Levels and sublevels of composition algebras over $\mathfrak{p}$-adic function fields

James O'Shea and Jan Van Geel

Abstract

In [O'S], the level and sublevel of composition algebras are studied, wherein these quantities are determined for those algebras defined over local fields. In this paper, the level and sublevel of composition algebras, of dimension 4 and 8 over rational function fields over local non-dyadic fields, are determined completely in terms of the local ramification data of the algebras. The proofs are based on the "classification" of quadratic forms over such fields, as is given in [PS1].

1 Introduction

The level and sublevel of composition algebras over local and global fields can be determined, cf. [Le, Proposition 3] and [O'S, Proposition 3.15]. Since the level of a non-dyadic local field is equal to the level of its residue field, cf. [L, Chap. 6, Corollary 2.6], the level of a composition algebra over an extension of such a field is less than or equal to 2. In particular, the level of octonion algebras over $K(t)$, where K is a local field of residue characteristic not 2, is either 1 or 2. In [PS1], Parimala and Suresh gave a full classification of the octonion algebras over $K(t)$, and indeed more generally over $K(C)$, where K is a local field of residue characteristic not 2 and C is a smooth projective curve over K. Using this classification, together with some results in [O'S], we will describe exactly the classes of octonion algebras over $K(t)$ of level 1 and 2. For the sake of completeness, we also consider the level of quaternion algebras.

Let F be a field of characteristic different from 2. A unital, not necessarily associative, F-algebra $\mathcal{C}$ is a *composition algebra* if it carries a non-degenerate quadratic form $q : \mathcal{C} \to F$ which allows composition, i.e. $q(x)q(y) = q(xy)$, cf. [J]. An important theorem of Hurwitz, cf. [J, p. 425], shows that the dimension of a composition algebra is necessarily equal to 1, 2, 4 or 8. The composition algebras of dimension 2 are exactly the quadratic étale F-algebras, while those of dimension 4 are precisely the quaternion algebras, and those of dimension 8 the octonion algebras.

Let $\mathcal{C}$ be a composition algebra over F with quadratic form q, and associated bilinear form $b_q(x,y) := q(x+y) - q(x) - q(y)$. The map

$$^{-} : \mathcal{C} \to \mathcal{C};\ x \mapsto \overline{x} := b_q(x,1)\cdot 1 - x$$

defines an *involution* on $\mathcal{C}$. It follows that

$$q(x)\cdot 1 = \overline{x}x = x\overline{x}.$$

The quadratic form $N_{\mathcal{C}}(x) := \overline{x}x$ is called the the *norm form* of $\mathcal{C}$.

A composition algebra that is not division is said to be *split*. A composition algebra is division if and only if its norm form is hyperbolic. Two composition algebras are isomorphic if and only if their norm forms are isometric, cf. [J, 7.6, Exercises 2,3]. We call the quadratic form $T_{\mathcal{C}}(x) := \frac{1}{2}b_q(x^2,1)\cdot 1$ the *trace form* of $\mathcal{C}$.

With respect to the standard basis of $\mathcal{C}$, the norm and trace forms have the following diagonalisations

$$N_{\mathcal{C}} \cong \langle 1\rangle \perp -T_P \text{ and } T_{\mathcal{C}} \cong \langle 1\rangle \perp T_P.$$

The above form T_P is called the *pure trace form* of $\mathcal{C}$. If $\mathcal{C}$ is the quaternion algebra $\left(\frac{a,b}{F}\right)$ (with F-basis $\{1,i,j,ij\}$, where $i^2=a$ and $j^2=b$), we have that

$$N_{\mathcal{C}} \cong \langle 1,-a,-b,ab\rangle \text{ and } T_{\mathcal{C}} \cong \langle 1,a,b,-ab\rangle.$$

Alternatively, if $\mathcal{C}$ is the octonion algebra $\left(\frac{a,b,c}{F}\right)$ (with F-basis $\{1,i,j,ij,e,ie,je,(ij)e\}$, where $i^2=a, j^2=b$ and $e^2=c$), we have that

$$N_{\mathcal{C}} \cong \langle 1,-a,-b,ab,-c,ac,bc,-abc\rangle \text{ and } T_{\mathcal{C}} \cong \langle 1,a,b,-ab,c,-ac,-bc,abc\rangle.$$

Definition 1.1. Let $\mathcal{A}$ be any F-algebra. The *level* of $\mathcal{A}$, denoted $s(\mathcal{A})$, is the least integer n such that -1 is a sum of n squares in $\mathcal{A}$. If no such integer exists, we say that $s(\mathcal{A})=\infty$.

The *sublevel* of $\mathcal{A}$, denoted $\underline{s}(\mathcal{A})$, is the least positive integer n such that 0 is a sum of $n+1$ squares in $\mathcal{A}$. If no such integer exists, we say that $\underline{s}(\mathcal{A})=\infty$.

It readily follows from these definitions that $\underline{s}(\mathcal{A}) \le s(\mathcal{A})$. In [O'S], the first author studied the level and sublevel of composition algebras. We recall the following:

Proposition 1.2. *Let $\mathcal{C}$ be a composition algebra over F.*

(a) *If $-1 \in F^{*2}$, then $s(\mathcal{C}) = \underline{s}(\mathcal{C}) = 1$.*

(b) *If $-1 \notin F^{*2}$, then $s(\mathcal{C}) = 1$ if and only if $T_{\mathcal{C}}$ is isotropic.*

(c) *If $-1 \notin F^{*2}$, then $\underline{s}(\mathcal{C}) = 1$ if and only if $T_{\mathcal{C}}$ or $2\times T_P$ is isotropic.*

Proof. (a) is trivial; (b) and (c) follow from [O'S, Proposition 3.12] and [O'S, Theorem 3.5]. ∎

2 Exact sequences from Galois cohomology.

We recall some of the basic facts from the theory of quadratic forms. We refer to the standard books, [L] and [S], for further details and proofs.

As above, F is a field of characteristic not equal to 2. A (regular) quadratic form φ over F is said to be *isotropic* if there exists a non-zero vector x such that $\varphi(x) = 0$. Otherwise, we say that φ is *anisotropic* over F. A 2-dimensional isotropic form is isometric to the quadratic form $\langle 1, -1\rangle$, which is called the *hyperbolic plane*. A quadratic form is *hyperbolic* if it is isometric to an orthogonal sum of hyperbolic planes. Since the characteristic of F is not 2, all quadratic forms can be represented by diagonal matrices. The *value set* of a quadratic form φ, denoted $D_F(\varphi)$, is the set of elements of F^* which are "represented by φ", i.e. $D_F(\varphi) = \{\alpha \in F^* \mid \text{ there exists } x \in V \text{ with } \varphi(x) = \alpha\}$.

Two quadratic forms that become isometric after the addition of a number of hyperbolic planes are *Witt equivalent*. The Witt equivalence classes of quadratic forms form a ring, called the *Witt ring* of F and denoted $W(F)$. The addition is given by the orthogonal sum and the multiplication by the Kronecker product of the matrices representing the forms. The classes of forms of even dimension constitute the *fundamental ideal* $I(F)$ in $W(F)$. The ideal $I(F)$ is generated by the forms $\langle 1, \alpha\rangle$, $\alpha \in F^*$. These forms $\langle 1, \alpha\rangle$ are called 1-*fold Pfister forms*. The higher powers of the fundamental ideal, $I^n(F)$, are additively generated by the *n-fold Pfister forms*,

$$\langle\langle \alpha_1, \dots, \alpha_n\rangle\rangle := \langle 1, \alpha_1\rangle \otimes \cdots \otimes \langle 1, \alpha_n\rangle.$$

Pfister forms have very special properties. They are isotropic if and only if they are hyperbolic. Moreover, a Pfister form π is a multiplicative form, i.e. $D_F(\pi)$ is a multiplicative subgroup of F^*.

Every Pfister form π is of type $\langle 1\rangle \perp \pi'$, where (the isometry class of) the quadratic form π' is known as the *pure subform* of π. We will invoke the following well-known fact concerning Pfister forms (cf. [S, Chap. 4, Theorem 1.4]):

Lemma 2.1. *Let π be an anisotropic n-fold Pfister for m over F, and let π' be its pure subform. Then $\beta \in D_F(\pi')$ i f and only if $\pi \cong \langle\langle \beta, \beta_2, \dots, \beta_n\rangle\rangle$ for suitable $\beta_i \in F^*$.*

Corollary 2.2. *Let F be a field, $-1 \notin F^{*2}$. Let $\mathcal{C}$ be a quaternion algebra, respectively an octonion algebra, over F. Then $s(\mathcal{C}) = 1$ if and only if $\mathcal{C} \cong \left(\frac{-1,b}{F}\right)$, respectively $\mathcal{C} \cong \left(\frac{-1,b,c}{F}\right)$.*

Proof. Since $i^2 = -1$ for $\mathcal{C} \cong \left(\frac{-1,b}{F}\right)$ or $\left(\frac{-1,b,c}{F}\right)$, we only have to prove the converse.

Since $-1 \notin F^{*2}$, Proposition 1.2 implies that $-1 \in D_F(T_P)$, whereby $1 \in D_F(-T_P)$. If $\mathcal{C}$ is a quaternion algebra, respectively an octonion algebra, the above lemma yields that the norm form is $\langle 1, 1, -b, -b\rangle$, respectively $\langle 1, 1, -b, -b, -c, -c, bc, bc\rangle$, for some $b, c \in F^*$. Since composition algebras are, up to isometry, determined by their norm forms, it follows that $\mathcal{C}$ is isomorphic to $\left(\frac{-1,b}{F}\right)$, respectively $\left(\frac{-1,b,c}{F}\right)$. ∎

Let K denote a non-dyadic local field. We will invoke the following exact sequences, which essentially yield a classification of the 2- and 3-fold Pfister forms over the rational function field $K(t)$.

$$0 \to I^2(K) \to I^2(K(t))/I^3(K(t)) \overset{\prod \partial_{2,x}}{\to} \prod_{x \in \mathbb{P}^1_K} \kappa(x)^*/\kappa(x)^{*2} \overset{\prod N_{\kappa(x)/\kappa}}{\to} K^*/K^{*2} \to 1 \quad (ES1)$$

and

$$0 \to I^3(K(t)) \overset{\oplus \partial_{2,x}}{\to} \oplus_{x \in \mathbb{P}^1_K} I^2(\kappa(x)) \overset{\sum}{\to} I^2(K) \to 0 \qquad (ES2)$$

where x runs over the closed points of $\mathbb{P}^1_K$, and $\kappa(x)$ is the residue field of the discrete valuation v_x associated to x. The map $\partial_{2,x}$ sends a quadratic form over $K(t)$ to its second residue form over $\kappa(x)$.

Both exact sequences are obtained from Galois cohomology. They are special cases of the sequence

$$0 \to H^i(K, \mathbb{Z}/2\mathbb{Z}) \to H^i(K(t), \mathbb{Z}/2\mathbb{Z}) \to \bigoplus_{x \in \mathbb{P}^1_K} H^{i-1}(\kappa(x), \mathbb{Z}/2\mathbb{Z}) \to H^{i-1}(K, \mathbb{Z}/2\mathbb{Z}) \to 0,$$

cf. [Se, page 122].

In formulating the exact sequences $(ES1)$ and $(ES2)$, we made use of the isomorphism $I^i(K(t))/I^{i+1}(K(t)) \cong H^i(K(t), \mathbb{Z}/2\mathbb{Z})$. For $i = 1$, this identification follows from Kummer theory. A well-known theorem of Merkurjev gives the isomorphism in the case where $i = 2$, with Rost establishing the cases where $i = 3$ and 4. (For all i, the isomorphism corresponds to the Milnor conjecture, as established by Voevodsky.) These important results also yield, in light of the Arason-Pfister main theorem, that $I^3(K) = 0$ and $I^4(K(t)) = 0$. In $(ES1)$, we additionally identified $I(K(t))/I^2(K(t))$ with $\kappa(x)^*/\kappa(x)^{*2}$. In $(ES2)$, identifying the group of order 2, $I^2(\kappa(x))$, with $\mathbb{Z}/2\mathbb{Z}$, allows one to define the map $\sum$ as addition in $\mathbb{Z}/2\mathbb{Z}$.

The exact sequence $(ES1)$ also describes the 2-torsion part of the Brauer group of $K(t)$ since $H^2(F, \mathbb{Z}/2\mathbb{Z}) \cong {}_2Br(F)$, with the isomorphism given by the map $(a, b) \mapsto \left(\frac{a,b}{F}\right)$, between the generators of both groups. (This fact is a reformulation of the aforementioned theorem of Merkurjev.) After this identification, the sequence becomes

$$0 \to {}_2Br(K) \to {}_2Br(K(t)) \overset{\prod \partial_x}{\to} \prod_{x \in \mathbb{P}^1_K} \kappa(x)^*/\kappa(x)^{*2} \overset{\prod N_{\kappa(x)/\kappa}}{\to} K^*/K^{*2} \to 1. \qquad (ES3)$$

The morphisms ∂_x now correspond to the ramification maps in the points x, which are defined on the generators by

$$\partial_x \left(\frac{f, g}{K(t)} \right) = (-1)^{v(f)v(g)} \overline{\left(\frac{f^{v(g)}}{g^{v(f)}} \right)} \bmod \kappa(x)^{*2},$$

with v the normalized discrete valuation on $K(t)$ corresponding to the point $x \in \mathbb{P}^1_K$. It follows from a theorem of Saltman, cf. [PS1, Corollary 2.2], that all the elements of ${}_2Br(K(t))$ are of index ≤ 4, so the division algebras of exponent 2 over $K(t)$ are quaternion or biquaternion algebras.

3 The level of composition algebras over $K(t)$, where K is a local non-dyadic field.

Throughout this section, we let K denote a *non-dyadic local field*, i.e. a finite extension of a p-adic field, with residue field $\mathbb{F}_q$, or a Laurent series field over $\mathbb{F}_q$, where $q = p^s$ and $p \neq 2$. The uniformizing element, for the non-dyadic valuation on K, is denoted by π. We recall that if ε is a non-square unit in the valuation ring, then $\{1, \pi, \varepsilon, \varepsilon\pi\}$ represent all square classes in K. As a consequence, the quaternion division algebra $\left(\frac{\pi,\varepsilon}{K(t)}\right)$ is the unique non-trivial element in ${}_2Br(K)$.

We will proceed to give explicit results on the level and sublevel of composition algebras $\mathcal{C}$ over $K(t)$. As was mentioned previously, both the level and sublevel of these algebras are $\leq$ 2. Also, in the case of commutative composition algebras, i.e. the one and two dimensional algebras, the level and sublevel are equal. If $\mathcal{C} = K(t)$, then $s(\mathcal{C}) = s(K) = s(\mathbb{F}_q)$, so the level is equal to 1 if and only $q \equiv 1 \bmod 4$. For $\mathcal{C}$ a quadratic extension of $K(t)$, the same holds true excepting the case where $\mathcal{C} = K(\sqrt{-1})(t)$, which is clearly always of level 1.

So we only have to consider the cases of quaternion and octonion algebras over $K(t)$. We start with the latter.

Theorem 3.1. *Let K be a local non-dyadic field such that $-1 \notin K^{*2}$.*

(a) *The sublevel of an octonion algebra O over $K(t)$ is equal to* 1.

(b) *The level of an octonion algebra O over $K(t)$ is equal to 2 if and only if there is an $x \in \mathbb{P}^1_K$ such that $K(\sqrt{-1}) \subset \kappa(x)$ and $\partial_{2,x}(N_O) \neq 0$.*

Proof. (a) The quadratic form $2 \times N_O$ is a 4-fold Pfister form over $K(t)$. Since $I^4(K(t)) = 0$, $2 \times N_O$ is hyperbolic. The 14-dimensional subform $-2 \times T_P$ is therefore isotropic, whereby $\underline{s}(O) = 1$ by proposition 1.2 (c).

(b) Assume firstly that for some $x \in \mathbb{P}^1_K$, $K(\sqrt{-1}) \subset \kappa(x)$ and $\partial_{2,x}(N_O) \neq 0 \in \mathbb{Z}/2\mathbb{Z}$. This means that $\partial_{2,x}(\langle 1 \rangle \perp -T_P)$ is a non-trivial element in $I^2(\kappa(x))$. Thus, we have that

$$\partial_{2,x}(\langle 1 \rangle \perp -T_P) \cong_{\kappa(x)} \langle 1, -\varepsilon_x, -\pi_x, \varepsilon_x \pi_x \rangle,$$

where ε_x is a non-square unit and π_x is a uniformizing element in $\kappa(x)$.

We consider the form

$$N_O \otimes_{K(t)} K(\sqrt{-1})(t) \cong (\langle 1 \rangle \perp -T_P) \otimes_{K(t)} K(\sqrt{-1})(t).$$

Since $K(\sqrt{-1}) \subset \kappa(x)$, x is not equal to the point at infinity in $\mathbb{P}^1_K$, so it corresponds to an irreducible polynomial $p(t) \in K[t]$. This irreducible polynomial $p(t)$ factorises over $K(\sqrt{-1})$ into two polynomials, $p_1(t)$ and $p_2(t)$, each of degree $[\kappa(x) : K(\sqrt{-1})]$. So there are two closed points $y_1, y_2 \in \mathbb{P}^1_{K(\sqrt{-1})}$ above x. For both points, we have that

$$\partial_{2,y_i}((\langle 1 \rangle \perp -T_P) \otimes_{K(t)} K(\sqrt{-1})(t)) = \langle 1, -\varepsilon_x, -\pi, \varepsilon_x \pi \rangle \otimes_{\kappa(x)} \kappa(y_i)$$

is anisotropic over $\kappa(y_i)$, since $\kappa(y_i) = \kappa(x)$ for $i = 1, 2$. It follows from the injectivity of $\oplus\partial_{2,x}$ in the exact sequence $(ES2)$ (over the field $K(\sqrt{-1})(t)$) that $(\langle 1\rangle \perp -T_P) \otimes K(\sqrt{-1})(t) \cong (\langle 1\rangle \perp T_P) \otimes K(\sqrt{-1})(t)$ is anisotropic. Hence $T_O \cong \langle 1\rangle \perp T_P$ is anisotropic over $K(t)$, implying that $s(O) \neq 1$ by Proposition 1.2 (b), whereby $s(O) = 2$.

Assume now that for all closed points $x \in \mathbb{P}^1_K$ such that $\partial_{2,x}(N_O) \neq 0$, one has that $K(\sqrt{-1}) \not\subset \kappa(x)$. We have to show that $s(O) = 1$.

Let $S = \{x \in \mathbb{P}^1_K \mid \partial_{2,x}(N_O) \neq 0\}$. Then, since -1 is not a square in $\kappa(x)$, for all $x \in S$,

$$\partial_{2,x}(\langle 1\rangle \perp -T_P) \cong_{\kappa(x)} \langle 1, 1, -\pi_x, -\pi_x\rangle,$$

where π_x is a uniformizing element in $\kappa(x)$. Again, we consider the form

$$(\langle 1\rangle \perp -T_P) \otimes_{K(t)} K(\sqrt{-1})(t).$$

We have the following exact diagram

$$\begin{array}{ccccccccc} 0 \to & I^3(K(t)) & \to & \oplus_{x\in\mathbb{P}^1_K} I^2(\kappa(x)) & \stackrel{\Sigma}{\to} & I^2(K) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & I^3(K(\sqrt{-1})(t)) & \to & \oplus_{y\in\mathbb{P}^1_{K(\sqrt{-1})(t)}} I^2(\kappa(y)) & \stackrel{\Sigma}{\to} & I^2(K(\sqrt{-1})) & \to 0, \end{array}$$

where the second vertical arrow is defined by $\varphi \mapsto \oplus_{y \text{ lying over } x} \varphi \otimes_{\kappa(x)} \kappa(y)$. For all $y \in \mathbb{P}^1_{K(\sqrt{-1})}$ lying over a point $x \notin S$, $\partial_{2,y}(\langle 1\rangle \perp -T_P \otimes_{K(t)} K(\sqrt{-1})(t)) = 0$, since $\partial_{2,x}(\langle 1\rangle \perp -T_P) = 0$. For all $y \in \mathbb{P}^1_{K(\sqrt{-1})}$ lying over a point $x \in S$, we have that

$$\partial_{2,y}(\langle 1\rangle \perp -T_P \otimes_{K(t)} K(\sqrt{-1})(t)) = \langle 1, 1, -\pi_x, -\pi_x\rangle \otimes_{\kappa(x)} \kappa(y) = 0,$$

since $\sqrt{-1} \in \kappa(y)$. So $\partial_{2,y}(\langle 1\rangle \perp -T_P \otimes_{K(t)} K(\sqrt{-1})(t))$ is trivial for all $y \in \mathbb{P}^1_{K(\sqrt{-1})}$. The injectivity of $\oplus\partial_{2,y}$ in the second line of the exact diagram implies that $(\langle 1\rangle \perp -T_P) \otimes_{K(t)} K(\sqrt{-1})(t)$ is hyperbolic over $K(\sqrt{-1})(t)$. Hence, we obtain that $1 \in D_{K(t)}(-T_P)$ (cf. [S, Chap. 2, Theorem 5.2]). Lemma 2.1 thus implies that $O \cong \left(\frac{-1,b,c}{F}\right)$, and consequently that $s(O) = 1$. ■

Given an octonion algebra over $K(t)$, Theorem 3.1 shows how to determine its level; namely by calculating the "ramification" of the norm form of the algebra. For non-dyadic local fields K, the octonion algebras over $K(t)$ are completely classified. One can use that classification, together with Theorem 3.1, to construct examples of octonion algebras with prescribed level. We recall the following result from [PS1]:

Theorem 3.2. [PS1, Theorem 3.9] *Let K be a non-dyadic local field. Then the elements of $I^3(K(t))$ are i n one to one correspondence with the 3-fold Pfister forms.*

Proof. Theorem 3.9 in [PS1] states that the elements of $H^3(K(t), \mathbb{Z}/2\mathbb{Z})$ are in one to one correspondence with the symbols $(f) \cdot (g) \cdot (h)$. As was previously mentioned, the group $I^3(K(t))$ can be identified with $H^3(K(t), \mathbb{Z}/2\mathbb{Z})$, and the 3-fold Pfister forms correspond with the symbols under this identification. ■

It follows that the elements of $I^3(K(t))$ are in one to one correspondence with the isomorphism classes of octonion algebras over $K(t)$. Moreover, the exact sequence $(ES2)$ tells us that the elements in $I^3(K(t))$ can be described as the set of functions

$$C(\mathbb{P}^1_K) := \{f : \mathbb{P}^1_K \to \mathbb{Z}/2\mathbb{Z} \mid f(x) = 0 \text{ for almost all } x, \sum_x f(x) = 0\}.$$

So there is a bijection between $C(\mathbb{P}^1_K)$ and the isomorphisms classes of octonions over $K(t)$. This bijection is given by

$$f \mapsto O_f,$$

with O_f the octonion algebra such that $\partial_{2,x}(N_{O_f}) = f(x) \in \mathbb{Z}/2\mathbb{Z}$. Theorem 3.1 can now be restated as follows:

Theorem 3.3. *Let K be a non-dyadic local field. The level of an octonion algebra O_f over $K(t)$, $f \in C(\mathbb{P}^1_K)$, is equal to 1 if and only if $\sqrt{-1} \notin \kappa(x)$ for all x such that $f(x) \neq 0$.*

Proof. This follows immediately from Theorem 3.1 (b). ∎

Examples 3.4. Theorems 3.1 and 3.3 imply the existence of octonion algebras over $K(t)$ of level 1, and of level 2. We will use them to obtain families of examples for both cases.

Let S be a set containing an even number of closed points, $x_i \in \mathbb{P}^1_K$, $i = 1, \dots 2n$, such that $\kappa(x_i)$ is an unramified extension of odd degree over K (for example, we can take S to be any set of an even number of K-rational points in $\mathbb{P}^1_K$). The following algebras O are octonion division algebras of level 1:

$$O = \begin{cases} \left(\frac{-1, \prod_{i=1}^{2n} p_i, \pi}{K(t)}\right) & \text{if } \infty \notin S, \\ \left(\frac{-1, \prod_{i=2}^{2n} p_i, \pi}{K(t)}\right) & \text{if } x_1 = \infty \in S, \end{cases}$$

where p_i is the irreducible polynomial corresponding the point x_i, if $x_i \neq \infty$. For both definitions of O, $\partial_{2,x_i}(N_O) = \langle 1, 1, \pi, \pi \rangle \otimes_K \kappa(x_i)$ is an anisotropic form over $\kappa(x_i)$ for all i, since the hypotheses on the x_i yield that -1 is not a square in $\kappa(x_i)$ and that π is a uniformizing element in $\kappa(x_i)$. For all the other points, $x \in \mathbb{P}^1_K$, $\partial_{2,x}(N_O) = 0$.

To obtain examples of octonion division algebras of level 2, let x be a closed point in $\mathbb{P}^1_K$ such that $\kappa(x)$ is an unramified extension of K containing $K(\sqrt{-1})$. Let $p_x(t)$ be the irreducible polynomial associated to x. Choose x so that a root γ_x of $p_x(t)$ is a non-square unit in the discrete valuation ring of $\kappa(x)$. Then theorem 3.1 implies that the algebra

$$O = \left(\frac{\pi, p_x, t}{K(t)}\right)$$

is an octonion division algebra of level 2, since the hypotheses on x imply that $\partial_{2,x}(N_O) = \langle 1, -\pi, -\gamma_x, \pi\gamma_x \rangle$ is an anisotropic quadratic form over $\kappa(x)$. We note that there are infinitely many $x \in \mathbb{P}^1_K$ satisfying the above hypotheses, since there are infinitely many unramified extensions of K containing $K(\sqrt{-1})$.

For example, $p_x = (t^2 - 2t + 2) \in \mathbb{Q}_3(t)$ corresponds to a point of degree 2 in $\mathbb{P}^1_{\mathbb{Q}_3}$ with residue field $\mathbb{Q}_3(i+1) = \mathbb{Q}_3(i)$, where $i^2 = -1$. The root $i+1$ of p_x is a unit for the 3-adic valuation on $\mathbb{Q}_3(i)$, but it is non-square, since $\overline{i+1}$ is not a square in $\mathbb{F}_9 = \mathbb{F}_3(i)$. By the above observations, the algebra

$$\left(\frac{3, t^2 - 2t + 2, t}{\mathbb{Q}_3(t)}\right)$$

is an octonion division algebra of level 2.

We now consider the case of quaternion algebras. If we identify $I^2(F)/I^3(F)$ as the two component of the Brauer group, via Merkurjev's theorem, then $(ES3)$ shows that every quaternion division algebra H over $K(t)$ is "almost" defined by local data, namely by its ramification data, i.e. the non-trivial residues $\partial_x(H)$. Theorem 3.6 shows how the level and the sublevel of a quaternion algebra H depend on this local data.

Definition 3.5. Let H be a quaternion algebra over $K(t)$. The *ramification data* of H is the set $R_H = \{\alpha_x \in \kappa(x)^*/\kappa(x)^{*2} \mid \alpha_x = \partial_x(H) \not\equiv 1 \bmod \kappa(x)^{*2}\}$. The *ramification locus* of H is the set $S_H = \{x \in \mathbb{P}^1_K \mid \alpha_x \in R_H\}$. The exact sequence $(ES3)$ tells us that S_H is a finite set. (We will also use the term "ramification data" to refer to the couple (S_H, R_H).)

In addition, we will employ the following subset of the ramification locus, $S_H^{-1} = \{x \in S_H \mid -1 \notin \kappa(x)^{*2}\}$.

Theorem 3.6. *Let K be a non-dyadic local field such t hat $-1 \notin K^{*2}$.*

(a) *The level of a quaternion algebra H over $K(t)$, with ramification data R_H, is equal to 2 if and only if there is an $x \in S_H$ such that $\alpha_x \not\equiv -1 \bmod \kappa(x)^{*2}$.*

(b) *The sublevel of a quaternion algebra H over $K(t)$ is equal to 2 if and only if there is an $x \in S_H^{-1}$ such that $\alpha_x \not\equiv -1 \bmod \kappa(x)^{*2}$.*

Proof. (a) In light of Corollary 2.2, we have to show that $K(t)(\sqrt{-1})$ is not a splitting field for H if and only if there is a point $x \in S_H$ with $\alpha_x \not\equiv -1 \bmod \kappa(x)^{*2}$.

Let $x \in S_H$ with $\alpha_x \not\equiv -1 \bmod \kappa(x)^{*2}$. Consider the exact sequence $(ES3)$ after extending from K to $K(\sqrt{-1})$. We claim that for every point y in $\mathbb{P}^1_{K(\sqrt{-1})}$ lying over x, we have

$$\partial_y(H \otimes_{K(t)} K(t)(\sqrt{-1})) = \alpha_x \not\equiv 1 \in \kappa(y)^*/\kappa(y)^{*2}.$$

If this is true, H is not split over $K(t)(\sqrt{-1})$, proving the "if" part of the statement.

Proof of the claim. If $\sqrt{-1} \in \kappa(x)$, the claim holds since there are two points y_1, y_2 over x with residue fields $\kappa(y_1), \kappa(y_2)$ equal to $\kappa(x)$. If $\sqrt{-1} \notin \kappa(x)$, then, by assumption, the square class of α_x, being non-trivial and not equal to that of -1, is not a unit in $\kappa(x)$, and hence must be the class of a uniformising element $\pi_x \in \kappa(x)$. For $y \in \mathbb{P}^1_{K(\sqrt{-1})}$ lying over x we have $\kappa(y) = \kappa(x)(\sqrt{-1})$. Since $\kappa(x)(\sqrt{-1})$ is an unramified quadratic extension of $\kappa(x)$

(the residue fields being non-dyadic fields), it follows that π_x is not a square in $\kappa(x)(\sqrt{-1})$ either.

We now assume that for all $x \in S_H$, $\alpha_x \equiv -1 \bmod \kappa(x)^{*2}$. The element $\partial_y(H \otimes_{K(t)} K(t)(\sqrt{-1}))$ can only be non-trivial for points $y \in \mathbb{P}^1_{K(\sqrt{-1})}$ lying above a point $x \in S_H$. Since $-1 \notin \kappa(x)^{*2}$ by assumption, it follows that for all points $x \in S_H$, there is one point y lying over x, with residue field $\kappa(y) = \kappa(x)(\sqrt{-1})$. But for such y, $\partial_y(H) \equiv \alpha_x \equiv -1 \bmod \kappa_y^{*2} \equiv 1 \bmod \kappa_y^{*2}$, implying that $\partial_y(H \otimes_{K(t)} K(t)(\sqrt{-1}))$ is trivial for all $y \in \mathbb{P}^1_{K(\sqrt{-1})}$. The exact sequence $(ES2)$ (viewed over $K(\sqrt{-1})$) implies that

$$\begin{array}{ll} & (1)\quad H \otimes_{K(t)} K(t)(\sqrt{-1}) \cong M_2(K(t)(\sqrt{-1})) \\ \text{or} & (2)\quad H \otimes_{K(t)} K(t)(\sqrt{-1}) \cong \left(\frac{\pi,\alpha}{K(\sqrt{-1})}\right) \otimes_{K(\sqrt{-1})} K(\sqrt{-1})(t), \end{array}$$

where $\left(\frac{\pi,\alpha}{K(\sqrt{-1})}\right)$ is the unique quaternion division algebra over $K(\sqrt{-1})$. If (1) holds, it follows that $K(t)(\sqrt{-1})$ is a splitting field of H, whereby $s(H) = 1$, proving part (a). We will show that the other alternative, (2), leads to a contradiction.

By local class field theory (cf. [N, Chap. 5, Corollary 1.2]), we may choose α such that $N_{K(\sqrt{-1})/K}(\alpha) = -1$. It follows, by a well-known formula (cf. [T, (3.2)]), that the corestriction of $H \otimes_{K(t)} K(t)(\sqrt{-1})$ is equal to $\left(\frac{\pi,-1}{K(t)}\right)$. However, $\left(\frac{\pi,-1}{K(t)}\right)$ is non-trivial in the Brauer group of $K(t)$, whereas the corestriction of $H \otimes_{K(t)} K(t)(\sqrt{-1})$ is trivial, since the composition, $\mathrm{cor} \circ \mathrm{res}$, is the zero map. This completes the proof of (a).

(b) While the following argument is similar in nature to that employed in (a), it is more efficient to invoke the exact sequence $(ES1)$ in this case.

Firstly, assume that for all $x \in S_H^{-1}$, $\alpha_x \equiv -1 \bmod \kappa(x)^{*2}$. For all $x \in S_H$, consider the second residue form $\partial_{2,x}(2 \times N_H) = 2 \times \partial_{2,x}(N_H) \in I(\kappa(x))$. If $x \in S_H^{-1}$, then, by hypothesis, we have that $2 \times \partial_{2,x}(N_H) \sim 2 \times \langle 1, 1\rangle \sim 0$ in $W(\kappa(x))$. If $x \in S_H \setminus S_H^{-1}$, then $2 \times \partial_{2,x}(N_H) \sim 0$, since it contains the subform $\langle 1, 1\rangle$. As all the second residue forms of $2 \times N_H$ are trivial, the exact sequence $(ES1)$ implies that the 3-fold Pfister form $2 \times N_H$ is hyperbolic. Thus $2 \times T_P$ is isotropic, since $2 \times (-T_P)$ is a subform of $2 \times N_H$, with $\dim 2 \times (-T_P) > \frac{1}{2} \dim 2 \times N_H$. Hence, $\underline{s}(H) = 1$ by Proposition 1.2 (c).

To prove the converse, we assume that there is an $x \in S_H^{-1}$ such that $\alpha_x \not\equiv -1 \bmod \kappa(x)^{*2}$. We have to show that $\underline{s}(H)$ must be 2. The existence of $x \in S_H^{-1}$ implies that the second residue form $\partial_{2,x}(2 \times N_H)$ is Witt equivalent to the 2-fold Pfister form $\langle 1, 1, u\pi_x, u\pi_x\rangle$, for some unit $u \in \kappa(x)$. Since $\sqrt{-1} \notin \kappa(x)^*$, it follows that this form is the unique anisotropic 2-fold Pfister form over $\kappa(x)$. So $2 \times N_H$ has a non-trivial second residue form, and therefore cannot be hyperbolic. Arguing as above, we see that $2 \times T_P$ is anisotropic. Moreover, part (a) of the theorem implies that T_H is also anisotropic. Invoking Proposition 1.2, we obtain that $\underline{s}(H) = 2$. This proves part (b) of the theorem. ∎

Corollary 3.7. *Let K be a non-dyadic local field such that $-1 \notin K^{*2}$.*

(a) *There exist quaternion division algebras H over $K(t)$ with $s(H) = 1$ (whereby $\underline{s}(H) = 1$).*

(b) *There exist quaternion division algebras H over $K(t)$ with $s(H) = 2$ and $\underline{s}(H) = 1$.*

(c) *There exist quaternion division algebras H over $K(t)$ with $\underline{s}(H) = 2$ (whereby $s(H) = 2$).*

Proof. (a) Any quaternion algebra of the form $\left(\frac{-1,p(t)}{K(t)}\right)$, where $p(t)$ is an irreducible polynomial such that $-1 \not\equiv 1 \bmod (K[t]/(p(t)))^{*2}$, is a quaternion division algebra of level 1, and therefore also of sublevel 1.

(b) Let L/K be any finite unramified extension of K containing $\sqrt{-1}$. Then, for some irreducible polynomial $q(t)$, $L \cong K[t]/(q(t))$, and $-1 \equiv 1 \bmod (q(t))$. Let x be the closed point in $\mathbb{P}^1_K$ associated to $q(t)$. The quaternion algebra $H = \left(\frac{\pi,q(t)}{K(t)}\right)$ is a quaternion division algebra, since $\partial_x(H) = \pi \not\equiv 1 \bmod \kappa(x)^{*2}$. As $q(t)$ is of even degree, it follows that $S_H = \{x\}$. Theorem 3.6 thus implies that $s(H) = 2$ and $\underline{s}(H) = 1$.

(c) We first note that it is possible to construct finite sets $S \subset \mathbb{P}^1_K$ and $R = \{\alpha_y \in \kappa(y)^*/\kappa(y)^{*2} \mid y \in S, \alpha_y \not\equiv 1 \bmod \kappa(y)^{*2}\}$ such that for some $x \in S$, $-1 \notin \kappa(x)^{*2}$ and $\alpha_x \not\equiv -1 \bmod \kappa(x)^{*2}$. In addition, we may assume that such S and R correspond to the ramification data of some element in ${}_2Br(K(t))$, in accordance with the exact sequence $(ES3)$. If this element in the Brauer group is the class of a quaternion division algebra, Theorem 3.6(b) implies that it must be a quaternion division algebra of sublevel 2, as desired.

Now assume that the element in the Brauer group is not the class of a quaternion division algebra over $K(t)$. Hence, it is the class of a biquaternion division algebra over $K(t)$, say $H_1 \otimes_K H_2$, where H_1 and H_2 are both quaternion division algebras. Let $y \in S$. Since $\partial_y(H_1) \cdot \partial_x(H_2) = \partial_x(H_1 \otimes_K H_2) \not\equiv 1 \bmod \kappa(x)^{*2}$, it follows that either $\partial_x(H_1) \not\equiv -1 \bmod \kappa(x)^{*2}$ or $\partial_x(H_2) \not\equiv -1 \bmod \kappa(x)^{*2}$. Thus, Theorem 3.6(b) implies that $\underline{s}(H_1) = 2$ or $\underline{s}(H_2) = 2$. ∎

Remark 3.8. Actually, it is possible to describe all quaternion division algebras over $K(t)$ of level 1 explicitly. From Theorem 3.6(a), it follows that there is a correspondence between quaternion division algebras over $K(t)$ of level 1 and ramification data (S, R) such that for all $x \in S$, $-1 \notin \kappa(x)^{*2}$ and for all $\alpha_x \in R$, $\alpha_x \equiv -1 \bmod \kappa(x)^{*2}$. The exact sequence $(ES3)$ implies that for such data (S, R), $\sum_{x\in S} \deg x \in 2\mathbb{Z}$ holds, because $1 \equiv \prod_{x\in S} N_{\kappa(x)/K}(-1) \equiv (-1)^{\sum_{x\in S} \deg x} \bmod K^{*2}$.

The exact sequence $(ES3)$ also implies that for every couple (S, R) with these properties, there are up to isomorphism two different division algebras of exponent 2 having (S, R) as ramification data. This is the case since ${}_2Br(K)$ is a group of order 2.

These division algebras can be explicitly described, and it turns out that they are both quaternion algebras. Let $S \subset \mathbb{P}^1_K$ be such that for all $x \in S$, $-1 \notin \kappa(x)^{*2}$, and such that

$\sum_{x\in S} \deg x \in 2\mathbb{Z}$. Let $S_f := S \setminus \{\infty\}$ be the set of finite points in S. Let $p_x(t)$ be the monic irreducible polynomials corresponding to $x \in S_f$. Then, it is easy to check that the quaternion division algebras

$$\left(\frac{-1, \prod_{x\in S} p(x)}{K(t)}\right) \text{ and } \left(\frac{-1, \pi\prod_{x\in S} p(x)}{K(t)}\right) \qquad \text{in the case where } \infty \notin S,$$

$$\text{and} \quad \left(\frac{-1, -\prod_{x\in S_f} p(x)}{K(t)}\right) \text{ and } \left(\frac{-1, -\pi\prod_{x\in S_f} p(x))}{K(t)}\right) \quad \text{in the case where } \infty \in S,$$

have ramification locus S and, for all $x \in S$, the residue $\alpha_x \equiv -1 \bmod \kappa(x)^{*2}$. Hence these algebras represent *all* quaternion division algebras of level 1.

It is not possible to similarly offer an exhaustive list of all the quaternion division algebras over $K(t)$ of level 2. This is the case since the ramification data, of the type given in Theorem 3.6(a), define division algebras in ${}_2Br(K)$, which need not to be quaternion algebras, but can instead be biquaternion algebras. (In [KRTY], the relation between the ramification data and the index of division algebras over rational function fields is investigated. There are some partial results in that paper, but a criterion for the index to be equal to 2, in terms of the ramification data, is not available, as is also the case when K is a local field.)

Still, it is possible, using Theorem 3.6, to list some families of quaternion division algebras over $K(t)$ of level 2.

Let $x \in \mathbb{P}^1_K$ be such that $\kappa(x)/K$ is an extension with odd ramification index. If x is a finite point and $p(t)$ the corresponding monic irreducible polynomial in $K[t]$, then for every $f(t) \in K[t]$, with $\gcd(p(t), f(t)) = 1$,

$$\left(\frac{\pi, p(t)f(t)}{K(t)}\right)$$

is a quaternion division algebra of level 2.

If $x = \infty$, then for $f(t)$ a polynomial of odd degree in $K[t]$,

$$\left(\frac{\pi, f(t)}{K(t)}\right)$$

is also a quaternion division algebra of level 2. Both cases follow from Theorem 3.6, since the residue map in x applied to such algebras equals $\pi \not\equiv -1 \bmod \kappa(x)^{*2}$. Moreover, if we take x such that $-1 \notin \kappa(x)^{*2}$, then the sublevel of these algebras also equals 2. On the other hand, it follows from the proof of Corollary 3.7(b) that Theorem 3.6 also implies that the algebra $\left(\frac{\pi, t^2+1}{K(t)}\right)$ is a quaternion division algebra of level 2 and sublevel 1. The latter can be seen directly by taking $\alpha = ti$ and $\beta = i + k \in \left(\frac{\pi, t^2+1}{K(t)}\right)$, whereby

$$\alpha^2 + \beta^2 = \pi t^2 + \pi - \pi(t^2 + 1) = 0.$$

Although our prime interest in this paper was the level and sublevel of composition algebras, the results we obtained do raise the question as to what can be said about the level and sublevel of biquaternion division algebras over $K(t)$. For the sublevel, the fact that all quadratic forms over $K(t)$ of dimension > 8 are isotropic (cf. [PS2]) immediately yields

Proposition 3.9. *The sublevel of a biquaternion algebra over $K(t)$ is equal to* 1.

Proof. Consider the biquaternion algebra $D = \left(\frac{a_1,b_1}{K(t)}\right) \otimes_{K(t)} \left(\frac{a_2,b_2}{K(t)}\right)$, with standard basis $\{x \otimes y \mid x \in \{1, i_1, j_1, k_1\}, y \in \{1, i_2, j_2, k_2\}\}$. Consider

$P = \{\alpha_1(i_1 \otimes 1) + \alpha_2(j_1 \otimes 1) + \alpha_3(k_1 \otimes i_2) + \alpha_4(k_1 \otimes j_2) + \alpha_5(k_1 \otimes k_2) \in D \mid \alpha_1, \dots, \alpha_5 \in K(t)\}$.

For $p \in P$,

$$p^2 = (a_1\alpha_1^2 + b_1\alpha_2^2 - a_1b_1a_2\alpha_3^2 - a_1b_1b_2\alpha_4^2 + a_1b_1a_2b_2\alpha_5^2)(1 \otimes 1).$$

Since $2 \times \langle a_1, b_1, -a_1b_1a_2, -a_1b_1b_2, a_1b_1a_2b_2 \rangle$ is of dimension > 8, the form is isotropic over $K(t)$, (cf. [PS2, Theorem 4.6]). Hence there exist p_1 and $p_2 \in P$ such that ${p_1}^2 + {p_2}^2 = 0$. ■

Examples 3.10. Let K be a local non-dyadic field such that $-1 \notin K^{*2}$. Let $a, b \in K$ such that $a + b\sqrt{-1}$ is a non-square unit in $K(\sqrt{-1})$.

Proposition 3.5 in [KRTY] states that the biquaternion algebra

$$A = \left(\frac{\pi, t^2 + 2}{K(t)}\right) \otimes_{K(t)} \left(\frac{a + bt, t^2 + 1}{K(t)}\right)$$

is of index 4 over the quadratic extension $K(\sqrt{-1})$. It follows that $K(\sqrt{-1})$ is not isomorphic to a subfield of A, and hence that $s(A) \neq 1$, whereby we may conclude that $s(A) = 2$. Clearly, the level of both factors must also equal 2, as can be verified via Theorem 3.6.

In general it is not the case that a biquaternion division algebra $H_1 \otimes_{K(t)} H_2$ over $K(t)$, with $s(H_1) = 2$ and $s(H_2) = 2$, has level 2. For example, consider

$$B = \left(\frac{t - \pi, -\pi}{K(t)}\right) \otimes_{K(t)} \left(\frac{(t + 1)(t - \pi), \pi}{K(t)}\right).$$

Invoking Theorem 3.6, we see that both factors have level 2, since they have residue $-\pi$, respectively π, in the point $t = \pi$. But B is a biquaternion division algebra over $K(t)$ of level 1, (cf. [KRTY, Lemma 3.10]), since considering classes in the Brauer group we have

$$\begin{aligned} [B] &= [\left(\tfrac{t-\pi,-\pi}{K(t)}\right) \otimes_{K(t)} \left(\tfrac{\pi,(t-\pi)}{K(t)}\right) \otimes_{K(t)} \left(\tfrac{\pi,t-\pi}{K(t)}\right) \otimes_{K(t)} \left(\tfrac{(t+1)(t-\pi),\pi}{K(t)}\right)] \\ &= [\left(\tfrac{t-\pi,-1}{K(t)}\right) \otimes_{K(t)} \left(\tfrac{\pi,t+1}{K(t)}\right)]. \end{aligned}$$

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James O'Shea, School of Mathematical Sciences, Science Centre, University College Dublin, Belfield, Dublin 4, Ireland.

email: james.oshea@ucd.ie

Jan Van Geel, Department of Pure Mathematics and Computer Algebra, Ghent University, Galglaan 2, B-9000 Ghent, Belgium

email: jvg@cage.ugent.be