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# Kummer subfields of tame division algebras over Henselian valued fields*†

Karim Mounirh

Département des Mathématiques
Faculté des Sciences
Université Catholique de Louvain
Belgium
e.mail: akamounirh@hotmail.com

**Abstract :** By generalizing the method used by Tignol and Amitsur in [TA85], we determine necessary and sufficient conditions for an arbitrary tame central division algebra $D$ over a Henselian valued field $E$ to have Kummer subfields [Corollary 2.11 and Corollary 2.12]. We prove also that if $D$ is a tame semiramified division algebra of prime power degree $p^n$ over $E$ such that $p \neq char(\bar{E})$ and $rk(\Gamma_D/\Gamma_F) \geq 3$ [resp., such that $p \neq char(\bar{E})$ and $p^3$ divides $exp(\Gamma_D/\Gamma_E)$], then $D$ is non-cyclic [Proposition 3.1] [resp., $D$ is not an elementary abelian crossed product [Proposition 3.2]].

## Introduction

Let $B$ be a tame central division algebra over a Henselian valued field $E$. We know by [JW90, Lemma 6.2] that $B$ is similar to some $S \otimes_E T$, where $S$ is an inertially split [resp., $T$ is a tame totally ramified] division algebra over $E$. By generalizing the method used by Tignol and Amitsur in [TA85], Morandi and Sethuraman determined in [MorSe95] necessary and sufficient conditions for $B$ to have Kummer subfields

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when $B = S \otimes_E T$. A good question was to see if we have the same results when $B$ is an arbitrary tame central division algebra over $E$. To deal with this question, we remarked that it will be the same if we can determine necessary and sufficient conditions for a graded central division algebra over a graded field to have Kummer graded subfields. Indeed, we know that if $char(\bar{E})$ does not divide $deg(B)$, then any result concerning graded subfields of $GB$ gives an analogous one for $B$.
A first key idea was the fact that if $D$ is a graded central division algebra over a graded field $F$, then there is a factor set $(\omega, f)$ of $\Gamma_D/\Gamma_F$ in $D_0F$ such that $D$ is the generalized graded crossed product $(D_0F, \Gamma_D/\Gamma_F, (\omega, f))$. Another important result consists in the fact that $f$ can be decompsed in a nice way. Indeed, we showed that for any $\bar{\gamma}, \bar{\gamma}' \in \Gamma_D/\Gamma_F$, we can write $f(\bar{\gamma}, \bar{\gamma}') = d(\bar{\gamma}, \bar{\gamma}')h(\bar{\gamma}, \bar{\gamma}')$, where $(\omega, d)$ is a factor set of $\Gamma_D/\Gamma_F$ in $D_0$ and $h \in Z^2(\Gamma_D/\Gamma_F, F^*)_{sym}$ [Lemma 1.6]. We show also in section 2 that if $K$ is a Kummer graded subfield of $D$, then there is an exact sequence of trivial $\Gamma_K/\Gamma_F$-modules $\alpha_K : 1 \to kum(K_0/F_0) \to kum(K/F) \to \Gamma_K/\Gamma_F \to 0$. We consider $\alpha_K$ as an element of $Z^2(\Gamma_D/\Gamma_F, kum(K_0/F_0))_{sym}$ and so applying the previous facts we get in [Corollary 2.10 and Corollary 2.11] necessary and sufficient conditions for $D$ to have Kummer graded subfields when $F_0$ contains enough roots of unity. This results are then applied to give necessary and sufficient conditions for a semiramified graded division algebra $D$ over a graded field $F$ to be cyclic [resp., to be an elementary abelian graded crossed product] when $F_0$ contains enough roots of unity. In section 3, and without assuming any root of unity to be in $\bar{E}$, we prove that if $E$ is a Henselian valued field and $B$ is a tame semiramified division algebra of prime power degree $p^n$ over $E$ such that $p \neq char(\bar{E})$ and $rk(\Gamma_B/\Gamma_F) \geq 3$ [resp., such that $p \neq char(\bar{E})$ and $p^3$ divides $exp(\Gamma_B/\Gamma_E)$], then $B$ is non-cyclic [Proposition 3.1] [resp., $B$ is not an elementary abelian crossed product [Proposition 3.2]].

Throughout this paper, we assume familiarity with the definitions and notations previously used in [M05] and [M07].

# 1 Generalized graded crossed products and graded division algebras

**(1.1)** Let $L$ be a field and $A$ a central simple algebra over $L$. We denote by $A^*$ the group of invertible elements of $A$ and by $Aut(A)$ the group of ring automorphisms of $A$. For any $c \in A^*$, we denote by $Inn(c)$ the ring automorphism of $A$ defined by $a \mapsto cac^{-1}$. Let $H$ be a finite group that acts by automorphisms on $L$ and let $\omega : H \to Aut(A)$ and $f : H \times H \to A^*$ be two maps. We say that $(\omega, f)$ is a factor set of $H$ in $A$ if the following conditions are satisfied :
(1) $\omega_\sigma(a) = \sigma(a)$ for all $a \in L$ and $\sigma \in H$,
(2) $\omega_\sigma \omega_\tau = Inn(f(\sigma,\tau))\omega_{\sigma\tau}$ for all $\sigma, \tau \in H$, and
(3) $f(\sigma,\tau)f(\sigma\tau,\mu) = \omega_\sigma(f(\tau,\mu))f(\sigma,\tau\mu)$ for all $\sigma,\tau,\mu \in H$.
If $(\omega, f)$ is a factor set of $H$ in $A$, then we define the generalized crossed product associated to $(\omega, f)$ to be the algebra $(A, H, (\omega, f)) = \oplus_{\sigma \in H} Ax_\sigma$, where $x_\sigma$ are independent indeterminates over $A$ satisfying the following multiplicative conditions (for all $\sigma \in H$ and $a \in A$) :
(4) $x_\sigma a = \omega_\sigma(a)x_\sigma$, and
(5) $x_\sigma x_\tau = f(\sigma,\tau)x_{\sigma\tau}$.

It is well-known that if $char(L)$ does not divide $card(H)$, then $(A, H, (\omega, f))$ is a semisimple algebra (see [MorSe95, p. 556]).

Let $(\omega, f)$ and $(\omega', f')$ be two factor sets of $H$ in $A$. We say that $(\omega, f)$ and $(\omega', f')$ are cohomologous if there is a family $(a_\sigma)_{\sigma \in H}$ of elements of $A^*$ such that for all $\sigma, \tau \in H$, $\omega'_\sigma = Inn(a_\sigma)\omega_\sigma$ and $f'(\sigma,\tau) = a_\sigma \omega_\sigma(a_\tau) f(\sigma,\tau) a_{\sigma\tau}^{-1}$. We write in this case $(\omega, f) \sim (\omega', f')$. The relation $\sim$ is an equivalence relation on the set of factor sets of $H$ in $A$. We denote the set of equivalence classes by $\mathcal{H}(H, A^*)$. If $A = L$ is a Galois field extension of some field $E$ and $H = Gal(L/E)$, then $\mathcal{H}(H, A^*)$ is the second Galois cohomology group $H^2(H, L^*)$.

Now, let $L$ be a graded field, $A$ a graded central simple algebra over $L$, $H$ a finite group that acts on $L$ by graded automorphisms (of grade 0), $GAut(A)_0$ the group of graded ring automorphisms (of grade 0) of $A$ (i.e. ring automorphisms of $A$ such that

$f(A_\delta) = A_\delta$). In the same way as above, if $\omega : H \to GAut(A)_0$ and $f : H \times H \to A^*$ are two maps that satisfy the conditions (1) to (3) above, then we say that $(\omega, f)$ is a graded factor set of $H$ in $A$. The corresponding graded generalized crossed product $(A, H, (\omega, f))$ is defined also as above. Namely, $(A, H, (\omega, f)) = \oplus_{\sigma \in H} Ax_\sigma$, where $x_\sigma$ are independent indeterminates on $A$ satisfying the multiplicative conditions : $x_\sigma a = \omega_\sigma(a)x_\sigma$ and $x_\sigma x_\tau = f(\sigma, \tau)x_{\sigma\tau}$ for all $a \in A$ and $\sigma, \tau \in H$. As we will see in the next lemma, $(A, H, (\omega, f))$ has a unique graded algebra structure extending that of $A$ and for which $x_\sigma$ are homogeneous elements (the proof of this lemma is inspired from [HW(2), Lemma 5.4]).

**Lemma 1. 2** *Let $L$ be a graded field, $A$ be a graded central simple algebra over $L$, $H$ a finite group that acts on $L$ by graded automorphisms, and $(\omega, f)$ a graded factor set of $H$ in $A$. Then, there is a unique graded algebra structure of $(A, H, (\omega, f))$ extending the grading of $A$ and for which $x_\sigma$ are homogeneous elements.*

*Proof.* Let $\Gamma_A$ (a totally ordered abelian group) be the support of $A$, $\Delta_A(= \Gamma_A \otimes_{\mathbb{Z}} \mathbb{Q})$ be the divisible hull of $\Gamma_A$ and consider the map $h : H \times H \to \Delta_A$, $(\sigma, \tau) \mapsto gr(f(\sigma, \tau))$. Then, it follows from condition (3) above that $h$ is a cocycle of $Z^2(H, \Delta_A)$ (for the trivial action of $H$ on $\Delta_A$). Since $H$ is finite and $\Delta_A$ is uniquely divisible, then $H^2(H, \Delta_A) = H^1(H, \Delta_A) = 0$. Therefore, there is a unique family $(\delta_\sigma)_{\sigma \in H}$ of elements of $\Delta_A$ such that $h(\sigma, \tau) = \delta_\sigma + \delta_\tau - \delta_{\sigma\tau}$ (the uniqueness follows from the fact that $H^1(H, \Delta_A) = 0$). The unique graded structure of $(A, H, (\omega, f))$ that extends that of $A$ and for which $x_\sigma$ are homogeneous elements is then defined by $gr(x_\sigma) = \delta_\sigma$.

In what follows, we will show that any graded division algebra can be represented as a generalized graded crossed product. This representation, will be applied in section 2 to determine necessary and sufficient conditions for the existence of Kummer graded subfields.

**(1.3)** Let $F$ be a graded field and $D$ a graded central division algebra over $F$. Then, the map $\theta_D : \Gamma_D/\Gamma_F \to Gal(Z(D_0)/F_0)$, defined by $\theta_D(gr(d) + \Gamma_F)(a) = dad^{-1}$ for

any $d \in D^*$ and $a \in Z(D_0)$, is a surjective group homomorphism. Since $HCq(D)$ is a tame central division algebra over $HFrac(F)$, then by [JW90, Proposition 1.7 and Definition p. 166] $Z(D_0)$ is an abelian field extension of $F_0$. For simplicity, we denote by $G$ the Galois group $Gal(Z(D_0)/F_0)$. So, by [HW(1)99, Remark 3.1] $Z(D_0)F$ is an abelian Galois graded field extension of $F$ with Galois group isomorphic to $G$. In what follows, we will consider the action of $\Gamma_D/\Gamma_F$ on $Z(D_0)F$ defined by $\theta_D$ (i.e., for any $\bar{\gamma} \in \Gamma_D/\Gamma_F$ and any $a \in Z(D_0)F$, we let $\bar{\gamma}(a) = d_{\bar{\gamma}} a d_{\bar{\gamma}}^{-1}$, where $d_{\bar{\gamma}}$ is an arbitrary homogeneous element of $D^*$ such that $gr(d_{\bar{\gamma}}) + \Gamma_F = \bar{\gamma}$).
We aim here to show that there is a graded factor set $(\omega, f)$ of $H := \Gamma_D/\Gamma_F$ in $D_0F$ such that $D = (D_0F, H, (\omega, f))$. For this, we fix a family of homogeneous elements $(z_{\bar{\gamma}})_{\bar{\gamma}\in H}$ of $D^*$ with $gr(z_{\bar{\gamma}}) + \Gamma_F = \bar{\gamma}$. Clearly, we have $D = \oplus_{\bar{\gamma}\in H} D_0 F z_{\bar{\gamma}}$ (because both graded algebras have the same 0-component and the same support). We define :

$$\omega : H \to GAut(D_0F)_0$$

and

$$f : H \times H \to (D_0F)^*$$

by $\omega_{\bar{\gamma}}(a) = z_{\bar{\gamma}} a z_{\bar{\gamma}}^{-1}$ and $f(\bar{\gamma}, \bar{\gamma}') = z_{\bar{\gamma}} z_{\bar{\gamma}'} z_{\bar{\gamma}+\bar{\gamma}'}^{-1}$. One can easily see that $(\omega, f)$ is a graded factor set of $H$ in $D_0F$. So, $D = \oplus_{\bar{\gamma}\in H} D_0 F z_{\bar{\gamma}} = (D_0F, H, (\omega, f))$

Let $B = \oplus_{\bar{\gamma}\in ker(\theta_D)} D_0 F z_{\bar{\gamma}}$ and for any $\sigma \in G$ choose a $\bar{\gamma}_\sigma \in H$ such that $\theta_D(\bar{\gamma}_\sigma) = \sigma$ and let $z_\sigma := z_{\bar{\gamma}_\sigma}$. Then, we have the following Proposition.

**Proposition 1. 4** *$B$ is the centralizer of $Z(D_0F)$ in $D$ and $D = \oplus_{\sigma\in G} B z_\sigma = (B, G, (w, g))$ for some graded factor set $(w, g)$ of $G$ in $B$.*

*Proof.* Let $C$ be the centralizer of $Z(D_0)F$ in $D$. Clearly, we have $B \subseteq C$. Moreover, by [HW(2)99, Proposition 1.5] we have $[C : F] = [D : F]/[Z(D_0)F : F] = [D_0 : F_0](\Gamma_D : \Gamma_F)/[Z(D_0) : F_0] = [D_0 : F_0]|ker(\theta_D)| = [B : F]$. Hence, $B = C$. Clearly, we have $\oplus_{\sigma\in G} B z_\sigma = \oplus_{\sigma\in G}(\oplus_{\bar{\gamma}\in ker(\theta_D)} D_0 F z_{\bar{\gamma}}) z_\sigma = \oplus_{\bar{\gamma}\in \Gamma_D/\Gamma_F} D_0 F z_{\bar{\gamma}} = D$.
Let

$$w : G \to GAut(B)_0$$

and

$$g : G \times G \to B^*$$

be the maps defined by $w_\sigma(b) = z_\sigma b z_\sigma^{-1}$ (for any $b \in B$ and $\sigma \in G$) and $g(\sigma, \tau) = z_\sigma z_\tau z_{\sigma\tau}^{-1}$ (for any $\sigma, \tau \in G$). Then, $(w, g)$ is a graded factor set of $G$ in $B$ and $(B, G, (w, g)) = \oplus_{\sigma \in G} B z_\sigma = D$.

**Remark 1.5** Remark that the existence of $(w, g)$ in Lemma 1.4 follows also by the graded version of [T87, Theorem 1.3(b)].

**(1.6)** Now, with the notations of (1.3) let $S = (\bar{\delta}_i := \delta_i + \Gamma_F)_{1 \le i \le r}$ a basis of $H$, $q_i = ord(\bar{\delta}_i)$ for $1 \le i \le r$ and $I = \{(m_1, ..., m_r) \in I\!N^r \mid 0 \le m_i < q_i \text{ for } 1 \le i \le r\}$. We fix a family $(x_i)_{1 \le i \le r}$ of elements of $F^*$ with $gr(x_i) = q_i \delta_i$, and we consider a family $(z_i)_{1 \le i \le r}$ of elements of $D^*$ with $gr(z_i) = \delta_i$. For $\bar{m} = (m_1, ..., m_r) \in I$, we let $\bar{m}\bar{\delta} = \sum_{1 \le i \le r} m_i \bar{\delta}_i$ and $z^{\bar{m}} = \prod_{i=1}^r z_i^{m_i}$. Remark that for any $\bar{\gamma} \in H$, there is a unique element $\bar{m} \in I$ such that $\bar{\gamma} = \bar{m}\bar{\delta}$. Henceforth, for any $\bar{\gamma} = \bar{m}\bar{\delta}$ (where $\bar{m} \in I$), we choose $z_{\bar{\gamma}} = z^{\bar{m}}$. Let $f : H \times H \to (D_0 F)^*$ be the map previously defined in (1.3) by $f(\bar{\gamma}, \bar{\gamma}') = z_{\bar{\gamma}} z_{\bar{\gamma}'} z_{\bar{\gamma}+\bar{\gamma}'}^{-1}$. Then, for any $\bar{m}, \bar{n} \in I$, $f(\bar{m}\bar{\delta}, \bar{n}\bar{\delta}) = z^{\bar{m}} z^{\bar{n}} z^{-\beta(\bar{m}+\bar{n})}$, where $\beta(\bar{m}+\bar{n}) \in I$ with $\bar{m}+\bar{n} \equiv \beta(\bar{m}+\bar{n}) \bmod \prod_{i=1}^r q_i \mathbb{Z}$. Write $m_i + n_i = \beta(\bar{m}+\bar{n})_i + t_i q_i$, where $t_i \in I\!N$, then $f(\bar{m}\bar{\delta}, \bar{n}\bar{\delta}) = d(\bar{m}\bar{\delta}, \bar{n}\bar{\delta}) h(\bar{m}\bar{\delta}, \bar{n}\bar{\delta})$, where $d(\bar{m}\bar{\delta}, \bar{n}\bar{\delta}) \in D_0^*$ and $h(\bar{m}\bar{\delta}, \bar{n}\bar{\delta}) = \prod_{i=1}^r x_i^{t_i}$. Consider the map $\omega$ defined in (1.3), we will denote also by $\omega$ the map : $H \to Aut(D_0)$ defined by $\bar{\gamma} \mapsto \omega_{\bar{\gamma}/D_0}$. We have the following lemma.

**Lemma 1. 7** *$(\omega, d)$ is a factor set of $H$ in $D_0$ and $h \in Z^2(H, F^*)_{sym}$.*

*Proof.* Let $\bar{m}, \bar{n}$ and $\bar{s}$ be elements of $I$. Since $H$ acts trivially on $F^*$, then

$$\bar{m}\bar{\delta} h(\bar{n}\bar{\delta}, \bar{s}\bar{\delta}) h(\bar{m}\bar{\delta}, \bar{n}\bar{\delta} + \bar{s}\bar{\delta}) = h(\bar{n}\bar{\delta}, \bar{s}\bar{\delta}) h(\bar{m}\bar{\delta}, \beta(\bar{n} + \bar{s})\bar{\delta}) = (\prod_{i=1}^r x_i{}^{\lambda_i})(\prod_{i=1}^r x_i{}^{\gamma_i})$$

where $\lambda_i = \frac{1}{q_i}(n_i + s_i - \beta(\bar{n} + \bar{s})_i)$ and $\gamma_i = \frac{1}{q_i}(m_i + \beta(\bar{n} + \bar{s})_i - \beta(\bar{m} + \beta(\bar{n} + \bar{s}))_i)$. We have $\beta(\bar{m} + \beta(\bar{n} + \bar{s})) = \beta(\bar{m} + \bar{n} + \bar{s})$, hence

$$\bar{m}\bar{\delta} h(\bar{n}\bar{\delta}, \bar{s}\bar{\delta}) h(\bar{m}\bar{\delta}, \bar{n}\bar{\delta} + \bar{s}\bar{\delta}) = (\prod_{i=1}^r x_i^{\xi_i}).$$

where $\xi_i = \frac{1}{q_i}m_i + n_i + s_i - \beta(\bar{m} + \bar{n} + \bar{s})_i$.

Likewise, we have :

$$h(\bar{m}\bar{\delta}, \bar{n}\bar{\delta})h(\bar{m}\bar{\delta} + \bar{n}\bar{\delta}, \bar{s}\bar{\delta}) = \prod_{i=1}^{r} x_i^{\xi_i}.$$

Moreover, it is clear that $h(\bar{m}\bar{\delta}, \bar{n}\bar{\delta}) = h(\bar{n}\bar{\delta}, \bar{m}\bar{\delta})$. Hence, $h \in Z^2(H, F^*)_{sym}$. The fact that $(\omega, f)$ is a graded factor set of $H$ in $D_0F$ and that $h \in Z^2(H, F^*)_{sym}$ imply $(\omega, d)$ is a factor set of $H$ in $D_0$.

**Remark 1.8** If $D$ is a semiramified graded division algebra over $F$, then using the same arguments as in the proof of Lemma 1.7, we prove that $d \in Z^2(H, D_0^*)$ (see that in this case $H \cong Gal(D_0/F_0)$).

## 2 Kummer graded subfields of graded division algebras

**(2.1)** Let $F$ be a graded field and $K$ is a finite-dimensional abelian graded field extension of $F$ (i.e., such that $Frac(K)/Frac(F)$ is an abelian Galois field extension [see HW(1)99]). We say that $K$ is a Kummer graded field extension of $F$ if $F_0$ contains a primitive $m^{th}$ root of unity, where $m$ is the exponent of $Gal(K/F)$. In such a case, as for ungraded Kummer field extensions, we set $KUM(K/F) = \{x \in K^* \mid x^m \in F\}$ and $kum(K/F) = KUM(K/F)/F^*$. One can easily see that $kum(K/F)$ is isomorphic to $Gal(K/F)$.

Now, let $K$ be a Kummer graded field extension of $F$, then we have $K = F[a \mid a \in KUM(K/F)]$, so $\Gamma_K/\Gamma_F$ is generated by $\{gr(a) + \Gamma_F \mid a \in KUM(K/F)\}$, therefore the group homomorphism $\psi : kum(K/F) \to \Gamma_K/\Gamma_F$, defined by $\psi(aF^*) = gr(a)+\Gamma_F$, for $a \in KUM(K/F)$, is surjective. Let $\phi : kum(K_0/F_0) \to kum(K/F)$ be the group homomorphism defined by $\phi(aF_0^*) = aF^*$, for every $a \in KUM(K_0/F_0)$. Clearly, $\phi$ is injective and $\psi \circ \phi = 0$. By comparing the cardinalities, we conclude that the following sequence of trivial $\Gamma_K/\Gamma_F$-modules :

$$\alpha_K : 1 \to kum(K_0/F_0) \xrightarrow{\phi} kum(K/F) \xrightarrow{\psi} \Gamma_K/\Gamma_F \to 0$$

is exact. Remark that since $kum(K/F)$ is abelian, then $\alpha_K \in Z^2(\Gamma_K/\Gamma_F, kum(K_0/F_0))_{sym}$.

**(2.2)** With the notations of (2.1), we have $KUM(K/F) \cap D_0 = KUM(K_0/F_0)$. Indeed, let $a \in KUM(K/F) \cap D_0$, then $\psi(aF^*) = 0$, so $aF^* \in im(\phi)$. Hence there is $b \in KUM(K_0/F_0)$ such that $aF^* = bF^*$. Since both $a$ and $b$ are in $D_0^*$, then $ab^{-1} \in F_0^* (= D_0^* \cap F^*)$. So, $a \in KUM(K_0/F_0)$. This shows that $KUM(K/F) \cap D_0 \subseteq KUM(K_0/F_0)$. The converse inclusion is trivial.

**2.3 Notations :** We precise here some notations needed for the next result :
(a) Let $e : KUM(K_0/F_0) \to kum(K_0/F_0)$ be the canonical surjective homomorphism. We denote by $e_* : H^2(\Gamma_K/\Gamma_F, KUM(K_0/F_0))_{sym} \to H^2(\Gamma_K/\Gamma_F, kum(K_0/F_0))_{sym}$ the corresponding homomorphism of cohomology groups (for the trivial action of $\Gamma_K/\Gamma_F$ on $KUM(K_0/F_0)$ and on $kum(K_0/F_0)$).
(b) Let $(\omega, d)$ be the factor set of $H$ in $D_0$ previously seen in Lemma 1.7, we denote by $res^H_{\Gamma_K/\Gamma_F}(\omega, d)$ its restriction when considering $\Gamma_K/\Gamma_F$ instead of $H$.
Obviously, $res^H_{\Gamma_K/\Gamma_F}(\omega, d)$ is a factor set of $\Gamma_K/\Gamma_F$ in $D_0$.
(c) Let $i : KUM(K_0/F_0) \to D_0^*$ be the inclusion map. For a cocycle $h \in Z^2(\Gamma_K/\Gamma_F, KUM(K_0/F_0))$ we denote by $i_*h$ the map : $\Gamma_K/\Gamma_F \times \Gamma_K/\Gamma_F \to D_0^*$, $(\bar{\gamma}, \bar{\gamma}') \mapsto i \circ h(\bar{\gamma}, \bar{\gamma}')$.

**Theorem 2. 4** *Let $F$ be a graded field, $D$ a graded central division algebra over $F$, $(\omega, d)$ the factor set of $\Gamma_D/\Gamma_F$ in $D_0$ seen in Lemma 1.7, $K$ a Kummer graded subfield of $D$ and $\alpha_K$ the cocycle of $Z^2(\Gamma_K/\Gamma_F, kum(K_0/F_0))_{sym}$ defined in (2.1), then there exists a cocycle $d' \in Z^2(\Gamma_K/\Gamma_F, KUM(K_0/F_0))_{sym}$ (for the trivial action of $\Gamma_K/\Gamma_F$ on $KUM(K_0/F_0)$) and a map $\omega' : \Gamma_K/\Gamma_F \to Aut(D_0)$ which satisfies $\omega'_{\bar{\gamma}}(a) = a$ for all $a \in K_0$ and $\bar{\gamma} \in \Gamma_K/\Gamma_F$, such that :*
*1. $(\omega', i_*d')$ is a factor set of $\Gamma_K/\Gamma_F$ in $D_0$ cohomologous to $res^{\Gamma_D/\Gamma_F}_{\Gamma_K/\Gamma_F}(\omega, d)$, and*
*2. $e_*([d']) = [\alpha_K]$.*

*Proof.* Let $H = \Gamma_D/\Gamma_F$ and write $D = \oplus_{\bar{\gamma} \in H} D_0 F x_{\bar{\gamma}}$, where $x_{\bar{\gamma}} a = \omega_{\bar{\gamma}}(a) x_{\bar{\gamma}}$ and $x_{\bar{\gamma}} x_{\bar{\gamma}'} = d(\bar{\gamma}, \bar{\gamma}') h(\bar{\gamma}, \bar{\gamma}') x_{\bar{\gamma}+\bar{\gamma}'}$ (where $h$ is the cocycle of $Z^2(\Gamma_D/\Gamma_F, F^*)_{sym}$ seen in Lemma 1.7). For any $\gamma \in \Gamma_K$, let $y_{\bar{\gamma}} \in KUM(K/F)$ such that $gr(y_{\bar{\gamma}}) + \Gamma_F = \bar{\gamma}$

$(= \gamma + \Gamma_F)$ and write $y_{\bar\gamma} = a_{\bar\gamma} x_{\bar\gamma}$, where $a_{\bar\gamma} \in (D_0F)^*$. Let $b_{\bar\gamma} \in D_0^*$ and $c_{\bar\gamma} \in F^*$ such that $a_{\bar\gamma} = b_{\bar\gamma} c_{\bar\gamma}$, then we have :

$$\begin{aligned} y_{\bar\gamma} y_{\bar\gamma'} &= a_{\bar\gamma} \omega_{\bar\gamma}(a_{\bar\gamma'}) d(\bar\gamma, \bar\gamma') a^{-1}_{\bar\gamma+\bar\gamma'} h(\bar\gamma, \bar\gamma') y_{\bar\gamma+\bar\gamma'} \\ &= b_{\bar\gamma} \omega_{\bar\gamma}(b_{\bar\gamma'}) d(\bar\gamma, \bar\gamma') b^{-1}_{\bar\gamma+\bar\gamma'} c_{\bar\gamma} c_{\bar\gamma'} c^{-1}_{\bar\gamma+\bar\gamma'} h(\bar\gamma, \bar\gamma') y_{\bar\gamma+\bar\gamma'} \\ &= d'(\bar\gamma, \bar\gamma') h'(\bar\gamma, \bar\gamma') y_{\bar\gamma+\bar\gamma'} \end{aligned}$$

where $d'(\bar\gamma, \bar\gamma') = b_{\bar\gamma} \omega_{\bar\gamma}(b_{\bar\gamma'}) d(\bar\gamma, \bar\gamma') b^{-1}_{\bar\gamma+\bar\gamma'}$ and $h'(\bar\gamma, \bar\gamma') = c_{\bar\gamma} \bar c_{\bar\gamma'} c^{-1}_{\bar\gamma+\bar\gamma'} h(\bar\gamma, \bar\gamma')$. Since $y_{\bar\gamma}$, $y_{\bar\gamma'}$ and $y_{\bar\gamma+\bar\gamma'}$ are in $KUM(K/F)$ and $h'(\bar\gamma, \bar\gamma') \in F^*$, then $d'(\bar\gamma, \bar\gamma') \in KUM(K/F) \cap D_0$ $(= KUM(K_0/F_0))$. One can easily check that $d' \in Z^2(\Gamma_K/\Gamma_F, KUM(K_0/F_0))_{sym}$ (this follows from the equality $(y_{\bar\gamma} y_{\bar\gamma'}) y_{\bar\gamma''} = y_{\bar\gamma}(y_{\bar\gamma'} y_{\bar\gamma''})$, the fact that $h' \sim res^H_{\Gamma_K/\Gamma_F}(h)$ is a symmetric 2-cocycle and the fact that $y_{\bar\gamma}$ are pairwise commuting for $\bar\gamma \in \Gamma_K/\Gamma_F$). Now, let $\omega' : \Gamma_K/\Gamma_F \to Aut(D_0)$ be the map defined by $\omega'_{\bar\gamma} = Inn(b_{\bar\gamma}) \omega_{\bar\gamma}$ (i.e., $\omega'_{\bar\gamma}(a) = b_{\bar\gamma} \omega_{\bar\gamma}(a) b^{-1}_{\bar\gamma}$ for all $a \in D_0$ and $\bar\gamma \in \Gamma_K/\Gamma_F$). Then, for any $a \in K_0$ and any $\bar\gamma \in \Gamma_K/\Gamma_F$, we have $\omega'_{\bar\gamma}(a) = b_{\bar\gamma} x_{\bar\gamma} a x^{-1}_{\bar\gamma} b^{-1}_{\bar\gamma} = a_{\bar\gamma} x_{\bar\gamma} a x^{-1}_{\bar\gamma} a^{-1}_{\bar\gamma} = y_{\bar\gamma} a y^{-1}_{\bar\gamma} = a$. One can easily see that $(\omega', i_* d')$ is a factor set of $\Gamma_K/\Gamma_F$ in $D_0$ cohomologous to $res^H_{\Gamma_K/\Gamma_F}(\omega, d)$. Moreover, the equality $y_{\bar\gamma} y_{\bar\gamma'} = d'(\bar\gamma, \bar\gamma') h'(\bar\gamma, \bar\gamma') y_{\bar\gamma+\bar\gamma'}$ yields, by considering classes modulo $F^*$ in $kum(K/F)$, $\bar y_{\bar\gamma} \bar y_{\bar\gamma'} = e(d'(\bar\gamma, \bar\gamma')) \bar y_{\bar\gamma+\bar\gamma'}$, where $e : KUM(K_0/F_0) \to kum(K_0/F_0)$ is the canonical surjective homomorphism (we identify here $kum(K_0/F_0)$ with its canonical image in $kum(K/F)$). Hence, $e_*([d']) = [\alpha_K]$.

**(2.5)** Let $F$ be a graded field, $D$ a graded division algebra over $F$, $A$ a finite abelian subgoup of $D^*/F^*$ with exponent $m$, and for any $a \in A$, let $d_a$ be a representative of $a$ in $D^*$. Assume that $F_0$ contains a primitive $m^{th}$ root of unity and let $F(A) = F[d_a \mid a \in A]$ be the subring of $D$ generated by $F$ and the elements $d_a$ $(a \in A)$. If $d_a$ are pairwise commuting, then as in the ungraded case $F(A)$ is a Kummer graded field extension of $F$ with $kum(F(A)) = A$ (it suffices to see that $F(A)$ is a graded field and that $Frac(F(A)) = Frac(F)(A)$ when $A$ is identified with its canonical image in $Cq(D)^*/Frac(F)^*$).

Conversely to Theorem 2.4, we have the following Theorem.

**Theorem 2. 6** *Let $F$ be a graded field, $D$ a graded central division algebra over $F$ and $(\omega, d)$ the factor set of $\Gamma_D/\Gamma_F$ in $D_0$ seen in Lemma 1.7. Assume $F_0$ contains*

*enough roots of unity and that there are :*
*1. a field extension $M$ of $F_0$ in $D_0$, and a subgroup $R$ of $\Gamma_D/\Gamma_F$ acting trivially on $M$,*
*2. a cocycle $d' \in Z^2(R, KUM(M/F_0))_{sym}$ and a map $\omega' : R \to Aut(D_0)$ such that $(\omega', i_*d')$ is a factor set of $R$ in $D_0$ cohomologous to $res_R^{\Gamma_D/\Gamma_F}(\omega, d)$ and such that $\omega'_{\bar{\gamma}}(a) = a$ for all $a \in M$ and $\bar{\gamma} \in R$.*
*Then, there exists a Kummer graded subfield $K$ of $D$ such that :*
*1. $K_0 = M$, $\Gamma_K/\Gamma_F = R$ and*
*2. $e_*([d']) = [\alpha_K]$.*

*Proof.* Let's denote by $H$ the quotient group $\Gamma_D/\Gamma_F$ and write $D = \oplus_{\bar{\gamma}\in H} D_0 F x_{\bar{\gamma}}$, where $x_{\bar{\gamma}} a = \omega_{\bar{\gamma}}(a) x_{\bar{\gamma}}$ and $x_{\bar{\gamma}} x_{\bar{\gamma}'} = d(\bar{\gamma}, \bar{\gamma}') h(\bar{\gamma}, \bar{\gamma}') x_{\bar{\gamma}+\bar{\gamma}'}$ ($h$ being the cocycle of $Z^2(H, F^*)_{sym}$ seen in Lemma 1.7). The fact that $(\omega', i_*d')$ is cohomologous to $res_R^H(\omega, d)$ means that there is a family $(b_{\bar{\gamma}})_{\bar{\gamma}\in R}$ of elements of $D_0^*$ such that for all $a \in D_0$ and $\bar{\gamma}, \bar{\gamma}' \in R$, we have $\omega'_{\bar{\gamma}}(a) = b_{\bar{\gamma}} \omega_{\bar{\gamma}}(a) b_{\bar{\gamma}}^{-1}$ and $d'(\bar{\gamma}, \bar{\gamma}') = b_{\bar{\gamma}} \omega_{\bar{\gamma}}(b_{\bar{\gamma}'}) d(\bar{\gamma}, \bar{\gamma}') b_{\bar{\gamma}+\bar{\gamma}'}^{-1}$. Let $y_{\bar{\gamma}} = b_{\bar{\gamma}} x_{\bar{\gamma}}$ for all $\bar{\gamma} \in R$. Then, we have $y_{\bar{\gamma}} y_{\bar{\gamma}'} = d'(\bar{\gamma}, \bar{\gamma}') h(\bar{\gamma}, \bar{\gamma}') y_{\bar{\gamma}+\bar{\gamma}'}$. Let $K = \oplus_{\bar{\gamma}\in R} MF y_{\bar{\gamma}} (\subseteq D)$. Since $d'$ and $h$ are symmetric, then $y_{\bar{\gamma}}$ are pairwise commuting. Moreover, by hypotheses $\omega'_{\bar{\gamma}}(a) = a$ for all $a \in M$ and $\bar{\gamma} \in R$, so $K$ is a commutative graded subring (hence a graded subfield) of $D$.
Let $A$ be the subgroup of $D^*/F^*$ generated by $kum(M/F_0)$ and the set $\{\bar{y}_{\bar{\gamma}}\}_{\bar{\gamma}\in R}$. One can easily see that up to a graded isomorphism we have $K = F(A)$. Therefore, $K$ is a Kummer graded field extension of $F$ with $kum(K/F) = A$. Considering classes in $kum(K/F)$, we have $\bar{y}_{\bar{\gamma}} \bar{y}_{\bar{\gamma}'} = e(d'(\bar{\gamma}, \bar{\gamma}')) \bar{y}_{\bar{\gamma}+\bar{\gamma}'}$, where $e : KUM(M/F_0) \to kum(M/F_0)$ is the canonical surjective homomorphism (we identify here $kum(M/F_0)$ with its canonical image in $kum(K/F)$), so $kum(K/F)$ is the extension of $kum(M/F_0)$ by $R$ with cocycle $e_*([d'])$.

**(2.7)** Let $F$ be a graded field, $D$ a semiramified graded division algebra over $F$ and $G = Gal(D_0/F_0)$. We know that $\Gamma_D/\Gamma_F \cong G$. Therefore, any subgroup of

$\Gamma_D/\Gamma_F$ can be identified to a subgoup of $G$. Let's consider the following diagram :

$$\begin{array}{ccc} H^2(\Gamma_K/\Gamma_F, KUM(K_0/F_0))_{sym} & \xrightarrow{i_*} & H^2(\Gamma_K/\Gamma_F, D_0^*) \\ e_* \downarrow & & \uparrow res^G_{\Gamma_K/\Gamma_F} \\ H^2(\Gamma_K/\Gamma_F, kum(K_0/F_0))_{sym} & & H^2(G, D_0^*) \end{array}$$

where $i_*$ is the homomorphism of cohomology groups induced by the inclusion map $KUM(K_0/F_0) \xrightarrow{i} D_0^*$, $e_*$ is the homomorphism of cohomology groups induced by the canonical surjective homomorphism $e : KUM(K_0/F_0) \to kum(K_0/F_0)$, and $res^G_{\Gamma_K/\Gamma_F}$ is the restriction map. As a consequence of Theorem 2.4, we have the following Corollary :

**Corollary 2. 8** *Let $F$ be a graded field, $D$ a semiramified graded division algebra over $F$, $G = Gal(D_0/F_0)$, $d$ the cocycle of $Z^2(G, D_0^*)$ seen in Remark 1.8, $K$ a Kummer graded subfield of $D$ and $\alpha_K$ the cocycle of $Z^2(\Gamma_K/\Gamma_F, kum(K_0/F_0))_{sym}$ defined in (2.1), then there exists a cocycle $d' \in Z^2(\Gamma_K/\Gamma_F, KUM(K_0/F_0))_{sym}$ such that :*
*(1) $i_*([d']) = res^G_{\Gamma_K/\Gamma_F}([d])$, and*
*(2) $e_*([d']) = [\alpha_K]$.*

Also, as a consequence of Theorem 2.6, we have the following Corollary.

**Corollary 2. 9** *Let $F$ be a graded field, $D$ a semiramified graded division algebra over $F$ and $d \in Z^2(G, D_0^*)$ the cocycles seen in Remark 1.8. Assume $F_0$ contains enough roots of unity and suppose there exist : a subfield $M$ of $D_0$ containing $F_0$, a subgroup $R$ of $\Gamma_D/\Gamma_F$ acting trivially on $M$, and a cocycle $d' \in Z^2(G, KUM(M/F_0))_{sym}$ such that $i_*([d']) = res^G_R([d])$. Then, there exists a Kummer graded subfield $K$ of $D$ such that :*
*(1) $M = K_0$, $R = \Gamma_K/\Gamma_F$, and*
*(2) $[\alpha_K] = e_*([d'])$.*

**(2.10)** Now let $E$ be a Henselian valued field and $D$ a tame central division algebra over $E$ such that $char(\bar{E})$ does not divide $deg(D)$. Since $GD$ is a graded central division algebra over $GE$, then we can define a graded factor set $(\omega, d)$ corresponding to $GD$ as made in Lemma 1.7. If $K$ is a Kummer subfield of $D$, then by [HW(1), Theorem 5.2] $GK$ is a Kummer graded subfield of $GD$. So, we can consider the symmetric cocycle $\alpha_{GK}$ of (2.1) corresponding to $GK$. For simplicity, we denote $\alpha_{GK}$ just by $\alpha_K$. As a direct consequence of Theorem 2.4, we have the following Corollary

**Corollary 2. 11** *Let $E$ be a Henselian valued field and $D$ a tame central division algebra over $E$ such that $char(\bar{E})$ does not divide $deg(D)$. Using the notations of (2.10), if $K$ is a Kummer subfield of $D$, then there is a cocycle $d' \in Z^2(\Gamma_K/\Gamma_E, KUM(\bar{K}/\bar{E}))_{sym}$ (for the trivial action of $\Gamma_K/\Gamma_E$ on $KUM(\bar{K}/\bar{E})$) and a map $\omega' : \Gamma_K/\Gamma_E \to Aut(\bar{D})$ which satisfies $\omega'_{\bar{\gamma}}(a) = a$ for all $a \in \bar{K}$ and $\bar{\gamma} \in \Gamma_K/\Gamma_E$, such that :*
*1. $(\omega', i_*d')$ is a factor set of $\Gamma_K/\Gamma_E$ in $\bar{D}$ cohomologous to $res^{\Gamma_D/\Gamma_E}_{\Gamma_K/\Gamma_E}(\omega, d)$, and*
*2. $e_*([d']) = [\alpha_K]$.*

Also, as a consequence of Theorem 2.6, we have the following Corollary :

**Corollary 2. 12** *Let $E$ be a Henselian valued field and $D$ a tame central division algebra over $E$ such that $char(\bar{E})$ does not divide $deg(D)$. Assume that $\bar{E}$ contains enough roots of unity and that (with the notations of (2.10)), there are :*
*1. a field extension $M$ of $\bar{E}$ in $\bar{D}$, and a subgroup $R$ of $\Gamma_D/\Gamma_E$ acting trivially on $M$,*
*2. a cocycle $d' \in Z^2(R, KUM(M/\bar{E}))_{sym}$ and a map $\omega' : R \to Aut(\bar{D})$ such that $(\omega', i_*d)$ is a factor set of $R$ in $\bar{D}$ cohomologous to $res^{\Gamma D/\Gamma E}_{R}(\omega, d)$ and such that $\omega'_{\bar{\gamma}}(a) = a$ for all $a \in M$ and $\bar{\gamma} \in R$.*
*Then, there exists a Kummer subfield $K$ of $D$ such that :*
*1. $\bar{K} = M$, $\Gamma_K/\Gamma_E = R$ and*
*2. $e_*([d']) = [\alpha_K]$.*

**Remark 2.13** (1) In the last two corollaries, we can use the group isomorphism $kum(K/E) \cong kum(GK/GE)$ and replace the exact sequence of trivial $\Gamma_K/\Gamma_E$-modules $\alpha_{GK}$ by another exact sequence of trivial $\Gamma_K/\Gamma_E$-modules

$$1 \to kum(\bar{K}/\bar{E}) \xrightarrow{\phi} kum(K/E) \xrightarrow{\psi} \Gamma_K/\Gamma_E \to 0$$

then use it to have necessary and sufficient condition for $D$ to have Kummer subfields.
(2) We have also analogous results to Corollary 2.8 and Corollary 2.9 for tame semi-

ramified division algebras over Henselian valued fields.
(3) We can drop the assumption that $E$ is Henselian in many results of this paper. Indeed, let $D$ be a valued central division algebra over a field $E$, $HE$ be the Henselization of $D$ with respect to the restriction of the valuation of $D$ and $HD = D \otimes_E HE$. Then, one can easily see that $GD = G(HD)$ and $GE = G(HE)$.

**Theorem 2. 14** *Let $F$ be a graded field, $D$ a semiramified graded division algebra over $F$ and $d$ the cocycle seen in Remark 1.8. If $F_0$ contains a primitive $deg(D)^{th}$ root of unity, then the following statements are equivalent :*
*(1) $D$ is cyclic,*
*(2) There is a field extension $M$ of $F_0$ in $D_0$ such that :*

*(i) the extensions $M/F_0$ and $D_0/M$ are cyclic, and*

*(ii) $(D_0/F_0, G, d) \otimes_{F_0} M \sim (D_0/M, \sigma, u)$ for some generator $\sigma$ of $Gal(D_0/M)$ and some $u \in M^*$ such that $uF_0^*$ generates $kum(M/F_0)$.*

*Proof.* This can be proved in the same way as [T86, Theorem 3.1].

**Theorem 2. 15** *Let $F$ be a graded field, $D$ a semiramified graded division algebra over $F$ and $d$ the cocycle seen in Remark 1.8. Suppose now that $deg(D)$ is a power of a prime $p$ and that $F_0$ contains a primitive $p^{th}$ root of unity. Then, the following statements are equivalent*
*(1) $D$ is an elementary abelian graded crossed product,*
*(2) there is a field extension $M$ of $F_0$ in $D_0$ such that $M/F_0$ and $D_0/M$ are elementary abelian, and $(D_0/F_0, G, d)$ represents in $Br(D_0/F_0)/Dec(D_0/F_0)$ an element of the image of the canonical group homomorphism $Br(M/F_0)/Dec(M/F_0) \to Br(D_0/F_0)/Dec(D_0/F_0)$,*
*(3) $exp(G) = p$ or $p^2$ and $(D_0/F_0, G, d)$ represents in $Br(D_0/F_0)/Dec(D_0/F_0)$ an element of the image of the canonical group homomorphism $Br(L/F_0)/Dec(L/F_0) \to Br(D_0/F_0)/Dec(D_0/F_0)$, where $L = Fix_{G^p}(D_0)$ ($G^p$ being the subgoup of $G$ consisting in $p$-powers of elements of $G$) (this last condition is void if $exp(G) = p$ since in this case $L = K$.)*

*Proof.* This can be proved in the same way as [T86, Theorem 4.1].

**Proposition 2. 16** *Let $E$ be a Henselian valued field, $D$ a division algebra over $E$ such that $char(\bar{E})$ does not divide $deg(D)$ and $H$ a finite group. Then, $D$ has a tame Galois subfield with Galois group isomorphic to $H$ if and only if $GD$ has a Galois graded subfield of Galois group isomorphic to $H$. Therefore, $D$ is cyclic [resp., an elementary abelian crossed product] if and only if $GD$ is cyclic [resp., an elementary abelian graded crossed product].*

*Proof.* Assume that $D$ has a Galois subfield of Galois group isomorphic to $H$, then by [HW(1), Theorem 5.2] $GK$ is a Galois graded subfield of $GD$ with Galois group isomorphic to $H$. Conversely, assume that $GD$ has a Galois graded subfield $L$ with Galois group isomorphic to $H$. Then, again by [HW(1), Theorem 5.2] there is a tame field extension $M$ of $E$ such that $GM \cong L$ and $Gal(M/E) \cong H$. By [HW(2)99, Theorem 5.9] $M$ is isomorphic to a subfield of $D$.

**Remark.** We recall that if $E$ is a Henselian valued field and $D$ is an inertially split division algebra over $E$ with $\bar{D}$ commutative, then $D$ is a tame semiramified division algebra over $E$ (see [M07, Proposition 2.6]). The reader can then see that similar results to Theorem 2.14, Theorem 2.15 in the case of tame semiramified division algebras over a Henselian valued field were proved in [MorSe95]. Using Theorem 2.14, Theorem 2.15, we get the next two Corollaries of [MorSe95]. In the next section, we will prove these two corollaries without assuming that $\bar{E}$ contains primitive roots of unity.

**Corollary 2. 17** [*MorSe95, Corollary 5.5*] *Let $E$ be a Henselian valued field and $D$ a tame semiramified division algebra of prime power degree over $E$. Suppose that $char(\bar{E})$ does not divide $deg(D)$ and $\bar{E}$ contains a primitive $deg(D)^{th}$ root of unity and that $rk(\Gamma_D/\Gamma_E) \geq 3$, then $D$ is non-cyclic.*

*Proof.* We have $rk(Gal(GD_0/GE_0)) = rk(Gal(\bar{D}/\bar{E})) = rk(\Gamma_D/\Gamma_E) \geq 3$. So by Theorem 2.14(2(i)) $GD$ is non-cyclic. Hence, by Proposition 2.16, $D$ is non-cyclic.

**Corollary 2. 18** [*MorSe95, Corollary 5.7*] *Let $E$ be a Henselian valued field and $D$ a tame semiramified division algebra of prime power degree $p^n$ over $E$ ($p$ being a prime integer and $n \in I\!N^*$). Suppose that $\bar{E}$ contains a primitive $p^{th}$ root of unity and that $p^3$ divides $exp(\Gamma_D/\Gamma_E)$, then $D$ has no elementary abelian maximal subfield.*

*Proof.* This follows by Theorem 2.15 and Proposition 2.16.

# 3 Non-cyclic and non-elementary abelian crossed product tame semiramified division algebras

Let $E$ be a Henselian valued field and $D$ a tame semiramified division algebra of prime power degree $p^n$ over a Henselian valued field $E$ such that $char(\bar{E}) \neq p$. In this section, we aim to show that if $rk(\Gamma_D/\Gamma_E) \geq 3$, then $D$ is non-cyclic [Proposition 3.1], and that if $p^3$ divides $exp(\Gamma_D/\Gamma_F)$, then $D$ has no elementary abelian maximal subfield [Proposition 3.2].

**Proposition 3. 1** *Let $E$ be a Henselian valued field and $D$ a semiramified division algebra of degree $n$ over $E$. Assume $char(\bar{E})$ does not divide $n$ and suppose $K$ is a cyclic maximal subfield of $D$. Then, $\Gamma_K/\Gamma_E$ and $\Gamma_D/\Gamma_K$ are cyclic. So, $\Gamma_D/\Gamma_E$ is generated by two elements. In particular, if $n$ is a prime power and $rk(\Gamma_D/\Gamma_E) \geq 3$, then $D$ is non-cyclic.*

*Proof.* Let $M$ be the inertial lift of $\bar{K}$ over $E$ in $K$ (see [JW90, Theorem 2.8 and Theorem 2.9]). Since $K$ is cyclic and totally ramified over $M$, then $\Gamma_K/\Gamma_E(= \Gamma_K/\Gamma_M)$ is cyclic. Furthermore, we have $\Gamma_D/\Gamma_K \cong (\Gamma_D/\Gamma_E)/(\Gamma_K/\Gamma_E) \cong Gal(\bar{D}/\bar{E})/Gal(\bar{D}/\bar{K}) \cong Gal(\bar{K}/\bar{E}) \cong Gal(M/E)$ (for the second equivalence, see that $K$ is a totally ramified maximal subfield of the semiramified division algebra $C_D^M$). So, $\Gamma_D/\Gamma_K$ is cyclic. Let $\gamma_1 + \Gamma_E$ be a generator of $\Gamma_K/\Gamma_E$ and $\gamma_2 + \Gamma_K$ a generator of $\Gamma_D/\Gamma_K$, then for any $\alpha \in \Gamma_D/\Gamma_E$, there are positive integers $n_1$ and $n_2$ such that $\alpha = n_1\gamma_1 + n_2\gamma_2 + \Gamma_E$. If $n$ is a prime power, then $rk(\Gamma_D/\Gamma_E) \leq 2$.

**Proposition 3. 2** *Let $E$ be a Henselian valued field and $D$ a tame semiramified division algebra of prime power degree $p^n$ over $E$ ($p$ being a prime integer and $n \in \mathbb{N}^*$). If $char(\bar{E}) \neq p$ and $p^3$ divides $exp(Gal(\bar{D}/\bar{E}))$, then $D$ has no elementary abelian maximal subfield.*

*Proof.* Suppose that $K$ is an elementary abelian maximal subfield of $D$, then $\bar{K}/\bar{E}$ is elementary abelian. Therefore, for any $\sigma \in Gal(\bar{D}/\bar{E})$, $\sigma^p \in Gal(\bar{D}/\bar{K})$. Let $M$ be the inertial lift of $\bar{K}$ over $E$ in $K$. Then, $K$ is a Galois totally ramified field extension of $M$ and $Gal(K/M) \cong \Gamma_K/\Gamma_M$. Moreover, since $C_D^M$ is tame semiramified, then $Gal(\bar{D}/\bar{K}) = Gal(\bar{D}/\bar{M}) \cong \Gamma_K/\Gamma_M (\cong Gal(K/M))$. Hence, $\sigma^{p^2} = id_{\bar{D}}$. A contradiction.

**Remark 3.3** (1) We recall that we saw in [M07, Proposition 4.6] that if $E$ is a Henselian valued field and $D$ is a nondegenerate tame semiramified division algebra of prime power degree over $E$, then $D$ has an elementary abelian maximal subfield if and only if $\Gamma_D/\Gamma_F$ is elementary abelian.
(2) As showed in [T86] with Malcev-Neumann division algebras, one can use Proposition 3.1 and Proposition 3.2 to prove the following result : Let $m$ and $n$ be integers which have the same prime factors and such that $m$ divides $n$, and let $k$ be an infinite field. If there is a prime $p \neq char(k)$ such that $p^2$ divides $m$ and $p^3$ divides $n$, then Saltman's universal division algebras of exponent $m$ and degree $n$ over $k$ are not crossed products.

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