On the relation of Voevodsky's algebraic cobordism to Quillen's K-theory

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Abstract

Quillen's algebraic K-theory is reconstructed via Voevodsky's algebraic cobordism. More precisely, for a ground field k the algebraic cobordism $\mathbf{P}^1$-spectrum MGL of Voevodsky is considered as a commutative ring $\mathbf{P}^1$-spectra. Setting $\mathrm{MGL}^i = \oplus_{2q-p=i} \mathrm{MGL}^{p,q}$ we regard the bigraded theory $\mathrm{MGL}^{p,q}$ as just a graded theory. There is a unique ring morphism $\phi\colon \mathrm{MGL}^0(k) \to \mathbb{Z}$ which sends the class $[X]_{\mathrm{MGL}}$ of a smooth projective k-variety X to the Euler characteristic $\chi(X, \mathcal{O}_X)$ of the structure sheaf $\mathcal{O}_X$. Our main result states that there is a canonical grade preserving isomorphism of ring cohomology theories on the category $\mathcal{S}m\mathcal{O}p/k$

$$\varphi\colon \mathrm{MGL}^*(X,U) \otimes_{\mathrm{MGL}^0(k)} \mathbb{Z} \to \mathrm{K}^{TT}_{-*}(X,U) = \mathrm{K}'_{-*}(X-U),$$

in the sense of [PS1], where K^{TT}_* is the Thomason-Trobaugh K-theory and K'_* is Quillen's K'-theory. In particular, the left hand side is a ring cohomology theory. Moreover both theories are oriented in the sense of [PS1] and φ respects the orientations. The result is an algebraic version of a theorem due to Conner and Floyd. That theorem reconstructs complex K-theory via complex cobordism [CF].

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1 A motivic version of a theorem by Conner and Floyd

Our main result relates Voevodsky's algebraic cobordism theory $\mathrm{MGL}^{*,*}$ to Quillen's K'-theory. We refer to [PPR1, Appendix] for the basic terminology, notation, constructions, definitions, results. Let S be a Noetherian separated finite-dimensional scheme S. One may think of S being the spectrum of a field or the integers. A *motivic space over* S is a functor

$$A\colon \mathcal{S}m/S^{op} \to \mathbf{sSet}$$

(see [PPR1, Appendix]). The category of motivic spaces over S is denoted $\mathbf{M}(S)$. This definition of a motivic space is different from the one considered by Morel and Voevodsky in [MV] – they consider only those simplicial presheaves which are sheaves in the Nisnevich topology on $\mathcal{S}m/S$. With our definition the Thomason-Trobaugh K-theory functor obtained by using big vector bundles is a motivic space on the nose. It is not a simplicial Nisnevich sheaf. This is why we prefer to work with the above notion of "space".

We write $\mathrm{H}^{\mathrm{cm}}_{\bullet}(S)$ for the pointed motivic homotopy category and $\mathrm{SH}^{\mathrm{cm}}(S)$ for the stable motivic homotopy category over S as constructed in [PPR1, A.3.9, A.5.6]. By [PPR1, A.3.11 resp. A.5.6] there are canonical equivalences to $\mathrm{H}_{\bullet}(S)$ of [MV] resp. $\mathrm{SH}(S)$ of [V1]. Both $\mathrm{H}^{\mathrm{cm}}_{\bullet}(S)$ and $\mathrm{SH}^{\mathrm{cm}}(S)$ are equipped with closed symmetric monoidal structures such that the $\mathbf{P}^1$-suspension spectrum functor is a strict symmetric monoidal functor

$$\Sigma^{\infty}_{\mathbf{P}^1}\colon \mathrm{H}^{\mathrm{cm}}_{\bullet}(S) \to \mathrm{SH}^{\mathrm{cm}}(S).$$

Here $\mathbf{P}^1$ is considered as a motivic space pointed by $\infty \in \mathbf{P}^1$. The symmetric monoidal structure $(\wedge, \mathbb{I}_S = \Sigma^{\infty}_{\mathbf{P}^1} S_+)$ on the homotopy category $\mathrm{SH}^{\mathrm{cm}}(S)$ is constructed on the model category level by employing the category $\mathbf{MSS}(S)$ of symmetric $\mathbf{P}^1$-spectra. It satisfies the properties required by Theorem 5.6 of Voevodsky congress talk [V1]. From now on we will usually omit the superscript $(-)^{\mathrm{cm}}$.

Given a $\mathbf{P}^1$-spectrum E one has a cohomology theory on the category of pointed spaces. Namely, for a pointed space (A, a) set $E^{p,q}(A, a) = Hom_{\mathrm{H}^{\mathrm{cm}}_{\bullet}(S)}(\Sigma^{\infty}_{\mathbf{P}^1}(A, a), \Sigma^{p,q}(E))$ and $E^{*,*}(A, a) = \oplus_{p,q} E^{p,q}(A, a)$. A cohomology theory on the category of non-pointed spaces is defined as follows. For a non-pointed space A set $E^{p,q}(A) = E^{p,q}(A_+, +)$ and $E^{*,*}(A) = \oplus_{p,q} E^{p,q}(A)$.

Each $X \in \mathcal{S}m/S$ defines a motivic space constant in the siplicial direction taking an S-smooth U to $\mathrm{Mor}_S(U, X)$. This motivic space is non-pointed. So we regard S-smooth varieties as motivic spaces (non-pointed) and set

$$E^{p,q}(X) = E^{p,q}(X_+, +).$$

Given a $\mathbf{P}^1$-spectrum E we will reduce the double grading on the cohomology theory $E^{*,*}$ to a grading. Namely, set $E^m = \oplus_{m=p-2q} E^{p,q}$ and $E^* = \oplus_m E^m$. *We often will write $E^*(k)$ for $E^*(\mathrm{Spec}(k))$ below in this text.*

A $\mathbf{P}^1$-ring spectrum is a monoid (E, μ, e) in $(\mathrm{SH(S)}, \wedge, \mathbb{I}_S)$. A commutative $\mathbf{P}^1$-ring spectrum is a commutative monoid (E, μ, e) in $(\mathrm{SH(S)}, \wedge, 1)$.

The cohomology theory E^* defined by a $\mathbf{P}^1$-ring spectrum is a ring cohomology theory. The cohomology theory E^* defined by a commutative $\mathbf{P}^1$-ring spectrum is a ring cohomology theory, however it is not necessary graded commutative. The cohomology theory E^* defined by an oriented commutative $\mathbf{P}^1$-ring spectrum is a graded commutative ring cohomology theory.

Occasionally a $\mathbf{P}^1$-ring spectrum (E, μ, e) might have a model (E', μ', e') which is a symmetric $\mathbf{P}^1$-ring spectrum, that is, a symmetric $\mathbf{P}^1$-spectrum E' equipped with a strict multiplication $\mu' \colon E' \wedge E' \to E'$ which is strictly associative and strictly unital for the unit $e' \colon \Sigma^\infty_{\mathbf{P}^1}(S_+) \to E'$. This is the case for the algebraic cobordism $\mathbf{P}^1$-ring spectrum $\mathbf{MGL}$, as described below. Such a model for the algebraic K-theory $\mathbf{P}^1$-ring spectrum BGL is currently not known to us.

For the rest of the paper let k be a field and $S = \mathrm{Spec}(k)$. Usually S will be replaced by k in the notation. We work in this text with the algebraic cobordism $\mathbf{P}^1$-spectrum MGL and the algebraic K-theory $\mathbf{P}^1$-spectrum BGL as described in [PPR1, Defn. 1.2.4] and [PPR2, Sect. 2.1] respectively. The spectrum MGL is a commutative ring $\mathbf{P}^1$-spectrum by that construction. The spectrum BGL is equipped with a structure of a commutative $\mathbf{P}^1$-ring spectrum as explained in [PPR1, Thm. 2.1.1]. Let K^{TT}_* be Thomason-Trobaugh K-theory functor [TT]. There is a canonical isomorphism

$$Ad\colon K^{TT}_{-*} \to \mathrm{BGL}^{*,0}$$

of ring cohomology theories on the category $\mathcal{S}m\mathcal{O}p/S$ in the sense of [PS1]. An invertible Bott element $\beta \in \mathrm{BGL}^{2,1}(\mathrm{Spec}(k))$ is constructed in [PPR1, Section 1.3]. For every pointed motivic space A the morphism

$$\mathrm{BGL}^{*,0}(A) \otimes \mathrm{BGL}^0(\mathrm{Spec}(k)) \to \mathrm{BGL}^{*,*}(A) \tag{1}$$

given by $a \otimes b \mapsto a \cup b$ is a ring isomorphism by [PPR1, Sect. 1.3]. Furthermore $\mathrm{BGL}^0(\mathrm{Spec}(k)) = \mathbb{Z}[\beta, \beta^{-1}]$ is the ring of Laurent polynomials on the Bott element β. To say the same in a different way,

$$\mathrm{BGL}^{*,0}(A)[\beta, \beta^{-1}] \cong \mathrm{BGL}^{*,*}(A). \tag{2}$$

The special case $A = X/(X \smallsetminus Z)$ where X is a smooth k-variety and $Z \subset X$ is a closed subset implies the following result [PPR1, Cor. 1.3.6].

Corollary 1.0.1. *Let X be a smooth k-scheme, Z a closed subset of X and $U = X \smallsetminus Z$ its open complement. Then there are isomorphisms*

$$\begin{aligned} K^{TT}_{-*,Z}(X)[\beta,\beta^{-1}] &\cong \mathrm{BGL}^{*,*}(X/U) = \mathrm{BGL}^{*}(X/U) \qquad (3) \\ K^{TT}_{-*,Z}(X) &\cong \mathrm{BGL}^{*,*}(X/U)/(\beta+1)\mathrm{BGL}^{*,*}(X/U) \qquad (4) \end{aligned}$$

of ring cohomology theories on $\mathcal{S}m\mathcal{O}p/k$ in the sense of [PS1].

We refer to [PPR2] for a construction of the commutative ring $\mathbf{P}^1$-spectrum MGL. For the purposes of the present preprint we will need to know only two properties of that spectrum. Those properties are: Quillen universality and MGL-cellularity (see Subsection 2.1 below).

1.1 Oriented commutative ring spectra

Following Adams and Morel we define an orientation of a commutative $\mathbf{P}^1$-ring spectrum. However we prefer to use Thom classes instead of Chern classes. Consider the pointed motivic space $\mathbf{P}^\infty = \mathrm{colim}_{n\geq 0}\,\mathbf{P}^n$ having base point $g_1\colon S = \mathbf{P}^0 \hookrightarrow \mathbf{P}^\infty$.

The tautological "vector bundle" $\mathcal{T}(1) = \mathcal{O}_{\mathbf{P}^\infty}(-1)$ is also known as the Hopf bundle. It has zero section $z\colon \mathbf{P}^\infty \hookrightarrow \mathcal{T}(1)$. The fiber over the point $g_1 \in \mathbf{P}^\infty$ is $\mathbb{A}^1$. For a vector bundle V over a smooth S-scheme X with zero section $z\colon X \hookrightarrow V$ consider a Nisnevich sheaf associated with the presheaf $Y \mapsto V(Y)/\big(V \smallsetminus z(X)\big)(Y)$ on the Nisnevich site $\mathcal{S}m/S$. The *Thom space* $\mathrm{Th}(V)$ of V is defined as that Nisnevich sheaf regarded as a presheaf. In particular $\mathrm{Th}(V)$ is a pointed motivic space in the sense of [PPR1, Defn. A.1.1]. Its Nisnevich sheafification coincides with Voevodsky's Thom space [V1, p. 422], since $\mathrm{Th}(V)$ is already a Nisnevich sheaf. The Thom space of the Hopf bundle is then defined as the colimit $\mathrm{Th}(\mathcal{T}(1)) = \mathrm{colim}_{n\geq 0}\,\mathrm{Th}\big(\mathcal{O}_{\mathbf{P}^n}(-1)\big)$. Abbreviate $T = \mathrm{Th}(\mathbf{A}^1_S) = \mathbf{A}^1_S/(\mathbf{A}^1_S \smallsetminus \{0\})$.

Let E be a commutative ring $\mathbf{P}^1$-spectrum. The unit gives rise to an element $1 \in E^{0,0}(\mathrm{Spec}(k)_+)$. Applying the $\mathbf{P}^1$-suspension isomorphism to that element we get an element $\Sigma_{\mathbf{P}^1}(1) \in E^{2,1}(\mathbf{P}^1/\{\infty\}))$. The canonical covering of $\mathbf{P}^1$ defines motivic weak equivalences

$$\mathbf{P}^1/\{\infty\} \xrightarrow{\sim} \mathbf{P}^1/\mathbf{A}^1 \xleftarrow{\sim} \mathbf{A}^1/\mathbf{A}^1 \smallsetminus \{0\} = T,$$

which in turn define pull-back isomorphisms $E(\mathbf{P}^1/\{\infty\}) \leftarrow E(\mathbf{A}^1/\mathbf{A}^1 \smallsetminus \{0\}) \to E(T)$. Denote $\Sigma_T(1)$ the image of $\Sigma_{\mathbf{P}^1}(1)$ in $E^{2,1}(T)$.

Definition 1.1.1. *Let E be a commutative ring $\mathbf{P}^1$-spectrum. A* Thom orientation *of E is an element $th \in E^{2,1}(\mathrm{Th}(\mathcal{T}(1))$ such that its restriction to* *the Thom space of the fibre over the distinguished point coincides with the element* $\Sigma_T(1) \in E^{2,1}(T)$*. A* Chern orientation *of* E *is an element* $c \in E^{2,1}(\mathbf{P}^\infty)$ *such that* $c|_{\mathbf{P}^1} = -\Sigma_{\mathbf{P}^1}(1)$*.* An orientation *of* E *is either a Thom orientation or a Chern orientation. Two Thom orientations of* E *coincide if respecting Thom elements coincides. Two Chern orientations of* E *coincide if respecting Chern elements coincides. One says that a Thom orientation* th *of* E *coincides with a Chern orientation* c *of* E *provided that* $c = z^*(th)$ *or equivalently the element* th *coincides with the one* $th(\mathcal{O}(-1))$ *given by (6) below.*

Remark 1.1.2. The element th should be regarded as a Thom class of the tautological line bundle $\mathcal{T}(1) = \mathcal{O}(-1)$ over $\mathbf{P}^\infty$. The element c should be regarded as a Chern class of the tautological line bundle $\mathcal{T}(1) = \mathcal{O}(-1)$ over $\mathbf{P}^\infty$.

Example 1.1.3. The following orientations given right below are relevant for our work. Here MGL denotes the $\mathbf{P}^1$-ring spectrum representing algebraic cobordism obtained in [PPR2, Defn 2.1.1] and BGL denotes the $\mathbf{P}^1$-ring spectrum representing algebraic K-theory constructed in [PPR1, Theorem 2.2.1].

- Let $u_1 : \Sigma^\infty_{\mathbf{P}^1}(\mathrm{Th}(\mathcal{T}(1)))(-1) \to \mathrm{MGL}$ be the canonical map of $\mathbf{P}^1$-spectra. Set $th^{\mathrm{MGL}} = u_1 \in \mathrm{MGL}^{2,1}(\mathrm{Th}(\mathcal{T}(1)))$. Since $th^{\mathrm{MGL}}|_{\mathrm{Th}(\mathbf{1})} = \Sigma_{\mathbf{P}^1}(1)$ in $\mathrm{MGL}^{2,1}(\mathrm{Th}(\mathbf{1}))$, the class th^{MGL} is an orientation of MGL.
- Set $c = (-\beta) \cup ([\mathcal{O}] - [\mathcal{O}(1)]) \in \mathrm{BGL}^{2,1}(\mathbf{P}^\infty)$. The relation (11) from [PPR1] shows that the class c is an orientation of BGL.

2 Oriented cohomology theories

Let (E, c) be an oriented commutative $\mathbf{P}^1$-ring spectrum. In this Section we compute the E-cohomology of infinite Grassmannians and their products. The results are the expected ones 2.0.6.

The oriented $\mathbf{P}^1$-ring spectrum (E, c) defines an oriented cohomology theory on $\mathcal{S}m\mathcal{O}p$ in the sense of [PS1, Defn. 3.1] as follows. The restriction of the functor $E^{*,*}$ to the category $\mathcal{S}m/S$ is a ring cohomology theory. By [PS1, Th. 3.35] it remains to construct a Chern structure on $E^{*,*}|_{\mathcal{S}m\mathcal{O}p}$ in the sense of [PS1, Defn.3.2]. Let $H(k)$ be the homotopy category of spaces over k. The functor isomorphism $\mathrm{Hom}_{\mathrm{H(k)}}(-, \mathbf{P}^\infty) \to \mathrm{Pic}(-)$ on the category $\mathcal{S}m/S$ provided by [MV, Thm. 4.3.8] sends the class of the identity map $\mathbf{P}^\infty \to \mathbf{P}^\infty$ to the class of the tautological line bundle $\mathcal{O}(-1)$ over $\mathbf{P}^\infty$. For a line bundle L over $X \in \mathcal{S}m/S$ let $[L]$ be the class of

L in the group $\operatorname{Pic}(X)$. Let $f_L\colon X \to \mathbf{P}^\infty$ be a morphism in H(k) corresponding to the class $[L]$ under the functor isomorphism above. For a line bundle L over $X \in \mathcal{S}m/S$ set $c(L) = f_L^*(c) \in E^{2,1}(X)$. Clearly, $c(\mathcal{O}(-1)) = c$. The assignment $L/X \mapsto c(L)$ is a Chern structure on $E^{*,*}|_{\mathcal{S}m\mathcal{O}p}$ since $c|_{\mathbf{P}^1} = -\Sigma_{\mathbf{P}^1}(1) \in E^{2,1}(\mathbf{P}^1,\infty)$. With that Chern structure $E^{*,*}|_{\mathcal{S}m\mathcal{O}p}$ is an oriented ring cohomology theory in the sense of [PS1]. In particular, (BGL, c^K) defines an oriented ring cohomology theory on $\mathcal{S}m\mathcal{O}p$.

Given this Chern structure, one obtains a theory of Thom classes $V/X \mapsto th(V) \in E^{2\operatorname{rank}(V),\operatorname{rank}(V)}\big(\mathrm{Th}_X(V)\big)$ on the cohomology theory $E^{*,*}|_{\mathcal{S}m\mathcal{O}p/S}$ in the sense of [PS1, Defn. 3.32] as follows. There is a unique theory of Chern classes $V \mapsto c_i(V) \in E^{2i,i}(X)$ such that for every line bundle L on X one has $c_1(L) = c(L)$. For a rank r vector bundle V over X consider the vector bundle $W := \mathbf{1} \oplus V$ and the associated projective vector bundle $\mathbf{P}(W)$ of lines in W. Set

$$\bar{th}(V) = c_r(p^*(V) \otimes \mathcal{O}_{\mathbf{P}(W)}(1)) \in E^{2r,r}(\mathbf{P}(W)). \tag{5}$$

It follows from [PS1, Cor. 3.18] that the support extension map

$$E^{2r,r}\big(\mathbf{P}(W)/(\mathbf{P}(W) \smallsetminus \mathbf{P}(\mathbf{1}))\big) \to E^{2r,r}\big(\mathbf{P}(W)\big)$$

is injective and $\bar{th}(E) \in E^{2r,r}\big(\mathbf{P}(W)/(\mathbf{P}(W) \smallsetminus \mathbf{P}(\mathbf{1}))\big)$. Set

$$th(E) = j^*(\bar{th}(E)) \in E^{2r,r}\big(\mathrm{Th}_X(V)\big), \tag{6}$$

where $j\colon \mathrm{Th}_X(V) \to \mathbf{P}(W)/(\mathbf{P}(W) \smallsetminus \mathbf{P}(\mathbf{1}))$ is the canonical motivic weak equivalence of pointed motivic spaces induced by the open embedding $V \hookrightarrow \mathbf{P}(W)$. The assignment V/X to $th(V)$ is a theory of Thom classes on $E^{*,*}|_{\mathcal{S}m\mathcal{O}p}$ (see the proof of [PS1, Thm. 3.35]). So the Thom classes are natural, multiplicative and satisfy the following Thom isomorphism property.

Theorem 2.0.4. *For a rank r vector bundle $p\colon V \to X$ on $X \in \mathcal{S}m/S$ with zero section $z\colon X \hookrightarrow V$, the map*

$$\cup th(V)\colon E^{*,*}(X) \to E^{*+2r,*+r}\big(V/(V \smallsetminus z(X))\big)$$

is an isomorphism of the two-sided $E^{*,*}(X)$-modules, where $- \cup th(V)$ is written for the composition map $\big(\cup th(V)\big) \circ p^*$.

Proof. See [PS1, Defn. 3.32.(4)]. □

Analogous to [V1, p. 422] one obtains for vector bundles $V \to X$ and $W \to Y$ in $\mathcal{S}m/S$ a canonical map of pointed motivic spaces $\mathrm{Th}(V) \wedge \mathrm{Th}(W) \to \mathrm{Th}(V \times_S W)$ which is a motivic weak equivalence as defined in [PPR1, Defn. 3.1.6]. In fact, the canonical map becomes an isomorphism after Nisnevich (even Zariski) sheafification. Taking $Y = S$ and $W = \mathbf{1}$ the trivial line bundle yields a motivic weak equivalence $\mathrm{Th}(V) \wedge T \to Th(V \oplus \mathbf{1})$. The canonical covering of $\mathbf{P}^1$ defines motivic weak equivalences

$$T = \mathbf{A}^1/\mathbf{A}^1 \smallsetminus \{0\} \xrightarrow{\sim} \mathbf{P}^1/\mathbf{A}^1 \xleftarrow{\sim} \mathbf{P}^1$$

and the arrow $T = \mathbf{A}^1/\mathbf{A}^1 \smallsetminus \{0\} \to \mathbf{P}^1/\mathbf{P}^1 \smallsetminus \{0\}$ is an isomorphism. Hence one may switch between T and $\mathbf{P}^1$ as desired.

Corollary 2.0.5. *For $W = V \oplus \mathbf{1}$ consider the composite motivic weak equivalence $\epsilon\colon \mathrm{Th}(V) \wedge \mathbf{P}^1 \to \mathrm{Th}(V) \wedge \mathbf{P}^1/\mathbf{A}^1 \leftarrow \mathrm{Th}(V) \wedge T \to \mathrm{Th}(W)$ in $\mathrm{H}_\bullet(S)$. Then the diagram*

$$\begin{array}{ccc}
E^{*+2r,*+r}(Th(V)) & \xrightarrow{\Sigma_{\mathbf{P}^1}} & E^{*+2r+2,*+r+1}(Th(V) \wedge \mathbf{P}^1) \\
{\scriptstyle id}\big\uparrow & & {\scriptstyle \epsilon^*}\big\uparrow \\
E^{*+2r,*+r}(Th(V)) & \xrightarrow{\Sigma_T} & E^{*+2r+2,*+r+1}(Th(W)) \\
{\scriptstyle \cup th(V)}\big\uparrow & & {\scriptstyle \cup th(W)}\big\uparrow \\
E^{*,*}(X) & \xrightarrow{id} & E^{*,*}(X).
\end{array}$$

commutes.

Theorem 2.0.6. *Let $c_i = c_i(\mathcal{T}(n)) \in E^{2i,i}(\mathrm{Gr}(n))$ be the i-th Chern class of the tautological bundle $\mathcal{T}(n)$. Then*

$$E^{*,*}(\mathrm{Gr}(n)) = E^{*,*}(k)[[c_1, c_2, \ldots, c_n]]$$

is the formal power series on the c_i's. The inclusion $i\colon \mathrm{Gr}(n) \hookrightarrow \mathrm{Gr}(n+1)$ satisfies $i^*(c_m) = c_m$ for $m < n+1$ and $i^*(c_{n+1}) = 0$.

2.1 A general result

The main result of this Section is Theorem 2.1.4. The complex cobordism spectrum, equipped with its natural orientation, is a universal oriented ring cohomology theory by Quillen's universality theorem [Qu1]. A motivic version of this universality theorem is proved in [PPR2] (see [Ve] for the original statement). We consider MGL with the commutative monoid structure described in [PPR2, Defn 2.1.1] and with the orientation th^{MGL} described in 1.1.3.

By a cofibration we mean below in the text a cofibration with respect to the closed model structure on the category $\mathbf{M}(S)$ (see [PPR1, Appendix]).

Recall that for a $\mathbf{P}^1$-spectrum E and a cofibration $Y \to X$ the group $E^{p,q}(X,Y)$ is defined as the cohomology $E^{p,q}(X/Y, Y/Y)$ of the pointed space $(X/Y, +)$ (if Y is the empty set, then one should take the group $E^{p,q}(X_+, +)$ for $E^{p,q}(X,Y)$).

Definition 2.1.1 (Universality Property)**.** *Let* (U, u) *be an oriented commutative ring* $\mathbf{P}^1$*-spectrum over a field* k*. We say that* (U, u) *satisfies Quillen universality property, if for each commutative ring* $\mathbf{P}^1$*-spectrum* E *over* k *the assignment* $\varphi \mapsto \varphi(\mathrm{u}) \in \mathrm{U}^{2,1}(\mathrm{Th}(\mathcal{T}(1)))$ *identifies the set of monoid morphisms*

$$\varphi\colon \mathrm{U} \to \mathrm{E} \tag{7}$$

in the motivic stable homotopy category $\mathrm{SH}^{\mathrm{cm}}(S)$ *with the set of orientations of* E*.*

Let (U, u) be an oriented commutative ring $\mathbf{P}^1$-spectrum over k. Let $(\mathrm{E}, \mathrm{th})$ oriented commutative ring $\mathbf{P}^1$-spectrum over k. Let

$$\varphi\colon \mathrm{U} \to \mathrm{E} \tag{8}$$

be a monoid morphism in $\mathrm{SH}^{cm}(k)$ such that $\varphi(u) = th$. For every space X over k and a cofibration $Y \to X$ and a unique morphism $f\colon X/Y \to \mathrm{Spec}(k)$ one has a commutative diagram of $U^0(k)$-module homomorphisms.

$$\begin{array}{ccc}
\mathrm{U}^*(X,Y) & \xrightarrow{\varphi_{X,Y}} & \mathrm{E}^*(X,Y) \\
{\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle f^*} \\
\mathrm{U}^0(k) & \xrightarrow{\varphi^0_S} & \mathrm{E}^0(k)
\end{array}$$

It is known that for each oriented commutative ring $\mathbf{P}^1$-spectrum (F, v) and each space A the ring $F^0(A)$ is contained in the center of $F^*(A)$. The last commutative diagram induces two homomorphisms

$$\bar{\varphi}_{X,Y}\colon \mathrm{U}^*(X,Y) \otimes_{\mathrm{U}^0(k)} \mathrm{E}^0(k) \to \mathrm{E}^*(X,Y) \tag{9}$$

$$\bar{\varphi}^0_{X,Y}\colon \mathrm{U}^0(X,Y) \otimes_{\mathrm{U}^0(k)} \mathrm{E}^0(k) \to \mathrm{E}^0(X,Y) \tag{10}$$

which are natural in a cofibration $Y \to X$.

Since this moment choose $(\mathrm{BGL}, \mathrm{th}^K)$ for $(\mathrm{E}, \mathrm{th})$ (see Example 1.1.3). Set $\bar{\mathrm{U}}^*(X,Y) = \mathrm{U}^*(X,Y) \otimes_{\mathrm{U}^0(k)} \mathrm{BGL}^0(k)$, $\bar{\mathrm{U}}^0(X,Y) = \mathrm{U}^0(X,Y) \otimes_{\mathrm{U}^0(k)} \mathrm{BGL}^0(k)$.

Definition 2.1.2 (Weakly MGL-Cellular)**.** *A Quillen universal oriented commutative ring* $\mathbf{P}^1$*-spectrum* (U, u) *is called weakly MGL-cellular if there exists an integer* N *such that the map* $\bar{\varphi}^0_{U_n,*}$ *is an isomorphism for* $n \geq N$ *.*

Remark 2.1.3. By the Universality Theorem [Ve] or [PPR2] the $\mathbf{P}^1$-spectrum MGL is Quillen universal. That is why we choose to write MGL-cellular in the definition above. The following theorem motivates two last Definitions.

Theorem 2.1.4. *Let* (U, u) *be an oriented commutative ring* $\mathbf{P}^1$*-spectrum over a field* k *satisfying the Quillen universality property. Suppose* (U, u) *is weakly* MGL*-cellular. Then for each cofibration* $Y \to X$ *of small spaces over the field* k *the homomorphism* $\bar{\varphi}_{X,Y}$ *is an isomorphism.*

Proof. The proof consists of several steps. Our first aim is to prove that homomorphisms $\bar{\varphi}^0_{X,Y}$ are isomorphisms. We beging with constructing a section of the natural transformation

$$\varphi^{0,0} \colon \mathrm{U}^{0,0} \to \mathrm{BGL}^{0,0}$$

of functors on the category of cofibrations of small spaces. To do this we begin with recalling that for every oriented commutative $\mathbf{P}^1$-ring spectrum (E, th) the ring cohomology theory $E^{*,*}|_{\mathcal{S}m\mathcal{O}p}$ is an oriented cohomology theory on the category $\mathcal{S}m\mathcal{O}p$ (see Section 2). Let $\mathbb{F}_{E,th}$ be the induced commutative formal group law over the ring $E^0(k)$. Let Ω be the complex cobordism ring and let $l_{E,th} \colon \Omega \to E^0(k)$ be the unique ring homomorphism, which takes the universal formal group $\mathbb{F}_\Omega$ to $\mathbb{F}_{E,th}$. Set

$$[\mathbf{P}^n]_E = l_{E,th}([\mathbb{CP}^n]), \tag{11}$$

where $[\mathbb{CP}^n]$ is the class of the complex projective space $\mathbb{CP}^n$ in Ω. Although the class $[\mathbf{P}^n]_E$ depends on the orientation class th, we use the notation $[\mathbf{P}^n]_E$ instead. If (E', th') is another oriented commutative $\mathbf{P}^1$-ring spectrum and $\psi \colon E \to E'$ is a monoid homomorphism in the category $\mathrm{SH}^{\mathrm{cm}}(S)$ which preserves orientation classes, then it sends the formal group law $\mathbb{F}_{E,th}$ to $\mathbb{F}_{E',th'}$. In particular $\psi([\mathbf{P}^n]_E) = [\mathbf{P}^n]_{E'}$. Applying this observation to the monoid homomorphism φ one obtains

$$\varphi([\mathbf{P}^1]_{\mathrm{U}}) = [\mathbf{P}^1]_{\mathrm{BGL}}.$$

To compute $[\mathbf{P}^1]_{\mathrm{BGL}}$ recall that the coefficient at XY in the formal group law $\mathbb{F}_\Omega$ coincides with the class $-[\mathbb{CP}^1]$ in Ω. The formal group law $\mathbb{F}_{\mathrm{BGL}}$ coincides with $X + Y + \beta^{-1}XY$, since $c^{\mathrm{BGL}}(L) = ([\mathbf{1}] - [\mathbf{L}^\vee])(-\beta)$. Thus one gets

$$[\mathbf{P}^1]_{\mathrm{BGL}} = -\beta^{-1}$$

We are ready to construct a section. Consider the map

$$s\colon \Sigma^\infty_{\mathbf{P}^1}(\mathbb{Z}\times \mathrm{Gr}) \to \mathrm{U} \tag{12}$$

in the stable homotopy category category $\mathrm{SH}^{\mathrm{cm}}(S)$ given by the element

$$c_1^{\mathrm{U}}(\infty - \tau_\infty^\vee) \cup [\mathbf{P}^1]_{\mathrm{U}} \in \mathrm{U}^{0,0}(\mathbb{Z}\times \mathrm{Gr}).$$

Claim 2.1.5. One has $\varphi(c_1^{\mathrm{U}}(\infty - \tau_\infty^\vee) \cup [\mathbf{P}^1]_{\mathrm{U}}) = \tau_\infty - \infty \in \mathrm{BGL}^{0,0}(\mathbb{Z}\times \mathrm{Gr})$.

In fact,

$$\varphi(c_1^{\mathrm{U}}(\infty - \tau_\infty^\vee) \cup [\mathbf{P}^1]_{\mathrm{U}}) = c_1^{\mathrm{BGL}}(\infty - \tau_\infty^\vee) \cup [\mathbf{P}^1]_{\mathrm{BGL}} = (\infty - \tau_\infty) \cup \beta \cup (-\beta^{-1})$$

$$= \tau_\infty - \infty.$$

Claim 2.1.5 shows that the composite map

$$\varphi\circ s\colon \Sigma^\infty_{\mathbf{P}^1}(\mathbb{Z}\times \mathrm{Gr}) \to \mathrm{BGL}$$

coincides with the adjoint of the motivic weak equivalence $i\colon \mathbb{Z}\times\mathrm{Gr} \to \mathcal{K} = \mathcal{K}_0$ from [PPR1, Lemma 1.2.2]. Thus for every cofibration $Y \to X$ of small motivic spaces the map

$$s_{X,Y}\colon \mathrm{BGL}^{0,0}(X/Y) = [X/Y, \mathcal{K}_0] = [X/Y, \mathbb{Z}\times\mathrm{Gr}] \to [\Sigma^\infty_{\mathbf{P}^1}(X/Y), \mathrm{U}] = \mathrm{U}^{0,0}(X/Y)$$

is a *section* of the map $\varphi^{0,0}_{X,Y}\colon \mathrm{U}^{0,0}(X,Y) \to \mathrm{BGL}^{0,0}(X,Y)$. Moreover, the section $s_{X,Y}$ is natural in the cofibration $Y \to X$.

Next we extend the section s to a section $\bar{s}^0\colon \mathrm{BGL}^0 \to \overline{\mathrm{U}}^0$ of the natural transformation $\bar{\varphi}^0\colon \overline{\mathrm{U}}^0 \to \mathrm{BGL}^0$ of functors on the category of cofibrations. To achieve this, recall that

$$\mathrm{BGL}^0 = \mathrm{BGL}^{0,0}[\beta, \beta^{-1}]$$

for the Bott element $\beta \in \mathrm{BGL}^{2,1}(k)$ (see (2)). Thus for every cofibration $Y \to X$ every element $\alpha \in \mathrm{BGL}^0(X,Y)$ can be presented in a unique way in the form $a\cup\beta^i$ with $a \in \mathrm{BGL}^{0,0}(X,Y)$. Define

$$\bar{s}^0_{X,Y}\colon \mathrm{BGL}^0 \to \overline{\mathrm{U}}^0 \tag{13}$$

by $\bar{s}^0_{X,Y}(a\cup\beta^i) = s_{X,Y}(a)\otimes\beta^i \in \overline{\mathrm{U}}^0(A)$, where $a \in \mathrm{BGL}^{0,0}(X,Y)$. It is immediate that s^0_A is natural in cofibration $Y \to X$. The following computation proves the claim which is right below the row of computation

$$\bar{\varphi}^0_A(\bar{s}^0(a\cup\beta^i)) = \bar{\varphi}^0_A(s(a)\otimes\beta^i) = \varphi(s(a))\cup\beta^i = a\cup\beta^i.$$

Claim 2.1.6. The map $\bar{s}^0_{X,Y}$ is a section of $\bar{\varphi}^0_{X,Y}$.

Now observe the following. If for a cofibration $Y \to X$ the map $\bar{\varphi}^0_{X,Y}$ is an isomorphism, then $\bar{s}^0_{X,Y}$ is an isomorphism inverse to $\bar{\varphi}^0_{X,Y}$. In particular, one has $\bar{s}^0_{X,Y} \circ \bar{\varphi}^0_{X,Y} = \mathrm{id}$.

The homomorphism $\bar{\varphi}^0_{X,Y}$ is an isomorphism for cofibrations of the form $* \to U_n$ with $n \geq N$, since U is weakly MGL-cellular. Taking $* \to U_n$ as a cofibration $Y \to X$ and the class $[u_n] \in \mathrm{U}^{2n,n}(\mathrm{U}_n, *)$ of the canonical morphism $u_n \colon \Sigma^\infty_{\mathbf{P}^1} \mathrm{U}_n(-n) \to \mathrm{U}$ we get the following relation:

$$(\bar{s}^0_{\mathrm{U}_n,*} \circ \varphi^0_{\mathrm{U}_n,*})([u_n]) = [u_n] \otimes 1 \in \overline{\mathrm{U}}^0(\mathrm{U}_n, *). \tag{14}$$

Now we are ready to check that $\bar{\varphi}^0_A$ is an isomorphism for all cofibrations $Y \to X$ of small motivic spaces. Recall that for a cofibrations $Y \to X$ of small motivic spaces there is a canonical isomorphism of the form

$$\mathrm{U}^{2i,i}(X,Y) = \mathrm{colim}_n[\Sigma^{2n,n}(X/Y, Y/Y), \mathrm{U}_{i+n}]_{\mathrm{H}_\bullet(S)} \tag{15}$$

where $\Sigma^{2n,n} = \Sigma^n_{\mathbf{P}^1}$ (if Y is empty then one should replace the pair $(X/Y, Y/Y)$ by the one $(X_+, +)$). This isomorphism implies that for every element $a \in \mathrm{U}^{2i,i}(X,Y)$ there exists an integer $n \geq 0$ such that $\Sigma^{2n,n}(a) = f^*([u_n])$ for an appropriate map $f \colon \Sigma^{2n,n}(X/Y) \to \mathrm{U}_{i+n}$ in the homotopy category $\mathrm{H}^{cm}_\bullet(S)$. Here $\Sigma^{2n,n}(a)$ is the n-fold $\Sigma_{\mathbf{P}^1}$-suspension of a.

The surjectivity of $\bar{\varphi}^0_{X,Y}$ is clear, since $\bar{s}^0_{X,Y}$ is its section. It remains to check the injectivity of $\bar{\varphi}^0_{X,Y}$. Take a homogeneous element $\alpha \in \overline{\mathrm{U}}^{2i,i}(X,Y) \subseteq \overline{\mathrm{U}}^0(X,Y)$ such that $\bar{\varphi}^0_{X,Y}(\alpha) = 0$. It has the form $\alpha = a \otimes \beta^m$ for a homogeneous element $a \in \mathrm{U}^{0,0}(X,Y)$. Since the element β is invertible in $\mathrm{BGL}^{*,*}(k)$, one concludes $\varphi^0_{X,Y}(a) = 0$.

Choose an integer $n \geq 0$ such that $\Sigma^{2n,n}(a) = f^*([u_n])$ and write A for X/Y to short the notation. The map φ of $\mathbf{P}^1$-spectra respects the suspension isomorphisms. Thus $\varphi_{\Sigma^{2n,n}A}(\Sigma^{2n,n}(a)) = \Sigma^{2n,n}(\varphi_A(a)) = 0$ and $(\bar{s}^0_{\Sigma^{2n,n}A} \circ \varphi_{\Sigma^{2n,n}A})(\Sigma^{2n,n}(a)) = 0$ too. The chain of relations in $\overline{\mathrm{U}}^0(\Sigma^{2n,n}A)$ given by

$$\begin{aligned} 0 &= \big(\bar{s}^0_{\Sigma^{2n,n}A} \circ \varphi_{\Sigma^{2n,n}A}\big)\big(\Sigma^{2n,n}(a)\big) = \big(\bar{s}^0_{\Sigma^{2n,n}A} \circ \varphi_{\Sigma^{2n,n}A}\big)\big(f^*([u_n])\big) \\ &= f^*\big((\bar{s}^0_{\mathrm{U}_{n+i}} \circ \varphi_{\mathrm{U}_{n+i}})([u_n])\big) = f^*([u_n] \otimes 1) = f^*([u_n]) \otimes 1 \\ &\qquad\qquad\qquad\qquad = \Sigma^{2n,n}(a) \otimes 1 \end{aligned}$$

implies that $\Sigma^{2n,n}(a \otimes 1) = \Sigma^{2n,n}(a) \otimes 1 = 0$. Because the n-fold suspension map

$$\Sigma^{2n,n} \colon \overline{\mathrm{U}}^0(X/Y, Y/Y) \to \overline{\mathrm{U}}^0(\Sigma^{2n,n}(X/Y, Y/Y))$$

is an isomorphism, $a \otimes 1 = 0$ in $\overline{\mathrm{U}}^0(X/Y) = \overline{\mathrm{U}}^0(X, Y)$. This proves the injectivity and hence the bijectivity of $\bar{\varphi}^0_{X,Y}$ for cofibrations of all small motivic spaces.

To prove that $\bar{\varphi}_{X,Y}$ is an isomorphism for cofibrations of all small motivic spaces we will use the fact that $\bar{\varphi}_{X,Y}$ respects the $\mathbf{P}^1$-suspension isomorphisms. Set $A = X/Y$.

For every integer $i \in \mathbb{Z}$ choose an integer $n \geq 0$ with $n \geq i$. Then for a pointed motivic space A one may form the suspension $\mathbb{G}_m^{\wedge n} \wedge S_s^{n-i} \wedge A = S^{n,n} \wedge S^{n-i,0} \wedge A$ in the category of pointed motivic spaces, which supplies the commutative diagram

$$\begin{tikzcd}
\mathrm{BGL}^i(A) \arrow[r, "\Sigma^{2n,n}", "\cong"'] & \mathrm{BGL}^i(S^{2n,n} \wedge A) & \mathrm{BGL}^0(S^{n,n} \wedge S^{n-i,0} \wedge A) \arrow[l, "\Sigma^{i,0}"', "\cong"] \\
\mathrm{U}^i(A) \arrow[u, "\bar{\varphi}^i_A"] \arrow[r, "\Sigma^{2n,n}", "\cong"'] & \mathrm{U}^i(S^{2n,n} \wedge A) \arrow[u, "\bar{\varphi}^i_{S^{2n,n} \wedge A}"] & \mathrm{U}^0(S^{n,n} \wedge S^{n-i,0} \wedge A) \arrow[l, "\Sigma^{i,0}"', "\cong"] \arrow[u, "\cong"', "\bar{\varphi}^0_{S^{n,n} \wedge S^{n-i,0} \wedge A}"]
\end{tikzcd}$$

with the suspension isomorphisms $\Sigma^{2n,n} = \Sigma^n_{\mathbf{P}^1}$ and $\Sigma^{i,0}$. The map $\bar{\varphi}^0_B$ is an isomorphism for B a small pointed motivic space, hence so is $\bar{\varphi}^i_A$. We proved that the map $\bar{\varphi}_{X,Y}$ is an isomorphism. Theorem 2.1.4 is proven.

□

2.2 The MGL-cellularity of the MGL

Theorem 2.2.1. *The oriented commutative ring $\mathbf{P}^1$-spectrum* $(\mathrm{MGL}, th^{\mathrm{MGL}})$ *from Example 1.1.3 is weakly* MGL*-cellular.*

Proof. We must check that the homomorphism $\bar{\varphi}^0_{X,Y}$ is an isomorphism for (X, Y) being $(Th(\mathcal{T}_n), *) = (\mathrm{MGL}_n, *)$. We check that inspecting step by step motivic spaces $\mathrm{Spec}(k)$, $\mathbf{P}^\infty$, $\mathrm{Gr}(n)$ and the pair $(Th(\mathcal{T}_n), *) = (\mathrm{MGL}_n, *)$.

The map $\bar{\varphi}^0_k$ is an isomorphism, since it is the identity map. By the case $n = 1$ of Theorem 2.0.6 one has $\overline{\mathrm{MGL}}^*(\mathbf{P}^\infty) = \overline{\mathrm{MGL}}^*(k)[[c^{\mathrm{MGL}}]]$, whence

$$\overline{\mathrm{MGL}}^0(\mathbf{P}^\infty) = \overline{\mathrm{MGL}}^0(k)[[c^{\mathrm{MGL}}]]$$

(the formal power series on the first Chern class c^{MGL} of the tautological line bundle $\mathcal{O}(-1)$). The same holds for BGL. Namely

$$\mathrm{BGL}^0(\mathbf{P}^\infty) = \mathrm{BGL}^0(k)[[c^{\mathrm{BGL}}]].$$

By its definition the morphism φ takes the orientation class th^{MGL} to the orientation class th^K and so it preserves the first Chern class. Whence the map

$\bar{\varphi}^0_{\mathbf{P}^\infty}$ coincides with a map of formal power series induced by the isomorphism $\bar{\varphi}^0_k$ of the coefficients rings. Hence $\bar{\varphi}^0_{\mathbf{P}^\infty}$ is an isomorphism as well.

Consider now $X = \mathrm{Gr}(n)$. By Theorem 2.0.6 its MGL-cohomology ring is the ring of formal power series on the Chern classes of the tautological bundle $\mathcal{T}_n$ over the coefficient ring $\mathrm{MGL}^{*,*}(k)$. The same holds for the BGL-cohomology ring. As observed above, the map φ preserves the first Chern class, thus it takes Chern classes to the Chern classes. Whence $\bar{\varphi}^0_{\mathrm{Gr}(n)}$ is an isomorphism as well.

Now consider $(X, Y) = (\mathrm{Th}(\mathcal{T}_n), *)$. The morphism φ respects Thom classes (see (5) and (6)). The vertical arrows in the commutative diagram

$$\begin{array}{ccc} \overline{\mathrm{MGL}}^0((\mathrm{Th}(\mathcal{T}_n), *)) & \xrightarrow{\bar{\varphi}^0_{\mathrm{Th}(\mathcal{T}_n),*)}} & \mathrm{BGL}^0(\mathrm{Th}(\mathcal{T}_n)) \\ \uparrow{\scriptstyle \overline{\mathrm{thom}}^{\mathrm{MGL}}} & & \uparrow{\scriptstyle \mathrm{thom}^{\mathrm{BGL}}} \\ \overline{\mathrm{MGL}}^0(\mathrm{Gr}(n)) & \xrightarrow{\bar{\varphi}^0_{\mathrm{Gr}(n)}} & \mathrm{BGL}^0(G(n)) \end{array}$$

are isomorphisms induced by the the Thom isomorphism 2.0.4. The map $\bar{\varphi}^0_{\mathrm{Gr}(n)}$ is an isomorphism by the preceding case, whence $\bar{\varphi}^0_{Th(\mathcal{T}_n),*)}$ is an isomorphism too.

□

2.3 Main Result

Let k be a field and $S = \mathrm{Spec}(k)$. By Theorem [PPR2, Theorem 2.2.1] and Example 1.1.3 there exists a unique monoid morphism

$$\varphi\colon \mathrm{MGL} \to \mathrm{BGL} \tag{16}$$

in $\mathrm{SH}^{\mathrm{cm}}(S)$ such that $\varphi(th^{\mathrm{MGL}}) = th^K$. For every cofibration $Y \to X$ of motivic spaces over k a unique morphism $f\colon X/Y \to S$ induces the homomorphism

$$\bar{\varphi}_{X,Y}\colon \overline{\mathrm{MGL}}^*(X,Y)\colon = \mathrm{MGL}^*(X,Y) \otimes_{\mathrm{MGL}^0(k)} \mathrm{BGL}^0(k) \to \mathrm{BGL}^*(X,Y) \tag{17}$$

which is natural in cofibration $Y \to X$. Recall that a space A is called small if the covariant functor $\Sigma^\infty_{\mathbf{P}^1} A$ represents on $\mathrm{SH}^{\mathrm{cm}}(S)$ commutes with arbitrary coproducts.

Theorem 2.3.1. *The homomorphism $\bar{\varphi}_{X,Y}$ is an isomorphism for all cofibrations $Y \to X$ of small motivic spaces.*

In fact, the $(\mathrm{MGL}, th^{\mathrm{MGL}})$ is Quillen universal by [Ve] or by Theorem 2.2.1 from [PPR2] and weakly MGL-cellular by Theorem 2.2.1 above. Theorem 2.1.4 completes the proof.

Remark 2.3.2. There is an unpublished result due to Morel and Hopkins, which states that there is a canonical isomorphism of the form

$$\mathrm{MGL}^{*,*}(X) \otimes_{\mathbb{L}} \mathbb{Z}[\beta, \beta^{-1}] \to \mathrm{BGL}^{*,*}(X)$$

where $\mathbb{L}$ denotes the Lazard ring carrying the universal formal group law. If the canonical homomorphism $\mathbb{L} \to \mathrm{MGL}^0(k)$ is an isomorphism, Theorem 2.3.1 implies their result.

Let X be a smooth k-scheme and $Z \subseteq X$ a closed subset, with open complement $U \subseteq X$. Consider the motivic space X/U and take the quotients of both sides of the isomorphism (17) modulo the principal ideal generated by the element $1 \otimes (\beta + 1)$. Corollary 1.0.1 then implies the following isomorphism

$$\bar{\bar{\varphi}}_{X/U} \colon \overline{\overline{\mathrm{MGL}}}^{*}(X, Y) = \mathrm{MGL}^*(X/U) \otimes_{\mathrm{MGL}^0(k)} \mathbb{Z} \to K^{TT}_{-*,Z}(X) \tag{18}$$

where $K^{TT}_{*,Z}(X)$ are the Thomason-Trobaugh K-groups with supports. This family of isomorphisms shows that the functor

$$(X, X \smallsetminus Z) \mapsto \mathrm{MGL}^*(X/(X \smallsetminus Z)) \otimes_{\mathrm{MGL}^0(k)} \mathbb{Z} =: \overline{\overline{\mathrm{MGL}}}^{*}(X/(X \smallsetminus Z))$$

is a ring cohomology theory in the sense of [PS1]. This implies the first part of our main result.

Theorem 2.3.3 (Main Theorem)**.** *Let $X \in \mathcal{S}m_k$ and $Z \subseteq X$ be a closed subset.*

- *The family of isomorphisms*

  $$\bar{\bar{\varphi}}_{X/(X-Z)} \colon \overline{\overline{\mathrm{MGL}}}^{*}(X/(X \smallsetminus Z)) \to K^{TT}_{-*,Z}(X) \tag{19}$$

  form an isomorphism $\bar{\bar{\varphi}}$ of ring cohomology theories on $\mathcal{S}m\mathcal{O}p/k$.

- *The $\bar{\bar{\varphi}}$ respects orientations provided that MGL^* and K^{TT}_{-*} are considered as oriented cohomology theories in the sense of [PS1] with orientations given by the Thom class $th^{\mathrm{MGL}} \otimes 1$ from 1.1.3 and the Chern structure $L/X \mapsto [\mathcal{O}] - [L^{-1}]$. In particular, the composition*

  $$\mathrm{MGL}^0(k) \longrightarrow \mathrm{MGL}^0(k) \otimes \mathbb{Z} \longrightarrow \mathbb{Z}$$

  $$a \longmapsto a \otimes 1 \quad b \otimes c \longmapsto \varphi(b) \cdot c$$

sends the class $[X] \in \mathrm{MGL}^0(X)$ of a smooth projective k-variety X to the Euler characteristic $\chi(X, \mathcal{O}_X)$ of the structure sheaf $\mathcal{O}_X$.

Proof. The first part is already proven. To prove the second one consider the orientations th^{MGL} and th^K from 1.1.3. Note that by the very definition of φ it sends th^{MGL} to th^K. Thus it respects the Chern structures on MGL^* and BGL^* described in Section 2.

The quotient map $\mathrm{BGL}^* \to K^{TT}_{-*}$ takes the Bott element β to (-1). Thus it takes the Chern structure on BGL^* to the Chern structure on K^{TT}_{-*} given by $L/X \mapsto [\mathcal{O}] - [L^{-1}] \in K_0(X)$. This shows that $\bar{\bar{\varphi}} \colon \overline{\overline{\mathrm{MGL}}}^* \to K^{TT}_{-*}$ respects the orientations described in the Theorem 2.3.3.

Let $f \mapsto f_{\mathrm{MGL}}$ resp. $f \mapsto f_K$ be the integrations on MGL^* resp. K^{TT}_{-*} given by these Chern structures via Theorem [PS3, Thm. 4.1.4]. By Theorem [PS2, Thm. 1.1.10] the composition $\mathrm{MGL}^* \to \mathrm{BGL}^* \to K^{TT}_{-*}$ respects the integrations on MGL^* and K^{TT}_{-*} since it preserves the Chern structures. In particular, given a smooth projective S-scheme $f \colon X \to \mathrm{Spec}(k)$, the diagram

$$\begin{array}{ccc} \overline{\overline{\mathrm{MGL}}}^0(X) & \xrightarrow{\bar{\bar{\varphi}}} & K_0^{TT}(X) \\ \downarrow{\scriptstyle f_{\mathrm{MGL}}} & & \downarrow{\scriptstyle f_K} \\ \overline{\overline{\mathrm{MGL}}}^0(k) & \xrightarrow{\bar{\bar{\varphi}}} & K_0^{TT}(k) \end{array}$$

commutes where f_{MGL} and f_K are the push-forward maps for MGL^* and K^{TT}_{-*} respectively. The integration $f \mapsto f_K$ on K^{TT}_{-*} respecting the Chern structure $L \mapsto [\mathcal{O}] - [L^{-1}]$ coincides with the one given by the higher direct images by Theorem [PS2, Thm. 1.1.11]. The last one sends the class $[V] \in K_0(X)$ of a vector bundle V over a smooth projective variety X to the Euler characteristic $\chi(X, \mathcal{V})$ of the sheaf $\mathcal{V}$ of sections of V.

Recall that for an oriented cohomology theory A with a Chern structure $L \mapsto c(L)$ and for a smooth projective variety $f \colon X \to \mathrm{Spec}(k)$ its class $[X]_A \in A^{\mathrm{even}}(\mathrm{Spec}(k))$ is defined as $f_A(1)$, where $f_A \colon A(X) \to A(\mathrm{Spec}(k))$ is the push-forward respecting the Chern structure (see [PS3, Thm. 4.1.4]). The f_A depends on the Chern structure. However we write just f_A for the push-forward operator. Taking the element $1 \in \mathrm{MGL}^{0,0}(X)$ and using the commutativity of the very last diagram we see that

$$\bar{\bar{\varphi}}([X]_{\mathrm{MGL}} \otimes 1) = \chi(X, \mathcal{O}_X).$$

Whence the Theorem.

□

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