Passage to the limit in non-abelian Čech cohomology

Benedictus Margaux

Abstract: We give a detailed proof of the good behaviour of non-abelian cohomology under passage to the limit on the base scheme.

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1 Introduction

One of the central technical results in étale cohomology ([SGA4] théorème VII 5.7 and its corollaries) is the good behaviour of $H^q(S, \mathbf{G})$ for $S = \varprojlim S_\lambda$ and $\mathbf{G} = \varprojlim \mathbf{G}_\lambda$ under reasonable assumptions on the schemes S_λ and the abelian group schemes $\mathbf{G}_\lambda$. It is also remarked without proof therein (*ibid.* remarque 5.14(a)) that similar results hold for non-abelian H^1 defined *à la* Čech, i.e., for sheaf torsors. This passage to the limit appears as a crucial ingredient in the study of Galois cohomology of local and henselian rings, as well as in the study of infinite dimensional Lie theory by cohomological methods (see [CTO], [GP1], [GP2] and [GP3] for example). We have considered it useful to write down a detailed proof of this important fact. For clarity of exposition we have chosen *not* to restrict our attention to the case when the S_λ are affine (which would be sufficient for the work of Gille and Pianzola under consideration).

2 Passage to to the limit for non-abelian H^1

If X is a scheme and $\mathbf{G}$ is a group scheme over X, then $H^1_{fppf}(X, \mathbf{G})$ will denote the pointed set of Čech cohomology for the $fppf$ topology of X (see [SGA3] Exp IV.6 for details. See also [Gi] and [M]). $H^1_{ét}$ and H^1_{Zar} are defined analogously.

Let S_0 be a scheme. Throughout we assume that $(S_\lambda)_{\lambda \in \Lambda}$ is a projective system of S_0-schemes based on some non-empty directed set Λ such that for all $\lambda \geq \mu$ the transition morphisms $u_{\lambda\mu} : S_\mu \to S_\lambda$ are affine. We can then form the projective limit $S = \varprojlim S_\lambda$ in the category of S_0-schemes ([EGAIV] §8.2). By construction, we have for each $\lambda \in \Lambda$ a canonical morphism $u_\lambda : S \to S_\lambda$.

Let $\mathbf{G}_0$ be a group scheme over S_0. For $\lambda \in \Lambda$ let $\mathbf{G}_\lambda = \mathbf{G}_0 \times_{S_0} S_\lambda$, and $\mathbf{G} = \mathbf{G}_0 \times_{S_0} S$. If $\mathcal{U}^\alpha = (U_i^\alpha \to S_\alpha)$ is a covering of S_α in the $fppf$ topology, then the base change

$$(U_i^\alpha \times_{S_\alpha} S) \times_S (U_j^\alpha \times_{S_\alpha} S) \simeq (U_i^\alpha \times_{S_\alpha} U_j^\alpha) \times_{S_\alpha} S \to U_j^\alpha \times_{S_\alpha} U_j^\alpha$$

maps cocycles in $Z^1_{fppf}(\mathcal{U}^\alpha, \mathbf{G}_\alpha)$ into cocycles in $Z^1_{fppf}(\mathcal{U}^\alpha \times_{S_\alpha} S, \mathbf{G})$. This leads to a map $H^1_{fppf}(\mathcal{U}^\alpha, \mathbf{G}_\alpha) \to H^1_{fppf}(\mathcal{U}^\alpha \times_{S_\alpha} S, \mathbf{G})$ which, by passing to the limit over all coverings of S_α, yields a map $\psi_\alpha : H^1_{fppf}(S_\alpha, \mathbf{G}_\alpha) \to H^1_{fppf}(S, \mathbf{G})$. By considering $\varinjlim \psi_\alpha$ we obtain a canonical map

$$\psi : \varinjlim_{\lambda \in \Lambda} H^1_{fppf}(S_\lambda, \mathbf{G}_\lambda) \to H^1_{fppf}(S, \mathbf{G}).$$

Completely analogous considerations hold for the Zariski and étale topology.

The main result is as follows.

Theorem 2.1. *Assume that S_0 and the $(S_\lambda)_{\lambda \in \Lambda}$ are all quasicompact and quasiseparated, and that the group $\mathbf{G}_0 \to S_0$ is locally of finite presentation. Then the canonical map*

$$\varinjlim_{\lambda \in \Lambda} H^1_{fppf}(S_\lambda, \mathbf{G}_\lambda) \to H^1_{fppf}(S, \mathbf{G})$$

is bijective. Similarly for $H^1_{\acute{e}t}$ and H^1_{Zar}.

We begin by establishing two preliminary results that will be used in the proof of the Theorem. The notation is chosen to closely match that of [EGAIV] (to which all references henceforth belong).

Lemma 2.2. *Let S be a quasicompact scheme. Then any covering $\mathcal{U} = (U_i \xrightarrow{\phi_i} S)_{i \in I}$ of S (for either of our three topologies) admits a refinement $\mathcal{V} = (V_\ell \xrightarrow{\psi_\ell} S)_{\ell \in L}$ where L is finite and the V_ℓ are affine. If in addition S is quasiseparated, then the morphisms ψ_ℓ of the refinement $\mathcal{V}$ may be assumed to be of finite presentation.*

Proof. The ϕ_i are open maps (being flat and locally of finite presentation. See théorème 2.4.6). Given that S is quasicompact, there exists a finite subcovering (in particular a refinement) of $\mathcal{U}$.

Assume henceforth that I is finite. Let $S = \bigcup_{j \in J} Y_j$ be a finite open covering of S by affine Y_j. Let $W_{ij} = \phi_i^{-1}(Y_j)$, and let $W_{ij} = \cup_{k \in K} V_{ijk}$ be an open affine cover of W_{ij} (where K is some index set). Consider the morphisms $\psi_{ijk} : V_{ijk} \to S$ defined by

$$\psi_{ijk} : V_{ijk} \hookrightarrow W_{ij} \xrightarrow{\phi_i|_{W_{ij}}} Y_j \hookrightarrow S.$$

The V_{ijk} form a covering of S. Since any covering admits a finite subcovering, we may assume that K is finite. Set $L = I \times J \times K$, and define $\tau : L \to I$ by $\tau : (i, j, k) \mapsto i$. Then

$\mathcal{V} = (V_{ijk} \overset{\psi_{ijk}}{\to} S)_{(i,j,k)\in L}$ together with τ yield a finite refinement by affine schemes of our original $\mathcal{U}$.

We claim that if S is quasiseparated, then the morphism ψ_{ijk} above are of finite presentation, i.e., quasicompact, quasiseparated, and locally of finite presentation. The ψ_{ijk} are locally of finite presentation since $\mathcal{V}$ is a covering in one of our three topologies. That ψ_{ijk} is quasiseparated is automatic since the V_{ijk}, being affine, are quasiseparated (cor. 1.2.3(i)). Finally, since any morphism from a quasicompact scheme into a quasiseparated scheme is quasicompact (prop. 1.2.4), the ψ_{ijk} are quasicompact.

□

Proposition 2.3. *Assume S_0 is quasicompact and quasiseparated. Let $f : X \to S$ be a morphism of S_0-schemes which is of finite presentation. Then.*

(i) There exist $\alpha \in \Lambda$, and a scheme morphism $f_\alpha : X_\alpha \to S_\alpha$ of finite presentation, such that $X_\alpha \times_{S_\alpha} S \simeq X$ as S-schemes.

(ii) If α is as in (i), then for f to be surjective (resp. an open immersion, flat, faithfully flat, étale), it is necessary and sufficient that there exist $\lambda \geq \alpha$ for which $f_\lambda = f_\alpha \times I_{S_\lambda} : X_\alpha \times_{S_\alpha} S_\lambda \to S_\alpha \times_{S_\alpha} S_\lambda \simeq S_\lambda$ is surjective (resp. an open immersion, flat, faithfully flat, étale).

Proof. The existence of $f_\alpha : X_\alpha \to S_\alpha$ as in (i) is given by théorème 8.8.2. In view of (i), to establish (ii) we may assume with no loss of generality that $X = X_\alpha \times_{S_\alpha} S$ and $f = f_\alpha \times I_S$. For all $\lambda \geq \alpha$ set

$$X_\lambda \quad = X_\alpha \times_{S_\alpha} S_\lambda \text{ and } \quad f_\lambda = f_\alpha \times I_{S_\lambda}.$$

The existence of λ as prescribed in (ii) follows from théorème 8.10.5 (for f surjective or an open immersion), théorème 11.2.6 (for f flat), and proposition 17.7.8 (for f étale). Combining surjectivity with flatness yields a λ for which f_λ is faithfully flat. □

We are now ready to establish our main result.

Proof (of Theorem 2.1). For future use we begin with an observation. For $\alpha \in \Lambda$ the morphism $u_\alpha : S \to S_\alpha$ is affine, hence quasicompact and quasiseparated. Thus S itself is quasicompact and quasiseparated.

In what follows we fix one of our three topologies on S. All coverings, cocycles, and H^1 will refer to this chosen topology.

ψ is surjective. Let $c \in H^1(S, \mathbf{G})$, and choose a covering $\mathcal{U} = (U_i \to S)_{i\in I}$ so that c corresponds to a cocycle $z \in Z^1(\mathcal{U}, \mathbf{G})$. Taking into account that Čech cohomology is defined by passing to the limit of all refinements of covers of S, we may by Lemma 2.2 assume with no loss of generality that I is finite and that the $U_i \to S$ are of finite presentation. Given that Λ is directed, Proposition 2.3 now yields the existence of an $\alpha \in \Lambda$ and a covering $\mathcal{U}^\alpha = (U_i^\alpha \to S_\alpha)$ which induces $\mathcal{U}$ under the base change $u_\alpha : S \to S_\alpha$. Thus,

$$
\begin{aligned}
z_{ij} &\in \mathbf{G}\big((U_i^\alpha \times_{S_\alpha} S) \times_S (U_j^\alpha \times_{S_\alpha} S)\big) \\
&\simeq \mathbf{G}(U_i^\alpha \times_{S_\alpha} U_j^\alpha \times_{S_\alpha} S) = \mathbf{G}(U_i^\alpha \times_{S_\alpha} U_j^\alpha \times_{S_\alpha} \varprojlim_{\lambda \geq \alpha} S_\lambda) \\
&= \varinjlim_{\lambda \geq \alpha} \mathbf{G}_\lambda(U_i^\alpha \times_{S_\alpha} U_j^\alpha \times_{S_\alpha} S_\lambda) \\
&= \varinjlim_{\lambda \geq \alpha} \mathbf{G}_\lambda\big((U_i^\alpha \times_{S_\alpha} S_\lambda) \times_{S_\lambda} (U_j^\alpha \times_{S_\alpha} S_\lambda)\big)
\end{aligned}
$$

(this penultimate equality of limits because $\mathbf{G}$ is locally of finite presentation. See théorème 8.8.2(i)). This yields the existence of a $\beta \geq \alpha$ for which there exists elements $z_{ij}^\beta \in \mathbf{G}_\beta\big((U_i^\alpha \times_{S_\alpha} S_\beta) \times_{S_\beta} (U_j^\alpha \times_{S_\alpha} S_\beta)\big)$ such that $z_{ij}^\beta \mapsto z_{ij}$ under the base change $u_\beta : S \to S_\beta$. We do not know whether the z_{ij}^β satisfy the cocycle condition, but since the z_{ij} do, we can again use the fact that $\mathbf{G}$ is locally of finite presentation to conclude that there exists $\gamma \geq \beta$ such that the image z_{ij}^γ of the z_{ij}^β under the base change $u_{\beta\gamma} : S_\gamma \to S_\beta$ form a cocyle. Since $z_{ij}^\gamma \mapsto z_{ij}$, our map ψ is surjective.

ψ is injective. Let $c_1 \in H^1(S_{\alpha_1}, \mathbf{G}_{\alpha_1})$ and $c_2 \in H^1(S_{\alpha_2}, \mathbf{G}_{\alpha_2})$ be such that $\psi(c_1) = \psi(c_2)$. We must show that c_1 and c_2 have the same image under the respective canonical maps

$$
H^1(S_{\alpha_n}, \mathbf{G}_{\alpha_n}) \to \varinjlim_{\lambda \geq \alpha_n} H^1(S_\lambda, \mathbf{G}_\lambda), \tag{2.2}
$$

where $n = 1, 2$. Since Λ is directed, we may assume with no loss of generality that $\alpha_1 = \alpha_2 = \alpha$ for some $\alpha \in \Lambda$. We may also assume, as explained above and after taking a common refinement, that c_n corresponds to a cocyle $z_n^\alpha \in Z^1(\mathcal{U}^\alpha, \mathbf{G}_\alpha)$ for some covering $\mathcal{U}^\alpha = (U_i^\alpha \to S_\alpha)_{i \in I}$ of S_α with I finite and U_i^α affine. Since $\psi(c_1) = \psi(c_2)$, there exists a refinement $\mathcal{V} = (V_j \to S)_{j \in J}$ of the cover $\mathcal{U}^\alpha \times_{S_\alpha} S = (U_i^\alpha \times_{S_\alpha} S \to S)_{i \in I}$ where the images of z_1^α and z_2^α become cohomologous. We may again assume J to be finite and the V_j to be affine.

Let $\Lambda' = \{\lambda \in \Lambda : \lambda \geq \alpha\}$. Assume $i \in I$ and $j \in J$ are such that a morphism $V_j \to U_i^\alpha \times_{S_\alpha} S$ is part of our refinement. The same reasoning used at the end of the proof of Lemma 2.2 shows that this morphism is of finite presentation.[1]

For $\lambda \in \Lambda'$ define $S'_\lambda = U_i^\alpha \times_{S_\alpha} S_\lambda$ and $S' = U_i^\alpha \times_{S_\alpha} S$. Then $S' = \varprojlim S'_\lambda$ where the limit is taken over Λ'. By Proposition 2.3 applied to S' , there exists $\lambda \in \Lambda'$ such that our (flat, étale...) morphism $V_j \to S'$ comes from a (flat, étale...) morphism $V_j^\lambda \to S'_\lambda$ by the base change $S' \to S'_\lambda$ arising from u_λ. In fact, since I and J are finite and Λ' is directed, there exists $\beta \geq \alpha$ such that our entire refinement $\mathcal{V}$ of $\mathcal{U}^\alpha$ comes from a refinement $\mathcal{V}^\beta$ of the covering $\mathcal{U}^\alpha \times_{S_\alpha} S_\beta$ by the base change $S \to S_\beta$. Replacing the z_n^α by their respective images $z_n^\beta \in Z^1(\mathcal{U}^\alpha \times_{S_\alpha} S_\beta, \mathbf{G}_\beta)$, and then $\mathcal{U}^\alpha \times_{S_\alpha} S_\beta$ by its refinement $\mathcal{V}^\beta$ does not change

[1] To see that $U_i^\alpha \times_{S_\alpha} S$ is quasiseparated observe that because U_i^α is affine, it is quasiseparated over S_α. Thus $U_i^\alpha \times_{S_\alpha} S$ is quasiseparated over S, and we can now conclude from the fact that S is quasiseparated.

our c_n. This allows us to reduce to the case where our original cocycles z_1^α and z_2^α are such that their images z_1 and z_2 in $Z^1(\mathcal{U}^\alpha \times_{S_\alpha} S, \mathbf{G})$ are cohomologous. Accordingly, there exists elements $g_i \in \mathbf{G}(U_i^\alpha \times_{S_\alpha} S)$ such that

$$g_i(z_1)_{ij} g_j^{-1} = (z_2)_{ij} \text{ for all } i, j \in I \tag{2.3}$$

(where in (2.3) the g_i's are restricted to the $U_i^\alpha \times_{S_\alpha} U_j^\alpha \times_{S_\alpha} S$ as usual). Because $\mathbf{G}$ is locally of finite presentation the g_i's may be assumed to come from some elements $g_i^\gamma \in \mathbf{G}_\gamma(U_i^\alpha \times_{S_\alpha} S_\gamma)$ for some $\gamma \geq \alpha$. Replacing z_1^α and z_2^α by their images z_1^γ and z_2^γ under the base change $u_{\alpha\gamma} : S_\gamma \to S_\alpha$ we see that $g_i^\gamma (z_1^\gamma)_{ij} (g_j^\gamma)^{-1}$ and $(z_2^\gamma)_{ij}$ have the same image under the base change $u_\gamma : S \to S_\gamma$. Again since $\mathbf{G}$ is of finite presentation, we obtain that $g_i^\delta (z_1^\delta)_{ij} (g_j^\delta)^{-1} = (z_2^\delta)_{ij}$ after a base change $u_{\gamma\delta} : S_\delta \to S_\gamma$ with $\delta \geq \gamma$ suitably chosen. $\square$

Remark 2.4. We assume throughout that S_0 and the S_λ are quasicompact and quasiseparated.

(a) If $\mathbf{G}_0$ is flat, affine and locally of finite presentation over S_0, then by descent theory the sheaf torsors whose isomorphism classes are measured by $H^1_{fppf}(S_\lambda, \mathbf{G}_\lambda)$ are representable. Similarly for $H^1_{fppf}(S, \mathbf{G})$. The surjectivity of the map $\varinjlim H^1_{fppf}(S_\lambda, \mathbf{G}_\lambda) \to H^1_{fppf}(S, \mathbf{G})$ is as in (10.16) of [SGA3] $\mathrm{VI_B}$ (where the case when the S_λ are affine and the $\mathbf{G}_\lambda$ are of finite presentation is studied). Groups which are locally of finite presentation but not finitely presented (which are covered by our result) arise naturally in the classification of reductive groups, in particular of tori.

(b) Let $\mathbf{G}$ be a finitely presented group scheme over S. By Proposition 2.3 the group $\mathbf{G}$ is obtained from a finitely presented group over S_α by base change. Replacing Λ by $\Lambda_\alpha = \{\lambda \in \Lambda : \lambda \geq \alpha\}$ puts us back within the assumptions of Theorem 2.1. Thus, $H^1(S, \mathbf{G})$ can be computed in terms of direct limits.

(c) If $\mathbf{G}$ is a flat affine and finitely presented group scheme over S, and if Y is a torsor over S under $\mathbf{G}$, then the twisted S-group ${}_Y\mathbf{G}$ is also finitely presented and the considerations of (b) above apply.

(d) In Theorem 2.1 the assumption that $\mathbf{G}_0$ be representable is not crucial. The proof goes through as long as $\mathbf{G}_0$ is a group functor on S_0 which is locally of finite presentation.[2] An important example is the case when $\mathbf{G}_0$ is the group of automorphisms $\mathbf{Aut}_{S_0}(X_0)$ of a finitely presented scheme X_0 over S_0 (see [A] for details).

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[2] I am thankful to B. Conrad for bringing this point to my attention.

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Benedictus Margaux
Laboratoire de Recherche "Princess Stephanie"
Monte Carlo 51840, Monaco
benedictus.margaux@gmail.com