LIFTING OF COEFFICIENTS FOR CHOW MOTIVES OF QUADRICS

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ABSTRACT. We prove that the natural functor from the category of Chow motives of smooth projective quadrics with integral coefficients to the category with coefficients modulo 2 induces a bijection on the isomorphism classes of objects.

CONTENTS

1. INTRODUCTION

A. Vishik has given a description of the Chow motives of quadrics with integral coefficients in [4]. It uses much subtler methods than the ones used to give a similar description with coefficients in $\mathbb{Z}/2$, found for example in [1], but the description obtained is the same. The result presented here allows to recover Vishik's results from the modulo 2 description.

In order to state the main result, we first define the categories involved. Let Λ be a commutative ring. We write $\mathcal{Q}_F$ for the class of smooth projective quadrics over a field F. We consider the additive category $\mathcal{C}(\mathcal{Q}_F, \Lambda)$, where objects are quadrics in $\mathcal{Q}_F$ and if X, Y are two such quadrics, $\mathrm{Hom}(X, Y)$ is the group of correspondences of degree 0, namely $\mathrm{CH}_{\dim X}(X \times Y, \Lambda)$. We write $\mathcal{CM}(\mathcal{Q}_F, \Lambda)$ for the idempotent completion of $\mathcal{C}(\mathcal{Q}_F, \Lambda)$. This is the category of graded Chow motives of smooth projective quadrics with coefficients in Λ. If $(X, \rho), (Y, \sigma)$ are two such motives then we have :

$$\mathrm{Hom}((X, \rho), (Y, \sigma)) = \sigma \circ \mathrm{CH}_{\dim X}(X \times Y, \Lambda) \circ \rho.$$

We will prove the following :

Theorem 1. *The functor $\mathcal{CM}(\mathcal{Q}_F, \mathbb{Z}) \to \mathcal{CM}(\mathcal{Q}_F, \mathbb{Z}/2)$ induces a bijection on the isomorphism classes of objects.*

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The proof mostly relies on the low rank of the homogeneous components of the Chow groups of quadrics when passing to a splitting field. These components are almost always indecomposable if we take into account the Galois action. The only exception is the component of rank 2 when the discriminant is trivial but in this case the Galois action on the Chow group is trivial which allows the proof to go through. It seems that this method doesn't generalize to other projective homogeneous varieties, at least to other quadratic grassmannians since in this case the ranks of the homogeneous components of the Chow groups are not bounded.

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2. Chow groups of quadrics

We first recall some facts and fix the notations that we will use.

Let F be a field and φ be a non-degenerate quadratic form on a F-vector space V of dimension $D+2$. The associated projective quadric X is smooth of dimension $D = 2d$ or $2d+1$. Let L/F be a splitting extension for X, *i.e.* an extension such that V_L has a totally isotropic space of dimension $d+1$. We write h^i, l_i for the usual basis of $\mathrm{CH}(X_L)$, where $0 \leq i \leq d$. The class h is the pull-back of the hyperplane class of the projective space of V_L, the class l_i is the class of a totally isotropic subspace of V_L of dimension $i+1$.

If D is even, then $\mathrm{CH}_d(X_L)$ is freely generated by h^d and l_d. There are exactly two classes of maximal totally isotropic spaces, l_d and $l_d{}'$. They correspond to spaces exchanged by a reflection and verify the relation $l_d + l_d{}' = h^d$.

The group $\mathrm{Aut}(L/F)$ acts on $\mathrm{CH}(X_L)$. It acts trivially on the i-th homogeneous components of $\mathrm{CH}(X_L)$, as long as $2i \neq D$.

Another scheme that may be associated to a quadratic form φ is $G(\varphi)$, the scheme of maximal totally isotropic subspaces of V. We can view it as a closed subscheme of the Grassmannian variety of V.

When D is even, if L is a splitting field for φ, then $G(\varphi)_L$ has two connected components exchanged by any reflection. We orient X and $G(\varphi)$ accordingly: a totally isotropic space whose class is l_d lies in the component G of $G(\varphi)_L$, we call G' the other component. See [1] for proofs of all these facts.

If Z is a scheme over a field F and L/F an extension we say that a cycle in $\mathrm{CH}(Z_L)$ is rational if it belongs to the image of the pull-back homomorphism $\mathrm{CH}(Z) \to \mathrm{CH}(Z_L)$.

In the next proposition, X is a smooth projective quadric of dimension $D = 2d$ associated to a quadratic space (V, φ), L/F is a splitting extension for X, and $\operatorname{disc} X$ is the discriminant algebra of φ.

Proposition 2. *Under the natural* $\mathrm{Aut}(L/F)$*-actions, we can identify the pair* $\{l_d, l_d{}'\}$ *and the connected components of* $\mathrm{Spec}(\operatorname{disc} X \otimes L)$.

Proof. Consider the closed subscheme E of $V \times G(\varphi)$ consisting of pairs (u, U) such that $u \in U$. It is vector bundle of rank d over $G(\varphi)$. The projective bundle $\mathbf{P}(E)$ can be seen as a closed subscheme of $X \times G(\varphi)$. There are unique elements

$e_k \in \mathrm{CH}^k(G(\varphi)_L)$, $e'_0 \in \mathrm{CH}^0(G(\varphi)_L)$ such that :

$$[\mathbf{P}(E)] = l_d \times [G] + {l_d}' \times [G'] + \sum_{k=1}^{d} h^{d-k} \times e_k$$

It is proven in [1] that the products $p = [G]e_1...e_d$ and $p' = [G']e_1...e_d$ are classes of rational points over L, and that they form a basis of $\mathrm{CH}_0(G(\varphi)_L)$. For $k = 1, .., d$, the cycle e_k is invariant under $\mathrm{Aut}(L/F)$. Therefore the group $\mathrm{Aut}(L/F)$ acts on $\{p, p'\}$ the way it acts on the connected components $\{G, G'\}$.

Taking pull-backs, we see that $[\mathbf{P}(E)]^*(p) = l_d$ and $[\mathbf{P}(E)]^*(p') = {l_d}'$. Moreover the correspondence $[\mathbf{P}(E)]$ is rational, therefore is equivariant under the action of $\mathrm{Aut}(L/F)$. Hence, as an $\mathrm{Aut}(L/F)$-set, the pair $\{l_d, {l_d}'\}$ is isomorphic to $\{p, p'\} \simeq \{G, G'\}$. But the scheme $G(\varphi)$ has a morphism to $\mathrm{Spec}(\mathrm{disc}\, X)$ (see [1]) and the classes of G and G' in $\mathrm{CH}(G(\varphi)_L)$ are the pull-backs of the classes of the connected components of $\mathrm{Spec}(\mathrm{disc}\, X \otimes L)$. The statement follows.

Here is another proof. Let γ be in $\mathrm{Aut}(L/F)$, W be a maximal totally isotropic space of V_L. Choose an isometry f sending W to ${}^{\gamma}W$. We may assume that the group of isometries of φ_L is generated by reflections : The only case where the Cartan-Dieudonné theorem does not apply is when $L = \mathbb{Z}/2\mathbb{Z}$ (and $D = 2$). In this case the quadric is already split over F and there is nothing to prove.

In any other case f can be written as a product of m reflections. The integer m does depend on f and on its decomposition, but its parity only depends on γ : If W lies in a connected component C of $G(\varphi)_L$, then ${}^{\gamma}W$ lies in the component ${}^{\gamma}C$. Since these components are exchanged by any reflection, ${}^{\gamma}C = C$ if and only if m is even.

As already noticed these components are in natural correspondence with those of $\mathrm{Spec}(\mathrm{disc}\, X \otimes L)$. On the other hand $[W] = [f(W)] \in \mathrm{CH}(X_L)$ if and only if m even, which completes the proof. □

3. Lifting of coefficients

We now give a useful caracterization of rational cycles (Proposition 7). The proof will rely on the following theorem ([3], Proposition 9):

Theorem 3 (Rost's nilpotence for quadrics)**.** *Let X be a smooth projective quadric over F, and let $\alpha \in \mathrm{End}_{\mathcal{CM}(\mathcal{Q}_F,\Lambda)}(X, \mathrm{id})$. If $\alpha_L \in \mathrm{CH}(X_L^2)$ vanishes for some field extension L/F, then α is nilpotent.*

Proofs for the following corollaries can be found in [1] :

Corollary 4. *Let X be a smooth projective quadric over F and L/F a field extension. Let π a projector in $\mathrm{End}_{\mathcal{CM}(\mathcal{Q}_L,\Lambda)}(X_L)$ that is the restriction of some element in $\mathrm{End}_{\mathcal{CM}(\mathcal{Q}_F,\Lambda)}(X)$. Then there exist a projector φ in $\mathrm{End}_{\mathcal{CM}(\mathcal{Q}_F,\Lambda)}(X)$ such that $\varphi_L = \pi$.*

Corollary 5. *Let $f : (X, \rho) \to (Y, \sigma)$ be a morphism in $\mathcal{CM}(\mathcal{Q}_F, \Lambda)$. If f_L is an isomorphism for some field extension L/F then f is an isomorphism*

Here we use the ideas found in [6] and [2].

Proposition 6. *For any $n \geq 1$, the functor $\mathcal{CM}(\mathcal{Q}_F, \mathbb{Z}/2^n) \to \mathcal{CM}(\mathcal{Q}_F, \mathbb{Z}/2)$ is bijective on the isomorphism classes of objects.*

Proof. This is just reformulating Corollary 2.7 of [2]. □

Any smooth projective quadric admits a (non-canonical) finite Galois splitting extension, of degree a power of 2. This will be used together with the following proposition in order to prove rationality of cycles.

Proposition 7. *Let $X, Y \in \mathcal{Q}_F$ and L/F be a splitting Galois extension of degree m for X and Y. A correspondence in* $\mathrm{CH}((X \times Y)_L, \mathbb{Z})$ *is rational if and only if it is invariant under the group* $\mathrm{Gal}(L/F)$ *and its image in* $\mathrm{CH}((X \times Y)_L, \mathbb{Z}/m)$ *is rational.*

Proof. We write Z for $X \times Y$. We first prove that if d is a $\mathrm{Gal}(L/F)$-invariant cycle in $\mathrm{CH}(Z_L)$ then $[L : F]d$ is rational.

Let $\tau : L \to \bar{F}$ be a separable closure so that we have a $\bar{F}$-isomorphism $L \otimes \bar{F} \to \bar{F} \times ... \times \bar{F}$ given by $u \otimes 1 \mapsto (\tau \circ \gamma(u))_{\gamma \in \mathrm{Gal}(L/F)}$.

We have a fibrered square :

$$\begin{tikzcd} Z_L & Z_{L\otimes\bar{F}} \arrow[l] \arrow[d] \\ Z \arrow[u, leftarrow] & Z_{\bar{F}} \arrow[l] \end{tikzcd}$$

It follows that we have a commutative diagram of pull-backs and push-forwards:

$$\begin{tikzcd} \mathrm{CH}(Z_L) \arrow[r] \arrow[d] & \mathrm{CH}(Z_{L\otimes\bar{F}}) \arrow[d] \\ \mathrm{CH}(Z) \arrow[r] & \mathrm{CH}(Z_{\bar{F}}) \end{tikzcd}$$

The top map followed by the map on the right is:

$$x \mapsto \sum_{\gamma \in \mathrm{Gal}(L/F)} t^*({}^{\gamma}x)$$

where $t : Z_{\bar{F}} \to Z_L$ is the map induced by τ. Using the commutativity of the diagram and the injectivity of t^*, we see that the composite $\mathrm{CH}(Z_L) \to \mathrm{CH}(Z) \to \mathrm{CH}(Z_L)$ maps x to $\sum {}^{\gamma}x$, where γ runs in $\mathrm{Gal}(L/F)$. The claim follows.

Now suppose that u is a cycle in $\mathrm{CH}(Z_L, \mathbb{Z})$ invariant under $\mathrm{Gal}(L/F)$, and that its image in $\mathrm{CH}(Z_L, \mathbb{Z}/m)$ is rational. We can find a rational cycle v in $\mathrm{CH}(Z_L, \mathbb{Z})$ and a cycle δ in $\mathrm{CH}(Z_L, \mathbb{Z})$ such that $m\delta = v - u$. Since $\mathrm{CH}(Z_L, \mathbb{Z})$ is torsion-free, δ is invariant under $\mathrm{Gal}(L/F)$. The first claim ensures that $v - u$ is rational, hence u is rational. □

Let us remark that if $X \in \mathcal{Q}_F$, L/F is a splitting extension, and $2i < \dim X$ then $2l_i = h^{\dim X - i} \in \mathrm{CH}(X_L)$ is always rational. It follows that $2\,\mathrm{CH}_i(X_L)$ consists of rational cycles when $2i \neq \dim X$.

4. Surjectivity in main Theorem

Proposition 8. *The functor* $\mathcal{CM}(\mathcal{Q}_F, \mathbb{Z}) \to \mathcal{CM}(\mathcal{Q}_F, \mathbb{Z}/2)$ *is surjective on the isomorphism classes of objects.*

Proof. Let $(X, \pi) \in \mathcal{CM}(\mathcal{Q}_F, \mathbb{Z}/2)$ and L/F a finite splitting Galois extension for X of degree 2^n. By Proposition 6, we can lift (X, π) to some $(X, \tau) \in \mathcal{CM}(\mathcal{Q}_F, \mathbb{Z}/2^n)$. It will be enough to build a projector $\rho \in \mathrm{CH}(X \times X, \mathbb{Z})$ which reduces modulo 2^n to τ. We will find a $\mathrm{Gal}(L/F)$-invariant projector in $\mathrm{CH}_{\dim X}(X \times X)_L$ which gives modulo 2^n the projector τ_L. Then Proposition 7 will give a cycle in $\mathrm{CH}(X \times X, \mathbb{Z})$ lifting τ. This cycle may be chosen to be a projector by Corollary 4.

For any commutative ring Λ, projectors in $\mathrm{CH}((X \times X)_L, \Lambda)$ are in bijective correspondence with ordered pairs of subgroups of $\mathrm{CH}(X_L, \Lambda)$ which form a direct sum decomposition. This bijection is compatible with the natural $\mathrm{Gal}(L/F)$-actions. A projector is of degree 0 if and only if the two summands in the associated decomposition are graded subgroups of $\mathrm{CH}(X_L, \Lambda)$.

When $\dim X$ is odd or when $\operatorname{disc} X$ is a field, each homogeneous component of $\mathrm{CH}(X_L, \Lambda)$ is $\mathrm{Gal}(L/F)$-indecomposable, hence a projector of degree 0 of $\mathrm{CH}(X_L, \Lambda)$ is just a subset of $\{0, ..., \dim X\}$. It follows that we can lift any such projector and that every projector is invariant under the action of $\mathrm{Gal}(L/F)$.

When $\dim X$ is even and $\operatorname{disc} X$ is trivial, $\mathrm{CH}_i(X_L, \Lambda)$ is indecomposable if $2i \neq 2d_X = \dim X$. The group $\mathrm{Gal}(L/F)$ acts trivially on $\mathrm{CH}_{d_X}(X_L, \Lambda)$. If the rank of the restriction of $(\tau_L)_*$ to $\mathrm{CH}_{d_X}(X_L, \mathbb{Z}/2^n)$ is 0 or 2, the projector τ_L clearly lifts to a $\mathrm{Gal}(L/F)$-invariant projector in $\mathrm{CH}_{\dim X}(X \times X)_L$.

The last case is when the rank is 1. Once we fix a decomposition of the group $\mathrm{CH}_{d_X}(X_L, \mathbb{Z}/2^n)$ as rank 1 summands, any other such decomposition is given by some element of $\mathrm{SL}_2(\mathbb{Z}/2^n)$. The next lemma ensures that we can lift such an element to $\mathrm{SL}_2(\mathbb{Z})$, thus that τ_L lifts to a projector with integral coefficients. It remains to notice that $\mathrm{Gal}(L/F)$ acts trivially on $\mathrm{CH}_{\dim X}(X \times X)_L$ since $\operatorname{disc} X$ is trivial, to conclude the proof. □

Lemma 9. *For any positive integers k and p, the reduction homomorphism $\mathrm{SL}_k(\mathbb{Z}) \to \mathrm{SL}_k(\mathbb{Z}/p\mathbb{Z})$ is surjective.*

Proof. Any matrix in $\mathrm{SL}_k(\mathbb{Z}/p\mathbb{Z})$ can be written as a product of elementary matrices, which correspond to elementary transformations. But elementary matrices lie in the image of $\mathrm{SL}_k(\mathbb{Z}) \to \mathrm{SL}_k(\mathbb{Z}/p\mathbb{Z})$. □

5. Injectivity in main Theorem

In order to prove injectivity in Theorem 1, we may assume that we are given two motives $(X, \rho), (Y, \sigma)$ in $\mathcal{CM}(\mathcal{Q}_F, \mathbb{Z})$ and an isomorphism between their images in $\mathcal{CM}(\mathcal{Q}_F, \mathbb{Z}/2)$. We will build an isomorphism with integral coefficients between the two motives.

We fix a finite Galois splitting extension L/F for X and Y of degree 2^n. Using Proposition 6 we may assume that there exists an isomorphism α between (X, ρ) and (Y, σ) in $\mathcal{CM}(\mathcal{Q}_F, \mathbb{Z}/2^n)$. By Proposition 7 and Corollary 5, it is enough to build an isomorphism $(X_L, \rho_L) \to (Y_L, \sigma_L)$ which reduces to a rational correspondence modulo 2^n and which is equivariant under $\mathrm{Gal}(L/F)$.

Let d_X be such that $\dim X = 2d_X$ or $2d_X + 1$ and d_Y defined similarly for Y. Let $r(X, \rho)$ be the rank of $\mathrm{CH}_{d_X}(X_L) \cap \operatorname{im}(\rho_L)_*$ if $\dim X$ is even and $r(X, \rho) = 0$ if $\dim X$ is odd. We define $r(Y, \sigma)$ in a similar fashion. We will distinguish cases using these integers.

A basis of $\mathrm{CH}(X_L) \cap \mathrm{im}\,(\rho_L)_*$ gives an isomorphism of (X_L, ρ_L) with twists of Tate motives, thus choosing bases for the groups $\mathrm{CH}(X_L) \cap \mathrm{im}\,(\rho_L)_*$ and $\mathrm{CH}(Y_L) \cap \mathrm{im}\,(\sigma_L)_*$, we can see morphisms between the two motives as matrices.

We fix a basis (e_i) of $\mathrm{CH}(X_L)$ as follows : we choose $e_i \in \mathrm{CH}_i(X_L)$ among the cycles $h^{\dim X - i}, l_i$ for $2i \neq \dim X$. We are done when $r(X, \rho) = 0$.

If $r(X, \rho) = 2$ we complete the basis with $e_{d_X} = l_{d_X}, e'_{d_X} = l'_{d_X} \in \mathrm{CH}_{d_X}(X_L)$.

If $r(X, \rho) = 1$ we choose a generator e_{d_X} of $\mathrm{CH}_{d_X}(X_L) \cap \mathrm{im}\,(\rho_L)_*$ to complete the basis.

We choose a basis (f_i) for $\mathrm{CH}(Y_L) \cap \mathrm{im}\,(\sigma_L)_*$ in a similar way.

If we write $\tilde{\rho}$ and $\tilde{\sigma}$ for the reduction modulo 2^n of ρ and σ, these bases reduce to bases $(\tilde{e}_i)$ of $\mathrm{CH}(X_L, \mathbb{Z}/2^n) \cap \mathrm{im}(\tilde{\rho}_L)_*$ and $(\tilde{f}_i)$ of $\mathrm{CH}(Y_L, \mathbb{Z}/2^n) \cap \mathrm{im}(\tilde{\sigma}_L)_*$. In these homogeneous bases the matrix of a correspondence of degree 0 is diagonal by blocks. The sizes of the blocks are the ranks of the homogeneous components of $\mathrm{im}\,(\rho_L)_*$.

Lemma 10. *If $r(X, \rho) = 1$ then $\operatorname{disc} X$ is trivial.*

Proof. Assume $\operatorname{disc} X$ is not trivial. The correspondence ρ induces a projection of $\mathrm{CH}_{d_X}(X_L)$ which is equivariant under the action of $\mathrm{Gal}(L/F)$. But $\mathrm{CH}_{d_X}(X_L)$ is indecomposable as a $\mathrm{Gal}(L/F)$-module. It follows that $(\rho_L)_*$ is either the identity or 0 when restricted to $\mathrm{CH}_{d_X}(X_L)$, hence $r(X, \rho) \neq 1$. □

Corollary 11. *If $r(X, \rho) \neq 2$ then $\mathrm{Gal}(L/F)$ acts trivially on $\mathrm{im}\,(\rho_L)_*$.*

Lemma 12. *If $r(X, \rho) = 2$ then $r(Y, \rho) = 2$, $\dim Y = \dim X$ and $\operatorname{disc} Y = \operatorname{disc} X$.*

Proof. Since the isomorphism $(\alpha_L)_*$ is graded, the d_X-th homogeneous component of $\mathrm{im}(\alpha_L)_*$ has rank 2. This image is a subgroup of the Chow group with coefficients in $\mathbb{Z}/2^n$ of a split quadric, thus the only possibility is that $\dim Y$ is even, $d_X = d_Y$ and $r(Y, \sigma) = 2$.

The isomorphism $(\alpha_L)_*$ is equivariant under the action of $\mathrm{Gal}(L/F)$. Therefore the pair $\{l_{d_X}, {l_{d_X}}'\} \subset \mathrm{CH}(X_L, \mathbb{Z}/2^n)$ is $\mathrm{Gal}(L/F)$-isomorphic to $\{l_{d_Y}, {l_{d_Y}}'\} \subset \mathrm{CH}(Y_L, \mathbb{Z}/2^n)$. Obviously, the same holds with integral coefficients. By proposition 2, we have a $\mathrm{Gal}(L/F)$-isomorphism between the split étale algebras $\operatorname{disc} X \otimes L$ and $\operatorname{disc} Y \otimes L$. Hence $\operatorname{disc} X$ and $\operatorname{disc} Y$ correspond to the same cocycle in $\mathrm{H}^1(\mathrm{Gal}(L/F), \mathbb{Z}/2)$, thus are isomorphic. □

We now proceed with the proof of the injectivity.

Let us first assume that $r(X, \rho) \neq 2$. Then $r(Y, \sigma) \neq 2$ by the preceding lemma. By Corollary 11 the group $\mathrm{Gal}(L/F)$ acts trivially on $\mathrm{im}\,(\rho_L)_*$ and on $\mathrm{im}\,(\sigma_L)_*$, therefore any morphism $(X_L, \rho_L) \to (Y_L, \sigma_L)$ is defined by a cycle invariant under $\mathrm{Gal}(L/F)$.

Since the isomorphism α_L is of degree 0, its matrix in our graded bases of the modulo 2^n Chow groups is diagonal. Let $\lambda_i \in (\mathbb{Z}/2^n)^\times$ be the coefficients in the diagonal so that we have $(\alpha_L)_*(\tilde{e}_i) = \lambda_i \tilde{f}_i$ for all i such that $\mathrm{CH}_i(X_L) \cap \mathrm{im}\,(\rho_L)_* \neq \emptyset$.

If $r(X, \rho) = 1$ then λ_{d_X} is defined and we consider the cycle $\beta = (\lambda_{d_X})^{-1}\alpha_L$. If $r(X, \rho) = 0$, we just put $\beta = \alpha_L$.

Now we take $k_i \in \mathbb{Z}/2^n$ such that $\lambda_i^{-1} = 2k_i + 1$. Let $\Delta \in \mathrm{End}(X_L, \tilde{\rho}_L)$ be the graph of the diagonal. We consider the rational cycle

$$\gamma = \Delta + 2 \sum k_i \tilde{e}_i \times \tilde{e}_{\dim X - i}$$

where the sum is taken over all i such that $\mathrm{CH}_i(X_L) \subset \mathrm{im}\,(\rho_L)_*$ (which implies in case $r(X,\rho) = 1$ that we do not take $i = d_X$). The composite $\beta \circ \gamma$ is rational, and its matrix in our bases is the identity matrix. This correspondence lifts to an isomorphism with integral coefficients $(X_L, \rho_L) \to (Y_L, \sigma_L)$.

Next assume that $r(X,\rho) = 2$. Then we have $\dim X = \dim Y$, $r(Y,\sigma) = 2$ and $\operatorname{disc} X = \operatorname{disc} Y$. The matrix of $(\alpha_L)_*$ is diagonal by blocks :

$$\begin{pmatrix} \nu_{i_1} & & & & & & \\ & \ddots & & & & & \\ & & \nu_{i_r} & & & & \\ & & & B & & & \\ & & & & \nu_{i_{r+1}} & & \\ & & & & & \ddots & \\ & & & & & & \nu_{i_p} \end{pmatrix}$$

where $\nu_i \in (\mathbb{Z}/2^n)^\times$ and $B \in \mathrm{GL}_2(\mathbb{Z}/2^n)$.

Now if $\operatorname{disc} X = \operatorname{disc} Y$ is a field, there is an element in $\mathrm{Gal}(L/F)$ that simultaneously exchanges the cycles in the bases $\{l_{d_X}, {l_{d_X}}'\}$ of $\mathrm{CH}_{d_X}(X_L, \mathbb{Z}/2^n)$ and $\{l_{d_Y}, {l_{d_Y}}'\}$ of $\mathrm{CH}_{d_Y}(Y_L, \mathbb{Z}/2^n)$. It follows that we may write B as :

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

for some a and b in $\mathbb{Z}/2^n$. The determinant of B is $(a+b)(a-b) \in (\mathbb{Z}/2^n)^\times$, hence $(a-b) \in (\mathbb{Z}/2^n)^\times$. Thus we may replace α_L by $(a-b)^{-1}\alpha_L$ and assume that $b = a+1$.

As before we may write $\nu_i^{-1} = 2k_i + 1$ and replace α_L by the rational cycle :

$$\alpha_L \circ (\Delta + 2\sum k_i \tilde{e}_i \times \tilde{e}_{\dim X - i})$$

the sum being taken over all i such that $\mathrm{CH}_i(X_L) \cap \mathrm{im}\,(\rho_L)_* \neq \emptyset$ and $i \neq d_X$. Therefore we may assume that $\nu_i = 1$ for all i and that we have a matrix of the shape :

$$\begin{pmatrix} \mathrm{I}_s & & & 0 \\ & a+1 & a & \\ & a & a+1 & \\ 0 & & & \mathrm{I}_t \end{pmatrix}$$

The matrix of the rational cycle $h^{d_X} \times h^{d_Y} \in \mathrm{CH}((X \times Y)_L, \mathbb{Z}/2^n)$ is :

$$\begin{pmatrix} 0 & & & 0 \\ & 1 & 1 & \\ & 1 & 1 & \\ 0 & & & 0 \end{pmatrix}$$

Now $\alpha_L - a(h^{d_X} \times h^{d_Y})$ is rational and its matrix is the identity. This cycle is invariant under $\mathrm{Gal}(L/F)$ and lifts to an isomorphism $(X_L, \rho_L) \to (Y_L, \sigma_L)$.

It remains to treat the case when $\operatorname{disc} X$ is trivial. In this case the group $\mathrm{Gal}(L/F)$ acts trivially on $\mathrm{CH}(X_L)$ (and on $\mathrm{CH}(Y_L)$ since $\operatorname{disc} Y$ is also trivial).

As before, composing with a rational cycle, we may assume that $\nu_i = 1$ for all i. We write $\det B^{-1} = 2k + 1$.

The cycle $\Delta + k(h^{d_X} \times h^{d_X}) \in \mathrm{End}(X_L, \tilde{\rho}_L)$ is rational and its matrix in our basis is :

$$\begin{pmatrix} \mathrm{I}_p & & & 0 \\ & {\scriptstyle 1+k} & {\scriptstyle k} & \\ & {\scriptstyle k} & {\scriptstyle 1+k} & \\ 0 & & & \mathrm{I}_r \end{pmatrix}$$

We see that the determinant of this matrix is $1 + 2k$. Therefore the composite $\alpha_L \circ (\Delta + k(h^{d_X} \times h^{d_X}))$ has determinant 1. We use Lemma 9 to conclude, which completes the proof of Theorem 1.

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