Strongly anisotropic involutions on central simple algebras

Amit Kulshrestha

Abstract

The classical theorem of Bröcker and Prestel on quadratic forms over formally real fields determines a valuation theoretic condition under which all totally indefinite forms are weakly isotropic. In this paper, we look for analogues of such results in a more general setting of algebras with involutions. We prove that for involutions of first kind over central simple algebras of index two, one indeed has a Bröcker-Prestel like statement. The connection between two conditions, namely total indefiniteness and weak isotropicity is made via so called gauge functions on central simple algebras.

1 Introduction

Let F be a formally real field and q be a quadratic form over F. Let Ω denote the set of real orderings of F. The form q is said to be *totally indefinite* if over all real closures $F_P; P \in \Omega$, it is isotropic. Clearly an isotropic form is totally indefinite, though the converse is not true. In fact, even if an orthogonal sum of copies of q is isotropic then q is totally indefinite. Quadratic forms q such that the m-fold orthogonal sum $m \times q$ is isotropic for some integer $m \geq 1$ are called *weakly isotropic*. Forms which are not weakly isotropic are termed as *strongly anisotropic* forms.

If F satisfies so called Strong Approximation Property (SAP), then weakly isotropic forms are precisely those which are totally indefinite (cf. [ELP],[P2]). This however, is not true for arbitrary formally real fields. For arbitrary formally real fields, there is a classical theorem proved independently by Bröcker and Prestel in 1974 (see [B] and [P1, Th. 8.13]) which connects the two conditions, namely weak isotropicity and total indefiniteness. This connection is via valuations on fields.

Theorem 1.1 (Bröcker - Prestel) *Let F be a formally real field and q be a quadratic form over F. Then following statements are equivalent:*

(i) The form q is weakly isotropic.

(ii) *The form q is indefinite at every ordering of F, and for every valuation v where q has more than one residue and the residue field F_0 is formally real, at least one residue class of q is weakly isotropic.*

Since quadratic forms correspond to orthogonal involutions on split central simple algebras, it is interesting to ask if an analogous result holds for arbitrary central simple algebras with involutions. In this paper, we prove such a result for involutions of first kind over central simple algebras of index at most 2. Our result is equivalent to Theorem 1.1 in the split orthogonal case, and in this sense it generalises Theorem 1.1. We first define some terminology.

Let A be a central simple algebra over F and σ be an involution on A. We say that (A, σ) is *weakly isotropic* if there exist non-zero $x_i \in A; 1 \leq i \leq r$ such that $\sum_{i=1}^{r} \sigma(x_i)x_i = 0$. Let t_n denote the transpose involution on the split algebra $M_n(F)$. It is easy to check that σ is weakly isotropic if and only if there exists $n \geq 1$ such that the involution $\sigma \otimes t_n$ is isotropic on $A \otimes M_n(F)$. We also remark that if F is not formally real then every involution is weakly isotropic. Thus throughout the paper, we shall be implicitly assuming that F is formally real.

Let (A, σ) a central simple algebra over F with an involution of the first kind. Let Trd_A denote the reduced trace of A. The *trace form* of (A, σ) is the quadratic form $T_\sigma : A \to F$ given by $x \mapsto \mathrm{Trd}_A(\sigma(x)x)$. If ad_{T_σ} is the adjoint involution on $\mathrm{End}_F(A)$ associated to the quadratic form T_σ then $(\mathrm{End}_F(A), ad_{T_\sigma}) \simeq (A \otimes_F A, \sigma \otimes \sigma)$. In [U, Ch. 5] there are examples of involutions σ which are weakly isotropic if and only the associated trace form T_σ is weakly isotropic.

If F is formally real and P is an ordering on F then by [KMRT, Prop. 11.7], the signature $\mathrm{sgn}_P(T_\sigma)$ of the quadratic form T_σ at P is square. We define the *signature* of σ at an ordering P as follows:

$$\mathrm{sgn}_P(A, \sigma) = \sqrt{\mathrm{sgn}_P(T_\sigma)}.$$

We say that (A, σ) is *totally indefinite* if $\mathrm{sgn}_P(A, \sigma) < \deg(A)$ for all orderings P on F.

One of the main difficulties in extending Theorem 1.1 is that valuations are very rare on central simple algebras (in fact impossible for non-division algebras), and even if they exist there is no clue of the relationship they have with orderings on the base field. However, an alternate notion of gauges on semisimple algebras was recently discovered by Tignol and Wadsworth [TW, TW1], and we use gauges to generalise Theorem 1.1 in several cases.

Let F be a formally real field and A be a central simple F-algebra. Let σ be an involution of first kind on A. We say that (A, σ) *satisfies BP-equivalence* if the following conditions are equivalent:

(BP1) The involution σ is weakly isotropic

(BP2) The involution σ is totally indefinite and at every gauge j on A which is ramified and has formally real residue field, the residue of σ is weakly isotropic.

The implication (BP1) $\Rightarrow$ (BP2) of BP-equivalence always holds (Th. 2.20). Interestingly, it was shown by Lewis, Scheiderer and Unger in [LSU] that if F satisfies Effective Diagonalisation (ED) property and (A, σ) is a central simple algebra over F with σ totally indefinite involution of the first kind then (A, σ) is weakly isotropic. In fact, they showed that this is a criterion to classify ED fields.

The main result of this article is the following:

Theorem 1.2 *Let F be a formally real field and A be a central simple F-algebra. Let σ be an involution of first kind on A. Then (A, σ) satisfies BP-equivalence in the following cases:*

(i) $\text{index}(A) \leq 2$.

(ii) $(A, \sigma) \simeq (H_1 \otimes H_2 \otimes \cdots \otimes H_r, \sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_r)$*, where each H_i is a quaternion algebra over H and σ_i is an involution of first kind and either type.*

Organisation of the paper is as follows: In section 2, we introduce several value functions on vector spaces and central simple algebras. The crucial notions of gauges and residue involutions are also introduced in this section. The section concludes with a proof of the implication (BP1) $\Rightarrow$ (BP2) of BP-equivalence without any assumption on the index of A. After section 2, the rest of the paper is devoted to prove the implication $(ii) \Rightarrow (i)$ of BP-equivalence in several cases. These cases are covered in section 3 (split case), section 4 (symplectic case) and section 5 (orthogonal case). Further, in section 6 we prove that if the trace form of (A, σ) determines its weak isotropicity then (A, σ) satisfies BP-equivalence.

2 Value functions on vector spaces and algebras

2.1 Norms on vector spaces

As already pointed out in the introduction, the absence of valuations on arbitrary central simple algebras needs to be taken care of to address the questions regarding the generalisation of Bröcker-Prestel in the set up of involutions on central simple algebras. In this section, we explain the kind of value functions used in the paper. These value functions were recently defined in [TW] and [TW1].

In what follows, D denotes a division algebra and Γ denotes a totally ordered abelian group. Let ∞ be a symbol such that in the union $\Gamma \cup \{\infty\}$, $\infty > \delta$ for every $\delta \in \Gamma$. A *valuation* $\bar{v}$ on D with values in Γ is a map $\bar{v} : D \to \Gamma \cup \{\infty\}$ such that:

(i) $\bar{v}(a) = \infty$ if and only if $a = 0$.

(ii) $\bar{v}(ab) = \bar{v}(a) + \bar{v}(b)$ for all $a, b \in D$.

(iii) $\bar{v}(a + b) \geq \min\{\bar{v}(a), \bar{v}(b)\}$ for all $a, b \in D$.

Such maps do not necessarily exist. In fact, by a Theorem of Morandi [M, Theorem 1.2], if D is finite dimensional over its centre $Z(D) = F$, then D carries a valuation $\bar{v}$ if and only if there is a valuation v on F such that $D \otimes_F F_v^h$ is a division algebra; here F_v^h denotes the henselisation of F at v. Moreover by [E] and [W], if these conditions hold then

$$\bar{v}(a) = \frac{1}{\deg_F(D)}.v(\mathrm{Nrd}(a)) \in \Gamma_F \otimes \mathbb{Q},$$

where Γ_F is the value group of v. It is clear from the formula above that $v = \bar{v}|_F$. The image $\bar{v}(D^*)$ is denoted by Γ_D. The valuation ring of $(D, \bar{v})$ is the set $D^{\geq 0} := \{a \in D : \bar{v}(a) \geq 0\}$. It is easy to see that $D^{\geq 0}$ is a ring and has a unique maximal two sided ideal which is given by $D^{>0} := \{a \in D : \bar{v}(a) > 0\}$. The quotient $D_0 := D^{\geq 0}/D^{>0}$ is called the *residue algebra* of D at $\bar{v}$ and the elements in $D^{\geq 0} \backslash D^{>0}$ are called *units*. It is evident that Γ_F is a subgroup of Γ_D. The index $[\Gamma_D : \Gamma_F]$ is called the *ramification index* of $\bar{v}$.

Let $\gamma \in \Gamma_D$. Define $D^{\geq\gamma} = \{a \in D : \bar{v}(a) \geq \gamma\}$ and $D^{>\gamma} = \{a \in D : \bar{v}(a) > \gamma\}$. Then the *associated graded ring* of D is defined as:

$$\mathrm{gr}_{\bar{v}}(D) := \bigoplus_{\gamma \in \Gamma_D} (D^{\geq\gamma}/D^{>\gamma}).$$

For $a \in D^*$, we denote the image of a in $D^{\geq\bar{v}(a)}/D^{>\bar{v}(a)}$ by a'. It is easy to see that $(ab)' = a'b'$ and $\mathrm{gr}_{\bar{v}}(D)$ is a graded division ring in the sense that every homogeneous element in $\mathrm{gr}_{\bar{v}}(D)$ is invertible.

Let V be a (right) vector space over D. By a $(D, \bar{v})$*-value function* (or simply a D*-value function*) on V with respect to $\bar{v}$, we mean a map $\alpha : V \to (\Gamma_D \otimes \mathbb{Q}) \cup \{\infty\}$ which satisfies:

(i) $\alpha(x) = \infty$ if and only if $x = 0$.

(ii) $\alpha(x\lambda) = \alpha(x) + \bar{v}(\lambda)$ for all $x \in V, \lambda \in D$.

(iii) $\alpha(x + y) \geq \min\{\alpha(x), \alpha(y)\}$ for all $x, y \in V$.

The image $\alpha(V \backslash \{0\})$ is denoted by Γ_V. We remark that Γ_V need not be a subgroup of $\Gamma_D \otimes \mathbb{Q}$ but a union of coset classes modulo Γ_D. Let Γ_V/Γ_D denote the set of these coset classes. If $|\Gamma_V/\Gamma_D| > 1$, then α is called *ramified*.

Example 2.1 Let τ be an involution on D and V be a finite dimensional (right) vector space over D with a basis $\{f_i\}_{i=1}^r$. Let $\gamma_1, \gamma_2, \cdots, \gamma_r \in \Gamma_D \otimes \mathbb{Q}$. Then the function α_γ defined by $\alpha_\gamma\left(\sum_i f_i \lambda_i\right) = \min_i\{\gamma_i + \bar{v}(\lambda_i)\}$ is a D-value function on V. □

A D-value function α is called a *D- norm* if there exists a D-basis $\{e_i\}$ of V such that if $x = \sum_i e_i \lambda_i$ for $\lambda_i \in D^*$, then $\alpha(x) = \min_i\{\alpha(e_i) + \bar{v}(\lambda_i)\}$. Such a basis is called a *splitting basis* of V for α. In the example above, the value function α_γ is indeed a D-norm with $\{f_i\}$ as a splitting basis. If $W \subseteq V$ is a D-vector subspace then the restriction $\alpha|_W$ is a D-norm on W [RTW, Prop. 2.5].

2.2 Gauges on semisimple algebras

In this section, we define some value functions on F-algebras. We use these functions for more specific central simple algebras in later sections.

Let F be a field with a valuation $v : F \to \Gamma_F \cup \{\infty\}$ and A be an algebra over F. A *surmultiplicative F-value function* j on A with respect to v is an F-value function $j : A \to (\Gamma_F \otimes \mathbb{Q}) \cup \{\infty\}$ on the F-vector space A such that $j(1) = 0$ and $j(ab) \geq j(a) + j(b)$ for all $a, b \in A$.

A surmultiplicative F-value function j defines a gradation on A as follows: Let $\Gamma_A = j(A)\backslash\{0\}$. For each $\gamma \in \Gamma_A$, define $A^{\geq\gamma} = \{a \in A : j(a) \geq \gamma\}$, $A^{>\gamma} = \{a \in A : j(a) > \gamma\}$ and $A_\gamma = A^{\geq\gamma}/A^{>\gamma}$. Further define

$$\mathrm{gr}_j(A) := \bigoplus_{\gamma \in \Gamma_A} A_\gamma.$$

The set Γ_A is the grade set of $\mathrm{gr}_j(A)$. Axioms for j imply the algebra structure on A induces a Γ_A-graded algebra structure on $\mathrm{gr}_j(A)$. For $a \in A$ we denote the image of a in $A_{j(a)}$ by a'. For $a, b \in A$, the induced multiplication on $\mathrm{gr}_j(A)$ is $a'b' = (ab)'$. Therefore it follows that $a'b' = 0$ if $j(ab) > j(a) + j(b)$.

A surmultiplicative value function j on an F-algebra A (not necessarily central simple), is called an *F-gauge* if the graded algebra $\mathrm{gr}_j(A)$ is a graded semisimple algebra over $\mathrm{gr}_v(F)$ with $[\mathrm{gr}_j(A) : \mathrm{gr}_v(F)] = [A : F]$. The latter condition is equivalent to demand that j is an F-norm on the F-vector space A. We remark that if A is not semisimple then it does not carry a gauge.

Let $Z(\mathrm{gr}_j(A))$ denote the center of $\mathrm{gr}_j(A)$ and $Z(A)$ denote the center of A. We say that j is a *tame F-gauge* if $Z(\mathrm{gr}_j(A))$ is separable over $\mathrm{gr}_v(F)$ and is equal to $\mathrm{gr}_j(Z(A))$. In this paper, we are interested in the gauges over central simple algebras only.

Lemma 2.2 *Let A be a central simple algebra over a valued field (F, v) with formally real residue field F_0. Let j be a F-gauge on A. Then $\mathrm{gr}_j(A)$ is a central simple graded algebra over $\mathrm{gr}_v(F)$. Hence j is a tame F-gauge.*

Proof Since F_0 is formally real, $char(F_0) = 0$ and thus every F-gauge on A is tame [TW1, Cor. 3.6]. Now it follows immediately from [TW1, Cor. 3.7] that $\mathrm{gr}_j(A)$ is a central simple graded algebra over $\mathrm{gr}_v(F)$. □

Let A be a central simple algebra over F and σ be an involution of the first kind on F. Let j be a surmultiplicative F-value function on A. We say that σ and j are *compatible* if $j \circ \sigma = j$. If j and σ are compatible then the map defined by $\sigma'(x') = \sigma(x)'$, $x \in A$ induces a graded involution σ' on $\mathrm{gr}_j(A)$, also called *residue* of σ at j. The residue σ' on $\mathrm{gr}_j(A)$ is *anisotropic* if there is no non-zero homogeneous element $x' \in \mathrm{gr}_j(A)$ such that $\sigma'(x')x' = 0 \in \mathrm{gr}_j(A)$, and *strongly anisotropic* if $\sum_i \sigma'(x_i')x_i' \neq 0$ for all homogeneous $x_i' \in \mathrm{gr}_j(A)$.

Proposition 2.3 [TW, Prop. 1.2] *For A, σ as above and a surmultiplicative F-value function j on A, the following are equivalent:*

(i) For every $x \in A$, $j(\sigma(x)x) = 2j(x)$.

(ii) j and σ are compatible and the graded involution σ' is anisotropic.

□

A surmultiplicative function j satisfying the equivalent conditions of Prop. 2.3 is called *σ-special*.

Theorem 2.4 [TW, Th. 5.1] *Let F_v^h denote the henselisation of F at v. Then the following are equivalent:*

(i) For a central simple algebra A over F with an involution σ, the involution $\sigma \otimes id$ is anisotropic over $A \otimes_F F_v^h$.

(ii) There exists a σ-special F-gauge j on A.

Further if these conditions hold, there is a unique gauge j satisfying (ii). □

We now mention two examples of gauges over central simple algebras.

Example 2.5 This example is rather a trivial one. On $A = M_r(F)$, where $r \geq 1$ is an integer, we define the gauge $g_r : M_r(F) \to \Gamma_F$ as follows:

$$g_r\left([a_{ij}]\right) = \min_{i,j}\{v(a_{ij})\}.$$

Proposition 2.6 *Let t_r denote the transpose involution on $M_r(F)$ and $\tilde{t_r}$ denote the transpose involution on $M_r\left(\mathrm{gr}_v(F)\right)$. Let g_r be the gauge on $M_r(F)$ as in example 2.5. Then we have the following:*

(i) *The gauge* g_r *is unramified and* $g_r \circ t_r = g_r$*.*

(ii) $\mathrm{gr}_{g_r}(M_r(F)) \simeq M_r(\mathrm{gr}_v(F))$*.*

(iii) *Under the isomorphism* $\mathrm{gr}_{g_r}(M_r(F)) \simeq M_r(\mathrm{gr}_v(F))$*, the graded involution* $(t_r)'$ *induced by* g_r *on* $\mathrm{gr}_{g_r}(M_r(F))$ *is the transpose involution* $\tilde{t}_r$ *on* $M_r(\mathrm{gr}_v(F))$*.*

(iv) *If the residue field* F_0 *is formally real then* g_r *is the* t_r*-special gauge.*

Proof The assertion (i) is clear from the definition of g_r. To prove (ii), let $\gamma \in \Gamma_F$ and $a = [a_{ij}] \in M_r(F)$. Then $g_r(a) > \gamma$ (resp. $\geq \gamma$) if and only if $v(a_{ij}) > \gamma$ (resp. $\geq \gamma$) for all i, j. This suggests that $a' + M_r(F)_\gamma \mapsto [a'_{ij} + F_\gamma]$ induces an isomorphism $\mathrm{gr}_{g_r}(M_r(F)) \simeq M_r(\mathrm{gr}_v(F))$. It is evident from this isomorphism that $(t_r)' = \tilde{t}_r$ and (iii) follows. Now to show (iv), we have to prove that $(t_r)'$ is anisotropic. In view of [TW, Prop. 1.6], it is enough to show that $t_{r_0} = (t_r)'|_{M_r(F)_0}$ is anisotropic. It follows from the isomorphism $\mathrm{gr}_{g_r}(M_r(F)) \simeq M_r(\mathrm{gr}_v(F))$ that $M_r(F)_0 \simeq M_r(F_0)$ and t_{r_0} corresponds to the transpose involution on $M_r(F_0)$, which is anisotropic because F_0 is formally real. This completes the proof. □

Example 2.7 Let D be a division algebra with a valuation $\bar{v}$, V be a (right) vector space over D and α be a D-norm on V. Let $A = \mathrm{End}_D(V)$. For $\phi \in A$, we set

$$j_\alpha(\phi) = \min_{x \in V \setminus \{0\}} \{\alpha(\phi(x)) - \alpha(x)\}.$$

If $\{e_i\}_{i=1}^r$ is a splitting basis of α and $\phi \in A$ then it follows from [TW1, Lemma 1.15] that $j_\alpha(\phi) = \min_{1 \leq i \leq r}\{\alpha(\phi(e_i)) - \alpha(e_i)\}$. In fact the function j_α thus defined is a gauge on A [TW1, Lemma 1.16]. If $\{e_i\}$ is an orthogonal basis of a hermitian form h over (D, τ) for some involution τ then the condition when j_α is ad_h-special is explained in §2.3. □

Let $\phi(e_j) = \sum_i e_i \phi_{ij}$, where $\phi_{ij} \in D^*$; then

$$j_\alpha(\phi) = \min_{1 \leq i, j \leq r} \{\alpha(e_i) + \bar{v}(\phi_{ij}) - \alpha(e_j)\}.$$

This description of j_α allows us to conclude the following:

Proposition 2.8 *With the notation as above, if* α *is ramified then* j_α *is ramified.*

Proof Let $\{e_i\}$ be a splitting basis of α. Since α is ramified, $|\Gamma_V/\Gamma_D| \geq 1$ and by a permutation of splitting basis we may assume that $\alpha(e_1) \neq \alpha(e_2) \in \Gamma_V/\Gamma_D$. We choose $\phi \in A = \mathrm{End}_D(V)$ such that $\phi_{12} = 1$ and $\phi_{ij} = 0$ otherwise. Then $j_\alpha(\phi) = \alpha(e_1) - \alpha(e_2) \notin \Gamma_D$. Thus $j_\alpha(\phi) \notin \Gamma_F$ and j_α is ramified. □

Lemma 2.9 [TW1, Lemma 1.10] *The graded algebras* $\mathrm{gr}_{j_\alpha}(\mathrm{End}_D(V))$ *and* $\mathrm{End}_{\mathrm{gr}(D)}(\mathrm{gr}_\alpha(V))$ *are* $\mathrm{gr}(F)$*-isomorphic.* □

We briefly explain the isomorphism in lemma 2.9. Let $\Gamma = \Gamma_D \otimes \mathbb{Q}$. Let $0 \neq f \in \mathrm{End}_D(V)$ and $\gamma := j_\alpha(f) \in \Gamma$. Then for each $\delta \in \Gamma$, f induces maps:

$$\widetilde{f_\delta} : V_\delta \to V_{\delta+\gamma}$$

given by $x' \mapsto f(x) + V^{>\delta+\gamma}$, where $x \in V$ is such that $\alpha(x) = \delta$. The map $\widetilde{f} = \bigoplus_{\delta\in\Gamma}$: $\mathrm{gr}_\alpha(V) \to \mathrm{gr}_\alpha(V)$ shifts graded components by γ and belongs to $(\mathrm{End}_{\mathrm{gr}(D)}(\mathrm{gr}_\alpha(V))_\gamma$. Further, if $f_1 \in (\mathrm{End}_D(V))^{>\gamma}$, then $\widetilde{f+f_1} = \widetilde{f}$ and this confirms that $f' \mapsto \widetilde{f}$ is a well-defined graded homomorphism from $\mathrm{gr}_{j_\alpha}(\mathrm{End}_D(V))$ to $\mathrm{End}_{\mathrm{gr}(D)}(\mathrm{gr}_\alpha(M))$. This is in fact an isomorphism [TW1, Lemma 1.10].

Let j_1 and j_2 be gauges on central simple F-algebras A_1 and A_2 respectively. Then we define the tensor product $j_1 \otimes j_2$ as follows:

$$j_1 \otimes j_2(x) = \sup\left\{ \min_{1\leq i\leq n} \{j_1(a_i) + j_2(b_i)\} : x = \sum_{i=1}^{n} a_i \otimes b_i \right\} \tag{1}$$

If $\{e_i\}$ and $\{f_k\}$ are splitting bases for j_1 and j_2 respectively, then $\{e_i \otimes f_k\}$ is a splitting basis for $j_1 \otimes j_2$ and $j_1 \otimes j_2(e_i \otimes f_k) = j_1(e_i) + j_2(f_k)$. It follows from [TW1, Cor. 1.25] that $j_1 \otimes j_2$ is a gauge and $\mathrm{gr}_{j_1\otimes j_2}(A_1 \otimes A_2) = \mathrm{gr}_{j_1}(A_1) \otimes \mathrm{gr}_{j_2}(A_2)$.

Proposition 2.10 *Let* (A_1, σ_1) *and* (A_2, σ_2) *be two central simple algebras with involutions of first kind. Let* j *be a* $\sigma_1 \otimes \sigma_2$*-special gauge on* $A_1 \otimes A_2$*. Let* $j_1 = j|_{A_1}$ *and* $j_2 = j|_{A_2}$*. Then* j_1 *is* σ_1*-special gauge on* A_1 *and* j_2 *is* σ_2*-special gauge on* A_2*. Moreover,* $j = j_1 \otimes j_2$*.*

Proof If j is a $\sigma_1 \otimes \sigma_2$-special gauge, then it is clear that j_1 and j_2 are surmultiplicative norms and satisfy the condition (i) of Prop. 2.3. Thus by [TW, Prop. 1.2], j_1 is the σ_1-special gauge and j_2 is the σ_2-special gauge. We now show that $j = j_1 \otimes j_2$. If $\{a_i\}$ is a splitting basis for j_1 and $b_i \in A_2$ then

$$j\left(\sum_i a_i \otimes b_i\right) \geq \min_i(j(a_i \otimes b_i)) \geq \min_i(j_1(a_i) + j_2(b_i)) = (j_1 \otimes j_2)\left(\sum_i a_i \otimes b_i\right)$$

where $a_i \in A_1$ and $b_i \in A_2$. Thus $j \geq j_1 \otimes j_2$. Therefore, there is a graded ring isomorphism

$$\mathrm{gr}_{j_1\otimes j_2}(A_1 \otimes A_2) \xrightarrow{\iota} \mathrm{gr}_j(A_1 \otimes A_2)$$

By [TW1, Prop. 1.20], it follows that $\mathrm{gr}_{j_1\otimes j_2}(A_1 \otimes A_2) \simeq \mathrm{gr}_{j_1}(A_1) \otimes \mathrm{gr}_{j_2}(A_2)$. Since both $\mathrm{gr}_{j_1}(A_1)$ and $\mathrm{gr}_{j_2}(A_2)$ are graded central simple, j_1 and j_2 being gauges, so is $\mathrm{gr}_{j_1\otimes j_2}(A_1 \otimes A_2)$. If the map ι above is not injective then the non-zero kernel contradicts the fact that $\mathrm{gr}_{j_1\otimes j_2}(A_1 \otimes A_2)$ is simple. Thus ι is injective and by [TW1, Prop. 1.21], $j = j_1 \otimes j_2$. □

Proposition 2.11 *Let A be a central simple algebra over a valued field F with an involution σ of the first kind. Let K/F be a field extension and w be an extension of v to K. Suppose $A \otimes_F K$ has a $\sigma \otimes id$-special w-gauge g and let $j = g|_A$. Then j is σ-special v-gauge on A and $g = j \otimes w$.*

Proof Consider a henselization (K_w^h, w_h) of (K, w). Then K_w^h contains a henselization (F_v^h, v_h) of (F, v). The product $g \otimes w_h$ is the $\sigma \otimes id$-special w_h-gauge on $A \otimes_K K_w^h$, hence $g \otimes w_h$ is the $\sigma \otimes id$-special v_h-gauge on $A \otimes F_v^h$. By [TW, Th. 5.1] it follows that $g \otimes w_h|_A$ is the σ-special v-gauge on A. Since $j = g \otimes w_h|_A$, the first claim is proved. The fact that $g = j \otimes w$ follows from [TW, Prop. 4.3]. □

Let j be a surmultiplicative F-value function on a central simple algebra A over F. Let j be compatible with an involution σ of the first kind on A and σ' be the induced graded involution on the graded algebra $\mathrm{gr}_j(A)$. We denote the restriction $\sigma'|_{A_0}$ by σ_0. With this notation we have the following

Proposition 2.12 *Let j be a surmultiplicative F-value function on a central simple algebra A with an involution σ of first kind. Then the following are equivalent:*

(i) For all $r \geq 1$ and all $x \in A \otimes_F M_r(F)$, $(j \otimes g_r)((\sigma \otimes t_r)(x)x) = 2(j \otimes g_r)(x)$.

(ii) j and σ are compatible and the graded involution σ' is strongly anisotropic.

(iii) j and σ are compatible and the involution σ_0 on A_0 is strongly anisotropic.

Proof We first show $(i) \Leftrightarrow (ii)$. With the notation as in example 2.5, $(i) \Leftrightarrow (ii)$ follows immediately in view of Prop. 2.3 provided the graded involution $(\sigma \otimes t_r)'$ on the graded algebra $\mathrm{gr}_{j \otimes g_r}(A \otimes_F M_r(F))$ goes to $\sigma' \otimes \tilde{t}_r$ under the isomorphism $\mathrm{gr}_{j \otimes g_r}(A \otimes_F M_r(F)) \simeq \mathrm{gr}_j(A) \otimes \mathrm{gr}_{g_r}(M_r(F)) \simeq \mathrm{gr}_j(A) \otimes M_r(\mathrm{gr}_v(F))$. Let $x \in A$ and $a = [a_{ij}] \in M_r(F)$ with $g_r(a) = \gamma \in \Gamma_F$. Then $(\sigma \otimes t_r)'((x \otimes a)') = (\sigma \otimes t_r)'(x' \otimes [a_{ij}]') = \sigma(x)' \otimes [a_{ji} + F_\gamma] = \sigma(x)' \otimes \tilde{t}_r([a_{ij} + F_\gamma]) = (\sigma' \otimes \tilde{t}_r)(x' \otimes [a_{ij}]')$, where equality is in fact the identification in the three isomorphic graded algebras above. This suggests that $(\sigma \otimes t_r)' = \sigma' \otimes \tilde{t}_r$ indeed, and $(i) \Leftrightarrow (ii)$ holds.

We now show $(ii) \Leftrightarrow (iii)$. The implication $(ii) \Rightarrow (iii)$ is trivial. To prove the other implication, let σ' be weakly isotropic. Let g_r be the gauge on $M_r(F)$ as in example 2.5. Since σ' is weakly isotropic, it follows that there exists $r \geq 1$ such that $\sigma' \otimes \tilde{t}_r$ is isotropic. As in the proof of $(ii) \Leftrightarrow (iii)$ above, we conclude that $\sigma' \otimes \tilde{t}_r = (\sigma \otimes t_r)'$ and hence $(\sigma \otimes t_r)'$ is isotropic on $\mathrm{gr}_{j \otimes g_r}(A \otimes M_r(F))$. Now by [TW, Prop. 1.6], the involution $(\sigma \otimes t_r)_0$ is isotropic. Let $\{a_i\}$ be a splitting basis of A for j and $\{e_{kl}\}$ be the standard basis of $M_r(F)$, then

$$j \otimes g_r \left(\sum a_i \otimes e_{kl} \right) = \min_{i,k,l} \{ j(a_i) + g_r(e_{kl}) \} = \min_i \{ j(a_i) \}$$

This suggests that $(A \otimes M_r(F))_0 = A_0 \otimes M_r(F_0)$ and $(\sigma \otimes t_r)_0 = \sigma_0 \otimes t_r$ and it follows that σ_0 is weakly isotropic. This completes the proof. □

A surmultiplicative F-value function j satisfying the above equivalent conditions is called *strongly σ-special*.

Corollary 2.13 *Let F_v^h denote the henselisation of F at v. Then the following are equivalent:*

(i) For a central simple algebra A over F with an involution σ, the involution $\sigma \otimes id$ is strongly anisotropic over $A \otimes_F F_v^h$.

(ii) There exists a strongly σ-special F-gauge j on A.

Further if these conditions hold, there is a unique gauge j satisfying (ii). □

Proof We prove $(i) \Rightarrow (ii)$, the other implication is just tracing the proof backwards. Let $(A, \sigma) \otimes_F F_v^h$ be strongly anisotropic. Then for all $r \geq 1$, $(A \otimes_F M_r(F), \sigma \otimes t_r) \otimes_F F_v^h$ is anisotropic. Now by Theorem 2.4, there exists a $\sigma \otimes t_r$-special gauge j_r on $A \otimes M_r(F)$. It is clear from Proposition 2.10 that $j = j_r|_A$ is σ-special and $j_r|_{M_r(F)}$ is t_r-special. Thus by example 2.5 and the uniqueness of t_r-special gauges, $j_r|_{M_r(F)} = g_r$ and by Proposition 2.10, $j_r = j \otimes g_r$. Now as in the proof of 2.12, $(\sigma \otimes t_r)' = \sigma' \otimes \tilde{t}_r$ and the anisotropicity of $(\sigma \otimes t_r)'$ confirms the strong anisotropicity of σ'. Further since j is σ-special, it follows from the unicity part of Theorem 2.4 that j is the unique gauge satisfying (ii). □

We conclude the section with the following analogue of Proposition 2.10 for strongly special gauges.

Proposition 2.14 *Let $\{(A_i, \sigma_i)\}_{i=1}^r$ be central simple algebras with involutions of first kind. Let j be a strongly $\otimes_{i=1}^r \sigma_i$-special gauge on $\otimes_{i=1}^r A_i$. Let $j_i = j|_{A_i}$. Then j_i is strongly σ_i-special gauge on A_i. Moreover, $j = \otimes_{i=1}^r j_i$.*

The proof is a repeated imitation of the proof of Proposition 2.10 where the use of Proposition 2.3 is replaced with the use of Proposition 2.12.

2.3 Hermitian forms and compatible norms

For a D-value function α on V and a hermitian form $h : V \times V \to D$ on (D, τ), we write that $\alpha \prec h$ if for all $x, y \in V$, $\alpha(x) + \alpha(y) \leq \bar{v}(h(x, y))$. Further, we write that $\alpha \preceq h$ if $\alpha \prec h$ and for each $x_0 \in V$, there exists $y_0 \in V$ such that $\alpha(x_0) + \alpha(y_0) = \bar{v}(h(x_0, y_0))$. Let α be a D-norm such that $\alpha \prec h$. We define the *dual norm* α^* on V as:

$$\alpha^*(x) = \inf_{y \in V \setminus \{0\}} \{\bar{v}(h(x, y)) - \alpha(y)\}.$$

In fact, if $\{e_i\}$ is a splitting basis for α then the dual basis $\{e_i^*\}$ given by $h(e_i^*, e_j) = \delta_{ij}$ is a splitting basis for α^*. Further, $\alpha^*(e_i^*) = -\alpha(e_i)$, $\alpha^{**} = \alpha$ and $\alpha = \alpha^*$ if and only if $\alpha \preceq h$.

The D-value functions on vector spaces too yield some graded objects. For $\delta \in \Gamma_V$, denote $V^{\geq \delta} := \{x \in V : \alpha(x) \geq \delta\}$ and $V^{>\delta} := \{x \in V : \alpha(x) > \delta\}$. The quotient $V_\delta := V^{\geq\delta}/V^{>\delta}$ is called the *residue* of (V, α) at δ. In fact, V_δ is a vector space over the residue algebra D_0 of D at $\bar{v}$. Further, the direct sum $\mathrm{gr}_\alpha(V) = \oplus_{\delta\in\Gamma_V} V_\delta$ is a graded vector space over the graded algebra $\mathrm{gr}_{\bar{v}}(D)$. The set $\{\delta \in \Gamma_V : V_\delta \neq 0\}$ is called the *grade set* of $\mathrm{gr}_\alpha(V)$. The image of $x \in V$ in $V_{\alpha(x)}$ is denoted by x'.

Let α be a value function on a vector space V over D. Let h be a hermitian form over (D, τ) such that $\alpha \preceq h$. For $x, y \in V$ we define:

$$h'_\alpha(x, y) = \begin{cases} h(x, y)' & \text{if } \bar{v}(h(x, y)) = \alpha(x) + \alpha(y), \\ 0 & \text{if } \bar{v}(h(x, y)) > \alpha(x) + \alpha(y). \end{cases}$$

The map h'_α can be extended bi-additively to $\mathrm{gr}_\alpha(V)$. In fact this extension (also denoted by h'_α) is a hermitian form on $\mathrm{gr}_\alpha(V)$ for the graded involution τ' on $\mathrm{gr}_{\bar{v}}(D)$. The form h'_α has a substantial information about the isotropicity of h, as reflects from the following:

Theorem 2.15 [RTW, Th. 3.11] *Let* $\mathrm{char}(D_0) \neq 2$. *Then* $h \mapsto h'_\alpha$ *gives a surjection* Θ *of the Witt group* $W(D, \tau)$ *into the graded Witt group* $W_g(\mathrm{gr}_{\bar{v}}(D), \tau')$. *Further if* v *is henselian, then* Θ *is an isomorphism.*

Let $h : V \times V \to D$ be a hermitian form over (D, τ) and α be a D-norm on V such that $\alpha \prec h$. The following proposition highlights a relation between j_α and j_{α^*}, where α^* is the dual of α.

Proposition 2.16 [TW, Prop. 1.7] *If* ad_h *is the involution on* $\mathrm{End}_D(V)$ *which is adjoint to the hermitian form* h *then for all* $\phi \in \mathrm{End}_D(V)$ *we have* $j_{\alpha^*}(ad_h(\phi)) = j_\alpha(\phi)$.

As a consequence of above proposition one can derive the following

Corollary 2.17 [TW, Prop. 1.7(iii)] *If* α *is a* D*-norm on* V *such that* $j_\alpha \circ ad_h = j_\alpha$ *then there exists a constant* $\gamma \in \frac{1}{2}\Gamma_V$ *such that* $\alpha - \gamma \preceq h$.

Let V be a D-vector space, α be a D-norm on V and h be a hermitian form over (D, τ) such that $\alpha \preceq h$. Let j_α be the gauge on $\mathrm{End}_D(V)$ as in example 2.7. Then we have the following:

Lemma 2.18 *Let* h'_α *be the graded hermitian form over* $\mathrm{gr}_\alpha(V)$ *and* $ad_{h'_\alpha}$ *be the adjoint involution on* $\mathrm{End}_{\mathrm{gr}(D)}(\mathrm{gr}_\alpha(V))$. *Further, let* ad_h *be the adjoint involution on* $\mathrm{End}_D(V)$ *and* $(ad_h)'$ *be the graded involution on* $\mathrm{gr}_{j_\alpha}(\mathrm{End}_D(V))$. *Then under the isomorphism of Lemma 2.9,* $ad_{h'_\alpha} = (ad_h)'$.

Proof Let $f \in \mathrm{End}_D(V)$. With the notation as in Lemma 2.9, it suffices to show that $ad_{h'_\alpha}(\widetilde{f}) = \widetilde{(ad_h)'}(f')$. Let $x, y \in V$. Then using the isomorphism $f' \mapsto \widetilde{f}$ we have

$$\begin{aligned} h'_\alpha\left(\widetilde{(ad_h)'}(f')(x'), y'\right) &= h'_\alpha\left((ad_h(f)'(x'), y'\right) \\ &= (h(ad_h(f(x), y)))' \\ &= (h(x, f(y)))' \\ &= h'_\alpha\left(x', \widetilde{f}(y')\right) \\ &= h'_\alpha\left(ad_{h'_\alpha}(\widetilde{f})(x'), y'\right) \end{aligned}$$

Thus $h'_\alpha\left(\widetilde{(ad_h)'}(f')(x'), y'\right) = h'_\alpha\left(ad_{h'_\alpha}(\widetilde{f})(x'), y'\right)$, which implies $ad_{h'_\alpha}(\widetilde{f}) = \widetilde{(ad_h)'}(f')$. □

We say that a gauge j over an algebra A over F is *residually real* if the residue field of the valuation $j|_F = v$ on F is formally real.

2.4 Consequences of weak isotropicity

We now prove the implication (BP1) ⇒ (BP2) of BP-equivalence. As usual, F denotes a formally real field with valuation v. We begin with the following:

Proposition 2.19 *Let (A, σ) be a central simple algebra with an involution σ of the first kind. If σ is weakly isotropic then it is totally indefinite.*

Proof Let P be an ordering on F and F_P be the real closure of F at P. Let D_P be the division algebra Brauer equivalent to $A_P := A \otimes_F F_P$. Then the involution $\sigma \otimes id$ is adjoint to a hermitian form $h_P : V \times V \to D_P$ over (D_P, τ_P), where τ_P is an involution on D_P of the same type as σ.

First assume that σ is orthogonal. If $D_P \neq F_P$ then by [KMRT, Cor. 11.11(1)(a)], $\mathrm{sgn}_P(\sigma) = 0 < \deg(A)$, otherwise h_P corresponds to a weakly isotropic quadratic form over F_P and therefore h_P is indefinite. Thus by [KMRT, Cor. 11.11(2)(a)] $\mathrm{sgn}_P(\sigma) = |\mathrm{sgn}_P(h_P)| < \deg_{F_P}(A_P) = \deg(A)$. This confirms that σ is totally indefinite.

Now we finish the proof for the case when σ is symplectic. If the algebra D_P is split then by [KMRT, Cor. 11.11(2)(b)] $\mathrm{sgn}_P(\sigma) = 0 < \deg(A)$. If D_P is not split then it is the quaternion division algebra over F_P. Let $t(h_P)$ be the quadratic form over F_P defined by $t(h_P)(x) = h_P(x, x)$, $x \in V$. Then $\dim_{F_P}(t(h_P)) = \dim_{F_P}(D_P).(\dim_{D_P}(V)) = 4.(\frac{1}{2}\deg_{F_P}(A_P)) = 2\deg_{F_P}(A_P) = 2\deg(A)$. Further, the weak isotropicity of σ implies the weak isotropicity of $t(h_P)$ and thus by [KMRT, Cor. 11.11(1)(b)] we have $\mathrm{sgn}_P(\sigma) = \frac{1}{2}|\mathrm{sgn}_P(t(h_P))| < \frac{1}{2}\dim(t(h_P)) = \deg(A)$. This establishes the total indefiniteness of σ. $\square$

Now we are ready to prove the implication (BP1) $\Rightarrow$ (BP2) of BP-equivalence.

Proposition 2.20 *Let F be a formally real field and A be a central simple algebra over F. Let σ be an involution of first kind on A. If σ is weakly isotropic then it is totally indefinite and at every residually real ramified gauge j on A which is compatible with σ, the residue σ' is weakly isotropic.*

Proof That σ is weakly isotropic implies it is totally indefinite follows from Prop. 2.19. Let j be a residually real ramified gauge on A which is compatible with σ. If σ' is not weakly isotropic then by Theorem 2.13, $(A \otimes_F F_v^h, \sigma \otimes id)$ is strongly anisotropic, which is a contradiction to the hypothesis that σ is weakly isotropic. This completes the proof. $\square$

3 The classical Bröcker-Prestel Theorem

This section is devoted to a brief introduction of the theorem of Bröcker and Prestel. For a detailed account of some parts of the section, one can refer to [S, Chapter 3, §7].

A *quadratic semiordering* on F is a subset $P \subseteq F$ such that: $P + P \subseteq P$, $F^2P \subseteq P$, $1 \in P$, $P \cup -P = F$ and $P \cap -P = \{0\}$. A quadratic semiordering $P \subseteq F$ is an *ordering* if $P.P \subseteq P$. A quadratic semiordering is P is called *archimedian* if for every $a \in F$, there exists $n \in \mathbb{N}$ (depending on a) such that $n - a \in P$.

Quadratic semiorderings connect orderings and valuations in the sense that archimedian quadratic semiorderings are in fact orderings, while the non-archimedian ones give rise to a valuation on F. A quadratic form is weakly isotropic if and only if it is indefinite with respect to all quadratic semiorderings [S, Th. 7.6]. Carefully looking at archimedian and non-archimedian quadratic semiorderings, many technical arguments yield the Theorem 1.1 as stated in the introduction.

We translate Theorem 1.1 in the language of value functions defined in the section 2.

Theorem 3.1 *Let F be a formally real field and V be a vector space over F. Let q be a quadratic form on V. Let v be a valuation on F with formally real residue field F_0. Then we have the following:*

1. *The quadratic form q is ramified at v if and only if there exists a ramified gauge j on $\mathrm{End}_F(V)$ which is compatible with ad_q.*

2. *Let j be a gauge on $\mathrm{End}_F(V)$ which is compatible with ad_q. Let $(ad_q)'$ be the graded involution with respect to j. Then the residues of q at v are strongly anisotropic over F_0 if and only if the involution $(ad_q)'$ is strongly anisotropic.*

Proof We first prove *1.* Let q be ramified. Let $\{e_i\}$ be an orthogonal basis of q and $a_i = q(e_i) \in F^*$. Since q is ramified at v, by a permutation of basis, we may assume that $v(a_1) \neq v(a_2) \in \Gamma_F/2\Gamma_F$, that is $v(a_1) - v(a_2) \notin 2\Gamma_F$. Let $\alpha = \alpha_q$ be the D-norm on V defined by $\alpha(e_i) = \frac{1}{2}v(a_i)$. It is clear that α is ramified and $\alpha \preceq q$. Let j_α be the gauge on $\mathrm{End}_F(V)$ as defined in 2.7. Since α is ramified, by Prop. 2.8 it follows that j_α is ramified. That j_α satisfies $j_\alpha \circ ad_q = j_\alpha$ follows from Proposition 2.16 in view of the fact that $\alpha \preceq q$ if and only if $\alpha = \alpha^*$.

Conversely, suppose j is a ramified gauge on $\mathrm{End}_F(V)$ such that $j \circ ad_q = j$. By [TW1, Cor. 1.15], $j \otimes v^h$ is a gauge on $\mathrm{End}_{F_v^h}(V \otimes_F F_v^h)$, where v^h is the extension of v to the henselisation F_v^h. Further by [TW1, Prop. 3.3], there exists a F_v^h-norm α on $V \otimes_F F_v^h$ such that $j \otimes v^h = j_\alpha$ where j_α is as in the example 2.7. From the hypothesis $j \circ ad_q = j$, it follows immediately that $j_\alpha \circ ad_{q^h} = j_\alpha$. In view of Corollary 2.17 we may choose α such that $\alpha \preceq h$. Further, under the composed map $W(F) \to W(F_v^h) \to W_{\mathrm{gr}}(\mathrm{gr}_v(F))$ the class $[q] \in W(F)$ of the quadratic form q goes to $[(q_{F_v^h})'_\alpha]$. Since j is ramified, $j \otimes v^h = j_\alpha$ is ramified. From 2.7, it is clear that α is ramified. This, together with the definition of the bilinear form $(q_{F_v^h})'_\alpha$ implies that $(q_{F_v^h})'_\alpha$ has more than one non-zero component. It follows from isomorphisms of [RTW, Prop. 1.5] that $W_{\mathrm{gr}}(\mathrm{gr}_v(F)) \simeq \bigoplus_{\Gamma_F/2\Gamma_F} W(F_0)$ and that under this isomorphism the graded form $(q_{F_v^h})'_\alpha$ corresponds to various residues of q at v. In view of this and the deduction above that $(q_{F_v^h})'_\alpha$ has more than one non-zero component, it follows that q is ramified at v.

We now prove *2.* By [TW1, Prop 3.3], $j \otimes F_v^h = j_\alpha$ for some F_v^h-norm α on $V \otimes F_v^h$ such that $\alpha \preceq q \otimes F_v^h$. First suppose that all residues of q are strongly anisotropic. Then by Springer's theorem, q is strongly anisotropic. In view of this and [TW1, Prop. 4.3, Th. 4.6], the graded bilinear form q'_α on the graded vector space $\mathrm{gr}_\alpha(V \otimes_F F_v^h)$ is strongly anisotropic. Thus the adjoint graded involution $ad_{q'_\alpha}$ is strongly anisotropic over the graded algebra $\mathrm{End}_{\mathrm{gr}(F_v^h)}(\mathrm{gr}_\alpha(V \otimes_F F_v^h))$. By [TW1, Lemma 1.10], we have an isomorphism

$$\mathrm{gr}_{j_\alpha}(\mathrm{End}_{F_v^h}(V \otimes_F F_v^h)) \simeq \mathrm{End}_{\mathrm{gr}(F_v^h)}(\mathrm{gr}_\alpha(V \otimes_F F_v^h)).$$

By Lemma 2.18, the graded involution $ad_{q'_\alpha}$ on the graded algebra $\mathrm{End}_{\mathrm{gr}(F_v^h)}(\mathrm{gr}_\alpha(V \otimes_F F_v^h))$ corresponds to the graded involution $(ad_q)'$ on $\mathrm{gr}_{j_\alpha}(\mathrm{End}_{F_v^h}(V \otimes_F F_v^h)) \simeq \mathrm{gr}_j(\mathrm{End}_F(V))$ under the isomorphism above. Thus $(ad_q)'$ is strongly anisotropic. The converse holds by tracing back the same arguments. □

Corollary 3.2 *Let (V, q) be a quadratic form over a formally real field F and ad_q be the involution on $\mathrm{End}_F(V)$ which is adjoint to q. Then BP-equivalence holds for $(\mathrm{End}_F(V), ad_q)$.*

Proof The implication (BP1) ⇒ (BP2) of BP-equivalence follows from Prop. 2.20. To prove (BP2) ⇒ (BP1), we assume that ad_q is strongly anisotropic and totally indefinite. By the classical

Bröcker-Prestel Theorem (Theorem 1.1) there exists a valuation v on F with formally real residue field such that q is ramified at v and all residues of q at v are strongly anisotropic. Thus by Theorem 3.1, there exists a ramified gauge j on $\mathrm{End}_F(V)$ which is residually real and strongly ad_q-special. This is a contradiction to the hypothesis (ii), and this completes the proof. □

4 Symplectic case

Let $(H, -)$ be a quaternion algebra with standard involution over F and $h : V \times V \to D$ be a hermitian form over $(H, -)$. Let ad_h denote the adjoint involution on $\mathrm{End}_H(V)$. The main result of this section is the following

Theorem 4.1 *Let A be a central simple algebra over a formally real field F. Let σ be an involution of symplectic type on A and* $\mathrm{index}(A) = 2$. *Then BP-equivalence holds for (A, σ).*

Proof The implication (BP1) ⇒ (BP2) of BP-equivalence follows from Prop. 2.20. We keep the same notation as above. To prove $(ii) \Rightarrow (i)$, assuming that the involution ad_h is strongly anisotropic and totally indefinite, we show the existence of a residually real ramified gauge j which is strongly ad_h-special.

To the hermitian form h over $(H, -)$, we associate the quadratic form $t(h) : V \to F$ given by $t(h)(x) = h(x, x)$. Let $H = (a, b)_F$; $a, b \in F^*$. Let $\{e_i\}$ be an orthogonal basis of V and $h(e_i, e_i) = a_i$. Since $h(e_i, e_i)$ are all symmetric, each $a_i \in F^*$ and the hermitian form h is represented by the diagonal $\langle a_1, a_2, \cdots, a_r\rangle$; $a_i \in F^*$. Further, $q_h \simeq \langle 1, -a, -b, ab\rangle \otimes \langle a_1, a_2, \cdots, a_r\rangle$.

Since h is strongly anisotropic and totally indefinite, by Theorem 1.1, there exists a valuation v on F with respect to which q_h has at least two residues and all of them are strongly anisotropic. Thus by Springer's theorem, the quadratic form q_h is strongly anisotropic over the henselisation F_v^h of F with respect to v. Since $\langle 1, -a, -b, ab\rangle$ is a subform of q_h, it follows that the algebra $H_v = H \otimes_F F_v^h$ is division. Thus by [M, Th. 1.2], the valuation $v : F \to \Gamma_F \cup \{\infty\}$ on F extends to a valuation on $H \otimes_F F_v^h$. We denote this valuation on $H_h = H \otimes_F F_v^h$ by $\bar{v}$. Now, with $\{e_i \otimes 1\}$ as a splitting basis we define a H_h-norm $\alpha : V \otimes_F F_v^h \to \Gamma_F \otimes \mathbb{Q} \cup \{\infty\}$ as follows:

$$\alpha\left(\sum_i (e_i \otimes \lambda_i)\right) = \min_i\{\frac{1}{2}\bar{v}(h(e_i, e_i)) + \bar{v}(\lambda_i)\}$$

It is clear that $\alpha \preceq h \otimes F_v^h$. Further, that α is ramified is immediate by the definition and the fact that $q_h \otimes F_v^h$ is ramified. Since $h \otimes F_v^h$ is strongly anisotropic, it follows from [RTW, Prop 4.3 and Prop 4.6] that $(h \otimes F_v^h)'_\alpha$ is strongly anisotropic. Therefore by [RTW, Prop. 4.2], $\alpha(x) = \frac{1}{2}\bar{v}(h(x, x))$ for all $x \in V \otimes_F F_h$. Now by the proofs of [TW, Th. 2.1] and Cor. 2.13, the gauge j_α on $\mathrm{End}_H(V) \otimes_F F_v^h$ is strongly $ad_{h\otimes F_v^h}$-special. Let $j = j_\alpha|_{\mathrm{End}_H(V)}$. Then $j \otimes v_h$ is strongly $ad_h \otimes id$-special gauge on $\mathrm{End}_H(V) \otimes_F F_v^h$ and by the unicity statement in 2.13, $j_\alpha = j \otimes v_h$. Now it follows from Prop. 2.14 that j is strongly ad_h-special, and hence the residue $(ad_h)'$ of ad_h at j is strongly anisotropic. □

5 Orthogonal case

Let h be a hermitian form over a quaternion algebra H with an orthogonal involution τ. Let ad_h be the adjoint involution on $\mathrm{End}_H(V)$. Then ad_h is an involution of orthogonal type and every involution on $\mathrm{End}_H(V)$ is obtained in this way. In this section we prove Theorem 1.2 for $(\mathrm{End}_H(V), ad_h)$.

Let C denote the associated conic of the quaternion algebra H and $K = F(C)$ denote the function field of C. If $H = (a, b)_F$ then $K = F(C) = \frac{F[u,v]}{(au^2+bv^2-1)}$ and $H \otimes_F K$ is split. Further, the hermitian form $h_K = h \otimes K$ corresponds by Morita theory to a quadratic form q over K. We recall the following:

Theorem 5.1 [PSS] *With the notation as above, if h is (strongly) anisotropic over (H, σ) then q is (strongly) anisotropic over K.*

Remark Let h be totally indefinite. Then the hermitian form h_K is also totally indefinite, because h_K is defined over F and is totally indefinite at F. Thus the quadratic form q is totally indefinite over K as well.

Lemma 5.2 *A gauge j on a central simple F-algebra A is ramified if and only if $[A_0 : F_0] < [A : F]$.*

Proof This follows from the dimension computation of $[\mathrm{gr}_j(A) : \mathrm{gr}(F)]$. Let the cosets of Γ_A modulo Γ_F be represented by $\gamma_A = \{\gamma_i\} \subseteq \Gamma_A$. We choose γ_A so that $0 \in \gamma_A$. Now by [TW, §2, eq. (2.4)] we have:

$$[\mathrm{gr}_j(A) : \mathrm{gr}_v(F)] = \sum_{\gamma_i \in \gamma_A} [A_{\gamma_i} : F_0]$$

By definition j is ramified if and only if $|\gamma_A| \geq 2$, which in view of the above equation is equivalent to $[A_0 : F_0] < [\mathrm{gr}_j(A) : \mathrm{gr}_v(F)]$. Since j is a gauge, we have $[\mathrm{gr}_j(A) : \mathrm{gr}_v(F)] = [A : F]$ and the lemma follows immediately. □

Lemma 5.3 *Let A be a central simple algebra over F and K/F be a field extension. Let j be a gauge on A and w be any extension of valuation $v = j|_F$ to K. If $j \otimes w$ is ramified on $A \otimes_F K$ then j is ramified on A.*

Proof Let A_0 denote the residue $\{a \in A : j(a) \geq 0\}/\{a \in A : j(a) > 0\}$ and $(A \otimes_F K)_0$ denote the residue $\{b \in A \otimes_F K : (j \otimes w)(b) \geq 0\}/\{b \in A \otimes_F K : (j \otimes w)(b) > 0\}$. Since $j \otimes w$ is ramified, by Lemma 5.2 we have

$$[A \otimes_F K : K] > [(A \otimes_F K)_0 : K_0]$$

We now observe that $A_0 \otimes_{F_0} K_0 \subseteq (A \otimes_F K)_0$. Thus

$$[A : F] = [A \otimes_F K : K] > [(A \otimes_F K)_0 : K_0] \geq [A_0 \otimes_{F_0} K_0 : K_0] = [A_0 : F_0]$$

and hence $[A : F] > [A_0 : F_0]$. This in view of Lemma 5.2 confirms that j is ramified. □

Theorem 5.4 *Let A be a central simple algebra over a formally real field F. Let σ be an involution of orthogonal type on A and* $\operatorname{index}(A) = 2$. *Then BP-equivalence holds for (A, σ).*

Proof The proof of the implication (BP1) $\Rightarrow$ (BP2) of BP-equivalence follows from Proposition 2.20. We prove the other implication (using the same notation as above). Let ad_h be totally indefinite and strongly anisotropic. It is clear from the remark above and Theorem 5.1 that the involution $ad_h \otimes id$ on $\operatorname{End}_H(V) \otimes_F K$ is totally indefinite and strongly anisotropic. Thus by Theorem 3.2 there exists valuation w on K and a ramified w-gauge g on $(\operatorname{End}_H(V), ad_h) \otimes_F K$ which has formally real residue field and is strongly $(ad_h \otimes id)$-special. It is clear from Theorem 2.13 that $(\operatorname{End}_H(V), ad_h) \otimes_F K_w^h$ is strongly anisotropic, where K_w^h denotes the henselisation of K at the valuation w. Let $v = w|_F$. Then v is a valuation on F with $F_v^h \subseteq K_w^h$ and clearly $(\operatorname{End}_H(V), ad_h) \otimes_F F_v$ is strongly anisotropic.

Now again by Theorem 2.13, it follows that there exists a strongly ad_h-special gauge on j on $\operatorname{End}_H(V)$. Further, from the proof of Theorem 2.4 and its unicity statement, it follows that $j = g|_{\operatorname{End}_H(V)}$ and by Prop. 2.11, $g = j \otimes w$. Now by Lemma 5.3, we conclude that j is ramified. Thus j is a ramified gauge on $\operatorname{End}_H(V)$ which has formally real residue field and the residue ad_h' is strongly anisotropic. □

6 Trace forms and involutions

It was pointed out by Unger in his thesis that there are central simple algebras A with involution σ which satisfy the following: $\mathfrak{C}$:*'σ is weakly isotropic if and only if its trace form T_σ is weakly isotropic'*. Such examples include multiquaternion algebras with totally decomposable involutions of first kind. For the algebras with involutions (A, σ) belonging to this class, we prove the following:

Theorem 6.1 *If (A, σ) satisfies $\mathfrak{C}$ then BP-equivalence holds for (A, σ).*

Proof The implication (BP1) ⇒ (BP2) follows from Theorem 2.20. For the rest, we first make a claim:

Claim: *Let (A, σ) be strongly anisotropic and totally indefinite. Then A carries a residually real ramified gauge which is strongly σ-special.*

Proof of the claim: The trace form T_σ of σ is adjoint to the involution $\sigma \otimes \sigma$ on $A \otimes_F A \simeq \mathrm{End}_F(A)$. Since σ is totally indefinite and $\mathrm{sgn}(\sigma \otimes \sigma) = \mathrm{sgn}(\sigma)^2$, it follows that $\sigma \otimes \sigma$ is totally indefinite. Further, since (A, σ) satisfies $\mathfrak{C}$, the form T_σ is strongly anisotropic. Thus, as in the proof of 3.1, there exists a ramified gauge g on $A \otimes A$ which is strongly $\sigma \otimes \sigma$-special.

Let A be embedded into $A \otimes_F A$ via embeddings $a \mapsto a \otimes 1$ and $a \mapsto 1 \otimes a$. Through these embedding, we restrict g to functions $j_1 = g|_{A\otimes 1}$ and $j_2 = g|_{1\otimes A}$, respectively. It follows from Prop. 2.10 that $j = j_1 \otimes j_2$. Now that one of j_1 and j_2 is ramified follows from [§2.2,(1)]:

$$g(x) = j_1 \otimes j_2(x) = \sup\{\min_{1\leq i\leq r} (j_1(a_i) + j_2(b_i)) : x = \sum_{i=1}^{r} a_i \otimes b_i\}$$

and the fact that g is ramified.

We now prove $(ii) \Rightarrow (i)$. Let (A, σ) be strongly anisotropic and totally indefinite. By the above discussion, there exists a ramified gauge j on A which is strongly σ-special and has formally real residue field. We consider the residue of σ at j. Now by Theorem 2.13 this residue σ' is strongly anisotropic. □

7 An induction step

Lemma 7.1 *Let (A, σ) be a central simple algebra with an involution of first kind and (H, θ) be a quaternion algebra with an involution of first kind, then $(A \otimes_F H, \sigma \otimes \theta)$ is strongly anisotropic if and only if $(A \otimes_F H \otimes_F H, \sigma \otimes \theta \otimes \theta) \simeq (A \otimes End_F(H), \sigma \otimes ad_{T_\theta})$ is strongly anisotropic.*

Proof It is clear that if $(A \otimes_F H, \sigma \otimes \theta)$ is weakly isotropic, then $(A \otimes_F H \otimes_F H, \sigma \otimes \theta \otimes \theta)$ is so. For the converse, assume that $(A \otimes_F H \otimes_F H, \sigma \otimes \theta \otimes \theta)$ is weakly isotropic. We make the following identification:

$$A \otimes_F H \otimes_F H = A \otimes_F \mathrm{End}_F(H) \simeq \mathrm{End}_A(H \otimes A)$$

via $a \otimes f \mapsto f_a$, where $f_a.(u \otimes b) = f(u) \otimes ab$. Via this identification, let $\sigma \otimes \theta \otimes \theta = \sigma \otimes ad_{T_\theta}$ correspond to an involution τ on $\mathrm{End}_A(H \otimes A)$. It follows that $\tau = ad_h$ for a weakly isotropic hermitian form $h : (H \otimes A) \times (H \otimes A) \to A$ over (A, σ). Thus $ad_h = \sigma \otimes ad_{T_\theta}$. We claim that the restriction $h|_{H\otimes 1} : (H \otimes 1) \times (H \otimes 1) \to A$ takes values in F. Indeed, if $a \in A$ and $x, y \in H$ are arbitrary, then we have:

$$\begin{aligned}
ah(x\otimes 1, y\otimes 1) &= h(x\otimes\sigma(a), y\otimes 1)\\
&= h((\sigma(a)\otimes id).(x\otimes 1), y\otimes 1)\\
&= h(x\otimes 1, ad_h(\sigma(a)\otimes id)(y\otimes 1))\\
&= h(x\otimes 1, \sigma\otimes ad_{T_\theta}(\sigma(a)\otimes id)(y\otimes 1))\\
&= h(x\otimes 1, (a\otimes id).(y\otimes 1))\\
&= h(x\otimes 1, y\otimes a)\\
&= h(x\otimes 1, y\otimes 1)a
\end{aligned}$$

Thus $h(x\otimes 1, y\otimes 1)\in Z(A)=F$. Further, for an arbitrary $f\in \mathrm{End}_F(H)$ we have

$$h|_{H\otimes 1}((1\otimes f)(x\otimes 1, y\otimes 1)) = h|_{H\otimes 1}(x\otimes 1, (1\otimes T_\theta(f))(y\otimes 1))$$

and it follows that $ad_{T_\theta} = ad_h|_{\mathrm{End}_F(H)} = ad_{h|_{H\otimes 1}}$. Thus $h|_{H\otimes 1}$ and T_θ are similar forms. Now let $\{1=i_0, i_1, i_2, i_3\}$ be a quaternion basis of H such that $\theta(i_k)=\pm i_k$ for $1\le k\le 4$. Let $x=\sum_r i_r\otimes a_r$, where $a_0, a_1, a_2, a_3\in A$ are arbitrary. Then (upto a scalar factor) we have:

$$\begin{aligned}
h(x,x) &= \sum_{r,s}\sigma(a_r)h(i_r\otimes 1, i_s\otimes 1)a_s\\
&= \sum_{r,s}\sigma(a_r)T_\theta(i_r, i_s)a_s
\end{aligned}$$

Since $T_\theta(i_r, i_s)=\mathrm{Trd}_H(\theta(i_r)i_s)$, we get

$$\begin{aligned}
h(x,x) &= 2\sigma(a_0)a_0 + 2\theta(i_1)i_1\sigma(a_1)a_1 + 2\theta(i_2)i_2\sigma(a_2)a_2 + 2\theta(i_3)i_3\sigma(a_3)a_3\\
&= 2((\sigma\otimes\theta)(a_0\otimes 1))(a_0\otimes 1) + 2((\sigma\otimes\theta)(a_1\otimes i_1))(a_1\otimes i_1)\\
&\quad + 2((\sigma\otimes\theta)(a_2\otimes i_2))(a_2\otimes i_2) + 2((\sigma\otimes\theta)(a_3\otimes i_3))(a_3\otimes i_3)
\end{aligned}$$

Since h is weakly isotropic, $\sum_i h(x_i, x_i)=0$ for some $x_i\in H\otimes A$. From the above computation it is clear that $\sigma\otimes\theta$ is weakly isotropic. This completes the proof. □

Theorem 7.2 *If BP-equivalence holds for all central simple algebras of index n with an involution of the first kind then it holds for every $(A\otimes H, \sigma\otimes\theta)$ where (A,σ) is an index n central simple algebra with σ of the first kind and (H,θ) is a quaternion algebra with θ of the first kind.*

Proof Let (A,σ) be an index n central simple algebra with σ of the first kind and (H,θ) be a quaternion algebra with θ of the first kind. We prove that $(A\otimes H, \sigma\otimes\theta)$ satisfies BP-equivalence. The implication $(i)\Rightarrow(ii)$ of BP-equivalence holds by Theorem 2.20. To prove $(ii)\Rightarrow(i)$, we assume that $(A\otimes H, \sigma\otimes\theta)$ is strongly anisotropic and totally indefinite and prove the existence of a residually real and ramified strongly $\sigma\otimes\theta$-special gauge. It is clear that $(A\otimes H\otimes H, \sigma\otimes\theta\otimes\theta)$

is totally indefinite and in view of Lemma 7.1, it is strongly anisotropic as well. Since index($A \otimes H \otimes H$) = index(A) = n, the pair $(A \otimes H \otimes H, \sigma \otimes \theta \otimes \theta)$ satisfies BP-equivalence by hypothesis and hence $A \otimes H \otimes H$ carries a strongly $\sigma \otimes \theta \otimes \theta$-special gauge g. Let $j = g|_{A \otimes H}$. Let $A \otimes H$ be embedded into $A \otimes H \otimes H$ via $a \otimes x \mapsto a \otimes x \otimes 1$ and $a \otimes x \mapsto a \otimes 1 \otimes H$. Let j_1 and j_2 be restrictions of g to $A \otimes H$ via these embeddings respectively. By Prop. 2.14, j is strongly σ-special and j_1 and j_2 are strongly θ-special and $j \otimes j_1$ is strongly $\sigma \otimes \theta$-special. Thus by Prop. 2.13, $j_1 = j_2$. Further by Prop. 2.14, $g = j \otimes j_1 \otimes j_1$. From this and the fact that g is ramified, we conclude that $j \otimes j_1$ is ramified. □

Corollary 7.3 *Let $\{(H_i, \sigma_i)\}_{i=1}^r$ be quaternion algebra over a formally real field with involutions of the first kind. Then $(\otimes_{i=1}^r H_i, \otimes_{i=1}^r \sigma_i)$ satisfies BP-equivalence.*

Proof Since quaternion algebras with an involution of first kind satisfy BP-equivalence (Theorems 4.1 and 5.4), the corollary follows from Theorem 7.2. □

Remark Since a totally decomposible involution on a multiquaternion algebra is weakly isotropic if and only if its trace form is so [U, Th. 5.20], the corollary 7.3 follows from Theorem 6.1 as well. Also the symplectic case of index 2 (Theorem 4.1) follows from Theorem 7.2 and Theorem 3.2.

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`amit@math.ucl.ac.be`
Département de mathématique
Université catholique de Louvain
B-1348 Louvain-la-Neuve, Belgium.