TRRT$] maximal subfield K' [resp. L']. So GK' [resp. GL'] is an inertial [resp. a totally ramified] maximal graded subfield of $D(\cong_g GHCq(D))$. Hence D is NSR. Therefore, by the above, $Cq(D)$ is NSR.

The following lemma is the analogue of [JW90, Theorem 4.4]. It gives equivalent statements for a graded central division algebra over a graded field to be NSR. In condition 3(i) of this lemma, the graded field extensions $L^{(i)}$ are said to be linearly disjoint if $L^{(1)} \otimes_F .... \otimes_F L^{(k)}$ is a graded field.

Lemma 2. 1 *Let F be a graded field and D a graded central division algebra of degree n over F. Then the following statements are equivalent :*
(1) D is NSR.
(2) D is split by an inertial and a totally ramified graded field extensions of F.

(3) $D \cong_g (L^{(1)}/F, \sigma_1, t_1) \otimes_F ... \otimes_F (L^{(k)}/F, \sigma_k, t_k)$, *where* $L^{(i)}$, σ_i *and* t_i *satisfy the following conditions :*

(i) $L^{(i)}$ *are linearly disjoint cyclic inertial graded field extensions of* F *with dimension* $[L^{(i)} : F] = n_i$ *and with Galois group generated by* σ_i $(1 \leq i \leq k)$, $\prod_{i=1}^{k} n_i = n$,

(ii) t_i *are nonzero homogeneous elements of* F *such that* $gr(t_i^{n/n_i}) + n\Gamma_F$ *are independent elements of* $\Gamma_F/n\Gamma_F$, *with order* n_i $(1 \leq i \leq k)$, *respectively.*

In such a case, $\Gamma_D/\Gamma_F = \oplus_{i=1}^{k}\langle gr(t_i) + \Gamma_F\rangle$.

Proof. [M05, Lemma 2].

Lemma 2. 2 *Let* E *be a Henselian valued field and* D *a defectless central division algebra over* E. *Then the following statements are equivalent :*
(1) D *is* NSR *over* E.
(2) GD *is* NSR *over* GE.

Proof. [M05, Lemma 3].

Theorem 2. 3 *Let* E *be a Henselian valued field and* D *a defectless central division algebra over* E. *Then the following statements are equivalent:*
(1) D *is* NSR.
(2) D *has an inertial and a totally ramified maximal subfields.*
(3) D *is split by an inertial and by a totally ramified field extensions of* E.

Proof. [M05, Theorem 4].

Corollary 2. 4 *Let* E *be a Henselian valued field and* D *a tame semiramified division algebra over* E, *then the following statements are equivalent.*

(1) D is nicely semiramified.
(2) D has a totally ramified maximal subfield.
(3) D is split by a totally ramified field extension of E.

Proof. Since D is tame semiramified, then $\bar{D}$ is an abelian field extension of $\bar{E}$. So, D has an inertial maximal subfield (the inertial lift of $\bar{D}$ over E in D). Therefore, our Lemma follows by Theorem 2.3.

Proposition 2. 5 *Let E be a Henselian valued field of characteristic $p > 0$ and D a tame semiramified division p-algebra over E. Then, the following statements are equivalent.*
(1) D is nicely semiramified.
(2) D is a tensor product of cyclic algebras.
(3) D has a purely inseparable maximal subfield.

Proof. (1) ⇒ (2) This follows by [JW90, Theorem 4.4].
(2) ⇒ (3) Assume that $D = D_1 \otimes_E ... \otimes_E D_r$, where D_i are cyclic algebras. By [A61, Theorem 26, p. 107], each D_i contains a (simple) purely inseparable maximal subfield K_i. So, $K := K_1 \otimes_E ... \otimes_E K_r$ is a purely inseparable maximal subfield of D.
(3) ⇒ (1) Let K be a purely inseparable maximal subfield of D. Since D is defectless over E and $\bar{D}$ is separable over $\bar{E}$, then K is totally ramified over E. So, D is nicely semiramified (by Corollary 2.4).

Proposition 2. 6 *Let E be a Henselian valued field and D an inertially split defectless central division algebra over E. Then the following statements hold :*
(1) D is semiramified if and only if $\bar{D}$ is a field.

(2) Let L be a field extension of E in D and let C be the centralizer of L in D. If D is semiramified, then C is also semiramified. Furthermore, if L is inertial over E and D is nicely semiramified, then C is nicely semiramified.
(3) Suppose D is semiramified and $D = D_1 \otimes_E D_2$ with D_1 and D_2 inertially split. Then, each D_i is semiramified, $\bar{D} = \bar{D}_1 \otimes_{\bar{E}} \bar{D}_2$, and $\Gamma_D/\Gamma_E = \Gamma_{D_1}/\Gamma_E \times \Gamma_{D_2}/\Gamma_E$ (via the canonical inclusion $\Gamma_{D_i}/\Gamma_E \to \Gamma_D/\Gamma_E$). Furthermore, if D_1 and D_2 are nicely semiramified, then D is nicely semiramified.
(4) If D is semiramified with a cyclic inertial maximal subfield, then D is nicely semiramified and Γ_D/Γ_E is cyclic.

Proof. (1) This follows easily from the equality $(\Gamma_D : \Gamma_E) = [Z(\bar{D}) : \bar{E}]$ obtained from the isomorphism θ_D, since $Z(\bar{D})$ is separable over $\bar{E}$ (see [JW90, Lemma 5.1]).
(2) Clearly, C is also inertially split. If D is semiramified, then by (1) C is also semiramified (because $\bar{C} \subseteq \bar{D}$). Assume L is inertial over E and D is nicely semiramified, and let K be a totally ramified field extension of E that splits D, then by [Mor89, Theorem 1], $K \otimes_E L$ is a totally ramified field extension of L that splits C, therefore by Corollary 2.4 C is nicely semiramified.
(3) By (1) D_i are semiramified. Let L_i be the inertial lift of the field $\bar{D}_i$ over E in D_i -see [JW90, Theorem 2.8]-, and let $L = L_1 \otimes_E L_2$, which is a subfield of D. Clearly, L is inertial over E, since it is a compositum of two inertial extensions of E. Hence, $\bar{L} = \bar{D}_1 \otimes_{\bar{E}} \bar{D}_2$, which by dimension count must be all of $\bar{D}$. Now, Γ_{D_1}/Γ_E acts trivially on $\bar{D}_2$ via θ_D, as D_1 centralizes D_2; likewise Γ_{D_2}/Γ_E acts trivially on $\bar{D}_1$. Hence, $(\Gamma_{D_1}/\Gamma_E) \cap (\Gamma_{D_2}/\Gamma_E)$ must be trivial, since it acts trivially on all of $\bar{D}_1 \otimes_{\bar{E}} \bar{D}_2 = \bar{D}$ and θ_D is injective. Hence, $(\Gamma_{D_1}/\Gamma_E) \times (\Gamma_{D_2}/\Gamma_E) \subseteq \Gamma_D/\Gamma_E$, and since these groups have the same cardinality, they must be equal. Suppose that D_1 and D_2 are nicely

semiramified, and let K_i be a totally ramified maximal subfield of D_i, then by [Mor89, Theorem 1] $K_1 \otimes_E K_2$ is a totally ramified maximal subfield of D. So, D is nicely semiramified.
(4) Let K be a cyclic inertial maximal subfield of D, then $\bar{K}(= \bar{D})$ is a cyclic field extension of $\bar{E}$. So, Γ_D/Γ_E is cyclic (since θ_D is an isomorphism). Let $\gamma + \Gamma_E$ be a generator of Γ_D/Γ_E and let x be an element of D with valuation γ. Then, $E[x]$ is a totally ramified maximal subfield of D. Hence, D is nicely semiramified.

Corollary 2. 7 *Let E be a Henselian valued field of residue characteristic $p > 0$ and D a tame central division algebra of degree a power of p over E. Then, D is semiramified if and only if $\bar{D}$ is a field.*

Proof. Let K be a tame maximal subfield of D. Since D has degree a power of p, then K is inertial over E. Therefore, our corollary follows by Proposition 2.6.

Remark 2.8 One can easily see that there are graded algebra analogues of Corollary 2.4, Proposition 2.5, Proposition 2.6 and Corollary 2.7. Indeed, let F be a graded field and D a graded central division algebra over F. Then D is semiramified if and only if $HCq(D)$ is semiramified, and D is nicely semiramified if and only if $HCq(D)$ is nicely semiramified.

Let A be a tame valued division algebra over a Henselian valued field E. We will say that A is t-indecomposable, if for any tame division algebras B and C over E such that $A \cong B \otimes_E C$, necessarily B or C is trivial.

Proposition 2. 9 *Let F be a graded field of characteristic $p > 0$, D a semirami-*

fied graded division p-algebra over F and K an immediate valued field extension of $Frac(F)$ for its canonical valuation (i.e., such that $\bar{K} = \overline{Frac(F)}$ and $\Gamma_K = \Gamma_{Frac(F)}$). Then, the following statements are equivalent :
(1) D is cyclic.
(2) D_0 is a cyclic field extension of F_0.
(3) Γ_D/Γ_F is a cyclic group.
(4) D has a simple purely wild maximal graded subfield.
(5) D is indecomposable and nicely semiramified.
(6) D is nicely semiramified and $exp(D) = deg(D)$.
(7) $Cq(D)\otimes_{Frac(F)}K$ is nicely semiramified and $exp(Cq(D)\otimes_{Frac(F)}K) = deg(Cq(D)\otimes_{Frac(F)} K)$.
(8) $Cq(D) \otimes_{Frac(F)} K$ is cyclic and indecomposable.
Furthermore, if $HFrac(F) \subseteq K$, then the above statements are equivalent to the following condition :
(9) $Cq(D) \otimes_{Frac(F)} K$ is cyclic and t-indecomposable.

Proof. (1) $\Rightarrow$ (2) Assume that D has a cyclic maximal graded subfield K. By [HW(1)99, Theorem 3.11] K is tame (hence inertial) over F. So, by [HW(1)99, Remark 3.1] $D_0(= K_0)$ is a cyclic field extension of F_0.
(2) $\Leftrightarrow$ (3) By [B98, Proposition 6.1].
(3) $\Rightarrow$ (4) Let $x \in D^*$ such that $gr(x) + \Gamma_F$ generates Γ_D/Γ_F. Then, $F[x]$ is a simple totally ramified -hence a purely wild- maximal graded subfield of D.
(4) $\Rightarrow$ (6) Clearly D_0F is an inertial maximal graded subfield of D. Let L be a simple purely wild maximal graded subfield of D. As D_0 is separable over F_0, then L is totally ramified over F. So, D is nicely semiramified, and $\Gamma_D/\Gamma_F(= \Gamma_L/\Gamma_F)$ is

cyclic (because L is simple over F). Therefore, by [B98, Proposition 6.5] $exp(D) = exp(\Gamma_D/\Gamma_F) = (\Gamma_D : \Gamma_F) = deg(D)$.

(6) $\Rightarrow$ (5) Clear.

(5) $\Rightarrow$ (1) By Lemma 2.1.

(6) $\Rightarrow$ (7) We have $G(Cq(D) \otimes_{Frac(F)} K) \cong_g D$. Indeed, let HK be the henselization of K with respect to the (considered) valuation of K, then by [HW(1)99, Corollary 5.7] the following diagram is commutative :

$$\begin{array}{ccc} TBr(HFrac(F)) & \stackrel{Ext}{\rightarrow} & TBr(HK) \\ \cong\downarrow & & \cong\downarrow \\ GBr(F) & \stackrel{Ext}{\rightarrow} & GBr(GHK) \end{array}$$

(where the horizontal maps are the scalar extension homomorphisms and the vertical ones are the canonical group isomorphisms seen in the Preliminaries). Since K is immediate over $Frac(F)$, then $GK \cong_g GHK \cong_g F$. Moreover, we have $G(Cq(D) \otimes_{Frac(F)} K) = G(Cq(D) \otimes_{Frac(F)} HK)$. Hence, $G(Cq(D) \otimes_{Frac(F)} K) \cong_g D$. Therefore, $Cq(D) \otimes_{Frac(F)} K$ is nicely semiramified and $deg(Cq(D) \otimes_{Frac(F)} K) = deg(D) = exp(D) = exp(Cq(D) \otimes_{Frac(F)} K)$.

(7) $\Rightarrow$ (8) Obviously, $Cq(D) \otimes_{Frac(F)} K$ is indecomposable, so D is indecomposable. Moreover, as seen above, we have $D \cong_g G(Cq(D) \otimes_{Frac(F)} HK)$. So, D is nicely semiramified. Hence, by Lemma 2.1, D (so $Cq(D)$) is cyclic. Therefore, $Cq(D) \otimes_{Frac(F)} K$ is cyclic.

Now, if $HFrac(F) \subseteq K$, then (8) $\Rightarrow$ (9) is evident.

(9) $\Rightarrow$ (1) Since $Cq(D) \otimes_{Frac(F)} K$ is t-indecomposable, $GK \cong_g F$ and $G(Cq(D) \otimes_{Frac(F)} K) \cong_g D$, then D is indecomposable (it suffices to use the canonical group isomorphism $TBr(K) \rightarrow GBr(GK)$). Moreover, by [A61, Theorem 26, p. 107] $Cq(D) \otimes_{Frac(F)} K$ has a purely inseparable maximal subfield L. See that $Cq(D) \otimes_{Frac(F)} K$ is defect-

less over K and $\overline{Cq(D) \otimes_{Frac(F)} K}(\cong D_0)$ is separable over $\bar{K}(\cong F_0)$, so L is totally ramified over K. Therefore, $Cq(D) \otimes_{Frac(F)} K$ (hence D) is nicely semiramified. So, by Lemma 2.1 D is cyclic.

Remark 2.10 More generally, if F is a graded field of characteristic $p > 0$ and D is a graded central division p-algebra over F, then in the same way as in [A61, Theorem 26, p. 107], one can show that D is cyclic if and only if D contains a simple purely wild maximal graded subfield (see Theorem C.6 comments below).

Let p be a prime positive integer and G a finite abelian p-group. Then, we can write $G = \langle \sigma_1 \rangle \oplus ... \oplus \langle \sigma_r \rangle$, where $\langle \sigma_i \rangle$ is the cyclic group generated by σ_i. We recall that the number r is called the rank of G. We write $r = rk(G)$.

(2.11) Now, let E be a Henselian valued field, D a nicely semiramified division algebra of prime power degree over E and L a totally ramified of radical type maximal subfield of D. Write $L = E[t_1^{1/n_1}, ..., t_r^{1/n_r}]$, where $v(t_i^{1/n_i}) + \Gamma_E$ $(1 \leq i \leq r)$ are independent elements of Γ_L/Γ_E, with order n_i, respectively. Then, the integer r depends only on D (because $\oplus_{i=1}^{r} \langle v(t_i^{1/n_i}) + \Gamma_E \rangle = \Gamma_L/\Gamma_E = \Gamma_D/\Gamma_E$). We call r the radical length of D and we write $r = rl(D)$. By definition, we have $rl(D) = rk(\Gamma_D/\Gamma_E)$.

In the same way, for a graded field F and a nicely semiramified graded division algebra D over F, we define $rl(D) = rk(\Gamma_D/\Gamma_F)$. Obviously, we have $rl(D) = rl(HCq(D))$.

Corollary 2. 12 *Let F be a graded field of characteristic $p > 0$ and D a semiramified* *graded division p-algebra over F. Then, D is nicely semiramified with radical length r if and only if D is a tensor product of r cyclic graded division algebras.*

Proof. If $D = D_1 \otimes_F ... \otimes_F D_r$, where D_i are cyclic graded division algebras, then by Proposition 2.5, $HCq(D)$ is nicely semiramified. So, D is nicely semiramified. Moreover, by the graded version of Proposition 2.6(3) and Proposition 2.9, each D_i is semiramified (since D_i are split by inertial graded field extensions of F) ; $\Gamma_D/\Gamma_F = \Gamma_{D_1}/\Gamma_F \oplus ... \oplus \Gamma_{D_r}/\Gamma_F$ and Γ_{D_i}/Γ_F are cyclic. So, $rl(D) = r$. The converse follows by Lemma 2.1.

3 Nondegenerate semiramified valued and graded division algebras

Let M/E be a field extension and D a central division algebra over E. We denote by D_M the central division algebra over M similar to $D \otimes_E M$ with respect to $Br(M)$. We recall that if M is a subfield of D, then up to an algebra isomorphism, we have $D_M = C_D^M$.

Definition 3.1 Let E be a Henselian valued field and D a central division algebra of prime power degree over E. We say that D is degenerate if there is an inertial field extension M of E in D such that D_M is nicely semiramified (over M) with radical length $r \geq 2$.

Proposition 3. 2 *Let E be a Henselian valued field and D an inertially split central division algebra of prime power degree over E. Then, D is degenerate if and only if* *there is a subfield K of D such that Γ_K/Γ_E is non-cyclic.*

Proof. Assume D is degenerate and let M be an inertial subfield of D such that D_M is nicely semiramified with $rl(D_M) \geq 2$. Let K be a totally ramified maximal subfield of C_D^M, then $\Gamma_{C_D^M}/\Gamma_M = \Gamma_K/\Gamma_E$. So Γ_K/Γ_E is non-cyclic.
Conversely, suppose that there is a subfield K of D such that Γ_K/Γ_E is non-cyclic. Let L be a maximal subfield of D that contains K and let M be 'the' inertial lift of $\bar{L}$ over E in L, then L is a totally ramified maximal subfield of the inertially split division algebra C_D^M. So, C_D^M is nicely semiramified with $rl(C_D^M) \geq 2$.

Corollary 3. 3 *Let E be a Henselian valued field such that $\bar{E}$ is finite. Then, any inertially split division algebra D of prime power degree over E is nondegenerate cyclic nicely semiramified (over E) with $exp(D) = deg(D)$.*

Proof. Since $\bar{E}$ is finite, then $\bar{D}$ is a cyclic field extension of $\bar{E}$. Moreover, since D is inertially split over E, then Γ_D/Γ_E is isomorphic to $Gal(\bar{D}/\bar{E})$. So, D is semiramified and by Proposition 3.2, D is nondegenerate. Let v be the extension of the valuation of E to D and let $x \in D$ such that $v(x)+\Gamma_E$ generates Γ_D/Γ_E. Then, $E[x]$ is a totally ramified maximal subfield of D. Hence, by Corollary 2.4 D is nicely semiramified. Therefore, D is cyclic (because it is nondegenerate and nicely semiramified). Moreover, by [JW90, Lemma 5.15], we have $exp(D) = exp(\Gamma_D/\Gamma_E) = (\Gamma_D : \Gamma_E) = deg(D)$.

Remark : By [P82, Proposition 17.6] any local field E is Henselian. Therefore, Corollary 3.3 can be considered as a generalization of [P82, Corollary 17.8(b) and Corollary 17.10(b)].
Henselian valued fields with finite residue fields are called generalized local fields. The

reader can see the thesis of F.H. Chang [Ch04] for more results concerning division algebras over these fields.

Corollary 3. 4 *Let E be a Henselian valued field of characteristic $p > 0$ and D a nondegenerate tame semiramified division p-algebra over E. Then, the following statements are equivalent :*
(1) D is cyclic.
(2) D is a tensor product of cyclic algebras.
(3) D is nicely semiramified.
(4) Γ_D/Γ_E is cyclic.
In such a case, we have $exp(D) = deg(D)$ (hence D is indecomposable) and all purely inseparable maximal subfields of D are simple (i.e. of the form $E[x]$ for some $x \in D^*$)

Proof. (1) $\Rightarrow$ (2) Clear.
(2) $\Rightarrow$ (3) By Proposition 2.5.
(3) $\Rightarrow$ (4) Obvious (since D is nondegenerate).
(4) $\Rightarrow$ (1) Since D is tame semiramified, then $\bar{D}$ is a Galois field extension of $\bar{E}$ with $Gal(\bar{D}/\bar{E})$ isomorphic to Γ_D/Γ_E. Therefore, the inertial lift of $\bar{D}$ over E in D is a cyclic maximal subfield of D.
Assume these conditions hold, then by [JW90, Lemma 5.15] $exp(D) = exp(\Gamma_D/\Gamma_E) = (\Gamma_D : \Gamma_E) = deg(D)$. Moreover, in this case, if K is a purely inseparable maximal subfield of D, then K is totally ramified over E (because D is defectless over E and $\bar{D}$ is separable over $\bar{E}$). D being nondegenerate, then by Proposition 3.2 Γ_K/Γ_E is cyclic. Let $x \in K^*$ with $v(x) + \Gamma_E$ generating Γ_K/Γ_E, then $K = E[x]$.

Corollary 3. 5 *Let E be a Henselian valued field of characteristic $p > 0$ and D*

a nondegenerate tame semiramified division p-algebra over E. Then, the following statements are equivalent :
(1) D is cyclic.
(2) GD is cyclic.
(3) D is nicely semiramified.

Proof. (1) $\Leftrightarrow$ (2) This follows by Corollary 3.4 and Proposition 2.9.
(1) $\Leftrightarrow$ (3) By Corollary 3.4.

Proposition 3. 6 *Let E be a Henselian valued field and D a tame semiramified division algebra of prime power degree over E. Then, the following statements are equivalent.*
(1) *D is degenerate.*
(2) *There is an inertial subfield M of D such D_M is the tensor product of two (non-trivial) nicely semiramified division algebras.*
(3) *There is an inertial subfield M of D such D_M is the tensor product of two (non-trivial) cyclic nicely semiramified division algebras.*
(4) There is an inertial subfield M of D such D_M is the tensor product of two (non-trivial) cyclic nicely semiramified division algebras D_1 and D_2 with Γ_{D_i}/Γ_M cyclic.

Proof. (2) $\Rightarrow$ (1) Assume there is an inertial subfield M of D such $D_M = D_1 \otimes_M D_2$, where D_i are non-trivial nicely semiramified division algebras. Then, by Proposition 2.6(3), $\Gamma_{D_M}/\Gamma_M = \Gamma_{D_1}/\Gamma_M \oplus \Gamma_{D_2}/\Gamma_M$. Let L_i be a totally ramified maximal subfield of D_i $(i = 1, 2)$. Since $\Gamma_{L_i}/\Gamma_M = \Gamma_{D_i}/\Gamma_M$, then by [Mor89, Theorem 1] $L_1 \otimes_M L_2$ is a totally ramified maximal subfield of D_M. So, D_M is nicely semiramified with $rl(D_M) \geq 2$.

(1) $\Rightarrow$ (4) If M is an inertial subfield of D such that D_M is nicely semiramified with radical length $r \geq 2$, then we can write $C_D^M = (M_1/M, \sigma_1, a_1) \otimes_M ... \otimes_M (M_r/M, \sigma_r, a_r)$, where M_i, σ_i and a_i satisfy the conditions of [JW90, Theorem 4.4(iii)]. We have $C_D^{MM_3...M_r} = C_{C_D^M}^{M_3...M_r}$, so $C_D^{MM_3...M_r} \cong (M_1M_3...M_r/M_3...M_r, \sigma_1, a_1) \otimes_{MM_3..M_r} (M_2M_3...M_r/M_3...M_r, \sigma_2, a_2)$. Let $D_i := (M_iM_3...M_r/M_3...M_r, \sigma_i, a_i)$ $(1 \leq i \leq 2)$, then by [JW90, Theorem 4.4] D_i are nicely semiramified and Γ_{D_i}/Γ_M are cyclic.
(4) $\Rightarrow$ (3) and (3) $\Rightarrow$ (2) Obvious.

Remark 3.7 Let E be a Henselian valued field of characteristic $p > 0$ and D a tame semiramified division p-algebra. Then, Proposition 2.6 and Proposition 3.4, D is degenerate if and only if there is an inertial subfield M of D such that D_M is the tensor product of more than two (non-trivial) cyclic tame division algebras.

(3.8) Let E be a Henselian valued field and D a tame division algebra over E, then D is similar to a tensor product $I \otimes_E N \otimes_E T$, where I is an inertial central division algebra over E, N is a nicely semiramified division algebra over E and T is a totally ramified central division algebra over E (by [JW90, Lemma 5.15 and Lemma 6.2]). D is semiramified if and only if T is trivial and $\bar{N}$ splits $\bar{I}$. Indeed, if D is semiramified, then T will be inertially split, hence trivial. Moreover, in this case, by [JW90, Lemma 5.15] $\bar{N}$ splits $\bar{I}$. The converse follows also by [JW90, Lemma 5.15]. In what follows, we aim to determine necessary and sufficient conditions for such tame semiramified division algebra D to be nondegenerate.

Proposition 3. 9 *Let E be a Henselian valued field, N a nicely semiramified division algebra over E, I an inertial central division algebra over E and D the central* *division algebra over E similar to $I \otimes_E N$ with respect to $Br(E)$. Assume that D is semiramified with prime power degree. Then, D is nondegenerate if and only if for any subfield K of $\bar{N}$, $\bar{N}/K$ is cyclic or $\bar{I}_K \notin Dec(\bar{N}/K)$.*

Proof. Let M be an inertial subfield of D. By [JW90, Lemma 5.15] we have $\bar{D} \cong \bar{N}$, so M is isomorphic to a subfield of N. Therefore, by Proposition 2.6(2) N_M is nicely semiramified. Moreover, since I is inertial, then by [JW90, Lemma 1.8] I_M is inertial (over M). Hence, by [JW90, Theorem 5.15] D_M is nicely semiramified if and only if $\bar{I}_{\bar{M}} \in Dec(\bar{N}/\bar{M})$ (see that $\bar{N}_{\bar{M}} = \bar{N}$). So, D is nondegenerate if and only if for any subfield K of $\bar{N}$, $\bar{N}/K$ is cyclic or $\bar{I}_K \notin Dec(\bar{N}/K)$.

(3.10) We recall that for a crossed product A over a field E and a Galois maximal subfield L of A, we say that L is degenerate in A if there is an intermediate field K of L/E such that :
(1) $Gal(L/K)$ is the direct sum of two cyclic subgroups $\langle\sigma_1\rangle$ and $\langle\sigma_2\rangle$, and
(2) the centralizer C_A^K decomposes into a tensor product of two cyclic algebras, split by the subfields K_1 and K_2 of L fixed respectively by σ_2 and σ_1 (i.e. $C_A^K = (K_1/K, \sigma_1, a_1) \otimes_K (K_2/K, \sigma_2, a_2)$ for some a_1, $a_2 \in K^*$) [BM00, Definition 0.12].

Proposition 3. 11 *Let E be a Henselian valued field, N a nicely semiramified division algebra of prime power degree over E, I an inertial central division algebra over E such that $\bar{N}$ is a maximal subfield of $\bar{I}$, and D the central division algebra over E similar to $I \otimes_E N$ with respect to $Br(E)$. Then, D is semiramified. Furthermore, D is degenerate if and only if $\bar{N}$ is degenerate in $\bar{I}$.*

Proof. Since $\bar{N}$ is a maximal subfield of $\bar{I}$, then by [JW90, Lemma 5.15] D is semiramified. Suppose that D is degenerate and let M be an inertial subfield of D such that D_M is nicely semiramified with $rl(D_M) \geq 2$. As previously seen in the proof of Proposition 3.9, I_M is inertial ; N_M is nicely semiramified and D_M is similar to $I_M \otimes_M N_M$, so again by [JW90, Lemma 1.8 and Lemma 5.15] $\bar{I}_{\bar{M}} \in Dec(\bar{N}/\bar{M})$. Let $r = rk(Gal(\bar{N}/\bar{M}))(= rk(Gal(\overline{D_M}/\bar{M})) = rl(D_M))$, and write $\bar{N} = K_1 \otimes_{\bar{M}} ... \otimes_{\bar{M}} K_r$, where $K_i/\bar{M}$ are cyclic field extensions $(1 \leq i \leq r)$. Since $\bar{I}_{\bar{M}} \in Dec(\bar{N}/\bar{M})$, then $\bar{I}_{\bar{M}} = (K_1/\bar{M}, \sigma_1, a_1) \otimes_{\bar{M}} ... \otimes_{\bar{M}} (K_r/\bar{M}, \sigma_r, a_r)$, where σ_i is a generator of the cyclic field extension $K_i/\bar{M}$ and $a_i \in \bar{M}^*$. Let $L = K_3...K_r$, then $\bar{I}_L = (K_1L/L, \sigma_1, a_1) \otimes_L (K_2L/L, \sigma_2, a_2)$. So, $\bar{N}$ is degenerate in $\bar{I}$.
Conversely, assume that $\bar{N}$ is degenerate in $\bar{I}$ and let L be an intermediate field of $\bar{N}/\bar{E}$ such that $\bar{I}_L = (L_1/L, \tau_1, b_1) \otimes_L (L_2/L, \tau_2, b_2)$, where $L_1 \otimes_L L_2 = \bar{N}$, τ_i is a generator of the cyclic field extension L_i/L and $b_i \in L^*$. Let M be the inertial lift of L over E in D, then $\bar{I}_{\bar{M}} \in Dec(\bar{N}/\bar{M})$ and $rl(D_M) = rk(Gal(\bar{N}/\bar{M})) = 2$. So, D is degenerate.

Proposition 3. 12 *Let E be a Henselian valued field, D a tame semiramified division algebra of prime power degree over E, and L 'the' inertial lift of $\bar{D}$ over E in D. Then, D is degenerate if and only if L is degenerate in D.*

Proof. If D is degenerate, then by Proposition 3.6 there is an inertial subfield M of D such that C_D^M is a tensor product of two cyclic nicely semiramified division algebras D_1 and D_2 with Γ_{D_i}/Γ_M cyclic. Write $D_i = (M_i/M, \sigma_i, a_i)$ $(1 \leq i \leq 2)$, where M_i are inertial cyclic field extensions of E with Galois group generated by σ_i, and $a_i \in M^*$. Clearly, this implies that $M_1 \otimes_M M_2$ is a degenerate maximal subfield of C_D^M. Moreover, by comparing the cardinalities, we have $\bar{M}_1 \otimes_{\bar{M}} \bar{M}_2 = \bar{D}$.

So, $M_1 \otimes_M M_2$ is isomorphic to L (because, $\bar{M} = \bar{M}_1 \otimes_{\bar{M}} \bar{M}_2 = \bar{D}$), whence L is degenerate in D.
Conversely, assume that L is degenerate in D and let K be a subfield of L satisfying the conditions of (3.10). Write $C_D^K = (K_1/K, \sigma_1, a_1) \otimes_K (K_2/K, \sigma_2, a_2)$, where K_1 and K_2 are the subfields of L fixed by σ_2 and σ_1 respectively and $a_1,\ a_2 \in K^*$, and let $D_i = (K_i/K, \sigma_i, a_i)$ $(i = 1, 2)$. By Proposition 2.6(2 and 3) C_D^K and D_i are semiramified. Moreover, K_i is a cyclic inertial maximal subfield of D_i, so again by Proposition 2.6(4) D_i is nicely semiramified and Γ_{D_i}/Γ_K is cyclic. Hence, by Proposition 2.6(3) C_D^K is nicely semiramified with radical length $r = 2$. Therefore, D is degenerate.

Proposition 3. 13 *Let E be a Henselian valued field and D a tame semiramified division algebra of prime power degree over E. If K is a subfield of D with $(\Gamma_K : \Gamma_E)$ maximal, then K is a maximal subfield of D. In particular, if K is totally ramified over E and D is nondegenerate, then D is cyclic.*

Proof. By Proposition 2.6(2) C_D^K is semiramified. Since $(\Gamma_K : \Gamma_F)$ is maximal, then $(\Gamma_{C_D^K} : \Gamma_K) = 1$. Hence, $C_D^K = K$. So, K is a maximal subfield of D. If K is totally ramified over E, then by Corollary 2.4 D is nicely semiramified. If, we suppose in addition that D is nondegenerate, then by Corollary 3.5 D is cyclic.

Remark 3.14 Let F be a graded field and D a graded central division algebra over F. As for the valuative case, we say that D is degenerate if there is an inertial graded subfield M of D such that D_M is nicely semiramified with $rl(D_M) \geq 2$. One can then use the same arguments as in Proposition 3.2 to show that for an inertially split graded central division algebra D over F, D is degenerate if and only if there is a graded subfield K of D such that Γ_K/Γ_F is non-cyclic. One can also see that there are analogous graded versions for all the results we are given in this work.

Lemma 3. 15 *Let E be a Henselian valued field and D a tame central division algebra of prime power degree over E. Then, D is degenerate if and only if GD is degenerate.*

Proof. Suppose that GD is degenerate and let K be an inertial graded subfield of GD such that GD_K is nicely semiramified with $rl(GD_K) \geq 2$. Let M be the inertial lift of K_0 over E in D. Then, $G(D_M) \cong GD_K$. So, by Lemma 2.2, D_M is nicely semiramified. Moreover, we have $rl(D_M) = rl(GD_K) \geq 2$. Hence, D is degenerate. Conversely, suppose that D is degenerate and let L be an inertial subfield of D such that D_L is nicely semiramified with $rl(D_L) \geq 2$. Then, clearly GD_{GL} is nicely semiramified and $rl(GD_{GL}) = rl(D_L) \geq 2$.

Proposition 3. 16 *Let F be a graded field, D a semiramified graded division algebra of prime power degree over F, (L, v) an immediate valued field extension of $Frac(F)$ (for its canonical valuation) and HL the henselization of L with respect to v. Then, the following statements are equivalent.*
(1) D is degenerate.
(2) $Cq(D) \otimes_{Frac(F)} HL$ is degenerate.
(3) D_0HL is degenerate in $Cq(D) \otimes_{Frac(F)} HL$.
(4) D_0L is degenerate in $Cq(D) \otimes_{Frac(F)} L$.
(5) $D_0Frac(F)$ is degenerate in $Cq(D)$.

Proof. By [Mor89, Theorem 1] $Cq(D) \otimes_{Frac(F)} HL$ is a division algebra (of center HL). Since L is an immediate valued field extension of $Frac(F)$, then so is HL.

Therefore, $GHL \cong F$ and $G(Cq(D) \otimes_{Frac(F)} HL) \cong_g D$. So, (1) $\Leftrightarrow$ (2) follows by Lemma 3.15.
(2) $\Leftrightarrow$ (3) By Proposition 3.12.
(1) $\Rightarrow$ (5) By the graded version of Proposition 3.6 there is an inertial graded subfield M of D such that C_D^M is a tensor product of two cyclic nicely semiramified graded division algebras $(M_i/M, \sigma_i, a_i)$, where M_i are inertial cyclic graded field extensions of M with Galois group generated by σ_i, and $a_i \in M^*$. So, up to an isomorphism, we have :

$$\begin{aligned} C_{Cq(D)}^{Frac(M)} &= C_D^M \otimes_M Frac(M) = ((M_1/M, \sigma_1, a_1) \otimes_M (M_2/M, \sigma_2, a_2)) \otimes_M Frac(M) \\ &= ((M_1/M, \sigma_1, a_1) \otimes_M Frac(M)) \otimes_{Frac(M)} ((M_2/M, \sigma_2, a_2) \otimes_M Frac(M)) \\ &= (Frac(M_1)/Frac(M), \sigma_1, a_1) \otimes_{Frac(M)} (Frac(M_2)/Frac(M), \sigma_2, a_2) \end{aligned}$$

By the graded version of Proposition 2.6(3), we have $M_1 \otimes_M M_2 = D_0 M$, so $Frac(M_1) \otimes_{Frac(M)} Frac(M_2) = D_0 Frac(F)$ (see that $D_0 Frac(F) = D_0 Frac(M)$). Moreover, it is clear that $Frac(M_1)$ [resp. $Frac(M_2)$] is the subfield of $D_0 Frac(F)$ fixed by σ_2 [resp. by σ_1]. This shows that $D_0 Frac(F)$ is degenerate in $Cq(D)$.
(5) $\Rightarrow$ (4) and (4) $\Rightarrow$ (3) Obvious.

Corollary 3. 17 *Let E be a field, K a non-cyclic abelian field extension of E with prime power dimension, G the Galois group of K over E, S a basis of G, U a nondegenerate matrix of $U_S(G, K^*)$, Z the center of the generic abelian crossed product (K, G, U) and L an immediate field extension of Z (for its canonical valuation), then $(K, G, U) \otimes_Z L$ is semiramified but not nicely semiramified. In particular, if $char(E) = p$ and G is a p-group, then $(K, G, U) \otimes_Z L$ is not a tensor product of cyclic algebras. If $p = 2$ and $exp((K, G, U) \otimes_Z L) = 2$, then $(K, G, U) \otimes_Z L$ has an involution of the first kind but is not a tensor product of 2-symbols.*

Proof. We have seen in [BM00, Theorem 1.1] that there is a graded field F and a semiramified graded division algebra D over F such that $(K, G, U) = Cq(D)$. Moreover, by [BM00, Proposition 0.13], U is nondegenerate (with respect to K/E or equivalently to $D_0Frac(F)/Frac(F)$) if and only if $D_0Frac(F)$ is nondegenerate in $Cq(D)$. So, by Proposition 3.16 D is nondegenerate. Therefore, D cannot be nicely semiramified (because $\Gamma_D/\Gamma_F(\cong G)$ is non-cyclic). The rest follows by Corollary 2.5.

Corollary 3. 18 *[AS, Theorem 3.2] Let E be a field of characteristic $p > 0$, K a finite dimensional non-cyclic abelian field extension of E with a Galois p-group G, S a basis of G and U a nondegenerate matrix of $U_S(G, K^*)$, then the generic abelian crossed product (K, G, U) is non-cyclic.*

Remark 3.19 (1) Let F be a graded field with support Γ_F ; $E = F_0$; M a finite-dimensional abelian field extension of E with Galois group G ; S a basis of G ; Δ a totally ordered abelian group that contains Γ_F with ordering extending that of Γ_F ; $\epsilon : \Delta \to G$ a group epimorphism with kernel Γ_F and (U, B) a cocycle of $Z^2_S(G, M^*)$. We have seen in the preliminaries how to construct the semiramified graded division algebra (MF, ϵ, G, U, B). Assume that U is nondegenerate in M/E and let L be an immediate field extension of $Frac(F)$. One can easily see that Corollary 3.17 is still true if we replace the generic abelian crossed product (K, G, U) by $Cq((MF, \epsilon, G, U, B))$. More generally, let F be a graded field and D a semiramified graded division algebra over F. Let $x_i \in D^*$ such that $(gr(x_i) + \Gamma_F)_{i=1}^r$ is a basis of Γ_D/Γ_F and let $u_{ij} = x_i x_j x_i^{-1} x_j^{-1}$ and $U = (u_{ij})_{1 \leq i,j \leq r}$. Then, by [BM00, Proposition 0.13] U is nondegenerate with respect to D_0/F_0 if and only if $D_0Frac(F)$ is nondegenerate in $Cq(D)$. Hence, by Proposition 3.16, U is nondegenerate with respecto to D_0/F_0 if and only if D is nondegenerate. It is easily seen that we can replace (K, G, U) in

Corollary 3.17 by $Cq(D)$ where D is a nondegenerate semiramified graded division algebra.

(2) Let E be a field, (D, v) a defectless valued finite-dimensional division algebra over E and HE the henselization of E with respect to the restriction of v to E. We say that D is tame over E if $D \otimes_E HE$ is so over HE. In particular, if we suppose that E is the center of D, then D is tame over E if and only if GE is the center of GD. Indeed, one can easily see that $GE = GHE$ and $GD = G(D \otimes_E HE)$. In the same way, we can define nondegenerate valued central division algebras over a field E (and generalize many results of this paper) without assuming that E is Henselian.

(3) In [McK05] the author defined what she called strongly degenerate matrices. The reader can see that many results of this paper can be proved for non-strongly-degenerate tame semiramified division algebras. In particular, this can be done for Corollary 3.17 and all the results of next sections, but we dont know if there exist really a non-strongly-degenerate tame semiramified division algebra over a Henselian valued field that is degenerate or if the two definitions (strongly degenerate and degenerate) coincide.

4 Subfields of nondegenerate tame semiramified division algebras

Lemma 4. 1 *Let F be a graded field and D a nondegenerate semiramified graded division algebra of prime power degree p^n over F with $rk(Gal(D_0/F_0)) \geq 2$. Let $d \in D^*$ such that $d^p \in F$. Then, $d \in D_0F^*$. If in addition $p = char(F)$, then $d \in F^*$.*

Proof. We will use the same argument as in the proof of [AS78, Lemma 3.1]. Let $x_i \in D^*$ such that $(gr(x_i)+\Gamma_F)_{i=1}^r$ is a basis of Γ_D/Γ_F. Then, $D = D_0F[x_1,...,x_r]$. Let $q_i = ord(gr(x_i)+\Gamma_F)$, $\sigma_i = \theta_D(gr(x_i)+\Gamma_F)$, where θ_D is the group isomorphism of [B95, p.4278] (see also, [HW(2)99, (2.2), p.86]) and $I = \{\bar{m} = (m_1,...,m_r) \in \mathbb{N}^r \mid 0 \leq m_i < q_i \text{ for all } 1 \leq i \leq r\}$. Then, we can write $d = ax^{\bar{m}}$, where $a \in D_0F^*$ and $\bar{m} \in I$. Suppose by contradiction that $\bar{m} \neq 0$ and let $D_0^{\bar{m}}$ be the subfield of D_0 fixed by $\sigma^{\bar{m}}$. Since $d^p \in F^*$, then $(\sigma^{\bar{m}})^p (= \theta_D(gr(d^p)+\Gamma_F) = id_{D_0}$. So, $[D_0 : D_0^{\bar{m}}] = p$. Let $N_{\bar{m}}$ be the norm of $D_0/D_0^{\bar{m}}$. Then, $(ax^{\bar{m}})^p = N_{\bar{m}}(a)(x^{\bar{m}})^p \in F^*$. Since $rk(Gal(D_0/F_0)) \geq 2$, then there is some $1 \leq j \leq r$ such that $\langle \sigma^{\bar{m}}, \sigma_j \rangle$ is non-cyclic. We have $N_{\bar{m}}(a)(x^{\bar{m}})^p = x_j(N_{\bar{m}}(a)(x^{\bar{m}})^p)x_j^{-1} = (x_jN_{\bar{m}}(a)x_j^{-1})(x_j(x^{\bar{m}})^px_j^{-1}) = N_{\bar{m}}(\sigma_j(a))N_{\bar{m}}(u_{j\bar{m}})(x^{\bar{m}})^p = N_{\bar{m}}(\sigma_j(a)u_{j\bar{m}})(x^{\bar{m}})^p$. So, $N_{\bar{m}}(\sigma_j(a)a^{-1}u_{j\bar{m}}) = 1$. Therefore, by Hilbert 90 Theorem, there is some $b \in D_0$ such that $\sigma_j(a)a^{-1}u_{j\bar{m}} = \sigma^{\bar{m}}(b)b^{-1}$. But this is not possible since D is nondegenerate. Hence, $\bar{m} = 0$. Remark that D_0 is separable over F_0, so D_0F is tame over F. Therefore, if $char(F) = p$, then $d \in F^*$.

Proposition 4. 2 *Let E be a Henselian valued field and D a nondegenerate tame semiramified division algebra of prime power degree over E. If $rk(\Gamma_D/\Gamma_E) \geq 2$, then D has no (non-trivial) totally ramified subfield (over E).*

Proof. Suppose first that $char(\bar{E}) = p > 0$ and $deg(D)$ is a power of p. If K is a totally ramified subfield of D, then GK is a purely wild graded field extension of GE. So, for any $x \in GK$, $x^{[GK:GE]} \in GE$. Hence, by Lemma 4.1, $x \in GE$. So, $GK = GE$. This proves $K = E$.

Now, assume $char(\bar{E})$ does not divide $deg(D)$. Suppose that K is a non-trivial totally ramified subfield of D. Let $x \in GK^*$ such that $gr(x) \notin \Gamma_{GE}$. If $ord(gr(x)+\Gamma_{GE}) = p^s$ (for some $s \geq 1$), then $x^{p^s} \in GE$ (since $GE[x]$ is totally ramified over GE). Therefore, by Lemma 4.1, $x^{p^{s-1}} \in \bar{D}GE$. A contradiction (since we will have $p^{s-1}gr(x)+\Gamma_{GE} = \Gamma_{GE}$).

Theorem 4. 3 *Let E be a Henselian valued filed of residue characteristic $p > 0$, D a nondegenerate tame semiramified division algebra of degree a power of p with Γ_D/Γ_E non-cyclic, and K a normal subfield of D. Then, K is a Galois inertial field extension of E. So, there is a subgroup H of $Gal(\bar{D}/\bar{E})$ such that $Gal(K/E) \cong Gal(\bar{D}/\bar{E})/H$. In particular, if K is a Galois maximal subfield of D, then $Gal(K/E) \cong Gal(\bar{D}/\bar{E})$.*

Proof. Let K be a normal subfield of D, then by Theorem 1.4, GK is a normal graded field extension of GE. Let L be the graded subfield of GK elementwise invariant by $Gal(GK/GE)$, then by Proposition 1.3 L is a purely wild graded field extension of GE. Therefore, by Lemma 4.1, $L = GE$. Hence, again by Proposition 1.3, GK is a Galois graded field extension of GE. It follows by Corollary 1.8, that K is a Galois field extension of E. Since $[GK : GE]$ is a power of p and GK is tame over GE, then GK is inertial over GE. So, K is inertial over E. The rest is obvious.

Corollary 4. 4 *Let E be a Henselian valued filed of residue characteristic $p > 0$ and D a nondegenerate tame semiramified division algebra of degree a power of p, then D is cyclic if and only if Γ_D/Γ_E is cyclic.*

Proof. Suppose that D is cyclic and let K be a cyclic maximal subfield of D, then by Theorem 4.3 $Gal(K/E) \cong Gal(\bar{D}/\bar{E})$. Moreover, since D is tame semiramified, then $Gal(\bar{D}/\bar{E}) \cong \Gamma_D/\Gamma_E$, so Γ_D/Γ_E is cyclic. Conversely, if Γ_D/Γ_E is cyclic, then 'the' inertial lift of $\bar{D}$ over E in D is a cyclic maximal subfiled of D.

Corollary 4. 5 [*S78, Theorem 3.2*] *Let E be a field of characteristic $p > 0$, K a* *non-cyclic abelian field extension of* E *with dimension a power of* p*,* G *the Galois group of* K *over* E*,* S *a basis of* G*,* U *a nondegenerate matrix of* $U_S(G, K^*)$*, and* L *a Galois subfiled of the generic abelian crossed product* (K, G, U)*, then there is a subgroup* H *of* G *such that* $Gal(L/E) \cong G/H$*. In particular, if* L *is a Galois maximal subfield of* (K, G, U)*, then* $Gal(L/E) \cong G$*.*

Proposition 4. 6 *Let* E *be a Henselian valued field and* D *a nondegenerate tame semiramified division algebra of prime power degree over* E*. Assume* $rk(\Gamma_D/\Gamma_E) \geq 2$*, and let* K *be an elementary abelian subfield of* D*. Then,* K *is inertial over* E*. Therefore,* D *is an elementary abelian crossed product if and only if* $Gal(\bar{D}/\bar{E})$ *is elementary abelian.*

Proof. Since K is an elementary abelian field extension of E, then we can write $K = K_1 \otimes_E K_2 \otimes_E ... \otimes_E K_r$, where K_i are cyclic field extensions of E with $[K_i : E]$ prime. By Proposition 4.2, K_i cannot be totally ramified over E. Hence K_i is inertial over E. So, K is inertial over E. If K is in addition a maximal subfield of D, then $\bar{D}(= \bar{K})$ is elementary abelian over $\bar{E}$. Conversely, suppose that $Gal(\bar{D}/\bar{E})$ is elementary abelian and let M be the inertial lift of $\bar{D}$ over E in D, then M is an elementary abelian maximal subfield of D.

Remark 4.7 Let F be a graded field and D a semiramified graded division algebra of degree q^2 over F, where q is a prime. Suppose $Gal(D_0/F_0)$ is cyclic and let $x \in D^*$ such that $gr(x)+\Gamma_F$ generates Γ_D/Γ_F, then $F[x]$ is a totally ramified maximal graded subfield of D. Hence, $T = F[x^q]$ is a totally ramified graded subfield of D with $[T : F] = q$. If we assume that F_0 contains a primitive q^{th} root of unity, then T is a cyclic graded field extension of F. Let σ be the F_0-automorphism of D_0 defined by $\sigma(d) = x^q d x^{-q}$, and let M_0 be the subfield of D_0 fixed by σ, then let $M = M_0 F$. Obviously, $M[x^q]$ is a graded subfield of D. Since M [resp. T] is inertial [resp. totally ramified] over F, then $M[x^q] = M \otimes_F T$. So, D has an elementary abelian maximal graded subfield. Hence $HCq(D)$ has an elementary abelian maximal subfield. This shows that Proposition 4.6 is not true if $deg(D) = q^2$ and Γ_D/Γ_E is cyclic.

Proposition 4. 8 *Let E be a Henselian valued field, D a nondegenerate tame semiramified division algebra of prime power degree over E, and K a subfield of D. If $Gal(\bar{D}/\bar{K})$ is non-cyclic, then K is inertial over E.*

Proof. Let M be the inertial lift of $\bar{K}$ over E in K. Clearly, C_D^M is a nondegenerate tame semiramified division algebra (over M) and K is a totally ramified subfield of C_D^M. So, by Proposition 4.2, $K = M$.

Proposition 4. 9 *Let E be a Henselian valued field and D a nondegenerate tame semiramified division algebra of prime power degree over E. Assume $rk(\Gamma_D/\Gamma_E) \geq 3$ and let K be an abelian subfield of D, then K is inertial over E.*

Proof. Write $K = K_1 \otimes_E K_2 \otimes_E ... \otimes_E K_r$, where K_i are cyclic field extensions of E. Let N_i be the inertial lift of $\bar{K}_i$ over E in K_i. Obviously, N_i is cyclic over E. So, $\bar{K}_i$ is cyclic over $\bar{E}$. Therefore, $\bar{D}$ cannot be cyclic over $\bar{K}_i$ (since $rk(Gal(\bar{D}/\bar{E})) \geq 3$). So, by Proposition 4.8 K_i is inertial over E.

Lemma 4. 10 *Let E be a Henselian valued field and D a nondegenerate tame semiramified division algebra of prime power degree over E. Consider $x \in D^*$ such that $v(x) \neq \Gamma_E$. Then, $Gal(\bar{D}/\overline{E[x]})$ is cyclic. In particular, if $exp(\Gamma_D/\Gamma_E)$ is prime, then $E[x]$ is a maximal subfield of D.*

Proof. Let M be the inertial lift of $\overline{E[x]}$ over E in $E[x]$. Clearly, $E[x]$ is a non-trivial totally ramified subfield of C_D^M. So, by Proposition 4.2, $Gal(\bar{D}/\bar{M})$ is cyclic. If $exp(\Gamma_D/\Gamma_E) = p$ is a prime, then so is $exp(Gal(\bar{D}/\bar{E}))$. Hence, $card(Gal(\bar{D}/\overline{E[x]})) \leq p$. See that we have $[E[x] : E] > [\overline{E[x]} : \bar{E}] = [\bar{D} : \bar{E}][\bar{D} : \overline{E[x]}]^{-1} \geq deg(D)p^{-1}$. Hence, $[E[x] : E] = deg(D)$.

Let H be a non-abelian group. We say that H is a quaternion group if H is of order 8 and is generated by two elements a and b satisfying the following conditions $a^4 = b^4 = 1$, $a^2 = b^2$ and $ba = a^{-1}b$. If K/E is a normal [resp., Galois] field extension with a quaternion Galois group, we say that K is a quaternion (normal [resp., Galois]) field extension of E.

Theorem 4. 11 *Let E be a Henselian valued field, p a prime integer and D a non-degenerate tame semiramified division algebra of degree p^n ($n \in \mathbb{N}^*$) over E. Assume $exp(\Gamma_D/\Gamma_E) = p$ and let K be a subfield of D, then the following statements hold :*
(1) if K is not maximal in D, then K is inertial over E.
(2) if K is a non-quaternion normal maximal subfield of D, then K is either cyclic with dimension $\leq p^2$ or inertial over E.

Remark that in the case of (1) and in the last case of (2) K is elementary abelian over E.

Proof. (1) By Lemma 4.10.
(2) Since K is normal over E, then in the same way as seen in the proof of Theorem 4.3, we prove that K is a Galois tame field extension of E. We will show first that if $p \neq 2$ and K is a Galois maximal subfield of D, then $Gal(K/E)$ is abelian. Suppose by con-

tradiction that $Gal(K/E)$ is not abelian, then by Theorem B.11 in comments below (or also by [Ha59, theorem 12.5.4, p. 190]) $Gal(K/E)$ has a non-normal subgroup H. Let L be the subfield of K elementwise fixed by H. Then, L is a non-normal field extension of E. Hence, by Proposition 4.8 both $Gal(\bar{D}/\bar{K})$ and $Gal(\bar{D}/\bar{L})$ are non-trivial cyclic groups. Moreover, since $exp(Gal(\bar{D}/\bar{E})) = exp(\Gamma_D/\Gamma_E) = p$, then $Gal(\bar{D}/\bar{K})$ and $Gal(\bar{D}/\bar{L})$ are of order p. We have $\bar{L} \subseteq \bar{K}$, so necessarily $\bar{L} = \bar{K}$. Let M be the inertial lift of $\bar{L}$ over E in L. Then, up to an isomorphism, M is also the inertial lift of $\bar{K}$ over E in K. So, K is totally ramified over M, and obviously $\Gamma_K/\Gamma_M (= \Gamma_K/\Gamma_E)$ is cyclic of order p. Hence, $p \leq [K : L] \leq [K : M] = (\Gamma_K : \Gamma_M) = p$. So, $L = M$. But this is not true because L is not normal over E. Therefore, $Gal(K/E)$ is abelian.
Assume now that $p = 2$ and let's prove that K is an abelian field extension of E. Suppose by contradiction that K is a non-abelian Galois maximal subfield of D. Then, in the same way as above, we show that all subgroups of $Gal(K/E)$ are normal. So, by [Ha59, Theorem 12.5.4, p. 190] $Gal(K/E)$ is the direct product of a quaternion group and an abelian group of exponent 2. Write $Gal(K/E) = H_1 \times H_2$, where H_1 is a non-trivial quaternion group and H_2 is a non-trivial abelian group of exponent 2, and let K_1 [resp., K_2] be the subfield of K elementwise invariant by H_2 [resp., H_1], then $K = K_1 \otimes_E K_2$, but by (1) above both K_1 and K_2 are inertial over E. Hence, K is an abelian field extension of E. A contradiction. This shows that K is necessarily an abelian field extension of E.
Remark that if $deg(D) > p^2$ and $Gal(K/E)$ is abelian, then by Proposition 4.9 K is inertial over E. Moreover, if $deg(D) \leq p^2$ and K is a non-cyclic abelian field extension of E, then $K = K_1 \otimes_E K_2$, where K_i are cyclic field extension of degree p over E. So, again by (1) above both K_1 and K_2 (and hence K) are inertial over E.

(4.12) Let F be a graded field and K a Galois finite-dimensional graded field extension of F. Then, K_0 is a Galois field extension of F_0. Indeed, let $\sigma \in Gal(K/K_0F)$ and $\tau \in Gal(K/F)$. For any $a \in (K_0F)^*$, we have $gr(\tau^{-1}(a)) = gr(a)$. So, $\tau^{-1}(a) = ca$ for some $c \in K_0$. Hence, $\tau \circ \sigma \circ \tau^{-1}(a) = a$. Therefore, $\tau \circ \sigma \circ \tau^{-1} \in Gal(K/K_0F)$. This shows that $Gal(K/K_0F)$ is a normal subgroup of $Gal(K/F)$. So, K_0F is a Galois graded field extension of F. Hence, by [HW(1)99, Remark 3.1] K_0 is a Galois field extension of F_0. In the same way, we prove that if L is a graded field extension of F in K such that $L_0 = K_0$, then L is a Galois graded field extension of F. Therefore, by Corollary 1.8 if E is a Henselian valued field, N a tame Galois finite-dimensional field extension of E, and M is a field extension of E in N such that $\bar{M} = \bar{N}$, then M is a Galois field extension of E.

Definition 4.13 Let G be an abelian group and H a non-trivial cyclic subgroup of G. We say that H is maximally cyclic in G if there is no cyclic subgroup H' of G such that $H \subset H'$.

Let E be a Henselian valued field and D a nondegenerate tame semiramified division algebra of prime power degree over E. We recall that we have previously seen in the proof of Theorem 4.3 and also in the proof of Theorem 4.11 that if K is a normal field extension of E, then K is a Galois field extension of E.

Proposition 4. 14 *Let E be a Henselian valued field, D a nondegenerate tame semiramified division algebra of prime power degree over E, and K a normal subfield of D such that $Gal(\bar{D}/\bar{K})$ is cyclic. If $Gal(\bar{D}/\bar{K})$ is maximally cyclic in $Gal(\bar{D}/\bar{E})$, then the following statements hold :*
(1) If $deg(D)$ is odd, then K is an abelian field extension of E.

(2) If $deg(D)$ is a power of 2 and Γ_D/Γ_E is non-cyclic, then K is either a quaternion or an abelian field extension of E.

Therefore, if $rk(\Gamma_D/\Gamma_E) \geq 3$ and K is not a quaternion field extension of E, then K is inertial over E.

Proof. (1) Suppose by contradiction that $Gal(K/E)$ is non-abelian. Then, by Theorem B.11, there is a non-normal subgroup H of $Gal(K/E)$. Let L be the subfield of K elementwise invariant by H. By Proposition 4.8, $Gal(\bar{D}/\bar{L})$ is cyclic (indeed, otherwise L would be inertial -hence normal- over E). Since $Gal(\bar{D}/\bar{K})$ is maximally cyclic in $Gal(\bar{D}/\bar{E})$ and $Gal(\bar{D}/\bar{K}) \subseteq Gal(\bar{D}/\bar{L})$, then $\bar{K} = \bar{L}$. So, by (4.12) $Gal(K/L)$ is a normal subgroup of $Gal(K/E)$. A contradiction.
(2) In the same way as above we prove that all subgroups of $Gal(K/E)$ are normal. Therefore, if K is non-abelian over E, then by [Ha59, Theorem 12.5.4, p.190] $Gal(K/E) = H_1 \times H_2$, where H_1 is a quaternion group and H_2 is an abelian group of exponent 2. Let K_1 [resp., K_2] be the subfield of K elementwise invariant by H_2 [resp., H_1]. Then, up to an isomorphism, we have $K = K_1 \otimes_E K_2$. Assume that $\bar{K}_i \neq \bar{E}$ (for both $i = 1, 2$), then $Gal(\bar{D}/\bar{K}_i)$ ($i = 1, 2$) are non-cyclic (because we have $Gal(\bar{D}/\bar{K}) \subset Gal(\bar{D}/\bar{K}_i)$ and $Gal(\bar{D}/\bar{K})$ is maximally cyclic in $Gal(\bar{D}/\bar{E})$). Therefore, by Proposition 4.8, K_1 and K_2 (and hence K) are inertial over E. But this contradicts the fact that K is non-abelian over E. Therefore, $\bar{K}_1$ or $\bar{K}_2$ is trivial. This means that K_1 or K_2 is totally ramified over E. So, by Proposition 4.2, either K_1 or K_2 is trivial. Remark that we cannot have $K = K_1$ because K is non-abelian over E. Hence, $K (= K_2)$ is a quaternion field extension of E.
The rest of the proposition follows by Proposition 4.9.

Corollary 4. 15 *Let E be a Henselian valued field and D a nondegenerate tame* *semiramified division algebra of prime power degree over E. Assume $deg(D)$ is odd and $char(\bar{E})$ does not divide $deg(D)$ and let K be a Galois maximal subfield of D. If $(\Gamma_K : \Gamma_E) = exp(\Gamma_D/\Gamma_E)$, then K is abelian over E and $rk(\Gamma_D/\Gamma_E) \leq 2$.*

Proof. Since D is nondegenerate, then Γ_K/Γ_E is cyclic. Moreover, since K is a maximal subfield of D, then $Gal(\bar{D}/\bar{K}) \cong \Gamma_K/\Gamma_E$. Hence, $Gal(\bar{D}/\bar{K})$ is maximally cyclic in $Gal(\bar{D}/\bar{E})$. So, by Proposition 4.14 K is abelian over E and $rk(\Gamma_D/\Gamma_E) \leq 2$ (because K is not inertial over E).

(4.16) As a consequence of Corollary 4.15, under the hypotheses of Corollary 4.15, if $rk(\Gamma_D/\Gamma_E) \geq 3$, $x \in D^*$ with $ord(v(x) + \Gamma_E) = exp(\Gamma_D/\Gamma_E)$, and L is a maximal subfield of D that contains x, then L cannot be Galois over E.

Proposition 4. 17 *Let F be a graded field, D a nondegenerate semiramified graded division algebra of prime power degree p^r over F with $rk(\Gamma_D/\Gamma_F) \geq 3$, and K a Galois graded subfield of D such that $[K : F] < pdeg(D)(exp(\Gamma_D/\Gamma_F))^{-1}$. Then, K is inertial over F.*

Proof. We will show that $Gal(K/F)$ is abelian. Our Proposition follows then by Proposition 4.9 -applied to $HCq(D)$- . Suppose that $Gal(K/F)$ is not abelian and suppose that $[K : F]$ is minimal. By Proposition 4.8 $Gal(D_0/K_0)$ is cyclic. So, $[K_0 : F_0] \geq [D_0 : F_0](exp(\Gamma_D/\Gamma_F))^{-1}$. Remark that necessary $(\Gamma_K : \Gamma_F) = p$. Indeed, if $(\Gamma_K : \Gamma_F) \neq p$, then there exists an intermediate graded field P such that $K_0F \subset P \subset K$. Since $P_0 = K_0$ and K is Galois over F, then by (4.12) P is also a Galois graded field extension of F. It follows by the minimality of $[K : F]$ that $Gal(P/F)$ is abelian, hence P is inertial over F. But this is not true (since $K_0F \subset P$).

So, $(\Gamma_K : \Gamma_F) = p$. Therefore, $[K : F] \geq pdeg(D)exp(\Gamma_D : \Gamma_F)^{-1}$. A contradiction.

Corollary 4. 18 *Let E be a Henselian valued field and D a nondegenerate tame semiramified division algebra of prime power degree p^r over E. Assume $char(\bar{E})$ does not divide $deg(D)$ and suppose K is a Galois subfield of D with $[K : E] < pdeg(D)exp(\Gamma_D : \Gamma_E)^{-1}$. Then, K is inertial over E.*

Definition 4.19 Let E be a Henselian valued field and K an abelian field extension of E. We say that K is nice over E if there is an inertial field extension M of E and a totally ramified field extension T of E such that $K = M \otimes_E T$. We define nice abelian graded field extensions in a similar way.

Proposition 4. 20 *Let E be a Henselian valued field and K a tame abelian field extension of E. Then, K is nice if and only if GK is nice.*

Proof. If K is nice, then $K = M \otimes_E T$, where M is inertial and T is totally ramified over E. So, by the graded version of [Mor89, Theorem 1] $GK = GM \otimes_{GE} GT$. Conversely, suppose that GK is nice. Then, $GK = \bar{K}GE \otimes_{GE} T'$ where T' is a totally ramified graded field extension of GE. Let M be the inertial lift of $\bar{K}$ over E in K and let $u_1, ..., u_r$ be homogeneous elements of T' such that $\Gamma_{T'}/\Gamma_{GE} = \langle gr(u_1) + \Gamma_{GE}\rangle \oplus ... \oplus \langle gr(u_r) + \Gamma_{GE}\rangle$. By Proposition 1.5, there are $x_i \in K$ such that $\tilde{x}_i = u_i$ and $G(E[x_i]) = GE[u_i]$. In particular, $E[x_i]$ are totally ramified over E, hence by [Mor89, Theorem 1] $T := E[x_1] \otimes_E ... \otimes_E E[x_r]$ is a totally ramified field extension of E. Once again, by applying [Mor89, Theorem 1] (and by comparing the dimensions), we have $K = M \otimes_E T$ (up to an isomorhism).

Lemma 4. 21 *Let E be a Henselian valued field and K a nice cyclic field extension* *of prime power dimension over E. Then, K is either inertial or totally ramified over E.*

Proof. Let M [resp. T] be an inertial [resp. totally ramified] field extension of E such that $K = M \otimes_E T$. Then, $Gal(K/E) = Gal(M/E) \oplus Gal(T/E)$. Remark that $Gal(K/F)$ is cyclic with prime power order. So, necessaraly $Gal(M/E)$ or $Gal(T/E)$ is a trivial group. Therefore, either M or T is trivial.

Lemma 4. 22 *Let E be a Henselian valued field with valuation v and D a semiramified division algebra of degree n over E. Assume $\bar{E}$ contains a primitive n^{th} root of unity. Then, the following statements are equivalent :*
(1) D has an abelian maximal subfield $K = M \otimes_E T$, where M [resp. T] is a cyclic inertial [resp. a cyclic totally ramified] field extension of E.
(2) There is a cyclic subfield M of D inertial over E and an element u of C_D^M such that $v(u) + \Gamma_E$ generates $\Gamma_{C_D^M}/\Gamma_E$ and $u^{[\bar{D}:\bar{M}]} \in E$.

Proof. (1) $\Rightarrow$ (2) By Proposition 2.6 C_D^M is semiramified (over M), so $[\bar{D} : \bar{M}] = (\Gamma_{C_D^M} : \Gamma_M)$. Moreover, K is a totally ramified maximal subfield of C_D^M, so $[\bar{D} : \bar{E}] = (\Gamma_K : \Gamma_M) = (\Gamma_T : \Gamma_E) = [T : E]$. Since T is cyclic over E, then $\Gamma_T/\Gamma_E (\cong Gal(T/E))$ is cyclic. Let $x \in T^*$ such that $\Gamma_T/\Gamma_E = \langle v(x) + \Gamma_E \rangle$, then $GT = GE[\tilde{x}]$ and the minimal polynomial of $\tilde{x}$ over $Frac(GE)$ is $X^{[T:E]} - \tilde{x}^{[T:E]}$. So, by Proposition 1.5 there is $u \in T$ such that $T = E[u]$, $u^{[T:E]} \in E^*$ and $v(u) + \Gamma_E = v(x) + \Gamma_E$.
(2) $\Rightarrow$ (1) Since $u^{[\bar{D}:\bar{M}]} \in E^*$, then $[E[u] : E] \leq [\bar{D} : \bar{M}]$. Moreover, since C_D^M is semiramified and $v(u) + \Gamma_E$ generates $\Gamma_{C_D^M}/\Gamma_E$, then $\bar{D}$ is a cyclic field extension of $\bar{M}$ and $[\bar{D} : \bar{M}] = card(\langle v(u) + \Gamma_E \rangle) \leq (\Gamma_{E[u]} : \Gamma_E) \leq [E[u] : E]$. Hence, $E[u]$ is totally ramified over E. Moreover, since $\Gamma_{E[u]}/\Gamma_E$ is cyclic and $\bar{E}$ contains a primitive

n^{th} root of unity, then $E[u]$ is cyclic over E. Therefore, $K := M \otimes_E E[u]$ is an abelian maximal subfield of D.

Proposition 4. 23 *Let E be a Henselian valued field with valuation v and D a non-degenerate tame semiramified division algebra of prime power degree over E. Then, the following statements are equivalent.*
(1) D has a nice abelian maximal subfield.
(2) D has an abelian maximal subfield $K = T \otimes_E M$, where T is cyclic totally ramified over E and M is cyclic inertial over E.
(3) D has a nice cyclic maximal subfield.
(4) D has a cyclic inertial maximal subfield.
(5) D has a non-trivial subfield which is totally ramified over E.
(6) $Gal(\bar{D}/\bar{E})$ is cyclic.

If $\bar{E}$ contains a primitive $deg(D)^{th}$ root of unity, then the above conditions are equivalent to the following :
(7) There a cyclic subfield M of D inertial over E and an element u of C_D^M such that $v(u) + \Gamma_E$ generates $\Gamma_{C_D^M}/\Gamma_E$ and $u^{[\bar{D}:\bar{M}]} \in E$.

Proof. (1) $\Rightarrow$ (6) Let K be a nice maximal subfield of D. If K is inertial over E, then $\bar{K} = \bar{D}$. So, $Gal(\bar{D}/\bar{E})$ is cyclic. If K is not inertial over E, then D contains a non-trivial totally ramified subfield. So, by Proposition 4.2, $Gal(\bar{D}/\bar{E})(\cong \Gamma_D/\Gamma_E)$ is cyclic.
(6) $\Rightarrow$ (3) Clear (indeed, the inertial lift of $\bar{D}$ over E in D is a nice cyclic maximal subfield of D).
(3) $\Rightarrow$ (2) By Lemma 4.21.
(2) $\Rightarrow$ (1) and (4) $\Leftrightarrow$ (6) Obvious.

(5) $\Rightarrow$ (6) By Proposition 4.2.

(6) $\Rightarrow$ (5) Let $d \in D^*$ such that $v(d) + \Gamma_E$ generates Γ_D/Γ_E, then $E[d]$ is a non-trivial totally ramified subfield of D.

If $\bar{E}$ contains a primitive $deg(D)^{th}$ root of unity, then (2) $\Leftrightarrow$ (7) follows by Lemma 4.22.

5 Indecomposable division algebras

Lemma 5. 1 *Let E be a Henselian valued field and D a nondegenerate tame semiramified division algebra of prime power degree over E. Assume $D = D_1 \otimes_E D_2$, where D_i ($i = 1, 2$) are non-trivial central division algebras over E, and let L_i be a maximal subfield of D_i. Then, $\bar{L}_i = \bar{D}_i$.*

Proof. Since D_i are non-trivial, then $rk(\Gamma_D/\Gamma_E) \geq 2$. Indeed, if Γ_D/Γ_E is cyclic, then D is nicely semiramified. So, by [JW90, Lemma 5.15] $exp(D) = exp(\Gamma_D/\Gamma_E) = deg(D)$. Hence, D is indecomposable. A contradiction. Moreover, by Proposition 4.2, $\bar{D}_i \neq \bar{E}$ (for otherwise, all subfields of D_i will be totally ramified over E). Let L_1 be a maximal subfield of D_1. Suppose by contradiction that $\bar{L}_1 \neq \bar{D}_1$, then $rk(Gal(\bar{D}_1 \otimes_{\bar{E}} \bar{D}_2/\bar{L}_1)) \geq 2$ (for if $\sigma \in Gal(\bar{D}_1/\bar{L}) \backslash \{id_{\bar{D}_1}\}$ and $\tau \in Gal(\bar{D}_2/\bar{E}) \backslash \{id_{\bar{D}_2}\}$, then $\sigma \otimes id_{\bar{D}_2}$ and $id_{\bar{D}_1} \otimes \tau$ are non-comparable elements of $Gal(\bar{D}_1 \otimes_{\bar{E}} \bar{D}_2/\bar{L}_1)$), so $rk(Gal(\bar{D}/\bar{L}_1)) \geq 2$. Hence, by Proposition 4.8, L_1 is inertial over E. So, by Proposition 2.6(1) D_1 is semiramified. Hence, $\bar{L}_1 = \bar{D}_1$. The contradiction obtained here shows that $\bar{L}_1 = \bar{D}_1$. In the same way, we prove that if L_2 is a maximal subfield of D_2, then $\bar{L}_2 = \bar{D}_2$.

Theorem 5. 2 *Let E be a Henselian valued field and D a nondegenerate tame semi-*

ramified division algebra of prime power degree over E. Then, D is indecomposable.

Proof. As seen above in the proof of Lemma 5.1, if Γ_D/Γ_E is cyclic, then D is indecomposable. Assume that $rk(\Gamma_D/\Gamma_E) \geq 2$ and suppose $D = D_1 \otimes_E D_2$, where D_1 and D_2 are non-trivial central division algebras over E. Choose a subfield K of D_i with $(\Gamma_K : \Gamma_E)$ maximal (among all subfields of D_1 and D_2). We can suppose $i = 1$. For an arbitrary element x_2 of D_2, $K[x_2]$ is a subfield of D. So, by Proposition 3.2, $\Gamma_{K[x_2]}/\Gamma_E$ is cyclic (with prime power order). Hence, the subgroups of $\Gamma_{K[x_2]}/\Gamma_E$ are totally ordered by inclusion. In particular, we have $\Gamma_{E[x_2]}/\Gamma_E \subseteq \Gamma_K/\Gamma_E$ (we can not have $\Gamma_K/\Gamma_E \subset \Gamma_{E[x_2]}/\Gamma_E$ because $(\Gamma_K : \Gamma_E)$ is maximal). So, Γ_{D_2}/Γ_E is cyclic. Let $y \in D_2$ such that $v(y) + \Gamma_E$ generates Γ_{D_2}/Γ_E (v being the extension of the valuation of E to D) and let L_2 be a maximal subfield of D_2 that contains y. By Lemma 5.1 $\bar{L}_2 = \bar{D}_2$. Moreover, we have $\Gamma_{L_2}/\Gamma_E = \Gamma_{D_2}/\Gamma_E$. Hence, $D_2 = L_2$. A contradiction.

Corollary 5. 3 *Let E be a field, K a finite dimensional non-cyclic abelian field extension of E with Galois group G, S a basis of G, U a nondegenerate matrix of $U_S(G, K^*)$, Z the center of the generic abelian crossed product (K, G, U) and L an immediate field extension of Z for its canonical valuation. Assume $card(G) = p^n$, where p is a prime positive integer and n is an arbirary positive integer, then $(K, G, U) \otimes_Z L$ is an indecomposable division algebra.*

Remark 5.4 Recently, Kelly L. Mckinnie has constructed tame semiramified indecomposable division p-algebras of arbitrary degree and of exponent p (for $p \neq 2$) [McK05, Corollary 6.15]. Using the same arguments and Theorem 5.2 -instead of [McK05, Corollary 5.4]-, we get examples of tame semiramified indecomposable division algebras of prime exponent q and of degree q^n for any prime $q \neq 2$ and any positive integer n (even for $q \neq char(\bar{E})$).

Appendix

Claim A. 1 *Let E/F be an algebraic graded field extension and g a homogenizable irreducible polynomial of $F[X]$. Assume g has a root $x \in E\backslash\{0\}$ and let $f_{x,Frac(F)}$ be the minimal polynomial of x over $Frac(F)$. Then, $g = af_{x,Frac(F)}$ for some $a \in F^*$. In particular, $x \in E^*$.*

Proof. Since x is a root of g, then $f_{x,Frac(F)}$ divides g in $Frac[X]$. Let $h \in Frac(F)[X]$ such that $g = hf_{x,Frac(F)}$. Obviously, we can write $h = b^{-1}h'$ [resp. $f_{x,Frac(F)} = c^{-1}f'$] with $h', f' \in F[X]$ and $b, c \in F$. Let $\lambda \in \Delta_F$ (where Δ_F is the divisible hull of Γ_F) such that $g \in (F[X]^{(\lambda)})^h$. It follows from the equality $(bc)g = h'f'$ that $(bc)_m g = h'_m f'_m$, where $(bc)_m$, h'_m and f'_m are the components of greater grade of bc, h' and f', respectively. Since g is irreducible in $F[X]$, then either h'_m or f'_m is in F. Assume that $h'_m \in F$, then $deg(g) = deg(f'_m) \leq deg(f_{x,Frac(F)})$. So, $deg(g) = deg(f_{x,Frac(F)})$ (since $f_{x,Frac(F)}$ divides g). If $f'_m \in F$, then $deg(g) = deg(h'_m) \leq deg(h)$. So, $deg(f_{x,Frac(F)}) = 0$. But this is not true. See that we can write $g = a_0 + a_1X + ... + a_nX^n$, where n is a positive integer and $a_i \in F^*$ for all $0 \leq i \leq n$ (Indeed, write $g = d_0 + d_1X + ... + d_nX^n$, where $d_i \in F$ and write $d_i = \sum(d_i)_\gamma$, where $(d_i)_\gamma \in F_{\bar{\gamma}}$. Since g is homogeneous (in $F[X]^{(\lambda)}$), then we can consider only the $(d_i)_\gamma$ where $gr((d_i)_\gamma X^i) = gr(g)$). Let $g' = a_n^{-1}g$. Then, $g' = f_{x,Frac(F)}$ (since $g'(x) = 0$, $deg(g') = deg(f_{x,Frac(F)})$). So, $g = a_n f_{x,Frac(F)}$. In particular, $f_{x,Frac(F)}$ is homogeneous in $F[X]^{(\lambda)}$. So, $x \in E^*$ (by [HW(1)99, Proposition 2.2 and corollary

2.5]).

B Normal subgroups

Lemma B. 1 *Let p be a prime positive integer and G a p-group, generated by two noncommutative elements x and y such that $ord(x) = p$. Then either $\langle x \rangle$ (i.e. the cyclic subgroup of G generated by x) or $\langle y \rangle$ is a non normal subgroup of G.*

Proof. Let $p^m = ord(y)$, then for every $0 < k < p$, $0 < k' < p^m$, $x^k \neq y^{k'}$. Indeed, otherwise we will have $\langle x \rangle = \langle x^k \rangle \subseteq \langle y \rangle$, which contradicts the noncommutativity of x and y. If $\langle x \rangle$ and $\langle y \rangle$ were both normal in G, then there would exist positive integers r and s with $1 < r < p$ and $1 < s < p^m$ such that $yx^r y^{-1} = x$ and $x^{-1}y^s x = y$. Therefore $(yx^r y^{-1})y = x(x^{-1}y^s x)$, hence $yx^r = y^s x$. Thus $x^{r-1} = y^{s-1}$. But this is not true.

In what follows, we proceed by contradiction to prove that if p is an odd prime integer and G is a non abelian p-group, then necessarily G has a subgroup H which is not normal in G.

B.2. Let G be a non abelian p-group such that all subroups of G are normal (in G). We may assume that G has minimal cardinal. It follows from Lemma B.1, that $card(G) > p^2$. Let $Z(G)$ be the center of G and $G^c := \{\alpha\beta\alpha^{-1}\beta^{-1} |\ \alpha, \beta \in G\}$ be the commutator of G.

Claim B. 3 *Under the hypotheses of (B.2), we have $G^c \subseteq Z(G)$.*

Proof. Since $card(G) > 1$, then by [L84, theorem 6.4, p. 25], $Z(G) \neq \{1\}$. If $G/Z(G)$ has a non normal subgroup A, then $H := \{\alpha \in G| \ \alpha Z(G) \in A\}$ would be a non normal subgroup of G. A contradiction. Therefore, by the minimality of $card(G)$, $G/Z(G)$ is necessarily abelian. So, $G^c \subseteq Z(G)$.

Claim B. 4 *Under the hypotheses of (B.2), we have* $card(G^c) = p$.

Proof. Assume that $card(G^c) > p$ and consider a subgroup G' of G^c with $card(G') = p$. Since the cardinal of G is minimal, then either G/G' is abelian or G/G' has a subgroup A which is not normal in G/G'. See that G/G' cannot be abelian since otherwise we would have $G^c \subseteq G'$. Also, if there was a non-normal subgroup A of G/G', then $H := \{\alpha \in G| \ \alpha G' \in A\}$ would be a non normal subgroup of G. A contradiction.

Claim B. 5 *Under the hypotheses of (B.2), if x and y are two non-commutative elements of G, then every element z of G can be written in the form $z = ax^m y^n$, with $a \in Z(G)$, $0 \leq m, n < p$. Thus, $card(G/Z(G)) = p^2$. Precisely, $G/Z(G) \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.*

Proof. Since $G/\langle x \rangle$ and $G/\langle y \rangle$ are abelian, then $G^c \subseteq \langle x \rangle \cap \langle y \rangle$. Let $p^r = card(\langle x \rangle / G^c)$ and $p^s = card(\langle y \rangle / G^c)$. Remark that $card(G)$ being minimal, then $G = \langle x, y \rangle$ (i.e. G is generated by x and y). Let $u = xyx^{-1}y^{-1}$, then $x^p y = u^p y x^p = yx^p$ (since $u \in G^c$ and $card(G^c) = p$.) So $x^p \in Z(G)$. By symmetry, we have also $y^p \in Z(G)$. Therefore, every element of G can be written in the form $ax^m y^n$, with $a \in Z(G)$ and $0 \leq m, n < p$. Thus, the elements of $G/Z(G)$ can be written as $\bar{x}^m \bar{y}^n$, with $0 \leq m, n < p$ (where $\bar{x} = xZ(G)$ and $\bar{y} = yZ(G)$.) So, $card(G/Z(G)) \leq p^2$. Remark that $G/Z(G)$ can not be cyclic since otherwise G would be abelian. Hence $card(G/Z(G)) = p^2$ and $G/Z(G) \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.

Claim B. 6 *Under the hypotheses of (B.2), if p is an odd prime integer and if we consider the map $\phi : G \to G$, defined by $\phi(z) = z^p$, then ϕ is a group homomorphism and $G^p := \phi(G) \subseteq Z(G)$.*

Proof. Let $x, y \in G$ and let $u = xyx^{-1}y^{-1}$. It is straightforword that $(xy)^p = uu^2..u^{p-1}x^py^p = (u^{\frac{p-1}{2}})^p x^py^p = x^py^p$ (since $u^{\frac{p-1}{2}} \in G^c$ and $card(G^c) = p$). So, ϕ is a group homomorphism. Since $x^py = u^pyx^p = yx^p$, then $\phi(G) \subseteq Z(G)$.

Obviously, Under the hypotheses of Claim B.6, $G/ker(\phi) \cong G^p$.

Claim B. 7 *under the hypotheses of Claim B.6, we have $ker(\phi) \subseteq Z(G)$.*

Proof. If $ker(\phi) \nsubseteq Z(G)$, then one can choose two non-commutative elements x and y of G with $x \in ker(\phi)$. Since $G = \langle x, y\rangle$ and $ord(x) = p$, then by Lemma B.1, either $\langle x\rangle$ or $\langle y\rangle$ is not normal in G. A contradiction.

Claim B. 8 *Under the hypotheses of (B.2), $Z(G)$ is cyclic.*

Proof. Let x and y be two non-commutative elements of G, and consider $w \in G$ with $ord(w) \geq p$. Then $\bar{x}\bar{y} = \bar{y}\bar{x}$ in $G/\langle w\rangle$. Hence $u := xyx^{-1}y^{-1} \in \langle w\rangle$. In particular, this is true for $1 \neq w \in Z(G)$. Therefore, $Z(G)$ is cyclic (for if $Z(G)$ were not cyclic, then it would have a decomposition $Z(G) = \langle w_1\rangle \times \langle w_2\rangle \times .. \times \langle w_k\rangle$ with $ord(w_i) \geq p$ $(1 \leq i \leq k)$ and $k > 1$. In particular, we have $\langle w_1\rangle \cap \langle w_2\rangle = \{1\}$, hence necessarily $u = 1$. But this is not true since x and y are non-commutative.)

Claim B. 9 *Under the hypotheses of Claim B.6, $ker(\phi) = G^c$.*

Proof. Since $G/ker(\phi)$ is abelian (otherwise G would have a non normal subgroup), then $G^c \subseteq ker(\phi)$. Moreover, by Claim B.7, we have $ker(\phi) \subseteq Z(G)$. So $ker(\phi)$ is cyclic (of exponent p). So $ker(\phi) = G^c$.

Claim B. 10 *Under the hypotheses of Claim B.6, G/G^p is cyclic with cardinal p.*

Proof. Since $G^p \cong G/ker(\phi)$, then $card(G/G^p) = p$, so G/G^p is cyclic.

Since $G^p \subseteq Z(G)$ and G/G^p is cyclic, then G is abelian. But this contradicts the hypotheses of Claim B.6. Therefore, we have the following Theorem.

Theorem B. 11 *Let $p > 2$ be a prime integer and G a non abelian p-group. Then G has a non normal subgroup.*

C Cyclic graded p-algebras

We recall first the following facts concerning cyclic graded simple algebras. All these results follow from analogous ones proved in the ungraded case (it suffices to use the canonical embedding $GBr(F) \to Br(Frac(F))$, defined by $[D] \mapsto [D \otimes_F Frac(F)]$) :

(C.1) Let F be a graded field and L/F a finite-dimensional cyclic graded field extension of degree n, then the following assertions hold (for arbitrary $a, b \in F^*$) :
(1) $(L/F, \sigma, a) = (L/F, \sigma^t, a^t)$, for any positive integer t coprime to n.
(2) $(L/F, \sigma, a) \otimes_F (L/F, \sigma, b)$ is similar to $(L/F, \sigma, ab)$ with respect to $GBr(F)$.

(C.2) (1) Let L/F be a finite-dimensional cyclic graded field extension with Galois group generated by σ, then the mapping $a \mapsto [(L/F, \sigma, a)]$ ($a \in F^*$) induces a group isomorphism $F^*/N_{L/F}(L^*) \to GBr(L/F)$.
(2) Let N/F be a cyclic graded field extension with Galois group generated by τ such that $F \subseteq L \subseteq N$ and let $s = [N : L]$ and $a \in F^*$, then $(N/F, \tau, a^s)$ is similar to $(L/F, \sigma, a)$ with respect to $GBr(F)$.

Theorem C. 3 *Let F be a graded field with characteristic $p > 0$ and a a non-zero homogeneous element of F such that $a^{1/p} \notin F$. If D is a non-trivial graded central division algebra split by $F[a^{1/p}]$, then there is a cyclic graded field extension L of dimension p over F such that $D = (L/F, \sigma, a)$, where σ is a generator of $Gal(L/F)$.*

Proof. Since $deg(D) = p$ and $F[a^{1/p}]$ splits D, then we can assume $F[a^{1/p}]$ is a maximal graded subfield of D. Let $x := a^{1/p}$ and consider the graded F-automorphism $\Phi : D \mapsto D$, defined by $z \mapsto xzx^{-1}$. Since $x \notin F$, then $\Phi \neq id_D$. So, $\phi = \Phi - id_D$ is a non-zero graded F-space homomorphism of D. Clearly, we have $\phi^p = 0$. Let r be the greater positive integer such that $\phi^r \neq 0$ and let $z \in D^*$ such that $\phi^r(z) \neq 0$. Put $v = \phi^{r-1}(z)$, $w = \phi^r(z)$ and $b = w^{-1}v$. One can easily check that $\phi(b) = b + 1$. Therefore, $L := F[b]$ is a cyclic graded field extension of F with Galois group generated by $\sigma := \phi_{/L}$, and $D = (L/F, \sigma, a)$.

Theorem C. 4 *Let F be a graded field with characteristic $p > 0$, a a non-zero homogeneous element of F and A a non-trivial graded central simple algebra which satisfies $[A] \in GBr(F[a^{1/p^e}]/F)$ and $[A] \notin GBr(F[a^{1/p^f}]/F)$ for all $f < e$, then $[A] = [(N/F, \sigma, a)]$, where N/F is a cyclic graded field extension of dimension p^e with Galois group generated by σ.*

Proof. Assume first that $F[a^{1/p^e}]$ has dimension p^e over F. We will proceed by induction on e. The case $e = 1$ was proved in Theorem C.3 above. Assume this is true for $e < n$ and let $e = n$. By the induction hypothesis there is a cyclic graded field extension M of dimension p^{n-1} over $F[a^{1/p}]$ such that $A \otimes_F F[a^{1/p}] \sim (M/F[a^{1/p}], \sigma, a^{1/p})$, where σ is a generator of the Galois group of $M/F[a^{1/p}]$. Clearly, $Frac(M)$ is a normal field extension of $Frac(F)$ with $Gal(Frac(M)/Frac(F)) = Gal(Frac(M)/Frac(F[a^{1/p}]))$. Moreover if N' is the tame closure of F in M, then by [HW(1)99, Proposition 3.7 and Definition 3.8] $Frac(N')$ is the separable closure of $Frac(F)$ in $Frac(M)$. Therefore by [Karp89, Proposition 7.7, p.283] $Frac(M) = Frac(N') \otimes_{Frac(F)} Frac(F[a^{1/p}])$ (up to an isomorphism). This shows that $Frac(N')$ is a cyclic field extension of $Frac(F)$, hence by [HW(1)99, Theorem 3.11] N' is a cyclic graded field extension of F (with Galois group generated by $\sigma_{/N'}$). Let K be a cyclic field extension of F_0 that contains N'_0 such that $[K : N'_0] = p$ (see [D82, Lemma 2, p. 107]) and let $N = KF$ (i.e. the graded field extension of F generated by K). By, [HW(1)99, Remark 3.1] N is a cyclic inertial graded field extension of F with Galois group generated by some τ such that $\tau_{/N'} = \sigma_{/N'}$. We have $A \otimes_F F[a^{1/p}] = [(M/F[a^{1/p}], \sigma, a^{1/p})] = [(NF[a^{1/p}]/F[a^{1/p}], \tau, a)] = [(N/F, \tau, a) \otimes_F F[a^{1/p}]]$. So, $[A] - [(N/F, \tau, a)] \in GBr(F[a^{1/p}]/F)$ and $a^{1/p} \notin F$. Hence, again by Theorem C.3, $[A] = [(N/F, \tau, a)] + [(T/F, \theta, a)]$, where T/F is a cyclic graded field extension of dimension p with galois group generated by θ. We have either, $N \cap T = F$ or $T \subseteq N$. For the first case, we consider the graded subfield L of $N \otimes_F T$ elementwise fixed by $\alpha := \tau^{p^{n-1}} \otimes \theta^{-1}$. By [D82, Lemma 9, p. 75] we have $[(Frac(N)/Frac(F), \tau, a)] + [(Frac(T)/Frac(F), \theta, a)] = [(Frac(L)/Frac(F), \alpha, a)]$. So, $[(N/F, \tau, a)] + [(T/F, \theta, a)] = [(L/F, \alpha, a)]$ (this follows from the canonical embedding $GBr(F) \to Br(Frac(F))$ defined by $[D] \mapsto [D \otimes_F Frac(F)]$).

Now, if we suppose that $T \subseteq N$, then there exists a positive integer t coprime to p such that $\tau_{/T} = \theta^t$. Since $1 + tp^{n-1}$ and p^n are coprime, then there is a positive integer s such that $s(1 + tp^{n-1}) \equiv 1(mod\ p^n)$. Hence, by (C.1 and C.2) above $[(T/F, \theta, a)] = [(T/F, \theta^t, a^t)] = [(N/F, \tau, a^{tp^{n-1}})]$. So, $[A] = (1 + tp^{n-1})[(N/F, \tau, a)] = (1 + tp^{n-1})[(N/F, \tau^s, a^s)] = [(N/F, \tau^s, a)]$.

If $F[a^{1/p^e}]$ has dimension $p^f < p^e$ over F, then $F[a^{1/p^e}] = F[b^{1/p^f}]$, where $a = b^{p^{e-f}}$. It follows by the above that $[A] = [(N'/F, \sigma', b)]$, where N'/F is a cyclic graded field extension with dimension p^f. We can then apply (C.2(2)) to conclude.

Corollary C. 5 *Let F be a graded field with characteristic $p > 0$, A a graded central division algebra of degree p^e over F. Assume that A has a simple purely wild maximal graded subfield, then A is cyclic.*

Theorem C. 6 *Let F be a graded field with characteristic $p > 0$ and D a graded central division algebra of degree a power of p over F. Then, D is cyclic if and only if it has a simple purely wild maximal graded subfield.*

Proof : If D is cyclic, then we can write $D = (L/F, \sigma, a)$, where L is a a cyclic maximal graded subfield of D, σ is a generating element of the Galois group $Gal(L/F)$ and a is a nonzero homogeneous element of F. Let n be the degree of D, then by definition, there is a nonzero homogeneous element x of D such that $D = \oplus_{i=1}^n Lx^i$, with $x^n = a$ and $xyx^{-1} = \sigma(y)$ for all $y \in L$. Hence $F[x]$ is a simple purely wild maximal graded subfield of D. The converse follows by Corollary C.5.

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References

[A61] AA Albert, Structure of algebras, Providence Amer Math Soc, 1961.

[AS78] SA. Amitsur and D. Saltman, Generic abelian crossed product and p-algebras, J. Algebra, V 51, (1978), p. 76-87.

[B92] M. Boulagouaz, The graded and tame extensions, pp. 27-40 in commutative ring theory (Fès, 1992), Lecture Notes in Pure and Appl. Math. Vol. 153, Dekker, New York, 1994.

[B] M. Boulagouaz, The graded and tame extensions. In : Lectures Notes pure app math 153. New York : Marcel Dekker, pp 27-40.

[B95] M. Boulagouaz, Le gradué d'une algèbre à division valuée. Comm. Alg. 23(1995), pp 4275-4300.

[B98] M. Boulagouaz, Algèbre à division graduée centrale. Comm. Alg. 26(1998), pp 2933-2947.

[BM00] M. Boulagouaz and K. Mounirh, Generic abelian crossed products and graded division algebras, pp. 33-47 in algebra and number theory (Fès, 1997), Lecture Notes in Pure and Appl. Math. Vol. 208, Dekker, New York, 2000.

[Ch04] F.H. Chang, Division algebras over generalized local fields, PHD Thesis, University of California, San Diego, 2004.

[D82] P. Draxl, Skew fields. London Math Soc Lectures Notes 81. Cambridge : Cambridge University Press, 1982.

[E72] O. Endler, Valuation theory, Springer, New York, 1972.

[JW90] B. Jacob and A. R. Wadsworth, Division algebras over Henselian Fields, J. Algebra, 128 (1990), pp. 126-179.

[Ha59] Hall jr. The theory of groups, The Macmillan company, New York 1959.

[HW(1)99] Y.-S. Hwang and A. R. Wadsworth, Algebraic extensions of graded and valued fields, Comm. Algebra, 27 (1999), pp. 821-840.

[HW(2)99] Y.-S. Hwang and A. R. Wadsworth, Correspondences between valued division algebras and graded division algebras, J. Algebra, 220 (1999), pp. 73-114.

[Karp89] G. Karpilovsky, Topics in field theory, North-Holand Mathematics studies 155, 1989.

[Ku72] Kuijk, Generic construction of non-cyclic division algebras, J. Pure App. Algebra 2 (1972), pp. 121-130.

[L84] S. Lang, Algebra, Addision-Wesley Publication Company, Inc. 1984.

[McK05] K. McKinnie, Galois subfields and indecomposable division algebras, University of Austin 2005.

[Mor89] P. Morandi, The Henselization of a valued division algebra, J. Algebra, 122 (1989), pp. 232-243.

[M05] K. Mounirh, Nicely semiramified division algebras over Henselian fields, , International Journal of Mathematics and Mathematical Sciences (2005), 571-577.

[NO82] C. Natasescu and F. Van Oystaeyen, Graded ring Theory, North-Holland, Library of Math., Vol. 82, 1982.

[P82] R-S. Pierce, Associative algebra, Spring-Verlag (1982).

[Pl76] V.P. Platonov, The Tannaka-Artin problem and reduced K-theory, Izv. Akad. Nauk SSSR Ser. Math. 40 (1976), 227-261; (Eng trans.) Math. USSR Izv. 10 (1976), 211-243.

[S78] D. Saltman, Noncrossed product p-algebra and Galois p-extensions, J. Algebra V52 (1978), pp.302-314

[Sch50] O.F.G. Shcilling. The thoery of valuations. Mathematical Survey, N4. American Mathematical Society, New York, 1950.

[T81] Produit croisé abelien, J. Algebra V70 (1981), pp. 420-436.