A universality theorem for Voevodsky's algebraic cobordism spectrum

I. Panin*† K. Pimenov† O. Röndigs‡

May 31, 2007§

Abstract

An algebraic version of a theorem due to Quillen is proved. More precisely, for a ground field k we consider the motivic stable homotopy category $\mathrm{SH}(k)$ of $\mathbf{P}^1$-spectra, equipped with the symmetric monoidal structure described in [PPR1]. The algebraic cobordism $\mathbf{P}^1$-spectrum MGL is considered as a commutative monoid equipped with a canonical orientation $th^{\mathrm{MGL}} \in \mathrm{MGL}^{2,1}(\mathrm{Th}(\mathcal{O}(-1)))$. For a commutative monoid E in the category $\mathrm{SH}(k)$ it is proved that assignment $\varphi \mapsto \varphi(th^{\mathrm{MGL}})$ identifies the set of monoid homomorphisms $\varphi\colon \mathrm{MGL} \to E$ in the motivic stable homotopy category $\mathrm{SH}(k)$ with the set of all orientations of E. The result was stated originally in a slightly different form by G. Vezzosi in [Ve].

1 Oriented commutative ring spectra

We refer to [PPR1, Appendix] for the basic terminology, notation, constructions, definitions, results. For the convenience of the reader we recall the basic definitions. Let S be a Noetherian scheme of finite Krull dimension. One may think of S being the spectrum of a field or the integers. Let $\mathcal{S}m/S$ be the category of smooth quasi-projective S-schemes, and let **sSet** be the category of simplicial sets. A *motivic space over* S is a functor

$$A\colon \mathcal{S}m/S^{op} \to \mathbf{sSet}$$

(see [PPR1, A.1.1]). The category of motivic spaces over S is denoted $\mathbf{M}(S)$. This definition of a motivic space is different from the one considered by Morel and Voevodsky in [MV] – they consider only those simplicial presheaves which are sheaves in the Nisnevich topology on $\mathcal{S}m/S$. With our definition the Thomason-Trobaugh K-theory functor obtained by using big vector bundles is a motivic space on the nose. It is not a simplicial Nisnevich sheaf. This is why we prefer to work with the above notion of "space".

*Universität Bielefeld, SFB 710, Bielefeld, Germany
†Steklov Institute of Mathematics at St. Petersburg, Russia
‡Institut für Mathematik, Universität Osnabrück, Osnabrück, Germany
§The authors thank the SFB-701 at the Universität Bielefeld, the RTN-Network HPRN-CT-2002-00287, the RFFI-grant 03-01-00633a, and INTAS-03-51-3251 for their support.

We write $\mathrm{H}^{\mathrm{cm}}_{\bullet}(S)$ for the pointed motivic homotopy category and $\mathrm{SH}^{\mathrm{cm}}(S)$ for the stable motivic homotopy category over S as constructed in [PPR1, A.3.9, A.5.6]. By [PPR1, A.3.11 resp. A.5.6] there are canonical equivalences to $\mathrm{H}_{\bullet}(S)$ of [MV] resp. $\mathrm{SH}(S)$ of [V1]. Both $\mathrm{H}^{\mathrm{cm}}_{\bullet}(S)$ and $\mathrm{SH}^{\mathrm{cm}}_{\bullet}(S)$ are equipped with closed symmetric monoidal structures such that the $\mathbf{P}^1$-suspension spectrum functor is a strict symmetric monoidal functor

$$\Sigma^{\infty}_{\mathbf{P}^1}\colon \mathrm{H}^{\mathrm{cm}}_{\bullet}(S) \to \mathrm{SH}^{\mathrm{cm}}(S).$$

Here $\mathbf{P}^1$ is considered as a motivic space pointed by $\infty \in \mathbf{P}^1$. The symmetric monoidal structure $(\wedge, \mathbb{I}_S = \Sigma^{\infty}_{\mathbf{P}^1} S_+)$ on the homotopy category $\mathrm{SH}^{\mathrm{cm}}(S)$ is constructed on the model category level by employing symmetric $\mathbf{P}^1$-spectra. It satisfies the properties required by Theorem 5.6 of Voevodsky congress talk [V1]. From now on we will usually omit the superscript $(-)^{\mathrm{cm}}$.

Given a $\mathbf{P}^1$-spectrum E one has a cohomology theory on the category of pointed spaces. Namely, for a pointed space (A,a) set $E^{p,q}(A,a) = Hom_{\mathrm{H}^{\mathrm{cm}}_{\bullet}(S)}(\Sigma^{\infty}_{\mathbf{P}^1}(A,a), \Sigma^{p,q}(E))$ and $E^{*,*}(A,a) = \oplus_{p,q} E^{p,q}(A,a)$. A cohomology theory on the category of non-pointed spaces is defined as follows. For a non-pointed space A set $E^{p,q}(A) = E^{p,q}(A_+, +)$ and $E^{*,*}(A) = \oplus_{p,q} E^{p,q}(A)$.

Each $X \in \mathcal{S}m/S$ defines a motivic space constant in the siplicial direction taking an S-smooth U to $\mathrm{Mor}_S(U,X)$. This motivic space is non-pointed. So we regard S-smooth varieties as motivic spaces (non-pointed) and set

$$E^{p,q}(X) = E^{p,q}(X_+, +).$$

Given a $\mathbf{P}^1$-spectrum E we will reduce the double grading on the cohomology theory $E^{*,*}$ to a grading. Namely, set $E^m = \oplus_{m=p-2q} E^{p,q}$ and $E^* = \oplus_m E^m$. *We often will write* $E^*(k)$ *for* $E^*(\mathrm{Spec}(k))$ *below in this text.*

To complete this section, note that for us a $\mathbf{P}^1$-*ring spectrum* is a monoid (E, μ, e) in $(\mathrm{SH(S)}, \wedge, \mathbb{I}_S)$. A *commutative* $\mathbf{P}^1$-ring spectrum is a commutative monoid (E, μ, e) in $(\mathrm{SH(S)}, \wedge, 1)$. The cohomology theory E^* defined by a $\mathbf{P}^1$-ring spectrum is a ring cohomology theory. The cohomology theory E^* defined by a commutative $\mathbf{P}^1$-ring spectrum is a ring cohomology theory, however it is not necessary graded commutative. The cohomology theory E^* defined by an oriented commutative $\mathbf{P}^1$-ring spectrum is a graded commutative ring cohomology theory.

1.1 Oriented commutative ring spectra

Following Adams and Morel we define an orientation of a commutative $\mathbf{P}^1$-ring spectrum. However we prefer to use Thom classes instead of Chern classes. Consider the pointed motivic space $\mathbf{P}^{\infty} = \mathrm{colim}_{n \geq 0} \mathbf{P}^n$ having base point $g_1\colon S = \mathbf{P}^0 \hookrightarrow \mathbf{P}^{\infty}$.

The tautological "vector bundle" $\mathcal{T}(1) = \mathcal{O}_{\mathbf{P}^{\infty}}(-1)$ is also known as the Hopf bundle. It has zero section $z\colon \mathbf{P}^{\infty} \hookrightarrow \mathcal{T}(1)$. The fiber over the point $g_1 \in \mathbf{P}^{\infty}$ is $\mathbb{A}^1$. For a vector bundle V over a smooth S-scheme X with zero section $z\colon X \hookrightarrow V$ consider a Nisnevich sheaf associated with the presheaf $Y \mapsto V(Y)/\big(V \smallsetminus z(X)\big)(Y)$ on the Nisnevich site $\mathcal{S}m/S$.

The *Thom space* $\mathrm{Th}(V)$ of V is defined as that Nisnevich sheaf regarded as a presheaf. In particular $\mathrm{Th}(V)$ is a pointed motivic space in the sense of [PPR1, Defn. A.1.1]. Its Nisnevich sheafification coincides with Voevodsky's Thom space [V1, p. 422], since $\mathrm{Th}(V)$ is already a Nisnevich sheaf. The Thom space of the Hopf bundle is then defined as the colimit $\mathrm{Th}(\mathcal{T}(1)) = \mathrm{colim}_{n\geq 0} \mathrm{Th}\big(\mathcal{O}_{\mathbf{P}^n}(-1)\big)$.

Abbreviate $T = \mathrm{Th}(\mathbf{A}^1_S) = \mathbf{A}^1_S/(\mathbf{A}^1_S \smallsetminus \{0\})$.

Let E be a commutative ring $\mathbf{P}^1$-spectrum. The unit gives rise to an element $1 \in E^{0,0}(\mathrm{Spec}(k))$. Applying the $\mathbf{P}^1$-suspension isomorphism to that element we get an element $\Sigma_{\mathbf{P}^1}(1) \in E^{2,1}(\mathbf{P}^1/\{\infty\}))$. The canonical covering of $\mathbf{P}^1$ defines motivic weak equivalences

$$\mathbf{P}^1/\{\infty\} \xrightarrow{\sim} \mathbf{P}^1/\mathbf{A}^1 \xleftarrow{\sim} \mathbf{A}^1/\mathbf{A}^1 \smallsetminus \{0\} = T,$$

which in turn define pull-back isomorphisms $E(\mathbf{P}^1/\{\infty\}) \leftarrow E(\mathbf{A}^1/\mathbf{A}^1 \smallsetminus \{0\}) \to E(T)$. Denote $\Sigma_T(1)$ the image of $\Sigma_{\mathbf{P}^1}(1)$ in $E^{2,1}(T)$.

Definition 1.1.1. Let E be a commutative ring $\mathbf{P}^1$-spectrum. A *Thom orientation of* E is an element $th \in E^{2,1}(\mathrm{Th}(\mathcal{T}(1))$ such that its restriction to the Thom space of the fibre over the distinguished point coincides with the element $\Sigma_T(1) \in E^{2,1}(T)$. A *Chern orientation of* E is an element $c \in E^{2,1}(\mathbf{P}^\infty)$ such that $c|_{\mathbf{P}^1} = -\Sigma_{\mathbf{P}^1}(1)$. *An orientation* of E is either a Thom orientation or a Chern orientation. Two Thom orientations of E coincide if respecting Thom elements coincides. Two Chern orientations of E coincide if respecting Chern elements coincides. One says that a Thom orientation th of E coincides with a Chern orientation c of E provided that $c = z^*(th)$ or equivalently the element th coincides with the one $th(\mathcal{O}(-1))$ given by (2) below.

Remark 1.1.2. The element th should be regarded as a Thom class of the tautological line bundle $\mathcal{T}(1) = \mathcal{O}(-1)$ over $\mathbf{P}^\infty$. The element c should be regarded as a Chern class of the tautological line bundle $\mathcal{T}(1) = \mathcal{O}(-1)$ over $\mathbf{P}^\infty$.

Example 1.1.3. The following orientations given right below are relevant for our work. Here MGL denotes the $\mathbf{P}^1$-ring spectrum representing algebraic cobordism obtained in Definition 2.1.1 below, and BGL denotes the $\mathbf{P}^1$-ring spectrum representing algebraic K-theory constructed in [PPR1, Theorem 2.2.1].

- Let $u_1 : \Sigma^\infty_{\mathbf{P}^1}(\mathrm{Th}(\mathcal{T}(1)))(-1) \to \mathrm{MGL}$ be the canonical map of $\mathbf{P}^1$-spectra. Set $th^{\mathrm{MGL}} = u_1 \in \mathrm{MGL}^{2,1}(\mathrm{Th}(\mathcal{T}(1)))$. Since $th^{\mathrm{MGL}}|_{\mathrm{Th}(\mathbf{1})} = \Sigma_{\mathbf{P}^1}(1)$ in $\mathrm{MGL}^{2,1}(\mathrm{Th}(\mathbf{1}))$, the class th^{MGL} is an orientation of MGL.

- Set $c = (-\beta) \cup ([\mathcal{O}] - [\mathcal{O}(1)]) \in \mathrm{BGL}^{2,1}(\mathbf{P}^\infty)$. The relation (11) from [PPR1] shows that the class c is an orientation of BGL.

2 Oriented ring spectra and infinite Grassmannians

Let (E, c) be an oriented commutative $\mathbf{P}^1$-ring spectrum. In this Section we compute the E-cohomology of infinite Grassmannians and their products. The results are the expected ones 2.0.6.

The oriented $\mathbf{P}^1$-ring spectrum (E, c) defines an oriented cohomology theory on $\mathcal{S}m\mathcal{O}p$ in the sense of [PS1, Defn. 3.1] as follows. The restriction of the functor $E^{*,*}$ to the category $\mathcal{S}m/S$ is a ring cohomology theory. By [PS1, Th. 3.35] it remains to construct a Chern structure on $E^{*,*}|_{\mathcal{S}m\mathcal{O}p}$ in the sense of [PS1, Defn.3.2]. Let $H(k)$ be the homotopy category of spaces over k. The functor isomorphism $\mathrm{Hom}_{\mathrm{H(k)}}(-, \mathbf{P}^\infty) \to \mathrm{Pic}(-)$ on the category $\mathcal{S}m/S$ provided by [MV, Thm. 4.3.8] sends the class of the identity map $\mathbf{P}^\infty \to \mathbf{P}^\infty$ to the class of the tautological line bundle $\mathcal{O}(-1)$ over $\mathbf{P}^\infty$. For a line bundle L over $X \in \mathcal{S}m/S$ let $[L]$ be the class of L in the group $\mathrm{Pic}(X)$. Let $f_L\colon X \to \mathbf{P}^\infty$ be a morphism in H(k) corresponding to the class $[L]$ under the functor isomorphism above. For a line bundle L over $X \in \mathcal{S}m/S$ set $c(L) = f_L^*(c) \in E^{2,1}(X)$. Clearly, $c(\mathcal{O}(-1)) = c$. The assignment $L/X \mapsto c(L)$ is a Chern structure on $E^{*,*}|_{\mathcal{S}m\mathcal{O}p}$ since $c|_{\mathbf{P}^1} = -\Sigma_{\mathbf{P}^1}(1) \in E^{2,1}(\mathbf{P}^1, \infty)$. With that Chern structure $E^{*,*}|_{\mathcal{S}m\mathcal{O}p}$ is an oriented ring cohomology theory in the sense of [PS1]. In particular, (BGL, c^K) defines an oriented ring cohomology theory on $\mathcal{S}m\mathcal{O}p$.

Given this Chern structure, one obtains a theory of Thom classes $V/X \mapsto th(V) \in E^{2\mathrm{rank}(V),\mathrm{rank}(V)}\big(\mathrm{Th}_X(V)\big)$ on the cohomology theory $E^{*,*}|_{\mathcal{S}m\mathcal{O}p/S}$ in the sense of [PS1, Defn. 3.32] as follows. There is a unique theory of Chern classes $V \mapsto c_i(V) \in E^{2i,i}(X)$ such that for every line bundle L on X one has $c_1(L) = c(L)$. For a rank r vector bundle V over X consider the vector bundle $W := \mathbf{1} \oplus V$ and the associated projective vector bundle $\mathbf{P}(W)$ of lines in W. Set

$$\bar{t}h(V) = c_r(p^*(V) \otimes \mathcal{O}_{\mathbf{P}(W)}(1)) \in E^{2r,r}(\mathbf{P}(W)). \tag{1}$$

It follows from [PS1, Cor. 3.18] that the support extension map

$$E^{2r,r}\big(\mathbf{P}(W)/(\mathbf{P}(W) \smallsetminus \mathbf{P}(\mathbf{1}))\big) \to E^{2r,r}\big(\mathbf{P}(W)\big)$$

is injective and $\bar{t}h(E) \in E^{2r,r}\big(\mathbf{P}(W)/(\mathbf{P}(W) \smallsetminus \mathbf{P}(\mathbf{1}))\big)$. Set

$$th(E) = j^*(\bar{t}h(E)) \in E^{2r,r}\big(\mathrm{Th}_X(V)\big), \tag{2}$$

where $j\colon \mathrm{Th}_X(V) \to \mathbf{P}(W)/(\mathbf{P}(W) \smallsetminus \mathbf{P}(\mathbf{1}))$ is the canonical motivic weak equivalence of pointed motivic spaces induced by the open embedding $V \hookrightarrow \mathbf{P}(W)$. The assignment V/X to $th(V)$ is a theory of Thom classes on $E^{*,*}|_{\mathcal{S}m\mathcal{O}p}$ (see the proof of [PS1, Thm. 3.35]). So the Thom classes are natural, multiplicative and satisfy the following Thom isomorphism property.

Theorem 2.0.4. *For a rank r vector bundle $p\colon V \to X$ on $X \in \mathcal{S}m/S$ with zero section $z\colon X \hookrightarrow V$, the map*

$$- \cup th(V)\colon E^{*,*}(X) \to E^{*+2r,*+r}\big(V/(V \smallsetminus z(X))\big)$$

is an isomorphism of the two-sided $E^{*,*}(X)$-modules, where $- \cup th(V)$ is written for the composition map $\big(- \cup th(V)\big) \circ p^*$.

Proof. See [PS1, Defn. 3.32.(4)]. □

Analogous to [V1, p. 422] one obtains for vector bundles $V \to X$ and $W \to Y$ in $\mathcal{S}m/S$ a canonical map of pointed motivic spaces $\mathrm{Th}(V) \wedge \mathrm{Th}(W) \to \mathrm{Th}(V \times_S W)$ which is a motivic weak equivalence as defined in [PPR1, Defn. 3.1.6]. In fact, the canonical map becomes an isomorphism after Nisnevich (even Zariski) sheafification. Taking $Y = S$ and $W = \mathbf{1}$ the trivial line bundle yields a motivic weak equivalence $\mathrm{Th}(V) \wedge T \to Th(V \oplus \mathbf{1})$. The canonical covering of $\mathbf{P}^1$ defines motivic weak equivalences

$$T = \mathbf{A}^1/\mathbf{A}^1 \smallsetminus \{0\} \xrightarrow{\sim} \mathbf{P}^1/\mathbf{A}^1 \xleftarrow{\sim} \mathbf{P}^1$$

and the arrow $T = \mathbf{A}^1/\mathbf{A}^1 \smallsetminus \{0\} \to \mathbf{P}^1/\mathbf{P}^1 \smallsetminus \{0\}$ is an isomorphism. Hence one may switch between T and $\mathbf{P}^1$ as desired.

Corollary 2.0.5. *For $W = V \oplus \mathbf{1}$ consider the composite motivic weak equivalence $\epsilon\colon \mathrm{Th}(V) \wedge \mathbf{P}^1 \to \mathrm{Th}(V) \wedge \mathbf{P}^1/\mathbf{A}^1 \leftarrow \mathrm{Th}(V) \wedge T \to \mathrm{Th}(W)$ in $\mathrm{H}_\bullet(S)$. Then the diagram*

$$\begin{array}{ccc}
E^{*+2r,*+r}(Th(V)) & \xrightarrow{\Sigma_{\mathbf{P}^1}} & E^{*+2r+2,*+r+1}(Th(V) \wedge \mathbf{P}^1) \\
{\scriptstyle id}\uparrow & & \uparrow{\scriptstyle \epsilon^*} \\
E^{*+2r,*+r}(Th(V)) & \xrightarrow{\Sigma_T} & E^{*+2r+2,*+r+1}(Th(W)) \\
{\scriptstyle \cup th(V)}\uparrow & & \uparrow{\scriptstyle \cup th(W)} \\
E^{*,*}(X) & \xrightarrow{id} & E^{*,*}(X).
\end{array}$$

commutes.

Let $\mathbf{Gr}(n, n+m)$ be the Grassmann scheme of n-dimensional linear subspaces of $\mathbf{A}^{n+m}_S$. The closed embedding $\mathbf{A}^{n+m} = \mathbf{A}^{n+m} \times \{0\} \hookrightarrow \mathbf{A}^{n+m+1}$ defines a closed embedding

$$\mathbf{Gr}(n, n+m) \hookrightarrow \mathbf{Gr}(n, n+m+1). \tag{3}$$

The tautological vector bundle is denoted $\mathcal{T}(n, n+m) \to \mathbf{Gr}(n, n+m)$. The closed embedding (5) is covered by a bundle map $\mathcal{T}(n, n+m) \hookrightarrow \mathcal{T}(n, n+m+1)$. Let $\mathbf{Gr}(n) = \mathrm{colim}_{m\geq 0} \mathbf{Gr}(n, n+m)$, $\mathcal{T}(n) = \mathrm{colim}_{m\geq 0} \mathcal{T}(n, n+m)$ and $\mathrm{Th}(\mathcal{T}(n)) = \mathrm{colim}_{m\geq 0} \mathrm{Th}(\mathcal{T}(n, n+m))$. These colimits are taken in the category of motivic spaces over S.

Theorem 2.0.6. *Let E be an oriented $\mathbf{P}^1$-ring spectrum. Then*

$$E^{*,*}(\mathbf{Gr}(n)) = E^{*,*}(k)[[c_1, c_2, \ldots, c_n]]$$

is the formal power series ring, where $c_i := c_i(\mathcal{T}(n)) \in E^{2i,i}(\mathbf{Gr}(n))$ denotes the i-th Chern class of the tautological bundle $\mathcal{T}(n)$. The inclusion $i\colon \mathbf{Gr}(n) \hookrightarrow \mathbf{Gr}(n+1)$ satisfies $i^*(c_m) = c_m$ for $m < n+1$ and $i^*(c_{n+1}) = 0$.

Proof. The case $n = 1$ is well-known (see for instance [PS1, Thm. 3.9]). For a finite dimensional vector space W and a positive m let $\mathbf{F}(m, W)$ be the flag variety of flags

$W_1 \subset W_2 \subset \cdots \subset W_m$ of linear subspaces of W such that the dimension of W_i is i. Let $\mathcal{T}^i(m,W)$ be the tautological rank i vector bundle on $\mathbf{F}(m,W)$.

Let $V = \mathbf{A}^\infty$ be an infinite dimensional vector bundle over S and set $e = (1,0,\dots)$. Then V_n denotes the n-fold product of V, and $e_i^n \in V_n$ the vector $(0,\dots,0,e,0,\dots,0)$ having e precisely at the ith position. Let $F(m) = \cup\mathbf{F}(m,W)$ and let $\mathcal{T}^i(m) = \cup\mathcal{T}^i(m,W)$, where W runs over all finite-dimensional vector subspaces of V_n. Thus we have a flag of vector bundles over $\mathcal{T}^1(m) \subset \mathcal{T}^2(m) \subset \cdots \subset \mathcal{T}^m(m)$ over $F(m)$. Set $L^i(m) = \mathcal{T}^i(m)/\mathcal{T}^{i-1}(m)$. It is a line bundle over $F(m)$.

Consider the morphism $p_m\ colon F(m) \to F(m-1)$ which takes a flag $W_1 \subset W_2 \subset \cdots \subset W_m$ to the flag $W_1 \subset W_2 \subset \cdots \subset W_{m-1}$. It is a projective vector bundle over $F(m-1)$ such that the line bundle $L^i(m)$ is its tautological line bundle. Thus there exists a tower of projective vector bundles $F(m) \to F(m-1) \to \cdots \to F(1) = \mathbf{P}(V_n)$. The projective bundle theorem implies that

$$E^{*,*}(F(n)) = E^{*,*}(k)[[t_1,t_2,\dots,t_n]]$$

(the formal power series in n variables), where $t_i = c(L^i(n))$ is the first Chern class of the line bundle $L^i(n)$ over $F(n)$.

Consider the morphism $q\colon F(n) \to \mathbf{Gr}(n)$, which takes a flag $W_1 \subset W_2 \subset \cdots \subset W_n$ to the space W_n. It can be decomposed as a tower of projective vector bundles. In particular, the pull-back map $q^*\colon E^{*,*}(\mathbf{Gr}(n)) \to E^{*,*}(F(n))$ is a monomorphism. It takes the class c_i to the symmetric polinomial $\sigma_i = t_1t_2\dots t_i + \cdots + t_{n-i+1}\dots t_{n-1}t_n$. So the image of q^* contains $E^{*,*}(k)[[\sigma_1,\sigma_2,\dots,\sigma_n]]$. It remains to check that the image of q^* is contained in $E^{*,*}(k)[[\sigma_1,\sigma_2,\dots,\sigma_n]]$. To do that consider another variety.

Namely, let V^0 be the n-dimensional subspace of V_n generated by the vectors e_i^n's. Let l_i^n be the line generated by the vector e_i^n. Let V_i^0 be a subspace of V^0 generated by all e_j^n's with $j \le i$. So one has a flag $V_1^0 \subset V_2^0 \subset \cdots \subset V_n^0$. We denote this flag F^0. For each vector subspace W in V_n containing V^0 consider thre algebraic subgroups of the general linear group $\mathbb{GL}_W$. Namely, set

$$P_W = Stab(V^0),\ B_W = Stab(F^0),\ T_W = Stab(l_1^n, l_2^n, \dots, l_n^n).$$

The group T_W stabilizes each the line l_i^n. Clearly, $T_W \subset B_W \subset P_W$ and $\mathbf{Gr}(n,W) = \mathbb{GL}_W/P_W$, $\mathbf{F}(n,W) = \mathbb{GL}_W/B_W$ Set $M(n,W) = \mathbb{GL}_W/T_W$. One has a tower of obvious morphisms

$$M(n,W) \xrightarrow{r_W} \mathbf{F}(n,W) \xrightarrow{q_W} \mathbf{Gr}(n,W).$$

Set $M(n) = \cup M(n,W)$, where W runs over all finite dimensional subspace W of V_n containing V^0. Now one has a tower of morphisms

$$M(n) \xrightarrow{r} F(n) \xrightarrow{q} \mathbf{Gr}(n).$$

The morphisms r_W can be decomposed in a tower of affine bundles. Whence it induces an isomorphism on the any cohomology theory. Thus the same holds for the morphism r and

$$E^{*,*}(M(n)) = E^{*,*}(k)[[t_1,t_2,\dots,t_n]].$$

Permuting vectors e_i^n's we get an inclusion $\Sigma_n \subset GL(V^0)$ of the symmetric group Σ_n in $\mathbb{GL}(V^0)$. The action of Σ_n by the conjugation on $\mathbb{GL}_W$ normalizes subgroups T_W and P_W. Thus Σ_n acts as on $M(n)$ so on $\mathbf{Gr}(n)$ and the morphism $q \circ r : M(n) \to \mathbf{Gr}(n)$ respects this action. Note that the action of Σ_n on $\mathbf{Gr}(n)$ is trivial and the action of Σ_n on $E^{*,*}(M(n))$ permutes the variable $t_1, t_2, \ldots, t_n$. Thus the image of $(q \circ r)^*$ is contained in $E^{*,*}(k)[[\sigma_1, \sigma_2, \ldots, \sigma_n]]$. Whence the same holds for the image of q^*. The Theorem is proven.

□

The projection from the product $\mathbf{Gr}(m) \times \mathbf{Gr}(n)$, to the j-th factor is called p_j. For every integer $i \geq 0$ set $c'_i = p_1^*(c_i(\mathcal{T}(m)))$ and $c''_i = p_2^*(c_i(\mathcal{T}(n)))$

Theorem 2.0.7. *Suppose E is an oriented commutative $\mathbf{P}^1$-ring spectrum. There is an isomorphism*

$$E^{*,*}\big((\mathbf{Gr}(m) \times \mathbf{Gr}(n))\big) = E^{*,*}(k)[[c'_1, c'_2, \ldots, c'_m, c''_1, c''_2, \ldots, c''_n]]$$

is the formal power series on the c'_i's and c''_j's. The inclusion $i_{m,n} \colon G(m) \times \mathbf{Gr}(n) \hookrightarrow G(m+1) \times G(n+1)$ satisfies $i^*_{m,n}(c'_r) = c'_r$ for $r < m+1$, $i^*_{m,n}(c'_{m+1}) = 0$, and $i^*_{m,n}(c''_r) = c''_r$ for $r < n+1$, $i^*_{m,n}(c''_{n+1}) = 0$.

Proof. Follows as in the proof of Theorem 2.0.6. □

2.1 The symmetric ring spectrum representing algebraic cobordism

To give a construction of the symmetric ring $\mathbf{P}^1$-spectrum MGL recall the notion of a Thom space. For a vector bundle V over a smooth S-scheme X with zero section $z \colon X \hookrightarrow V$ let the *Thom space* $\mathrm{Th}(V)$ of V be the Nisnevich sheaf associated to the presheaf $Y \mapsto V(Y)/\big(V \smallsetminus z(X)\big)(Y)$ on the Nisnevich site $\mathcal{S}m/S$. Since sheaves are presheaves, $\mathrm{Th}(V)$ is a pointed motivic space in the sense of [PPR1, Defn. A.1.1] which coincides with Voevodsky's Thom space [V1, p. 422]. Analogous to [V1, p. 422] one obtains for vector bundles $V \to X$ and $W \to Y$ in $\mathcal{S}m/S$ a canonical map of pointed motivic spaces $\mathrm{Th}(V) \wedge \mathrm{Th}(W) \to \mathrm{Th}(V \times_S W)$ which is a motivic weak equivalence as defined in [PPR1, Defn. 3.1.6]. In fact, the canonical map becomes an isomorphism after Nisnevich (even Zariski) sheafification.

Define the pointed motivic space T as the Thom space $\mathrm{Th}(\mathbf{1})$ of the trivial rank one vector bundle $\mathbf{1}$ over S. The algebraic cobordism spectrum appears naturally as a T-spectrum, not as a $\mathbf{P}^1$-spectrum. Hence we describe it as a symmetric T-ring spectrum and obtain a symmetric $\mathbf{P}^1$-ring spectrum (and in particular a $\mathbf{P}^1$-ring spectrum) by switching the suspension coordinate (see [PPR1, A.6.9]). For $m \geq n \geq 0$ let $\mathcal{T}(n, mn) \to \mathbf{Gr}(n, mn)$ denote the tautological vector bundle over the Grassmann scheme of n-dimensional linear subspaces of $\mathbf{A}_S^{mn} = \mathbf{A}_S^m \times_S \cdots \times_S \mathbf{A}_S^m$. Permuting the copies of $\mathbf{A}_S^m$ induces a Σ_n-action on $\mathcal{T}(n, mn)$ and $\mathbf{Gr}(n, mn)$ such that the bundle projection is equivariant. The closed embedding $\mathbf{A}_S^m = \mathbf{A}_S^m \times \{0\} \hookrightarrow \mathbf{A}_S^{m+1}$ defines a closed

Σ_n-equivariant embedding $\mathbf{Gr}(n, mn) \hookrightarrow \mathbf{Gr}(n,(m+1)n)$. In particular, $\mathbf{Gr}(n, mn)$ is pointed by $g_n\colon S = \mathbf{Gr}(n,n) \hookrightarrow \mathbf{Gr}(n, mn)$. The fiber of $\mathbf{Gr}(n, mn)$ over g_n is $\mathbf{A}^n_S$. Let $\mathbf{Gr}(n)$ be the colimit of the sequence

$$\mathbf{Gr}(n,n) \hookrightarrow \mathbf{Gr}(n,2n) \hookrightarrow \cdots \hookrightarrow \mathbf{Gr}(n,mn) \hookrightarrow \cdots$$

in the category of pointed motivic spaces over S. The pullback diagram

$$\begin{tikzcd} \mathcal{T}(n,mn) \arrow[r] \arrow[d] & \mathcal{T}(n,(m+1)n) \arrow[d] \\ \mathbf{Gr}(n,mn) \arrow[r] & \mathbf{Gr}(n,(m+1)n) \end{tikzcd}$$

induces a Σ_n-equivariant inclusion of Thom spaces

$$\mathrm{Th}(\mathcal{T}(n,mn)) \hookrightarrow \mathrm{Th}(\mathcal{T}(n,(m+1)n)).$$

Let $\mathbb{MGL}_n$ denote the colimit of the resulting sequence

$$\mathbb{MGL}_n = \operatorname*{colim}_{m\geq n} \mathrm{Th}(\mathcal{T}(n,mn)) \tag{4}$$

with the induced Σ_n-action. There is a closed embedding

$$\mathbf{Gr}(n,mn) \times \mathbf{Gr}(p,mp) \hookrightarrow \mathbf{Gr}(n+p, m(n+p)) \tag{5}$$

which sends the subspaces $V \hookrightarrow \mathbf{A}^{mn}$ and $W \hookrightarrow \mathbf{A}^{mp}$ to the subspace $V \times W \hookrightarrow \mathbf{A}^{mn} \times \mathbf{A}^{mp} = \mathbf{A}^{m(n+p)}$. In particular (g_n, g_p) maps to g_{n+p}. The inclusion (5) is covered by a map of tautological vector bundles and thus gives a canonical map of Thom spaces

$$\mathrm{Th}(\mathcal{T}(n,mn)) \wedge \mathrm{Th}(\mathcal{T}(p,mp)) \to \mathrm{Th}(\mathcal{T}(n+p, m(n+p))) \tag{6}$$

which is compatible with the colimit (4). Furthermore, the map (6) is $\Sigma_n \times \Sigma_p$-equivariant, where the product acts on the target via the standard inclusion $\Sigma_n \times \Sigma_p \subseteq \Sigma_{n+p}$. The result is a $\Sigma_n \times \Sigma_p$-equivariant map

$$\mathbb{MGL}_n \wedge \mathbb{MGL}_p \to \mathbb{MGL}_{n+p} \tag{7}$$

of pointed motivic spaces (see [V1, p. 422]). The inclusion of the fiber $\mathbf{A}^p$ over g_p in $\mathcal{T}(p)$ induces an inclusion $\mathrm{Th}(\mathbb{A}^p) \subset \mathrm{Th}(\mathcal{T}(p)) = \mathbb{MGL}_p$. Precomposing it with the canonical Σ_p-equivariant map of pointed motivic spaces

$$\mathrm{Th}(\mathbb{A}^1) \wedge \mathrm{Th}(\mathbb{A}^1) \wedge \cdots \wedge \mathrm{Th}(\mathbb{A}^1) \to \mathrm{Th}(\mathbb{A}^p)$$

defines a family of maps $e_p\colon (\Sigma^\infty_T S_+)_p = \mathcal{T}^{\wedge p} \to \mathbb{MGL}_p$. Inserting it in the inclusion (7) yields $\Sigma_n \times \Sigma_p$-equivariant structure maps

$$\mathbb{MGL}_n \wedge \mathrm{Th}(\mathbb{A}^1) \wedge \mathrm{Th}(\mathbb{A}^1) \wedge \cdots \wedge \mathrm{Th}(\mathbb{A}^1) \to \mathbb{MGL}_{n+p} \tag{8}$$

of the symmetric T-spectrum $\mathbb{MGL}$. The family of $\Sigma_n \times \Sigma_p$-equivariant maps (7) form a commutative, associative and unital multiplication on the symmetric T-spectrum $\mathbb{MGL}$ (see [J, Sect. 4.3]). Regarded as a T-spectrum it is weakly equivalent to Voevodsky's spectrum **MGL** described in [V1, 6.3].

Let $\overline{T}$ be the Nisnevich sheaf associated to the presheaf $X \mapsto \mathbf{P}^1(X)/(\mathbf{P}^1 - \{0\})(X)$ on the Nisnevich site $\mathcal{S}m/S$. The canonical covering of $\mathbf{P}^1$ supplies an isomorphism

$$T = \mathrm{Th}(\mathbf{A}^1_S) \xrightarrow{\cong} \overline{T}$$

of pointed motivic spaces. This isomorphism induces an isomorphism $\mathbf{MSS}_T(S) \cong \mathbf{MSS}_{\overline{T}}(S)$ of the categories of symmetric T-spectra and symmetric $\overline{T}$-spectra. In particular, $\mathbb{MGL}$ may be regarded as a symmetric $\overline{T}$-spectrum by just changing the structure maps up to an isomorphism. Note that the isomorphism of categories respects both the symmetric monoidal structure and the model structure. The canonical projection $p\colon \mathbf{P}^1/(\mathbf{P}^1 - \{0\}) \to \overline{T}$ is a motivic weak equivalence, because $\mathbf{A}^1$ is contractible. It induces a Quillen equivalence

$$\mathbf{MSS}(S) = \mathbf{MSS}_{\mathbf{P}^1}(S) \underset{p^*}{\overset{p_\sharp}{\rightleftarrows}} \mathbf{MSS}_{\overline{T}}(S)$$

when equipped with model structures as described in [J] (see [PPR1, A.6.9]). The right adjoint p^* is very simple: it sends a symmetric $\overline{T}$-spectrum E to the symmetric $\mathbf{P}^1$-spectrum having terms $\big(p^*(E)\big)_n = E_n$ and structure maps

$$E_n \wedge \mathbf{P}^1 \xrightarrow{E_n \wedge p} E \wedge \overline{T} \xrightarrow{\text{structure map}} E_{n+1} .$$

In particular $\mathrm{MGL} := p^*\mathbb{MGL}$ is a symmetric $\mathbf{P}^1$-spectrum by just changing the structure maps. Since p^* is a lax symmetric monoidal functor, MGL is a commutative monoid in a canonical way. Finally, the identity is a left Quillen equivalence from the model category $\mathbf{MSS}^{\mathrm{cm}}(S)$ used in [PPR1] to Jardine's model structure by the proof of [PPR1, A.6.4]. Let $\gamma\colon \mathrm{Ho}(\mathbf{MSS}^{\mathrm{cm}}(S)) \to \mathrm{SH}(S)$ denote the equivalence obtained by regarding a symmetric $\mathbf{P}^1$-spectrum just as a $\mathbf{P}^1$-spectrum.

Definition 2.1.1. Let $(\mathrm{MGL}, \mu_{\mathrm{MGL}}, e_{\mathrm{MGL}})$ denote the commutative $\mathbf{P}^1$-ring spectrum which is the image $\gamma(\mathrm{MGL})$ of the commutative symmetric $\mathbf{P}^1$-ring spectrum MGL in the motivic stable homotopy category $\mathrm{SH}(S)$.

2.2 A universality theorem for the algebraic cobordism spectrum

The complex cobordism spectrum, equipped with its natural orientation, is a universal oriented ring cohomology theory by Quillen's universality theorem [Q]. In this section we prove a motivic version of Quillen's universality theorem. The statement is contained already in [Ve]. Recall that the $\mathbf{P}^1$-ring spectrum MGL carries a canonical orientation th^{MGL} as defined in 1.1.3. It is the canonical map $th^{\mathrm{MGL}}\colon \Sigma^\infty_{\mathbf{P}^1}(Th(\mathcal{O}(-1)))(-1) \to \mathrm{MGL}$ of $\mathbf{P}^1$-spectra.

Theorem 2.2.1 (Universality Theorem)**.** *Let E be a commutative $\mathbf{P}^1$-ring spectrum and let $S = \operatorname{Spec}(k)$ for a field k. The assignment $\varphi \mapsto \varphi(th^{\mathrm{MGL}}) \in E^{2,1}(\mathrm{Th}(\mathcal{T}(1)))$ identifies the set of monoid homomorphisms*

$$\varphi\colon \mathrm{MGL} \to E \tag{9}$$

in the motivic stable homotopy category $\mathrm{SH}^{\mathrm{cm}}(S)$ with the set of orientations of E. The inverse bijection sends an orientation $th \in E^{2,1}(\mathrm{Th}(\mathcal{T}(1)))$ to the unique morphism

$$\varphi \in E^{0,0}(\mathrm{MGL}) = \mathrm{Hom}_{\mathrm{SH}(S)}(\mathrm{MGL}, E)$$

such that $u_i^*(\varphi) = th(\mathcal{T}(i)) \in E^{2i,i}(\mathrm{Th}(\mathcal{T}(i)))$, where $th(\mathcal{T}(i))$ is given by (2) and $u_i\colon \Sigma^\infty_{\mathbf{P}^1}(\mathrm{Th}(\mathcal{T}(i)))(-i) \to \mathrm{MGL}$ is the canonical map of $\mathbf{P}^1$-spectra.

Proof. Let $\varphi\colon \mathrm{MGL} \to E$ be a homomorphism of monoids in $\mathrm{SH}(S)$. The class $th := \varphi(th^{\mathrm{MGL}})$ is an orientation of E, because

$$\varphi(th)|_{Th(\mathbf{1})} = \varphi(th|_{Th(\mathbf{1})}) = \varphi(\Sigma_{\mathbf{P}^1}(1)) = \Sigma_{\mathbf{P}^1}(\varphi(1)) = \Sigma_{\mathbf{P}^1}(1).$$

Now suppose $th^E \in E^{2i,i}(\mathrm{Th}(\mathcal{O}(-1)))$ is an orientation of E. We are going to construct a unique monoid homomorphism $\varphi\colon \mathrm{MGL} \to E$ in $\mathrm{SH}(S)$ such that $u_i^*(\varphi) = th(\mathcal{T}(i))$. To do so, we compute $E^{*,*}(\mathrm{MGL})$. By [PPR1, Cor. 2.1.4], this group fits into the short exact sequence

$$0 \to \varprojlim{}^1 E^{*+2i-1,*+i}(\mathrm{Th}(\mathcal{T}(i))) \to E^{*,*}(\mathrm{MGL}) \to \varprojlim E^{*+2i,*+i}(\mathrm{Th}(\mathcal{T}(i))) \to 0$$

where the connecting maps in the tower are given by the top line of the commutative diagram

$$\begin{tikzcd}
E^{*+2i-1,*+i}(\mathrm{Th}_i) & E^{*+2i+1,*+i+1}(\mathrm{Th}_i \wedge \mathbf{P}^1) \arrow[l, "\Sigma^{-1}_{\mathbf{P}^1}"'] & E^{*+2i+1,*+i+1}(\mathrm{Th}_{i+1}) \arrow[l] \\
E^{*,*}(\mathbf{Gr}(i)) \arrow[u, "-\cup th(\mathcal{T}(i)"] & E^{*,*}(\mathbf{Gr}(i)) \arrow[l, "\mathrm{id}"'] \arrow[u, "\epsilon^*\circ(-\cup th(\mathcal{T}(i)\oplus\mathbf{1})"'] & E^{*,*}(\mathbf{Gr}(i+1)) \arrow[l, "\mathrm{inc}_i^*"'] \arrow[u, "-\cup th(\mathcal{T}(i+1))"']
\end{tikzcd}$$

Here $\epsilon\colon \mathrm{Th}(V) \wedge \mathbf{P}^1 \to Th(V \oplus \mathbf{1})$ is the canonical map. The pull-backs inc_i^* are all surjective by Theorem 2.0.4. So we proved the following

Claim 2.2.2. *The canonical map*

$$E^{*,*}(\mathrm{MGL}) \to \varprojlim E^{*+2i,*+i}(Th(\mathcal{T}(i))) = E^{*,*}(k)[[c_1, c_2, c_3, \dots]]$$

is an isomorphism of $E^{*,*}(k)$-modules.

The family of elements $th(\mathcal{T}(i))$ is an element in the $\varprojlim$-group, thus there is a unique element $\varphi \in E^{0,0}(\mathrm{MGL})$ with $u_i^*(\varphi) = th(\mathcal{T}(i))$. We claim that φ is a monoid homomorphism. To check that it respects the multiplicative structure, consider the diagram

$$\begin{tikzcd} \Sigma^\infty_{\mathbf{P}^1}(Th(\mathcal{T}(i)))(-i) \wedge \Sigma^\infty_{\mathbf{P}^1}(Th(\mathcal{T}(j)))(-j) \arrow[r, "\Sigma^\infty_{\mathbf{P}^1}(in_{ij})"] \arrow[d, "u_i \wedge u_j"'] & \Sigma^\infty_{\mathbf{P}^1}(Th(\mathcal{T}(i+j)))(-i-j) \arrow[d, "u_{i+j}"'] \\ \mathrm{MGL} \wedge \mathrm{MGL} \arrow[r, "\mu_{\mathrm{MGL}}"] \arrow[d, "\varphi\wedge\varphi"'] & \mathrm{MGL} \arrow[d, "\varphi"'] \\ E \wedge E \arrow[r, "\mu_E"] & E. \end{tikzcd}$$

Its enveloping square commutes in $\mathrm{SH}(S)$ since one has a chain of relations

$$\varphi \circ u_{i+j} \circ \Sigma^\infty_{\mathbf{P}^1}(in_{ij}) = in_{ij}^*(th(\mathcal{T}(i+j))) = th(in_{ij}^*(\mathcal{T}(i+j))) = th(\mathcal{T}(i) \times \mathcal{T}(j)) =$$

$$th(\mathcal{T}(i)) \times (\mathcal{T}(j)) = \mu_E(th(\mathcal{T}(i)) \wedge th(\mathcal{T}(j))) = \mu_E \circ ((\varphi \circ u_i) \wedge (\varphi \circ u_j)).$$

To obtain the relation $\mu_E \circ (\varphi \wedge \varphi) = \varphi \circ \mu_{\mathrm{MGL}}$ in $\mathrm{SH}(k)$ consider the short exact sequence of the form

$$0 \to \varprojlim{}^1 E^{*+4i-1,*+2i}(Th(\mathcal{T}(i)) \wedge Th(\mathcal{T}(i))) \to E^{*,*}(\mathrm{MGL} \wedge \mathrm{MGL})$$

$$\to \varprojlim E^{*+4i,*+2i}(Th(\mathcal{T}(i)) \wedge Th(\mathcal{T}(i))) \to 0.$$

Note that $Th(\mathcal{T}(i)) \wedge Th(\mathcal{T}(i)) = Th(\mathcal{T}(i) \times \mathcal{T}(i))$, the group $E^{*+4i-1,*+2i}(Th(\mathcal{T}(i) \times \mathcal{T}(i)))$ is isomorphic to $E^{*-1,*}(\mathbf{Gr}(i) \times \mathbf{Gr}(i))$ via the Thom isomorphisms 2.0.4. Now the $\varprojlim^1$-group is trivial since the connecting maps coincide with the pull-back maps

$$E^{*-1,*}(\mathbf{Gr}(i+1) \times \mathbf{Gr}(i+1)) \to E^{*-1,*}(\mathbf{Gr}(i) \times \mathbf{Gr}(i))$$

which are surjective by Theorem 2.0.7. So we proved the following

Claim 2.2.3. *The canonical map*

$$E^{*,*}(\mathrm{MGL} \wedge \mathrm{MGL}) \to \varprojlim E^{*+2i,*+i}(Th(\mathcal{T}(i)) \wedge Th(\mathcal{T}(i))) =$$

$$E^{*,*}(k))[[c_1', c_1'', c_2', c_2'', \dots]]$$

is an isomorphism of $E^{*,*}(k)$-modules. Here c_i' is the i-th Chern class coming from the first factor of Gr × Gr *and c_2'' is the i-th Chern class coming from the second factor.*

Now the family of relations

$$\varphi \circ u_{i+i} \circ \Sigma^\infty_{\mathbf{P}^1}(in_{ii}) = \mu_E \circ ((\varphi \circ u_i) \wedge (\varphi \circ u_i))$$

shows that $\mu_E \circ (\varphi \wedge \varphi) = \varphi \circ \mu_{\mathrm{MGL}}$ in $\mathrm{SH}(k)$.

To prove the Theorem it remains to check that the two assignment described in the Theorem are inverse of each other. If we begin with an orientation $th \in E^{2,1}(Th(\mathcal{O}(-1)))$

we get a morphism φ such that for each i one has $\varphi \circ u_i = th(\mathcal{T}_i)$. And the new orientation $th' := \varphi(th^{\mathrm{MGL}})$. coincides with the original one, due to the chain of relations

$$th' = \varphi(th^{\mathrm{MGL}}) = \varphi(u_1) = \varphi \circ u_1 = th(\mathcal{T}_1) = th(\mathcal{O}(-1)) = th.$$

On the other hand if we begin with a monoid homomorphism φ we get an orientation $th := \varphi(th^{\mathrm{MGL}})$ of E. Then monoid homomorphism φ' we obtain then satisfies $u_i^*(\varphi') = th(\mathcal{T}_i)$. for every $i \geq 0$. To check that $\varphi' = \varphi$, recall that MGL is oriented, so we may use Claim 2.2.2 with $E = \mathrm{MGL}$ to get an isomorphism

$$\mathrm{MGL}^{*,*}(\mathrm{MGL}) \to \varprojlim \mathrm{MGL}^{*+2i,*+i}(Th(\mathcal{T}(i))).$$

This isomorphism shows that the identity $\varphi' = \varphi$ will follow from the identities $u_i^*(\varphi') = u_i^*(\varphi)$ for every $i \geq 0$. Since $u_i^*(\varphi') = th(\mathcal{T}_i)$ it remains to check the relation $u_i^*(\varphi) = th(\mathcal{T}_i)$. It follows from the

Claim 2.2.4. $u_i = th^{\mathrm{MGL}}(\mathcal{T}_i) \in \mathrm{MGL}^{2i,i}(Th(\mathcal{T}(i)))$.

In fact, $u_i^*(\varphi) = \varphi \circ u_i = \varphi(u_i) = \varphi(th^{\mathrm{MGL}}(\mathcal{T}(i))) = th(\mathcal{T}(i))$. The very last relation in this chain of relations holds since φ is a monoid homomorphism which takes th^{MGL} to th. It remains to prove the Claim. To do that, consider the case $i = 2$. The general case can be proved in the same manner. The commutative diagram in $\mathrm{SH}(k)$

$$\begin{tikzcd}
\Sigma^\infty_{\mathbf{P}^1} Th(\mathcal{T}(1))(-1) \wedge \Sigma^\infty_{\mathbf{P}^1} Th(\mathcal{T}(1))(-1) \arrow[r, "\Sigma^\infty(in_{11})"] \arrow[d, "u_1 \wedge u_1"'] & \Sigma^\infty_{\mathbf{P}^1} Th(\mathcal{T}(2))(-2) \arrow[d, "u_2"] \\
\mathrm{MGL} \wedge \mathrm{MGL} \arrow[r, "\mu_{\mathrm{MGL}}"] & \mathrm{MGL}
\end{tikzcd}$$

implies that

$$in_{11}^*(u_2) = u_1 \times u_1 \in \mathrm{MGL}^{4,2}(Th(\mathcal{T}(1)) \wedge Th(\mathcal{T}(1))) = \mathrm{MGL}^{4,2}(Th(\mathcal{T}(1) \times \mathcal{T}(1))).$$

Now the chain of relations

$$in_{11}^*(th^{\mathrm{MGL}}(\mathcal{T}(2))) = th^{\mathrm{MGL}}(in_{11}^*(\mathcal{T}(2))) = th^{\mathrm{MGL}}(\mathcal{T}(1) \times \mathcal{T}(1)) = th^{\mathrm{MGL}}(\mathcal{T}(1)) \times th^{\mathrm{MGL}}(\mathcal{T}(1))$$

shows that it remains to prove the injectivity of the map in_{11}^*. To do that consider the commutative diagram

$$\begin{tikzcd}
\mathrm{MGL}^{*,*}(Th(\mathcal{T}(1) \times \mathcal{T}(1))) & \mathrm{MGL}^{*,*}(Th(\mathcal{T}(2))) \arrow[l, "in_{11}^*"'] \\
\mathrm{MGL}^{*,*}(\mathbf{Gr}(1) \times \mathbf{Gr}(1)) \arrow[u, "thom"] & \mathrm{MGL}^{*,*}(\mathbf{Gr}(2)) \arrow[l, "i_{11}^*"'] \arrow[u, "thom"']
\end{tikzcd}$$

where the vertical arrows are the Thom isomorphisms from Theorem 2.0.4 and $i_{11} : \mathbf{Gr}(1) \times \mathbf{Gr}(1) \hookrightarrow \mathbf{Gr}(2)$ is the embedding described in the very beginning of Section 2.1. For an oriented commutative ring $\mathbf{P}^1$-spectrum (E, th) one has $E^{*,*}(\mathbf{Gr}(2)) =$

$E^{*,*}(k)[[c_1, c_2]]$ (the formal power series on c_1, c_2) by Theorem 2.0.6. From the other hand

$$E^{*,*}(\mathbf{Gr}(1) \times \mathbf{Gr}(1)) = E^{*,*}(k)[[t_1, t_2]]$$

(the formal power series on t_1, t_2) by Theorem 2.0.7 and the map i^*_{11} takes c_1 to $t_1 + t_2$ and c_2 to $t_1 t_2$. Whence i^*_{11} is injective. The proofs of the Claim and of the Theorem are completed.

$\square$

References

[Ad] *J. F. Adams.* Stable homotopy and generalised cohomology. University of Chicago Press, 1974.

[J] *J. F. Jardine.* Motivic symmetric spectra. Doc. Math. 5 (2000), 445–553.

[LM1] *M. Levine, F. Morel.* Algebraic cobordism I. Preprint, 116 pages (2002). http://www.math.uiuc.edu/K-theory/0547/2002

[MV] *F. Morel, V. Voevodsky.* $\mathbf{A}^1$-homotopy theory of schemes. Publ. IHES 90, (1999), 45–143 (2001).

[PPR1] *I. Panin, K. Pimenov, O. Röndigs.* On Voevodsky's algebraic K-theory spectrum BGL. Preprint, 47 pages (2007). Available via http://www.math.uiuc.edu/K-theory/0838/

[PS1] *I. Panin.* (*After I. Panin and A. Smirnov*) Oriented cohomology theories on algebraic varieties, Special issue in honor of H. Bass on his seventieth birthday. Part III. K-Theory, 30 (2003), no. 3, 265–314.

[PY] *I. Panin, S. Yagunov.* Rigidity for orientable functors. J. Pure and Appl. Algebra, 172, (2002), 49–77.

[Q] *D. Quillen.* On the formal group laws of unoriented and complex cobordism theory. Bull. Am. Math. Soc. 75, 1293-1298 (1969)

[TT] *R. Thomason, T. Trobaugh.* Higher algebraic K-theory of schemes and of derived categories. In The Grothendieck festschift, 3, 247-436, Birkhauser, Boston, 1990.

[Ve] *G. Vezzozi.* Brown-Peterson spectra in stable $\mathbf{A}^1$-homotopy theory. Rend. Semin. Mat. Univ. Padova 106, 47-64 (2001).

[V1] *V. Voevodsky.* $\mathbf{A}^1$-Homotopy theory. Doc. Math., Extra Vol. ICM 1998(I), 417-442.