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Bloch-Ogus Sequence in Degree Two

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1 Introduction

Let X be a smooth variety. Let n be an integer prime to the characteristic of the base field and r be any integer. Let $X^{(d)}$ be the set of codimension d points of X, and denote by $\Bbbk(x)$ the residue field of $x \in X$. Define

$$\mathcal{H}^q(X, \mu_n^{\otimes r}) \stackrel{\text{df}}{=} \text{Zariski sheaf associated to the presheaf } U \mapsto H^q_{\acute{e}t}(U, \mu_n^{\otimes r})$$

In the seminal paper [BO74], Bloch and Ogus showed that filtration by codimension gives rise to an exact sequence of Zariski sheaves

$$0 \to \mathcal{H}^q(X, \mu_n^{\otimes r}) \to \bigoplus_{x \in X^{(0)}} i_{x*} H^q_{\acute{e}t}(\Bbbk(x), \mu_n^{\otimes r}) \to \bigoplus_{x \in X^{(1)}} i_{x*} H^{q-1}_{\acute{e}t}(\Bbbk(x), \mu_n^{\otimes (r-1)}) \to \cdots \to \bigoplus_{x \in X^{(q)}} i_{x*} H^0_{\acute{e}t}(\Bbbk(x), \mu_n^{\otimes (r-q)}) \to 0$$

Here $i_x \colon \operatorname{Spec} \Bbbk(x) \to X$ denotes the inclusion map of x. The étale cohomology groups $H^i_{\acute{e}t}(\Bbbk(x), \mu_n^{\otimes r})$ are viewed as constant sheaves on $\operatorname{Spec} \Bbbk(x)$.

The following result follows directly. Let A be a 2-dimensional regular local ring which is essentially of finite type over a field. Denote by k its residue field and write $K = \operatorname{Frac} A$. Let n be an integer prime to $\operatorname{char} k$ and r be any integer. Then the sequence

$$0 \to H^2_{\acute{e}t}(A, \mu_n^{\otimes r}) \to H^2_{\acute{e}t}(K, \mu_n^{\otimes r}) \to \bigoplus_{\operatorname{ht} \mathfrak{p} = 1} H^1_{\acute{e}t}(\Bbbk(\mathfrak{p}), \mu_n^{\otimes (r-1)}) \to \mu_n^{\otimes (r-2)}(k) \to 0$$

is exact, where the sum runs over the height 1 prime ideals of A.

The purpose of this paper is to show that the last sequence remains exact when A is allowed to be any 2-dimensional excellent regular local ring, $r = 2$, and n is prime to the characteristic of the residue field. Some parts of the proof work for arbitrary r and regular local rings of arbitrary dimension. When A contains all the n-th roots of unity we obtain as a corollary an exact sequence expressing the relation between the n-torsion of the Brauer groups of A and K and the character groups of $\Bbbk(\mathfrak{p})$:

$$0 \to {}_n \operatorname{Br}(A) \to {}_n \operatorname{Br}(K) \to \bigoplus_{\operatorname{ht} \mathfrak{p} = 1} H^1\big(\Bbbk(\mathfrak{p}), \mathbb{Z}/n\big) \to \mu_n^{-1} \to 0$$

Using different methods, the exactness of this sequence of Brauer groups is also shown by Saltman ([`saltman`], Theorem 6.12) even without the assumption of roots of unity on A.

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2 Conventions and background

Unless otherwise stated all cohomology groups are étale cohomology groups. As above, we write $X^{(d)}$ for the set of codimension d points of a scheme X and $\Bbbk(x)$ for the residue field of $x \in X$. For an integer n invertible in X (i.e. prime to all the residue characteristics of X), we write μ_n for the étale sheaf of n-th roots of unity; if r is an arbitrary integer, we denote by $\mu_n^{\otimes r}$ the r-th Tate twist of μ_n, as defined in [Mil80], p. 78.

If R is a local ring, we denote its henselization and strict henselization by R_h and R_{sh}, respectively. When R is a normal local domain, both R_h and R_{sh} are domains ([Mil80], I.4.10, p. 37) and we write $K_h = \operatorname{Frac} R_h$ and $K_{sh} = \operatorname{Frac} R_{sh}$ for their fields of fractions. Finally, for a field k we write G_k for its absolute Galois group, i.e., $G_k = \operatorname{Gal}(k_{sep}/k)$, where k_{sep} denotes the separable closure of k.

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2.1 The ramification map

In this section we review some general facts about the ramification map. A more complete discussion can be found in [GMS03], chapter II, p. 15.

Let K be a field, and let $v\colon K^\times \to \mathbb{Z}$ be a discrete valuation on K; let R be the associated valuation ring and k be its residue field. Let n be a positive integer prime to $\operatorname{char} k$. Since R_{sh} is strictly henselian and n is prime to the residue characteristic, $R_{sh}^\times$ is n-divisible, and hence $\mu_n \subset R_{sh}$. Therefore, using the isomorphism $G_k = G_{K_h}/G_{K_{sh}}$ ([Mil80], I.4.10, p. 37), we can view $H^q(K_{sh}, \mu_n^{\otimes r})$ as a G_k-module by [NSW00], 1.6.2, p. 58. We recall the proof of

Lemma 2.1.1 *There is an isomorphism of G_k-modules*

$$H^q(K_{sh}, \mu_n^{\otimes r}) = \begin{cases} \mu_n^{\otimes r} & \text{if } q = 0 \\ \mu_n^{\otimes(r-1)} & \text{if } q = 1 \\ 1 & \text{if } q \geq 2 \end{cases}$$

Proof. Since $\mu_n \subset R_{sh}$, $H^0(K_{sh}, \mu_n^{\otimes r}) = \mu_n^{\otimes r}$. Moreover the cup product is G_k-linear by [NSW00], 1.5.3, p. 47, and induces an isomorphism

$$H^1(K_{sh}, \mu_n) \otimes \mu_n^{\otimes(r-1)} = H^1(K_{sh}, \mu_n) \otimes H^0(K_{sh}, \mu_n^{\otimes(r-1)}) \xrightarrow{\cup} H^1(K_{sh}, \mu_n^{\otimes r})$$

and the Kummer isomorphism $\delta\colon K_{sh}^\times/n \to H^1(K_{sh}, \mu_n)$ is G_k-linear by [NSW00], 1.5.2, p. 46, and gives an isomorphism

$$K_{sh}^\times/n \otimes \mu_n^{\otimes(r-1)} \xrightarrow{\delta\otimes 1} H^1(K_{sh}, \mu_n) \otimes \mu_n^{\otimes(r-1)}$$

Let v' denote the valuation associated to R_{sh}. From the exact sequence of G_k-modules

$$0 \longrightarrow R_{sh}^\times \longrightarrow K_{sh}^\times \xrightarrow{v'} \mathbb{Z} \longrightarrow 0$$

and the fact that $R_{sh}^\times$ is n-divisible, we obtain an isomorphism

$$K_{sh}^\times/n \otimes \mu_n^{\otimes(r-1)} = K_{sh}^\times \otimes \mu_n^{\otimes(r-1)} \xrightarrow{v'\otimes 1} \mu_n^{\otimes(r-1)}$$

Composing the maps above, we obtain the desired isomorphism $H^1(K_{sh}, \mu_n^{\otimes r}) = \mu_n^{\otimes(r-1)}$, which sends $\delta a \cup w$ to $w^{v'(a)}$ for all $a \in K_{sh}^\times/n$ and $w \in \mu_n^{\otimes(r-1)}$.

Finally we show that $H^q(K_{sh}, \mu_n^{\otimes r})$ is trivial for $q \geq 2$. Let $p = \operatorname{char} k$. We have an exact sequence (see for instance [Neu99], II.9.15, p. 175)

$$1 \to P \to G_{K_{sh}} \to T \to 1$$

Here P denotes the “wild” part, a pro-p-group (or trivial if $p = 0$), and T denotes the “tame” part,

$$T = \prod_{l \neq p} \mathbb{Z}_l(1)$$

where l runs through all primes different from p and $\mathbb{Z}_l(1) = \varprojlim_{m\geq 1} \mu_{l^m}$.

Since n is prime to p, $H^j(P, \mu_n^{\otimes r}) = 0$ for $j > 0$, so by the Hochschild-Serre spectral sequence (see [NSW00], 2.1.5, p. 82)

$$H^i(T, H^j(P, \mu_n^{\otimes r})) \Rightarrow H^{i+j}(G_{K_{sh}}, \mu_n^{\otimes r})$$

we have that $H^i(G_{K_{sh}}, \mu_n^{\otimes r}) = H^i(T, \mu_n^{\otimes r})$ for all i. But for all primes l dividing n, we have that $\operatorname{cd}_l T = \operatorname{cd}_l \hat{\mathbb{Z}} = 1$, where cd_l denotes the cohomological l-dimension ([NSW00], 3.3.5, p. 140), and therefore $H^i(T, \mu_n^{\otimes r})$ is trivial for $i \geq 2$, as required. □

We have a Hochschild-Serre spectral sequence

$$H^p\big(k, H^q(K_{sh}, \mu_n^{\otimes r})\big) \Rightarrow H^{p+q}(K_h, \mu_n^{\otimes r}) \tag{$*$}$$

Since $H^q(K_{sh}, \mu_n^{\otimes r})$ is trivial for $q \geq 2$, we obtain a long exact sequence

$$\begin{aligned} 0 &\to H^1\big(k, H^0(K_{sh}, \mu_n^{\otimes r})\big) \to H^1(K_h, \mu_n^{\otimes r}) \to H^0\big(k, H^1(K_{sh}, \mu_n^{\otimes r})\big) \\ &\to H^2\big(k, H^0(K_{sh}, \mu_n^{\otimes r})\big) \to H^2(K_h, \mu_n^{\otimes r}) \to H^1\big(k, H^1(K_{sh}, \mu_n^{\otimes r})\big) \to \cdots \end{aligned}$$

Using the computation of the lemma, we can rewrite it as

$$\begin{aligned} 0 &\to H^1(k, \mu_n^{\otimes r}) \to H^1(K_h, \mu_n^{\otimes r}) \to H^0(k, \mu_n^{\otimes (r-1)}) \\ &\to H^2(k, \mu_n^{\otimes r}) \to H^2(K_h, \mu_n^{\otimes r}) \to H^1(k, \mu_n^{\otimes (r-1)}) \to \cdots \end{aligned}$$

where the maps $H^1(k, \mu_n^{\otimes r}) \to H^1(K_h, \mu_n^{\otimes r})$ and $H^2(k, \mu_n^{\otimes r}) \to H^2(K_h, \mu_n^{\otimes r})$ are inflation (see [NSW00], p. 83). But since

$$1 \to G_{K_{sh}} \to G_{K_h} \to G_k \to 1$$

splits ([GMS03], 7.6, p. 17), the inflation maps $H^p(k, \mu_n^{\otimes r}) \to H^p(K_h, \mu_n^{\otimes r})$ are injective, and hence the long exact sequence above breaks into short exact sequences

$$0 \longrightarrow H^p(k, \mu_n^{\otimes r}) \longrightarrow H^p(K_h, \mu_n^{\otimes r}) \xrightarrow{\rho} H^{p-1}(k, \mu_n^{\otimes (r-1)}) \longrightarrow 0$$

for all $p \geq 1$.

Definition 2.1.2 Let v be a discrete valuation on a field K, let R be the associated dvr and let k be the residue field. Let n be prime to $\operatorname{char} k$. The **p-th ramification map with respect to** v, $\mathrm{ram}_v \colon H^p(K, \mu_n^{\otimes r}) \to H^{p-1}(k, \mu_n^{\otimes (r-1)})$, is defined to be the composition

$$H^p(K, \mu_n^{\otimes r}) \xrightarrow{\text{res}} H^p(K_h, \mu_n^{\otimes r}) \xrightarrow{\rho} H^{p-1}(k, \mu_n^{\otimes (r-1)})$$

where ρ is given by the above spectral sequence.

In particular, for $p = 1$, ρ is given by restriction ([NSW00], p. 83), and the ramification map is the composition

$$H^1(K, \mu_n^{\otimes r}) \xrightarrow{\text{res}} H^1(K_h, \mu_n^{\otimes r}) \xrightarrow{\text{res}} H^1(K_{sh}, \mu_n^{\otimes r})^{G_k} = H^0(k, \mu_n^{\otimes (r-1)})$$

where the last equality follows from the lemma. This fact makes it easy to compute ram_v in the following special case.

Lemma 2.1.3 *Let $\delta \colon K^\times/n \to H^1(K, \mu_n)$ be the Kummer isomorphism. Then for any $a \in K^\times/n$ and any $w \in H^0(K, \mu_n^{\otimes (r-1)})$*

$$\mathrm{ram}_v(\delta a \cup w) = w^{v(a)}$$

Proof. By abuse of language, we shall also write δ for the Kummer isomorphism from $K_{sh}^\times/n$ to $H^1(K_{sh}, \mu_n^{\otimes r})$. Now observe that the restriction of $\delta a \cup w \in H^1(K, \mu_n^{\otimes r})$ to $H^1(K_{sh}, \mu_n^{\otimes r})$ is just $\delta a \cup w$, where a and w are now viewed as elements of $K_{sh}^\times/n$ and $H^0(K_{sh}, \mu_n^{\otimes (r-1)})$. But from the proof of the last lemma the isomorphism $H^1(K_{sh}, \mu_n^{\otimes r}) = \mu_n^{\otimes (r-1)}$ sends $\delta a \cup w$ to $w^{v'(a)}$. Since v' is an extension of v, $v'(a) = v(a)$ and we are done. ◻

Next we state a functorial property of the ramification map. See [CT95], 3.3.1, p. 21, and also [GMS03], Proposition 8.2, p. 19.

Lemma 2.1.4 *Let $R \subset S$ be an inclusion of dvrs with fraction fields $K \subset L$ and residue fields $k \subset l$. Let n be prime to* $\operatorname{char} k$*, and e denote the ramification index of S over R. Then the following diagram commutes*

$$\begin{tikzcd}
H^p(L, \mu_n^{\otimes r}) \arrow[r, "\mathrm{ram}_S"] & H^p(l, \mu_n^{\otimes(r-1)}) \\
H^p(K, \mu_n^{\otimes r}) \arrow[r, "\mathrm{ram}_R"] \arrow[u, "\mathrm{res}"] & H^p(k, \mu_n^{\otimes(r-1)}) \arrow[u, "e\,\mathrm{res}"']
\end{tikzcd}$$

Now let X be an integral noetherian normal scheme with function field $K(X)$. Since X is normal, each $x \in X^{(1)}$ defines a dvr $\mathcal{O}_{X,x}$. We denote by

$$\mathrm{ram}_x \colon H^p(K(X), \mu_n^{\otimes r}) \to H^{p-1}(\Bbbk(x), \mu_n^{\otimes(r-1)})$$

the ramification map corresponding to the dvr defined by x. Varying x, we obtain a map

$$H^p(K(X), \mu_n^{\otimes r}) \xrightarrow{\bigoplus_{x \in X^{(1)}} \mathrm{ram}_x} \bigoplus_{x \in X^{(1)}} H^{p-1}(\Bbbk(x), \mu_n^{\otimes(r-1)})$$

which is well-defined by the next lemma.

Lemma 2.1.5 *Let X be an integral noetherian normal scheme. Given an element $\alpha \in H^p(K(X), \mu_n^{\otimes r})$, there are only finitely many $x \in X^{(1)}$ for which $\mathrm{ram}_x(\alpha)$ is not trivial.*

Proof. Since X is noetherian, it is covered by finitely many affine noetherian schemes and thus we may assume that $X = \operatorname{Spec} A$ is affine with A noetherian. Let $K = K(X) = \operatorname{Frac} A$. By [NSW00], 1.2.6, p. 22, given $\alpha \in H^p(K, \mu_n^{\otimes r})$, we can find a finite Galois extension L of K so that α is split by L, i.e., α is in the image of the inflation $H^p(G, \mu_n^{\otimes r}) \to H^p(K, \mu_n^{\otimes r})$ where $G = \mathrm{Gal}(L/K)$. Let B be the integral closure of A in L. We claim that if all points in the fibre of $x \in X^{(1)}$ in $\operatorname{Spec} B$ are unramified over x then $\mathrm{ram}_x(\alpha) = 1$. This finishes the proof since there are only finitely many height 1 primes of B in the support of $\Omega_{B/A}$ since B is noetherian.

To prove the claim, let $\mathfrak{p}$ be a height 1 prime ideal of A and suppose that all prime ideals $\mathfrak{q} \in \operatorname{Spec} B$ over $\mathfrak{p}$ are unramified. Let $R = A_{\mathfrak{p}}$ be the corresponding dvr and let $k = \Bbbk(\mathfrak{p})$. Then by construction of R_{sh} we may assume that $B_{\mathfrak{q}} \subset R_{sh}$ for some prime ideal $\mathfrak{q}$ lying over $\mathfrak{p}$. Hence the restriction of α to K_h belongs to the image of the inflation map $H^p(k, \mu_n^{\otimes r}) = H^p(G_{K_h}/G_{K_{sh}}, \mu_n^{\otimes r}) \to H^p(K_h, \mu_n^{\otimes r})$, which equals the kernel of the map of $\rho \colon H^p(K_h, \mu_n^{\otimes r}) \to H^{p-1}(k, \mu_n^{\otimes(r-1)})$. Hence $\mathrm{ram}_{\mathfrak{p}}(\alpha) = 1$, as required. □

2.2 Gysin sequence for dvr's

Let R be a dvr with $K = \operatorname{Frac} R$ and residue field k. Let n be prime to $\operatorname{char} k$, and write $i \colon \operatorname{Spec} k \to \operatorname{Spec} R$ and $j \colon \operatorname{Spec} K \to \operatorname{Spec} R$ for the inclusions of the closed and generic points of $\operatorname{Spec} R$, respectively. We have a Leray spectral sequence (see [Mil80], III.1.18, p. 89)

$$H^p(R, R^q j_* \mu_n^{\otimes r}) \Rightarrow H^{p+q}(K, \mu_n^{\otimes r})$$

We now compute the higher direct images $R^q j_* \mu_n^{\otimes r}$. For $q \geq 2$, we have that the stalks at the generic and closed points are respectively

$$H^q(K_{sep}, \mu_n^{\otimes r}) = 1 \qquad \text{and} \qquad H^q(K_{sh}, \mu_n^{\otimes r}) = 1$$

by Lemma 2.1.1. Hence $R^q j_* \mu_n^{\otimes r}$ is trivial for $q \geq 2$. The cases $q = 0$ and $q = 1$ are special cases of the next two theorems.

Theorem 2.2.1 *Let X be an integral normal noetherian scheme with function field K, and let $j\colon \operatorname{Spec} K \to X$ be the inclusion of the generic point. Suppose that n is prime to the characteristic of any residue field of X. Then $j_*\mu_{n,K}^{\otimes r} = \mu_{n,X}^{\otimes r}$.*

Proof. For a scheme S, let P_S denote the presheaf

$$V \mapsto \underbrace{\mu_{n,S}(V) \otimes_{\mathbb{Z}/n(V)} \cdots \otimes_{\mathbb{Z}/n(V)} \mu_{n,S}(V)}_{r \text{ times}} \qquad (V \text{ étale over } S)$$

so that the associated sheaf is $aP_S = \mu_{n,S}^{\otimes r}$.

The canonical injection $\mathbb{G}_m \to j_p\mathbb{G}_{m,K}$ induces an inclusion $\mu_{n,X}(V) \subset \mu_{n,K}(j^{-1}V) = (j_p\mu_{n,K})(V)$ for each étale $V \to X$, and gives a map $P_X \to j_pP_K \to j_*aP_K$, hence a map $\mu_{n,X}^{\otimes r} \to j_*\mu_{n,K}^{\otimes r}$. We prove that this map is an isomorphism by looking at stalks. Let $x \in X$, $\bar{x} = \operatorname{Spec} \Bbbk(x)_{sep}$, $\tilde{X} = \operatorname{Spec} \mathcal{O}_{X,\bar{x}}$, $K_{\bar{x}} = \operatorname{Frac} \mathcal{O}_{X,\bar{x}}$. Since X is normal, $\tilde{X}$ is integral and $K_{\bar{x}}$ is a field. From [Mil80] Theorem II.3.2(b), p. 70, and the fact that for any presheaf P the sheafification map $P \to aP$ induces an isomorphism on stalks, we have that

$$(\mu_{n,X}^{\otimes r})_{\bar{x}} \to (j_*\mu_{n,K}^{\otimes r})_{\bar{x}}$$

is equivalent to the morphism

$$\Gamma(\tilde{X}, P_{\tilde{X}}) \to \Gamma(\operatorname{Spec} K_{\bar{x}}, P_{\operatorname{Spec} K_{\bar{x}}})$$

But the latter morphism is induced by the inclusion

$$\Gamma(\tilde{X}, \mu_n) \to \Gamma(\operatorname{Spec} K_{\bar{x}}, \mu_n) = \mu_n(K_{\bar{x}})$$

so it suffices to show that this map is an isomorphism. But this is easy: since the domain $\mathcal{O}_{X,\bar{x}}$ is strictly henselian and n is prime to the characteristic of its residue field by hypothesis, both $\Gamma(\tilde{X}, \mu_n)$ and $\mu_n(K_{\bar{x}})$ consist of all the n-th roots of unity. $\square$

Theorem 2.2.2 *Let X be an integral regular noetherian scheme, and let n be a positive integer relatively prime to all the residue characteristics of X. For each point $x \in X$, let $\iota_x\colon \operatorname{Spec} \Bbbk(x) \to X$ be the inclusion of x into X. Denote by η the generic point of X, write $g = \iota_\eta$, and let P be the presheaf on X given by*

$$U \mapsto H^1(g^{-1}U, \mu_n^{\otimes r}), \qquad U \text{ étale over } X$$

Then there is a map $\phi\colon P \to \bigoplus_{x \in X^{(1)}} \iota_{x*}\mu_n^{\otimes(r-1)}$ of presheaves given by

$$\begin{array}{ccc}
\Gamma(U, P) & \xrightarrow{\phi_U} & \Gamma\big(U, \bigoplus_{x \in X^{(1)}} \iota_{x*}\mu_n^{\otimes(r-1)}\big) \\
\| & & \| \\
H^1(K(U), \mu_n^{\otimes r}) & \xrightarrow{\bigoplus_{u \in U^{(1)}} \mathrm{ram}_u} & \bigoplus_{u \in U^{(1)}} H^0(\Bbbk(u), \mu_n^{\otimes(r-1)})
\end{array}$$

inducing an isomorphism

$$aP = R^1g_*\mu_n^{\otimes r} \xrightarrow{\sim} \bigoplus_{x \in X^{(1)}} \iota_{x*}\mu_n^{\otimes(r-1)}$$

Proof. Observe that, for U étale over X, $g^{-1}(U) = U \times_X \eta = K(U)$, hence $P(U) = H^1(K(U), \mu_n^{\otimes r})$. On the other hand, $U_x \overset{\text{def}}{=} \iota_x^{-1}U = U \times_X \operatorname{Spec} \Bbbk(x)$ is the fibre of x in U, and since U is étale over X, it is flat over X of relative dimension 0, and hence $U^{(1)} = \coprod_{x \in X^{(1)}} U_x$. Furthermore, X is noetherian, hence

$$\begin{aligned}\Gamma\big(U, \bigoplus_{x \in X^{(1)}} \iota_{x*}\mu_n^{\otimes(r-1)}\big) &= \bigoplus_{x \in X^{(1)}} \Gamma(U, \iota_{x*}\mu_n^{\otimes(r-1)}) \\ &= \bigoplus_{x \in X^{(1)}} \bigoplus_{u \in U_x} H^0(\Bbbk(u), \mu_n^{\otimes(r-1)}) \\ &= \bigoplus_{u \in U^{(1)}} H^0(\Bbbk(u), \mu_n^{\otimes(r-1)})\end{aligned}$$

Now let $V \to U$ be étale. The fact that ϕ is a map of presheaves boils down to the commutativity of

$$\begin{tikzcd} H^1(K(U), \mu_n^{\otimes r}) \arrow[r, "\mathrm{ram}_u"] \arrow[d, "\mathrm{res}"] & H^0(\Bbbk(u), \mu_n^{\otimes(r-1)}) \arrow[d, "\mathrm{res}"] \\ H^1(K(V), \mu_n^{\otimes r}) \arrow[r, "\bigoplus \mathrm{ram}_v"] & \bigoplus_{v \mapsto u} H^0(\Bbbk(v), \mu_n^{\otimes(r-1)}) \end{tikzcd}$$

where v runs over the fibre of u in V (apply Lemma 2.1.4 with $e = 1$).

Now we compare stalks. Let $z \in X$ be an arbitrary point, and let $\bar{z}$ be a geometric point over z. Write $\tilde{Z} = \operatorname{Spec} \mathcal{O}_{X,\bar{z}}$. By [Mil80], III.1.15, p. 88, and the above, we have that

$$\begin{tikzcd} P_{X,\bar{z}} \arrow[r, "\phi_{\bar{z}}"] \arrow[d, equal] & \bigoplus_{x \in X^{(1)}} (\iota_{x*}\mu_n^{\otimes(r-1)})_{\bar{z}} \arrow[d, equal] \\ H^1(K(\tilde{Z}), \mu_n^{\otimes r}) \arrow[r, "\bigoplus \mathrm{ram}"] & \bigoplus_{y \in \tilde{Z}^{(1)}} H^0(\Bbbk(y), \mu_n^{\otimes(r-1)}) \end{tikzcd}$$

which is an isomorphism by the next lemma. ∎

Lemma 2.2.3 *Let A be a strictly henselian regular ring with $K = \operatorname{Frac} A$, and write $Z = \operatorname{Spec} A$. Let $z \in Z^{(1)}$. The following diagram is commutative:*

$$\begin{tikzcd} H^1(K, \mu_n^{\otimes r}) \arrow[r, "\mathrm{ram}_z"] \arrow[d, equal] & H^0(\Bbbk(z), \mu_n^{\otimes(r-1)}) \arrow[d, equal] \\ \bigoplus_{x \in Z^{(1)}} \mu_n^{\otimes(r-1)} \arrow[r, "z\text{-th projection}"] & \mu_n^{\otimes(r-1)} \end{tikzcd}$$

Proof. Let $R = \mathcal{O}_{Z,\bar{z}}$ and $F = \operatorname{Frac} R$. Since A is regular, by the Auslander-Buchsbaum theorem it is a UFD (see [Mat89], 19.8, p. 160). Hence A is normal, and R is a dvr. We have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A^\times \arrow[r] \arrow[d] & K^\times \arrow[r] \arrow[d] & \bigoplus_{x \in Z^{(1)}} \mathbb{Z} \arrow[r] \arrow[d, "z\text{-th}\ \textit{projection}"] & 0 \\ 0 \arrow[r] & R^\times \arrow[r] & F^\times \arrow[r] & \mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

Since A and R are strictly henselian and n is prime to the residue characteristics, we have that both $A^\times$ and $R^\times$ are n-divisible. Hence tensoring with $\mu_n^{\otimes(r-1)}$ we obtain another commutative diagram

$$\begin{tikzcd}
K^\times/n \otimes \mu_n^{\otimes(r-1)} \arrow[r, "\sim"] \arrow[d] & \bigoplus_{x \in Z^{(1)}} \mu_n^{\otimes(r-1)} \arrow[d, "z\text{-th}\ projection"] \\
F^\times/n \otimes \mu_n^{\otimes(r-1)} \arrow[r, "\sim"] & \mu_n^{\otimes(r-1)}
\end{tikzcd}$$

where the horizontal arrows are isomorphisms (compare with the proof of Lemma 2.1.1). Since $\mu_n \subset K$ and $\mu_n \subset F$, by [NSW00], 1.5.2, p. 46,

$$\begin{tikzcd}
H^1(K, \mu_n^{\otimes r}) \arrow[r, "\sim"] \arrow[d, "\mathrm{res}"'] & K^\times/n \otimes \mu_n^{\otimes(r-1)} \arrow[d] \\
H^1(F, \mu_n^{\otimes r}) \arrow[r, "\sim"] & F^\times/n \otimes \mu_n^{\otimes(r-1)}
\end{tikzcd}$$

commutes, where again the horizontal arrows are isomorphisms. The result now follows by combining the two last diagrams and observing that the composition

$$H^1(K, \mu_n^{\otimes r}) \xrightarrow{\mathrm{res}} H^1(F, \mu_n^{\otimes r}) \xrightarrow{\sim} \mu_n^{\otimes(r-1)} = H^0(\Bbbk(z), \mu_n^{\otimes(r-1)})$$

is none other than ram_z. $\square$

We summarize the results for a dvr.

Theorem 2.2.4 *Let R be a dvr with $K = \operatorname{Frac} R$ and residue field k. Let n be prime to $\operatorname{char} k$, and write $i : \operatorname{Spec} k \to \operatorname{Spec} R$ and $j : \operatorname{Spec} K \to \operatorname{Spec} R$ for the inclusions. Then*

$$R^q j_* \mu_n^{\otimes r} = \begin{cases} \mu_n^{\otimes r} & \text{if } q = 0 \\ i_* \mu_n^{\otimes(r-1)} & \text{if } q = 1 \\ 0 & \text{if } q \geq 2 \end{cases}$$

Since $R^q j_* \mu_n^{\otimes r} = 0$ for $q \geq 2$, the Leray spectral sequence yields a long exact sequence

$$\begin{aligned}
0 \to H^1(R, \mu_n^{\otimes r}) &\to H^1(K, \mu_n^{\otimes r}) \to H^0(R, R^1 j_* \mu_n^{\otimes r}) \\
\to H^2(R, \mu_n^{\otimes r}) &\to H^2(K, \mu_n^{\otimes r}) \to H^1(R, R^1 j_* \mu_n^{\otimes r}) \\
\to H^3(R, \mu_n^{\otimes r}) &\to H^3(K, \mu_n^{\otimes r}) \to H^2(R, R^1 j_* \mu_n^{\otimes r}) \to \cdots
\end{aligned}$$

On the other hand, since i is a closed immersion, i_* is exact ([Mil80], II.3.6, p. 72), hence $R^q i_* \mu_n^{\otimes(r-1)} = 0$ for $q \geq 1$. Now $R^1 j_* \mu_n^{\otimes r} = i_* \mu_n^{\otimes(r-1)}$, and from the Leray spectral sequence applied to $i\colon \operatorname{Spec} k \to \operatorname{Spec} R$, we have that

$$H^p(R, R^1 j_* \mu_n^{\otimes r}) = H^p(R, i_* \mu_n^{\otimes(r-1)}) = H^p(k, \mu_n^{\otimes(r-1)})$$

Replacing $H^p(R, R^1 j_* \mu_n^{\otimes r})$ by $H^p(k, \mu_n^{\otimes(r-1)})$ in the long exact sequence above yields the so-called **Gysin sequence** for dvrs:

$$\begin{aligned}
0 \to H^1(R, \mu_n^{\otimes r}) &\to H^1(K, \mu_n^{\otimes r}) \to H^0(k, \mu_n^{\otimes(r-1)}) \\
\to H^2(R, \mu_n^{\otimes r}) &\to H^2(K, \mu_n^{\otimes r}) \to H^1(k, \mu_n^{\otimes(r-1)}) \\
\to H^3(R, \mu_n^{\otimes r}) &\to H^3(K, \mu_n^{\otimes r}) \to H^2(k, \mu_n^{\otimes(r-1)}) \to \cdots
\end{aligned}$$

Here the maps

$$H^p(R, \mu_n^{\otimes r}) \to H^p(K, \mu_n^{\otimes r})$$

are the natural maps induced by the inclusion of R into K. Moreover, from the explicit description of the isomorphism $R^1 j_* \mu_n^{\otimes r} = i_* \mu_n^{\otimes(r-1)}$, it can be shown that the maps

$$H^p(K, \mu_n^{\otimes r}) \to H^{p-1}(k, \mu_n^{\otimes r-1})$$

are none other than the ramification maps with respect to the dvr R.

2.3 K-theory

There is still another important case where we can easily compute the ramification maps. First recall that given a field F and an integer n prime to $\operatorname{char} F$, the **Galois symbol** is the map

$$h_F\colon K_r^M(F) \to H^r(F, \mu_n^{\otimes r})$$

from the Milnor K-group $K_r^M(F)$ of F to $H^r(F, \mu_n^{\otimes r})$ ([NSW00], 6.4.2, p. 306), induced by the map

$$F^\times \otimes \cdots \otimes F^\times \xrightarrow{\delta \otimes \cdots \otimes \delta} H^1(F, \mu_n) \otimes \cdots \otimes H^1(F, \mu_n) \xrightarrow{\cup} H^r(F, \mu_n^{\otimes r})$$

obtained by composing the Kummer isomorphism $\delta\colon F^\times/n \to H^1(F, \mu_n)$ and the cup product.

Now we return to the setup of the beginning of section 2.1. The next theorem allows us to define a map from $K_r^M(K)$ to the $K_{r-1}^M(k)$, called the **tame symbol** with respect to the valuation v. See [FV02], IX.2, p. 286 for more details.

Theorem 2.3.1 *Let K be a field, v be a valuation on K. Denote R be the associated dvr, and let π be a uniformiser for R. For each $r \geq 1$ there is a unique morphism*

$$\partial_v\colon K_r^M(K) \to K_{r-1}^M(k)$$

satisfying

$$\partial_v\{\pi, u_2, \ldots, u_r\} = \{\bar{u}_2, \ldots, \bar{u}_r\}$$

where $u_i \in R^\times$ and $\bar{u}_i$ denote the image of u_i in k. For $r = 2$,

$$\partial_v\{a, b\} = (-1)^{v(a)v(b)} \overline{\left(\frac{a^{v(b)}}{b^{v(a)}}\right)}$$

for all $a, b \in K^\times$.

Now we can state the connection between the tame symbol and the ramification map. See [Kat86] for more details.

Theorem 2.3.2 *Assume the setup of the last theorem, and let n be prime to $\operatorname{char} k$. The diagram*

$$\begin{tikzcd} K_r^M(K) \arrow[r, "\partial_v"] \arrow[d, "h_K"'] & K_{r-1}^M(k) \arrow[d, "h_k"] \\ H^r(K, \mu_n^{\otimes r}) \arrow[r, "\mathrm{ram}_v"] & H^{r-1}(k, \mu_n^{\otimes(r-1)}) \end{tikzcd}$$

commutes.

Next, if $(R, \mathfrak{m}, k)$ is a 1-dimensional local domain with $F = \operatorname{Frac} R$, we define a group homomorphism

$$\operatorname{ord}_R\colon K_1^M(F) = F^\times \to \mathbb{Z} = K_0^M(k)$$

called **order function**, which agrees with the discrete valuation associated to R when the latter is a dvr. We set for $r \in R - \{0\}$

$$\operatorname{ord}_R(r) = \operatorname{length}_R(R/r)$$

which is finite since R/r is 0-dimensional. When R is excellent, there is a better way to describe ord_R. Let S be the normalisation of R in F; since R is excellent, S is finite over R and therefore S is a regular semilocal ring of dimension 1. Then every maximal ideal $\mathfrak{q}$ of S defines a discrete valuation $v_{\mathfrak{q}}\colon F^\times \to \mathbb{Z}$ and we may write (see [Fulton], A.3.1, p. 412)

$$\operatorname{ord}_R(r) = \sum_{\mathfrak{q}} [\mathbb{k}(\mathfrak{q}) : k] \cdot v_{\mathfrak{q}}(r) \qquad \text{for all } r \in F^\times$$

where $\mathfrak{q}$ runs over all the maximal ideals of S.

In our application, given a 2-dimensional regular local ring A with $X = \operatorname{Spec} A$, we will assign to each $x \in X^{(1)}$ the order map $\operatorname{ord}_x\colon \Bbbk(x)^\times \to \mathbb{Z}$ associated to the 1-dimensional domain $A/\mathfrak{p}$, where $\mathfrak{p}$ is the height 1 prime ideal of A corresponding to x. This can be used to explicitly describe the map

$$\bigoplus_{x\in X^{(1)}} H^1(\Bbbk(x), \mu_n) \to \mathbb{Z}/n$$

in the Bloch-Ogus sequence for $r = 2$: by Kummer theory, we have an isomorphism $H^1(\Bbbk(x), \mu_n) = \Bbbk(x)^\times/n$, and the map $H^1(\Bbbk(x), \mu_n) \to \mathbb{Z}/n$ is the one induced by ord_x. In particular, when $A/\mathfrak{p}$ is a dvr, the above map is the one induced by the corresponding discrete valuation.

We will also need a particular case of Gersten's deep conjecture. Let A be a noetherian regular local ring with $K = \operatorname{Frac} A$ and write $X = \operatorname{Spec} A$. Gersten's conjecture predicts the existence of an exact sequence of Quillen's K-groups

$$0 \to K_q^Q(A) \to K_q^Q(K) \to \bigoplus_{x\in X^{(1)}} K_{q-1}^Q(\Bbbk(x)) \to \bigoplus_{x\in X^{(2)}} K_{q-2}^Q(\Bbbk(x)) \to \cdots$$

for each integer $q \geq 0$. Gersten's conjecture was first proved by Quillen when A is essentially of finite type over a field ([Quillen]). This result has been extended by Panin (see [Pan03]) to the equicharacteristic case.

We will only need the case $q = 2$ of the above conjecture applied to an excellent 2-dimensional regular local ring $(A, \mathfrak{m}, k)$. Since Quillen's and Milnor's K groups coincide in degrees 0, 1, 2 and $K_0(R) = \mathbb{Z}$, $K_1(R) = R^\times$ for any local noetherian ring R (see [srinivas], chapters 2 and 7), the above sequence reads

$$0 \longrightarrow K_2(A) \longrightarrow K_2(K) \xrightarrow{\bigoplus \partial_x} \bigoplus_{x\in X^{(1)}} \Bbbk(x)^\times \xrightarrow{\bigoplus \operatorname{ord}_x} \mathbb{Z} \longrightarrow 0 \tag{$\dagger$}$$

where the maps are as follows: $K_2(A) \to K_2(K)$ is the natural map induced by the inclusion $A \hookrightarrow K$, ∂_x is the tame symbol with respect to the valuation defined by $x \in X^{(1)}$, and ord_x is the order function associated to x as described above. The next lemma is proved in [Kat86], remark 6.14, p. 173. For completeness and in order to describe the maps explicitly, we include the short proof.

Theorem 2.3.3 *Let $(A, \mathfrak{m}, k)$ be an excellent 2-dimensional regular local ring. Let $K = \operatorname{Frac} A$ and $X = \operatorname{Spec} A$. The sequence ($\dagger$) is exact.*

Proof. Let $\mathcal{M}^i$ be the category of $\mathcal{O}_X$-modules whose support has codimension at least i. We have localisation sequences ([srinivas], p. 65)

$$\begin{aligned}\cdots \to K_q(\mathcal{M}^{i+1}) \to K_q(\mathcal{M}^i) \to K_q(\mathcal{M}^i/\mathcal{M}^{i+1}) &= \bigoplus_{x\in X^{(i)}} K_q(\Bbbk(x)) \\ \to K_{q-1}(\mathcal{M}^{i+1}) \to K_{q-1}(\mathcal{M}^i) \to K_{q-1}(\mathcal{M}^i/\mathcal{M}^{i+1}) &= \bigoplus_{x\in X^{(i)}} K_{q-1}(\Bbbk(x)) \to \cdots\end{aligned}$$

Since A and k are regular, $K_q(\mathcal{M}^0) = G_q(A) = K_q(A)$, $K_q(\mathcal{M}^2) = G_q(k) = K_q(k)$. Applying this to $i = 0, 1$ and $q = 2$ we obtain exact sequences

$$K_2(A) \to K_2(K) \to K_1(\mathcal{M}^1) \to A^\times \to K^\times$$

and

$$\bigoplus_{x\in X^{(1)}} K_2(\Bbbk(x)) \to k^\times \to K_1(\mathcal{M}^1) \to \bigoplus_{x\in X^{(1)}} \Bbbk(x)^\times \to \mathbb{Z}$$

By [vdK1], $K_2(A) \to K_2(K)$ is injective. Also, $A^\times \to K^\times$ is the inclusion map, hence is injective. On the other hand, if $x \in X^{(1)}$ is such that $R = A/\mathfrak{p}$ is a dvr for the corresponding height 1 prime ideal $\mathfrak{p}$, then by the naturality of the localisation sequence and the next lemma the x-th coordinate of the map

$\bigoplus_{x\in X^{(1)}} K_2(\Bbbk(x)) \to k^\times$ is given by the tame symbol, while the x-th coordinate of $\bigoplus_{x\in X^{(1)}} \Bbbk(x)^\times \to \mathbb{Z}$ is given by the corresponding valuation map. This shows that both maps are surjective. The exactness of Gersten's sequence (†) now follows by splicing together the above two exact sequences.

The fact that the map $\bigoplus_{x\in X^{(1)}} \Bbbk(x)^\times \to \mathbb{Z}$ is given by the order is a consequence of the next lemma and the naturality of the localisation sequence. To show that $K_2(K) \to \bigoplus_{x\in X^{(1)}} \Bbbk(x)^\times$ is given by the tame symbol, observe that for each height 1 prime ideal $\mathfrak{p}$ of A the localisation map $A \to A_\mathfrak{p}$ is flat and hence we have a commutative diagram

$$\begin{tikzcd} K_2(A) \arrow[r] \arrow[d] & K_2(K) \arrow[r] \arrow[d, equal] & K_1(\mathcal{M}^1) \arrow[d] \\ K_2(A_\mathfrak{p}) \arrow[r] & K_2(K) \arrow[r, "\partial_\mathfrak{p}"] & K_1(\Bbbk(\mathfrak{p})) \end{tikzcd}$$

where the rows are localisation sequences for A and $A_\mathfrak{p}$ respectively and $\partial_\mathfrak{p}$ is the tame symbol map by the next lemma. But since $-\otimes_A A_\mathfrak{p}$ kills any module in $\mathcal{M}^2$, we have that the right vertical map factors as

$$K_1(\mathcal{M}^1) \to K_1(\mathcal{M}^1/\mathcal{M}^2) = \bigoplus_{x\in X^{(1)}} K_1(\Bbbk(x)) \to K_1(\Bbbk(\mathfrak{p}))$$

Here the last map is the x-th projection map where $x \in X^{(1)}$ is the point corresponding to the prime $\mathfrak{p}$. Hence the x-th coordinate of the map $K_2(K) \to K_1(\mathcal{M}^1) \to \bigoplus_{x\in X^{(1)}} K_1(\Bbbk(x))$ is given by the tame symbol, as required. □

Lemma 2.3.4 *Let $(R, \mathfrak{m}, k)$ be an excellent 1-dimensional local domain with $F = \operatorname{Frac} R$. In the localisation sequence*

$$\cdots \to K_2(F) \to K_1(k) \to G_1(R) \to K_1(F) \to K_0(k) \to \cdots$$

the map $F^\times = K_1(F) \to K_0(k) = \mathbb{Z}$ is the order map corresponding to R. Moreover if R is regular (i.e., a dvr) then $K_2(F) \to K_1(k) = k^\times$ is the tame symbol map with respect to the discrete valuation associated to R.

Proof. For R regular this is proved in [`snaith`]. In general, let S be the normalisation of R in F and denote by $\pi\colon \operatorname{Spec} S \to \operatorname{Spec} R$ be the normalisation map. Since R is excellent, π is a finite map and hence S is a regular semi-local domain. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$ be the maximal ideals of S, let $v_i\colon F^\times \to \mathbb{Z}$ be the corresponding discrete valuations and let $f_i \overset{\text{df}}{=} [\Bbbk(\mathfrak{q}_i) : k]$ be the residue degrees. Since π is finite we have an exact functor π_* from the category of coherent modules over $\operatorname{Spec} S$ to that of coherent modules over $\operatorname{Spec} R$, which respects the filtration by codimension. Hence by the naturality of the localisation sequence we obtain a commutative diagram

$$\begin{tikzcd} G_1(S) \arrow[r] \arrow[d, "\pi_*"'] & K_1(F) \arrow[r, "\bigoplus v_i"] \arrow[d, equal] & \bigoplus_{1\le i\le n} K_0(\Bbbk(\mathfrak{q}_i)) \arrow[d, "\bigoplus f_i"] \\ G_1(R) \arrow[r] & K_1(F) \arrow[r] & K_0(k) \end{tikzcd}$$

The fact that the map $F^\times = K_1(F) \to K_0(\Bbbk(\mathfrak{q}_i)) = \mathbb{Z}$ is given by v_i again follows by [`snaith`]. On the other hand, since π_* takes the generator $\Bbbk(\mathfrak{q}_i)$ of $K_0(\Bbbk(\mathfrak{q}_i)) = \mathbb{Z}$ to $\Bbbk(\mathfrak{q}_i)$, viewed as a k-vector space, we have that this map is just multiplication by f_i. Hence $F^\times = K_1(F) \to K_0(k) = \mathbb{Z}$ is given by $a \mapsto \sum_{1\le i\le n} f_i \cdot v_i(a) = \operatorname{ord}(a)$. □

3 A Bloch-Ogus sequence

We are now ready to prove the main result of this paper. We work in the following

Setup 3.0.1 Let $(A, \mathfrak{m}, k)$ be a noetherian regular local ring. Write $K = \operatorname{Frac} A$ for its fraction field and let $X = \operatorname{Spec} A$. Let n be an integer prime to $\operatorname{char} k$.

Theorem 3.0.2 *In setup 3.0.1, assume further that A is 2-dimensional and excellent. Then the sequence*

$$0 \to H^2(A, \mu_n^{\otimes 2}) \to H^2(K, \mu_n^{\otimes 2}) \to \bigoplus_{x \in X^{(1)}} H^1(\Bbbk(x), \mu_n) \to \mathbb{Z}/n \to 0$$

is exact.

As mentioned in the introduction, some parts of the proof hold in greater generality. We divide the proof up accordingly.

3.1 Injectivity

We start with exactness on the left.

Theorem 3.1.1 *Assume setup 3.0.1.*

1. *The following sequence is exact.*

$$0 \longrightarrow H^1(A, \mu_n^{\otimes r}) \longrightarrow H^1(K, \mu_n^{\otimes r}) \xrightarrow{\text{ram}} \bigoplus_{x \in X^{(1)}} H^0(\Bbbk(x), \mu_n^{\otimes(r-1)}) \longrightarrow 0$$

2. *The following map is injective.*

$$H^2(A, \mu_n^{\otimes r}) \to H^2(K, \mu_n^{\otimes r})$$

Proof. Let $g\colon \operatorname{Spec} K \to X$ be the inclusion of the generic point. From the Leray spectral sequence

$$H^p(A, R^q g_* \mu_n^{\otimes r}) \Rightarrow H^{p+q}(K, \mu_n^{\otimes r})$$

we have a 5-term exact sequence

$$\begin{aligned} 0 &\to H^1(A, g_*\mu_n^{\otimes r}) \to H^1(K, \mu_n^{\otimes r}) \to H^0(A, R^1 g_* \mu_n^{\otimes r}) \\ &\to H^2(A, g_*\mu_n^{\otimes r}) \to H^2(K, \mu_n^{\otimes r}) \end{aligned}$$

By Theorem 2.2.1 and Theorem 2.2.2 we have that $g_*\mu_n^{\otimes r} = \mu_n^{\otimes r}$ and $R^1 g_* \mu_n^{\otimes r} = \bigoplus_{x \in X^{(1)}} \iota_{x*} \mu_n^{\otimes(r-1)}$, and thus 1 and 2 will be proved as long as we can show that

$$H^1(K, \mu_n^{\otimes r}) \longrightarrow H^0(A, R^1 g_* \mu_n^{\otimes r}) = \bigoplus_{x \in X^{(1)}} H^0(\Bbbk(x), \mu_n^{\otimes(r-1)})$$

is surjective, which is done in the next theorem.

Theorem 3.1.2 *Assume setup 3.0.1. Then*

$$H^1(K, \mu_n^{\otimes r}) \xrightarrow{\text{ram}} \bigoplus_{x \in X^{(1)}} H^0(\Bbbk(x), \mu_n^{\otimes(r-1)})$$

is surjective.

Proof. Fix $z \in X^{(1)}$, and let w be a generator of $H^0(\Bbbk(z), \mu_n^{\otimes(r-1)})$. We shall construct an element $f \in H^1(K, \mu_n^{\otimes r})$ such that

$$\operatorname{ram}_x(f) = \begin{cases} w & \text{if } x = z \\ 1 & \text{if } x \neq z \end{cases}$$

Let ζ be a primitive n-th root of 1, write

$$A' = A[\zeta] \qquad K' = K(\zeta) \qquad G = \operatorname{Gal}(K'/K)$$

Observe that A' is finite étale over A; in particular, A' is a regular semi-local ring, and hence a UFD ([Mil80], I.3.17, p. 27), so A' is the integral closure of A in K'. Choose a prime $z' \in \operatorname{Spec} A'$ lying over z, and let

$$D_z = \text{decomposition group of } z' \text{ in } G$$

Observe that $D_z \subset G$ is independent of the choice of z' since G is abelian. Finally set

$$\begin{aligned} K'' &= K'^{D_z} \\ A'' &= \text{integral closure of } A \text{ in } K'' \end{aligned}$$

and write $z'' \in \operatorname{Spec} A''$ for the image of $z' \in \operatorname{Spec} A'$. By [SGA71], exposé V, 3.4, p. 117, A'' is finite étale over the regular ring A. Hence A'' is a semi-local regular ring and therefore a UFD.

Since there is an isomorphism $D_z = \operatorname{Gal}(\Bbbk(z')/\Bbbk(z))$ we have an isomorphism

$$H^0(K'', \mu_n^{\otimes(r-1)}) = H^0(\Bbbk(z), \mu_n^{\otimes(r-1)})$$

Hence we may identify w with a generator of $H^0(K'', \mu_n^{\otimes(r-1)})$.

Now let $\delta\colon K''^{\times} \to H^1(K'', \mu_n)$ be the coboundary map in the Kummer sequence. Let $t \in A''$ be a prime element defining z'', which exists since A'' is a UFD. Cupping $\delta t \in H^1(K'', \mu_n)$ with $w \in H^0(K'', \mu_n^{\otimes(r-1)})$ we obtain $\delta t \cup w \in H^1(K'', \mu_n^{\otimes r})$. Applying the corestriction map, we may now define the required element

$$f = \operatorname{cores}(\delta t \cup w) \in H^1(K, \mu_n^{\otimes r})$$

Now let $x \in X^{(1)}$ be an arbitrary point, and choose a prime $x'' \in A''$ lying over x. We compute $\operatorname{ram}_x(f)$. Since A'' is unramified over A at x'', by Lemma 2.1.4 we have a commutative diagram

$$\begin{tikzcd} H^1(K'', \mu_n^{\otimes r}) \arrow[r, "\operatorname{ram}_{x''}"] & H^0(\Bbbk(x''), \mu_n^{\otimes(r-1)}) \\ H^1(K, \mu_n^{\otimes r}) \arrow[u, "\operatorname{res}"] \arrow[r, "\operatorname{ram}_x"] & H^0(\Bbbk(x), \mu_n^{\otimes(r-1)}) \arrow[u, hook, "\operatorname{res}"'] \end{tikzcd}$$

and since the restriction map on the right hand side is nothing else than the inclusion $H^0(\Bbbk(x), \mu_n^{\otimes(r-1)}) \subset H^0(\Bbbk(x''), \mu_n^{\otimes(r-1)})$ we have that, as an element of $\mu_n^{\otimes(r-1)}$,

$$\operatorname{ram}_x(f) = \operatorname{ram}_{x''}\big(\operatorname{res}(f)\big)$$

On the other hand, from [NSW00], 1.5.7, p. 49, we have that

$$\operatorname{res}(f) = \operatorname{res}(\operatorname{cores}(\delta t \cup w)) = N(\delta t \cup w) = \prod_{\sigma \in \operatorname{Gal}(K''/K)} \delta t^{\sigma} \cup w^{\sigma}$$

where N denotes the norm map of $\operatorname{Gal}(K''/K)$-modules. Hence by Lemma 2.1.3 we have that

$$\operatorname{ram}_x(f) = \prod_{\sigma \in \operatorname{Gal}(K''/K)} \operatorname{ram}_{x''}(\delta t^{\sigma} \cup w^{\sigma}) = \prod_{\sigma \in \operatorname{Gal}(K''/K)} (w^{\sigma})^{v_{x''}(t^{\sigma})}$$

Clearly $v_{x''}(t^{\sigma}) = 0$ if $x \neq z$ and thus $\operatorname{ram}_x(f) = 1$ in this case. On the other hand, for $x = z$, as σ runs over $\operatorname{Gal}(K''/K)$, t^{σ} runs over generators of the distinct primes in A'' lying over z, with z'' corresponding to $\sigma = 1$. Hence $v_{z''}(t^{\sigma}) = 1$ for $\sigma = 1$ and $v_{z''}(t^{\sigma}) = 0$ otherwise, and thus $\operatorname{ram}_z(f) = w$, as required. □

3.2 Purity

In this section, we show

Theorem 3.2.1 *In setup 3.0.1, assume further that A is excellent. Then*

$$H^2(A,\mu_n^{\otimes r}) \longrightarrow H^2(K,\mu_n^{\otimes r}) \xrightarrow{\text{ram}} \bigoplus_{x\in X^{(1)}} H^1(\Bbbk(x),\mu_n^{\otimes(r-1)})$$

is exact.

For each $x \in X^{(1)}$ we have a Gysin exact sequence

$$H^2(\operatorname{Spec}\mathcal{O}_{X,x},\mu_n^{\otimes r}) \longrightarrow H^2(K,\mu_n^{\otimes r}) \xrightarrow{\text{ram}_x} H^1(\Bbbk(x),\mu_n^{\otimes(r-1)})$$

and thus

$$\begin{aligned}&\ker\Big(H^2(K,\mu_n^{\otimes r}) \to \bigoplus_{x\in X^{(1)}} H^1(\Bbbk(x),\mu_n^{\otimes(r-1)})\Big)\\ &= \bigcap_{x\in X^{(1)}} \operatorname{im}\Big(H^2(\operatorname{Spec}\mathcal{O}_{X,x},\mu_n^{\otimes r}) \to H^2(K,\mu_n^{\otimes r})\Big)\end{aligned}$$

and we have to show that this subgroup of $H^2(K,\mu_n^{\otimes r})$ coincides with the image of $H^2(A,\mu_n^{\otimes r})$ in $H^2(K,\mu_n^{\otimes r})$. This is a consequence of the deep absolute purity conjecture, which has been proved by Gabber. Before we can state it, we need some terminology.

Definition 3.2.2 Let X be a scheme. Let Y be a closed subscheme of X, define $U = X - Y$, and let $i\colon Y \to X$ and $j\colon U \to X$ be the corresponding closed and open immersions. We say that (Y,X) is a **regular couple** of codimension c if X and Y are regular locally noetherian schemes such that for each $y \in Y$ $\operatorname{codim}_y(Y,X) = c$. Hence each $y \in Y$ has an affine open neighborhood X' such that $Y \cap X'$ is defined by an ideal $(x_1,\dots,x_c)$, where $x_1,\dots,x_c \in \Gamma(X',\mathcal{O}_X)$ are part of a regular system of parameters of X at each point of $Y \cap X'$.

Theorem 3.2.3 (Absolute cohomological purity) *Let (Y,X) be a regular couple of codimension c, where X is an excellent scheme. Let n be an invertible integer on X, and F be a locally constant sheaf of cyclic groups of order n on X. With the above notation, the equivalent conditions*

$$\begin{array}{lll}\mathcal{H}^q_Y(X,F) \overset{\text{def}}{=} R^q i^! F = 0 & \textit{if} & q \neq 2c\\ R^q j_*(j^*F) = 0 & \textit{if} & q \neq 0,\ 2c-1\end{array}$$

hold, and

$$\mathcal{H}^{2c}_Y(X,F) = i^*(R^{2c-1}j_*)j^*F$$

is a locally constant sheaf of cyclic groups of order n on Y.

Proof. See [Fuj02]. ◻

The next corollary and lemma follow [CT95], 3.4.2, p. 23 and 3.8.2, p. 29.

Corollary 3.2.4 *Let X be an excellent regular scheme, and Y be a closed subscheme of X such that* $\operatorname{codim}_y(Y,X) \geq 2$ *for each $y \in Y$. Let $U = X - Y$, and n be prime to all residue characteristics. Then*

$$H^2(X,\mu_n^{\otimes r}) = H^2(U,\mu_n^{\otimes r})$$

Proof. Suppose first that (Y,X) is a regular couple of codimension at least 2. By the absolute cohomological purity we have that $R^q j_*\mu_n^{\otimes r} = 0$ for $q = 1,2$. From the Leray spectral sequence

$$H^p(X,R^q j_*\mu_n^{\otimes r}) \Rightarrow H^{p+q}(U,\mu_n^{\otimes r})$$

we conclude that $H^2(X, j_*\mu_n^{\otimes r}) = H^2(U, \mu_n^{\otimes r})$, and since X is normal, $j_*\mu_n^{\otimes r} = \mu_n^{\otimes r}$ by Theorem 2.2.1 and the result follows.

The general case is done by descending induction on the codimension c of Y in X. The result holds if $c = \dim X \geq 2$ by the special case above. Now suppose that the result holds for c, and let Y be a closed subset of X of codimension $c-1$. Since X is excellent, so is Y, and hence the set of non-regular points of Y is closed and has codimension at least c in X. Let Y' be the closed subset of Y consisting of the non-regular points and the components with strictly smaller dimension than that of Y. Then the codimension of Y' in X is at least c and $(Y - Y', X - Y')$ is a regular couple of codimension c. Hence, by the special case above applied to $(Y - Y', X - Y')$, we conclude that

$$H^2(X - Y', \mu_n^{\otimes r}) = H^2(X - Y, \mu_n^{\otimes r})$$

On the other hand, by induction hypothesis, we have that

$$H^2(X - Y', \mu_n^{\otimes r}) = H^2(X, \mu_n^{\otimes r})$$

The result follows by combining the two equalities. □

Remark 3.2.5 When $X = \operatorname{Spec} A$ for a 2-dimensional noetherian regular local ring A (not necessarily excellent) and Y equals its closed point, one may obtain the previous result in a more elementary fashion, using the classical purity for the Brauer group. It is enough to show that $(R^q j_*\mu_n^{\otimes r})_{\bar{x}} = 0$ for all $x \in X$ and $q = 1, 2$. Since j is an isomorphism on U this is clear when $x \in U$. On the other hand, if x is the closed point of X, then $(R^q j_*\mu_n^{\otimes r})_{\bar{x}} = H^q(U_{sh}, \mu_n) \otimes \mu_n^{\otimes(r-1)}$, where $U_{sh} \overset{\text{df}}{=} U \times_X \operatorname{Spec} A_{sh}$ is the complement of the closed point in $\operatorname{Spec} A_{sh}$. Now we have that

1. $\Gamma(U_{sh}, \mathbb{G}_m) = A_{sh}^\times$ is n-divisible;
2. $\operatorname{Pic}(U_{sh}) = \operatorname{Pic}(A_{sh}) = 0$ by the proof of [EGAIVd], 21.11.1, p. 302;
3. $\operatorname{Br}(U_{sh}) = \operatorname{Br}(A_{sh}) = 0$ by [Mil80], IV.1.7, p. 139, and the proof of IV.2.16, p. 149 (which is based on the purity for the Brauer group of 2-dimensional regular noetherian rings of [AG]).

The result then follows from the Kummer sequence $1 \longrightarrow \mu_n \longrightarrow \mathbb{G}_m \xrightarrow{n} \mathbb{G}_m \longrightarrow 1$.

Now we return to the setup of the beginning of this section. Let $x \in X^{(1)}$. Since A and $\mathcal{O}_{X,x}$ are regular rings, by the results in the last section the maps $H^2(A, \mu_n^{\otimes r}) \to H^2(K, \mu_n^{\otimes r})$ and $H^2(\operatorname{Spec} \mathcal{O}_{X,x}, \mu_n^{\otimes r}) \to H^2(K, \mu_n^{\otimes r})$ are both injective. Identifying the groups $H^2(A, \mu_n^{\otimes r})$ and $H^2(\operatorname{Spec} \mathcal{O}_{X,x}, \mu_n^{\otimes r})$ with their images in $H^2(K, \mu_n^{\otimes r})$, we have the following

Lemma 3.2.6 *Assume setup 3.0.1 with A excellent. Then*

$$H^2(A, \mu_n^{\otimes r}) = \bigcap_{x \in X^{(1)}} H^2(\operatorname{Spec} \mathcal{O}_{X,x}, \mu_n^{\otimes r})$$

viewed as subgroups of $H^2(K, \mu_n^{\otimes r})$.

Proof. By the previous corollary, it is enough to show that

$$H^2(U, \mu_n^{\otimes r}) = \bigcap_{x \in X^{(1)}} H^2(\operatorname{Spec} \mathcal{O}_{X,x}, \mu_n^{\otimes r}) \qquad (*)$$

where U is an open set of X containing all its codimension 1 points.

First, we show that for any open set U of X containing $X^{(1)}$, the left hand side is contained in the right hand side in $(*)$. For any $x \in X^{(1)}$, the inclusion $\operatorname{Spec} K \to U$ of the generic point factors as $\operatorname{Spec} K \to \operatorname{Spec} \mathcal{O}_{X,x} \to U$. Hence by functoriality $H^2(U, \mu_n^{\otimes r}) \to H^2(K, \mu_n^{\otimes r})$ factors as

$$H^2(U, \mu_n^{\otimes r}) \to H^2(\operatorname{Spec} \mathcal{O}_{X,x}, \mu_n^{\otimes r}) \to H^2(K, \mu_n^{\otimes r})$$

and therefore the image of $H^2(U,\mu_n^{\otimes r}) \to H^2(K,\mu_n^{\otimes r})$ is contained in the image of $H^2(\operatorname{Spec}\mathcal{O}_{X,x},\mu_n^{\otimes r}) \to H^2(K,\mu_n^{\otimes r})$, as required.

To prove the reverse inclusion, let $\alpha \in H^2(K,\mu_n^{\otimes r})$ be an element in the right hand side of $(*)$. We will say that α is "defined" on an open set V of X if it belongs to the image of $H^2(V,\mu_n^{\otimes r}) \to H^2(K,\mu_n^{\otimes r})$. We now prove that α is defined on an open set U containing $X^{(1)}$.

By [Mil80], III.1.16, p. 88,

$$H^2(K,\mu_n^{\otimes r}) = \varinjlim_{f\neq 0} H^2(A_f,\mu_n^{\otimes r})$$

Hence there is an affine open set $U' = \operatorname{Spec} A_f$ on which α is defined. If U' does not contain some $x \in X^{(1)}$, we will show how to replace it with a bigger open set on which α is defined. But since X is noetherian, this process must eventually stop, yielding the desired open set U containing $X^{(1)}$ and on which α is defined.

Since

$$H^2(\operatorname{Spec}\mathcal{O}_{X,x},\mu_n^{\otimes r}) = \varinjlim_{D(g)\ni x} H^2(A_g,\mu_n^{\otimes r})$$

and α is in the image of $H^2(\operatorname{Spec}\mathcal{O}_{X,x},\mu_n^{\otimes r}) \to H^2(K,\mu_n^{\otimes r})$ by hypothesis, we can find an affine open neighborhood U'' of x such that α is defined on U''. Let $\alpha_{U'} \in H^2(U',\mu_n^{\otimes r})$ and $\alpha_{U''} \in H^2(U'',\mu_n^{\otimes r})$ be elements mapping to $\alpha \in H^2(K,\mu_n^{\otimes r})$. By [Mil80], III.2.24, p. 110, we have a Mayer-Vietoris exact sequence

$$H^2(U'\cup U'',\mu_n^{\otimes r}) \to H^2(U',\mu_n^{\otimes r}) \oplus H^2(U'',\mu_n^{\otimes r}) \to H^2(U'\cap U'',\mu_n^{\otimes r})$$

Hence α will be defined on $U'\cup U''$ as long as the images of $\alpha_{U'}$ and $\alpha_{U''}$ agree on $H^2(U'\cap U'',\mu_n^{\otimes r})$. But since both elements map to $\alpha \in H^2(K,\mu_n^{\otimes r})$, this can be accomplished by shrinking U'' to some U'''. More explicitly, we show that

$$\varprojlim_{U'''\ni x} U'\cap U''' = \operatorname{Spec} K, \tag{$**$}$$

where U' and x are as above and U''' runs over the affine open neighborhoods of x. Then by [Mil80] III.1.16, p. 88 again, we will be able to conclude that

$$\varinjlim_{U'''\ni x} H^2(U'\cap U''',\mu_n^{\otimes r}) = H^2(K,\mu_n^{\otimes r})$$

Hence there exists a neighborhood $U''' \subset U''$ of x such that the images $\alpha_{U'}$ and $\alpha_{U''}|_{U'''}$ agree on $H^2(U'\cap U''',\mu_n^{\otimes r})$.

To prove $(**)$, we may restrict our attention to open sets U''' of the form $D(g)$. Then, since $D(f)\cap D(g) = D(fg)$, we have to show that

$$\varinjlim_{D(g)\ni x} A_{fg} = K$$

Let $v\colon K^\times \to \mathbb{Z}$ be the valuation corresponding to x, and let t be a uniformiser of v. Then there exists $g \in A$ such that $x \in D(g)$ and $t \in A_g$. Moreover, since $f \in A$ and $x \notin U = D(f)$, we have that $n = v(f) > 0$.

Now let $a \in K^\times$ be an arbitrary element, we have to show that $a \in A_{fg''}$ for some g'' such that $x \in D(g'')$. Let $m = v(a)$; then $u = a\cdot(f/t^{n+1})^m$ has value 0, hence $u \in A_{g'}$ for some $g' \in A$ with $x \in D(g')$. Therefore

$$a = u\cdot\left(\frac{t^{n+1}}{f}\right)^m \in A_{fgg'}$$

and since $x \in D(gg')$, we may take $g'' = gg'$. □

3.3 Gersten's sequence

Using the special case of Gersten's conjecture discussed before, we complete the proof of exactness at the rightmost terms of the Bloch-Ogus sequence.

Theorem 3.3.1 *Assume setup 3.0.1 with A excellent and 2-dimensional. Then the sequence*

$$H^2(K, \mu_n^{\otimes 2}) \to \bigoplus_{x \in X^{(1)}} H^1(\Bbbk(x), \mu_n) \to \mathbb{Z}/n \to 0$$

is exact.

Proof. We have a commutative diagram

$$\begin{tikzcd} K_2(K) \arrow[r] \arrow[d] & \bigoplus_{x \in X^{(1)}} K_1(\Bbbk(x)) \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d] \\ H^2(K, \mu_n^{\otimes 2}) \arrow[r] & \bigoplus_{x \in X^{(1)}} H^1(\Bbbk(x), \mu_n) \arrow[r] & \mathbb{Z}/n \end{tikzcd}$$

where the top row is exact by Theorem 2.3.3, and the middle and right vertical arrows are surjective (and so is the left one, by the Merkurjev-Suslin theorem, but we will not need this fact).

We first show that the sequence is exact at $\mathbb{Z}/n$. In fact, for any $x \in X^{(1)}$ such that $A/\mathfrak{p}$ is a dvr for the corresponding height 1 prime $\mathfrak{p}$, the map $H^1(\Bbbk(x), \mu_n) \to \mathbb{Z}/n$ is induced by the associated valuation, and hence is surjective.

Next we check exactness at the sum. Let (α_x) be an element in the kernel of $\bigoplus_{x \in X^{(1)}} H^1(\Bbbk(x), \mu_n) \to \mathbb{Z}/n$. Choose $(a_x) \in \bigoplus_{x \in X^{(1)}} K_1(\Bbbk(x))$ mapping to (α_x); the image of (a_x) in $\mathbb{Z}$ must then be a multiple of n. Hence by modifying (a_x) at an entry corresponding to a height 1 prime $\mathfrak{p}$ such that $A/\mathfrak{p}$ is a dvr, we can assume that (a_x) maps to 0 in $\mathbb{Z}$, so (a_x) is the image of some $b \in K_2(K)$. But then the image of b in $H^2(K, \mu_n^{\otimes 2})$ maps to (α_x), as required. □

The proof of Theorem 3.0.2 now follows by piecing together Theorem 3.1.1, Theorem 3.2.1, and Theorem 3.3.1.

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