IRREGULAR AND SINGULAR LOCI OF COMMUTING VARIETIES

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To Bertram Kostant on the occasion of his 80*th birthday*

Abstract. Let $\mathcal{C}$ be the commuting variety of the Lie algebra $\mathfrak{g}$ of a connected noncommutative reductive algebraic group G over an algebraically closed field of characteristic zero. Let $\mathcal{C}^{\rm sing}$ be the singular locus of $\mathcal{C}$ and let $\mathcal{C}^{\rm irr}$ be the locus of points whose G-stabilizers have dimension $> \operatorname{rk} G$. We prove that: (a) $\mathcal{C}^{\rm sing}$ is a nonempty subset of $\mathcal{C}^{\rm irr}$; (b) $\operatorname{codim}_{\mathcal{C}}\mathcal{C}^{\rm irr} = 5 - \max\, l(\mathfrak{a})$ where the maximum is taken over all simple ideals $\mathfrak{a}$ of $\mathfrak{g}$ and $l(\mathfrak{a})$ is the "lacety" of $\mathfrak{a}$; (c) if $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$ and $\alpha \in \mathfrak{t}^*$ a root of $\mathfrak{g}$ with respect to $\mathfrak{t}$, then $\overline{G(\operatorname{Ker}\alpha \times \operatorname{Ker}\alpha)}$ is an irreducible components of $\mathcal{C}^{\rm irr}$ of codimension 4 in $\mathcal{C}$. This yields the bound $\operatorname{codim}_{\mathcal{C}}\mathcal{C}^{\rm sing} \geqslant 5 - \max\, l(\mathfrak{a})$ and, in particular, $\operatorname{codim}_{\mathcal{C}}\mathcal{C}^{\rm sing} \geqslant 2$. The latter may be regarded as an evidence in favor of the known long-standing conjecture that $\mathcal{C}$ is always normal. We also prove that the algebraic variety $\mathcal{C}$ is rational.

1. Introduction

1.1. Let $\mathfrak{g}$ be a noncommutative reductive Lie algebra over an algebraically closed field k of characteristic zero with adjoint group G. Let $\mathcal{C} = \mathcal{C}(\mathfrak{g})$ be the commuting variety of $\mathfrak{g}$,

$$\mathcal{C} = \mathcal{C}(\mathfrak{g}) := \{(x, y) \in \mathfrak{g} \times \mathfrak{g} \mid [x, y] = 0\}.$$

The known long-standing conjectures assert that

- (N) $\mathcal{C}$ is normal;
- (R) the ideal of regular functions on $\mathfrak{g} \times \mathfrak{g}$ vanishing on $\mathcal{C}$ is generated by the coordinates of $[x, y]$ for a generic point (x, y) of $\mathfrak{g} \times \mathfrak{g}$.

Let $\mathcal{C}^{\rm sing}$ be the singular locus of $\mathcal{C}$. If Conjecture (N) is true, then, according to the known theorem in algebraic geometry (see, e.g., [Sh, Ch. II, §5, Theorem 3]),

$$\operatorname{codim}_{\mathcal{C}} \mathcal{C}^{\rm sing} \geqslant 2. \tag{1}$$

1.2. In this paper we study $\operatorname{codim}_{\mathcal{C}} \mathcal{C}^{\rm sing}$. Our approach is based on the comparison of $\mathcal{C}^{\rm sing}$ with the *irregular locus* $\mathcal{C}^{\rm irr}$ whose codimension we manage to compute. The subvariety $\mathcal{C}^{\rm irr}$ is determined by the natural action of G on $\mathcal{C}$ as follows.

Let X be an algebraic variety endowed with an action of an algebraic group H. For a point $x \in X$, denote by $H(x)$ and H_x respectively the H-orbit and H-stabilizer of x. If Y is a subset of X, we put

$$Y^{\rm reg} := \{x \in Y \mid \dim H(x) \geqslant \dim H(y) \text{ for every } y \in Y\}, \quad X^{\rm irr} := X \setminus X^{\rm reg}$$

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(although the action is not reflected in this notation, below it is always clear from the contents what action is meant). The set $Y^{\rm reg}$ is open in Y.

As a first step we prove the following

Theorem 1.3.

(i) $(0,0)\in\mathcal{C}^{\rm sing}$, *so* $\mathcal{C}^{\rm sing}\neq\varnothing$*;*

(ii) $\mathcal{C}^{\rm sing}\subseteq\mathcal{C}^{\rm irr}$*;*

(iii) *If Conjecture* (R) *is true, then* $\mathcal{C}^{\rm sing}=\mathcal{C}^{\rm irr}$.

1.4. Then we compute $\operatorname{codim}_{\mathcal{C}}\mathcal{C}^{\rm irr}$. To this end we first prove that

$$2\leqslant\operatorname{codim}_{\mathcal{C}}\mathcal{C}^{\rm irr}\leqslant 4. \tag{2}$$

We give a direct proof of (2) in the framework of decomposition classes of $\mathfrak{g}$. Actually we deduce the upper bound in (2) from the following

Theorem 1.5. *Let* $\mathfrak{t}$ *be a Cartan subalgebra of* $\mathfrak{g}$ *and let* $\alpha\in\mathfrak{t}^*$ *be a root of* $\mathfrak{g}$ *with respect to* $\mathfrak{t}$*. Then*

$$\overline{G(\operatorname{Ker}\alpha\times\operatorname{Ker}\alpha)},$$

is an irreducible component of $\mathcal{C}^{\rm irr}$ *of codimension* 4 *in* $\mathcal{C}$*.*

Then we apply bounds (2) to computing $\operatorname{codim}_{\mathcal{C}}\mathcal{C}^{\rm irr}$. The latter problem is immediately reduced to that for simple Lie algebras $\mathfrak{g}$. Indeed, the decomposition

$$\mathfrak{g}=\mathfrak{g}_1\oplus\cdots\oplus\mathfrak{g}_d\oplus\mathfrak{z},$$

where $\mathfrak{g}_1,\ldots,\mathfrak{g}_d$ are simple ideals and $\mathfrak{z}$ is the center of $\mathfrak{g}$, clearly, implies the decomposition

$$\mathcal{C}(\mathfrak{g})=\mathcal{C}(\mathfrak{g}_1)\times\cdots\times\mathcal{C}(\mathfrak{g}_d)\times(\mathfrak{z}\times\mathfrak{z}) \tag{3}$$

that, in turn, implies that

$$\mathcal{C}^{\rm reg}=C(\mathfrak{g}_1)^{\rm reg}\times\cdots\times C(\mathfrak{g}_d)^{\rm reg}\times(\mathfrak{z}\times\mathfrak{z})$$

and hence

$$\operatorname{codim}_{\mathcal{C}}\mathcal{C}^{\rm irr}=\min_i\operatorname{codim}_{\mathcal{C}(\mathfrak{g}_i)}\mathcal{C}^{\rm irr}(\mathfrak{g}_i).$$

For simple $\mathfrak{g}$, we obtain the following complete answer:

Theorem 1.6. *Let* $\mathfrak{g}$ *be a simple Lie algebra. Then*

$$\operatorname{codim}_{\mathcal{C}}\mathcal{C}^{\rm irr}=5-l,$$

where l *is the "lacety" of* $\mathfrak{g}$*, i.e.,*

$$l=\begin{cases}1 & \text{if } \mathfrak{g} \text{ is of type } \mathsf{A}_r,\ \mathsf{D}_r,\ \mathsf{E}_6,\ \mathsf{E}_7, \text{ or } \mathsf{E}_8,\\ 2 & \text{if } \mathfrak{g} \text{ is of type } \mathsf{B}_r,\ \mathsf{C}_r, \text{ or } \mathsf{F}_4,\\ 3 & \text{if } \mathfrak{g} \text{ is of type } \mathsf{G}_2.\end{cases}$$

The proof of Theorem 1.6 is reduced by means of (2) to finding dimensions of certain subvarieties in the centralizers of some nilpotent elements of some semisimple Lie algebras of rank $\leqslant 3$. To tackle the latter problem we go case-by-case and utilize in our arguments some computations.

1.7. The formulated results yield the following information about $\operatorname{codim}_{\mathcal{C}}\mathcal{C}^{\rm sing}$. Clearly, (3) implies that

$$\mathcal{C}\setminus\mathcal{C}^{\rm sing}=\big(\mathcal{C}(\mathfrak{g}_1)\setminus\mathcal{C}(\mathfrak{g}_1)^{\rm sing}\big)\times\cdots\times\big(\mathcal{C}(\mathfrak{g}_d)\setminus\mathcal{C}(\mathfrak{g}_d)^{\rm sing}\big)\times(\mathfrak{z}\times\mathfrak{z}), \tag{4}$$

that, in turn, yields

$$\operatorname{codim}_{\mathcal{C}}\mathcal{C}^{\rm sing}=\min_i\operatorname{codim}_{\mathcal{C}(\mathfrak{g}_i)}\mathcal{C}(\mathfrak{g}_i)^{\rm sing}. \tag{5}$$

Thereby computing $\operatorname{codim}_{\mathcal{C}}\mathcal{C}^{\rm sing}$ is reduced to that for simple algebras $\mathfrak{g}$. Theorems 1.3 and 1.6 imply the following

Theorem 1.8. *Let $\mathfrak{g}$ be a simple Lie algebra. Then*

$$\operatorname{codim}_{\mathcal{C}} \mathcal{C}^{\rm sing} \geqslant 5 - l,$$

where l is the "lacety" of $\mathfrak{g}$.

The lower bound in (2) and Theorem 1.3 show that inequality (1) indeed holds for every algebra $\mathfrak{g}$. Moreover, from Theorem 1.8 and (5) we deduce that for some algebras $\mathfrak{g}$ a stronger inequality holds:

Corollary 1.9. *For every noncommutative reductive Lie algebra $\mathfrak{g}$ we have:*

(i) $\operatorname{codim}_{\mathcal{C}} \mathcal{C}^{\rm sing} \geqslant 2$;
(ii) *If $\mathfrak{g}$ contains no simple ideals of type G_2, then* $\operatorname{codim}_{\mathcal{C}} \mathcal{C}^{\rm sing} \geqslant 3$;
(iii) *If every simple ideal of $\mathfrak{g}$ is of type A, then* $\operatorname{codim}_{\mathcal{C}} \mathcal{C}^{\rm sing} \geqslant 4$.

Example 1.10. Let $\mathfrak{g} = \mathfrak{sl}_2$. Since SL_2-stabilizer of every nonzero element of $\mathfrak{sl}_2$ is one-dimensional, we have $\mathfrak{g}^{\rm irr} = \{0\}$. This implies that $\mathcal{C}^{\rm irr} = \{(0,0)\}$. By Theorem 1.3 this, in turn, yields that $\mathcal{C}^{\rm sing} = \{(0,0)\}$. As $\dim \mathcal{C} = 4$ by (10) below, we have $\operatorname{codim}_{\mathcal{C}}\mathcal{C}^{\rm sing} = \operatorname{codim}_{\mathcal{C}}\mathcal{C}^{\rm irr} = 4$ that agrees with Theorems 1.6 and 1.8.

This case is simple enough for obtaining this and further information directly, exploring the equations. Namely, take an $\mathfrak{sl}_2$-triple e, f, h, see [Bo$_2$, §11, 1], as a basis of $\mathfrak{g}$. Then the coordinates of $[x_1e + x_2f + x_3h, y_1e + y_2f + y_3h]$ in this basis generate in the polynomial algebra $k[x_1, x_2, x_3, y_1, y_2, y_3]$ the ideal

$$J := (x_2y_3 - x_3y_2,\ x_1y_2 - x_2y_1,\ x_1y_3 - x_3y_1). \tag{6}$$

Using (6) and, e.g., a computer algebra system (we used MAGMA), one immediately verifies that J is prime. Hence Conjecture (R) is true in this case. This makes it possible to prove that $\mathcal{C}^{\rm sing} = \{(0,0)\}$ directly exploring the rank of the Jacobi matrix of the system of generators of J given by (6).

Consider $x_1, x_2, x_3, y_1, y_2, y_3$ as the standard coordinate functions on the algebra

$$D := \{\operatorname{diag}(a_1, \ldots, a_6) \in \operatorname{Mat}_{6\times 6} \mid a_i \in k\}.$$

Then $\mathcal{C}$ is a closed subset of D. The group D^* of invertible elements of D is a 6-dimensional torus and (6) implies that the intersection of kernels of the characters $x_1x_2^{-1}y_1^{-1}y_2$ and $x_1x_3^{-1}y_1^{-1}y_3$ of D^* coincides with the open subset $T := \mathcal{C} \cap D^*$ of $\mathcal{C}$. Since $\mathcal{C}$ is irreducible, this means that T is a 4-dimensional subtorus of D^* and $\mathcal{C}$ is its closure in D. Hence $\mathcal{C}$ is the affine toric variety, namely, the closure of T-orbit of the identity matrix $I_6 \in D$ with respect to the action of T on D by the left multiplication. Applying the criterion of normality of such orbit closures [PV$_1$, Theorem 10], one easily verifies that the variety $\mathcal{C}$ is normal, i.e., in this case Conjecture (N) is true as well. □

Corollary 1.9(i) may be regarded as an evidence in favor of Conjecture (N).

1.11. Finally, we prove the following

Theorem 1.12. *The algebraic variety $\mathcal{C}$ is rational.*

1.13. We close this introduction by noting that the arguments and techniques of this paper are suitable, mutatis mutandis, for obtaining analogous results on commuting varieties associated with symmetric spaces and, more generally, cyclically graded semisimple Lie algebras, cf. [PV$_2$, 8.5].

1.14. *Notational conventions.*

$x = x_s + x_n$ is the Jordan decomposition of an element $x \in \mathfrak{g}$ with x_s semisimple and x_n nilpotent.
$\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$.
$\Phi \subseteq \mathfrak{t}^*$ is the root system of $\mathfrak{g}$ with respect to $\mathfrak{t}$.

Δ is a system of simple roots of Φ.
Φ^+ is the set of positive roots of Φ with respect to Δ.
$m := \dim\, \mathfrak{g}$.
$r := \dim\, \mathfrak{t}$.
$\mathfrak{a}_\mathfrak{b}$ is the centralizer of a subset $\mathfrak{b}$ of $\mathfrak{g}$ in a subset $\mathfrak{a}$ of $\mathfrak{g}$,

$$\mathfrak{a}_\mathfrak{b} := \{x \in \mathfrak{a} \mid [x, y] = 0 \text{ for all } y \in \mathfrak{b}\}.$$

$\mathfrak{z}(\mathfrak{a}) := \mathfrak{a}_\mathfrak{a}$ is the center of $\mathfrak{a}$.
Lie H is the Lie algebra of an algebraic groups H.
$\mathrm{T}_x(X)$ is the tangent space to an algebraic variety X at a point $x \in X$.
$|M|$ is the number of elements of a set M.
$\langle\, S\,\rangle$ is the k-linear span a subset S of a vector space over k.
I_n is the identity $n \times n$ matrix.

We say that a property holds for *points in general position* of an algebraic variety X if it holds for every point laying off a proper closed subset of X.

Our numeration of simple roots is that of Bourbaki [Bo$_1$].

2. Morphism $\boldsymbol{\mu}$

2.1. If $x \in \mathfrak{g}$, then $\mathrm{Lie}\, G_x = \mathfrak{g}_x$. Hence, for every $(x, y) \in \mathfrak{g} \times \mathfrak{g}$,

$$\text{Lie } G_{(x,y)} = \mathfrak{g}_x \cap \mathfrak{g}_y = (\mathfrak{g}_x)_y = (\mathfrak{g}_y)_x. \tag{7}$$

By [R] we have $\mathcal{C} = \overline{G(\mathfrak{t} \times \mathfrak{t})}$. This immediately implies that $\mathcal{C}$ is an irreducible variety,

$$\begin{aligned} \mathcal{C}^{\mathrm{reg}} &= \{(x, y) \in \mathcal{C} \subset \mathfrak{g} \times \mathfrak{g} \mid \dim(\mathfrak{g}_x)_y = r\},\\ \mathcal{C}^{\mathrm{irr}} &= \{(x, y) \in \mathcal{C} \subset \mathfrak{g} \times \mathfrak{g} \mid \dim(\mathfrak{g}_x)_y > r\}, \end{aligned} \tag{8}$$

and the fibers of the morphism

$$\pi_1\colon \mathcal{C} \to \mathfrak{g}, \quad (x, y) \mapsto x, \tag{9}$$

over points in general position in $\mathfrak{g}$ are isomorphic to $\mathfrak{t}$. Since π_1 is surjective, the latter property yields, by theorem on dimension of fibres (see, e.g., [Sh, Ch. I, §6, Theorem 7]), that

$$\dim\, \mathcal{C} = m + r. \tag{10}$$

2.2. Consider the morphism

$$\mu\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \quad (x, y) \mapsto [x, y]. \tag{11}$$

We have $\mathcal{C} = \mu^{-1}(0)$.

Lemma 2.3. *Let $z = (a, b)$ be a point of $\mathfrak{g} \times \mathfrak{g}$. Then*

(i) $\dim\, \mathrm{Ker}\, d_z\mu = m + \dim\, G_z$;
(ii) $\dim\, ([\mathfrak{g}, a] + [\mathfrak{g}, b]) = \dim\, G(z)$.

Proof. We identify in the natural way $\mathfrak{g} \oplus \mathfrak{g}$ (respectively, $\mathfrak{g}$) with $\mathrm{T}_z(\mathfrak{g} \times \mathfrak{g})$ (respectively, $\mathrm{T}_{\mu(z)}(\mathfrak{g})$). Then (11) implies that the differential $d_z\mu\colon \mathfrak{g} \oplus \mathfrak{g} \to \mathfrak{g}$ is given by the formula

$$d_z\mu((x, y)) = [x, b] + [a, y] \;\text{ for every } (x, y) \in \mathfrak{g} \oplus \mathfrak{g}. \tag{12}$$

Let $\mathfrak{g} \times \mathfrak{g} \to k, (x, y) \mapsto \langle x, y\rangle$, be a nondegenerate G-invariant symmetric bilinear form (since $\mathfrak{g}$ is reductive, it exists). Then from (12) we deduce that for every $t \in \mathfrak{g}$, $(x, y) \in \mathfrak{g} \oplus \mathfrak{g}$, we have

$$\langle t, d_z\mu((x, y))\rangle = \langle t, [x, b]\rangle + \langle t, [a, y]\rangle = \langle [b, t], x\rangle + \langle [t, a], y\rangle.$$

This means that if $\mathfrak{g}^*$ and $(\mathfrak{g}\oplus\mathfrak{g})^* = \mathfrak{g}^*\oplus\mathfrak{g}^*$ are identified by means of $\langle\ ,\ \rangle$ with, respectively, $\mathfrak{g}$ and $\mathfrak{g}\oplus\mathfrak{g}$, then the map $(d_z\mu)^*\colon \mathfrak{g}\to\mathfrak{g}\oplus\mathfrak{g}$ dual to $d_z\mu$ is given by the formula

$$(d_z\mu)^*(t) = ([b,t],[t,a]). \tag{13}$$

¿From (13) we deduce that

$$\mathrm{Ker}\,(d_z\mu)^* = \mathfrak{g}_a\cap\mathfrak{g}_b,$$

which, together with (7), imply that

$$\mathrm{rk}\ (d_z\mu)^* = m - \dim(\mathfrak{g}_a\cap\mathfrak{g}_b) = m - \dim G_z. \tag{14}$$

Now (i) follows from (14) because

$$\mathrm{rk}\ d_z\mu = \mathrm{rk}\ (d_z\mu)^*. \tag{15}$$

Further, by (12) the image of $d_z\mu$ is $[\mathfrak{g},a]+[\mathfrak{g},b]$. Since $\dim G(z) = m-\dim G_z$, this, (14), and (15) imply (ii). $\square$

Lemma 2.4. *Let z be a point of $\mathcal{C}$. Then*

(i) $\dim \mathrm{Ker}\, d_z\mu \geqslant \dim \mathcal{C}$;
(ii) *The following properties are equivalent:*
  (a) $\dim \mathrm{Ker}\, d_z\mu = \dim\mathcal{C}$;
  (b) $z\in\mathcal{C}^{\mathrm{reg}}$.

Proof. This follows from Lemma 2.3(i), (7), (8), and (10). $\square$

2.5. *Proof of Theorem* 1.3. By (4), proving (i), we may (and shall) assume that $\mathfrak{g}$ is simple. The point $(0,0)$ is fixed under the action of G on $\mathfrak{g}\times\mathfrak{g}$ and the G-module $\mathrm{T}_{(0,0)}(\mathfrak{g}\times\mathfrak{g})$ is isomorphic to $\mathfrak{g}\oplus\mathfrak{g}$. Since $\mathfrak{g}$ is simple, this implies that every proper submodule of $\mathrm{T}_{(0,0)}(\mathfrak{g}\times\mathfrak{g})$ is isomorphic to $\mathfrak{g}$ and, in particular, its dimension is m. But $\mathrm{T}_{(0,0)}(\mathcal{C})$ is a nonzero submodule of $\mathrm{T}_{(0,0)}(\mathfrak{g}\times\mathfrak{g})$ and $\dim\mathrm{T}_{(0,0)}(\mathcal{C})\geqslant\dim\mathcal{C} = m+r > m$. Hence $\mathrm{T}_{(0,0)}(\mathcal{C}) = \mathrm{T}_{(0,0)}(\mathfrak{g}\times\mathfrak{g})$ and therefore $\dim\mathrm{T}_{(0,0)}(\mathcal{C}) = 2m$. Since $m>r$,we deduce from here that $\dim\mathrm{T}_{(0,0)}(\mathcal{C}) > \dim\mathcal{C}$. This proves (i).

As $\mathcal{C}=\mu^{-1}(0)$, we have, for every point $z\in\mathcal{C}$, the inclusion

$$\mathrm{T}_z(\mathcal{C})\subseteq \mathrm{Ker}\ d_z\mu. \tag{16}$$

Let $z\in\mathcal{C}^{\mathrm{sing}}$, i.e., $\dim\mathrm{T}_z(\mathcal{C})>\dim\mathcal{C}$. Then inclusion (16) yields that $\dim\mathrm{Ker}\ d_z\mu > \dim\mathcal{C}$; whence $z\in\mathcal{C}^{\mathrm{irr}}$ by Lemma 2.4(ii). This proves (ii).

Assume that Conjecture (R) is true. Then $\mathrm{T}_z(\mathcal{C}) = \mathrm{Ker}\ d_z\mu$ for every point $z\in\mathcal{C}$ and hence the inclusion $z\in\mathcal{C}^{\mathrm{sing}}$ is equivalent to the inequality $\dim\mathrm{Ker}\ d_z\mu>\dim\mathcal{C}$. By Lemma 2.4(ii) the latter inequality is equivalent to the inclusion $z\in\mathcal{C}^{\mathrm{irr}}$. This proves (iii). $\square$

2.6. *Remark.* It is claimed in [NS, Theorem 1.1] that $\mathcal{C}^{\mathrm{sing}} = \mathcal{C}^{\mathrm{irr}}$ for $\mathfrak{g}=\mathfrak{gl}_n$. Unfortunately, the proof of this claim given in [NS] is incorrect since the arguments on p. 548 are based on the implicit assumption that Conjecture (R) is true. However, these arguments do prove the inclusion $\mathcal{C}^{\mathrm{sing}}\subseteq\mathcal{C}^{\mathrm{irr}}$ for $\mathfrak{g}=\mathfrak{gl}_n$ that is the paricular case of Theorem 1.3(ii).

Similar mistake is made in paper [Bre] aimed to explore the singular locus of the commuting variety of pairs of symmetric matrices.

3. Lower and upper bounds for $\mathrm{codim}_{\mathcal{C}}\,\mathcal{C}^{\mathrm{irr}}$

3.1. Our further arguments are based on consideration of decomposition classes ("Zerlegungsklassen") [BK], [Bro$_1$], [Bro$_2$] (a.k.a. "packets" [Sp], [E] and "Jordan classes" [TY]).

Recall that two elements $x,y\in\mathfrak{g}$ are called *decomposition equivalent* if there is a $g\in G$ such that $\mathfrak{g}_{x_s}=\mathfrak{g}_{g(y_s)}$ and $x_n=g(y_n)$. This defines an equivalence relation on $\mathfrak{g}$ whose equivalence classes are called *decomposition classes*. Every decomposition class $\mathcal{D}$ is an irreducible locally closed smooth G-stable subvariety of $\mathfrak{g}$. All G-orbits in $\mathcal{D}$ are of the same dimension and $\mathcal{D} \subseteq \overline{\mathcal{D}}^{\rm reg}$. If $z \in \mathcal{D}$, then

$$\begin{aligned} \mathcal{D} &= G(\mathfrak{z}(\mathfrak{g}_{z_s})^{\rm reg} + z_n), \\ \dim\, \mathcal{D} &= \dim \mathfrak{z}(\mathfrak{g}_{z_s}) + \dim G(z). \end{aligned} \tag{17}$$

3.2. As $\mathfrak{z}(\mathfrak{g}_{z_s})$ is a Levi subalgebra, it is conjugate to a standard one with respect to Δ. This yields the following description of decomposition classes. Let I be a subset of Δ (may be empty), let $\Phi(I)$ be the set elements of Φ that are linear combinations of elements of I, and let $\Phi(I)^+ := \Phi^+ \cap \Phi(I)$. Consider the Levi subalgebra

$$\mathfrak{g}(I) := \mathfrak{t} \oplus \textstyle\sum_{\alpha\in\Phi(I)} \mathfrak{g}^\alpha.$$

We have

$$\mathfrak{g}(I) = \mathfrak{t}(I) \oplus \mathfrak{s}(I) \quad \text{where } \ \mathfrak{t}(I) := \mathfrak{z}(\mathfrak{g}(I)),\ \mathfrak{s}(I) := [\mathfrak{g}(I), \mathfrak{g}(I)],$$

and

$$\mathfrak{t}(I) = \textstyle\bigcap_{\alpha\in I} \operatorname{Ker} \alpha, \qquad \mathfrak{t}(I)^{\rm reg} = \mathfrak{t}(I) \setminus \textstyle\bigcup_{\alpha\notin\Phi(I)} \operatorname{Ker} \alpha. \tag{18}$$

Let x be a nilpotent element of $\mathfrak{s}(I)$. Then

$$\mathcal{D}(I, x) := G(\mathfrak{t}(I)^{\rm reg} + x) \tag{19}$$

is a decomposition class, every decomposition class coincides with some $\mathcal{D}(I, x)$, and $\mathcal{D}(I, x) = \mathcal{D}(J, y)$ if and only if $I = J$ and x and y lay in the same orbit of the adjoint group of $\mathfrak{s}(I)$. In particular, since the number of nilpotent orbits in $\mathfrak{s}(I)$ is finite, there are only finitely many decomposition classes.

If I consists of a single element α, we shall write $\mathfrak{t}(\alpha)$ in place of $\mathfrak{t}(\{\alpha\})$ etc.

Lemma 3.3.

(i) $\dim \mathcal{D}(I, x) = m - \dim \mathfrak{s}(I)_x$;

(ii) *The following are equivalent:*

(a) $\mathcal{D}(I, x) \subseteq \mathfrak{g}^{\rm irr}$;

(b) $\dim \mathfrak{s}(I)_x > |I|$;

(c) $x \in \mathfrak{s}(I)^{\rm irr}$.

Proof. If $t \in \mathfrak{t}(I)^{\rm reg}$, then $\mathfrak{g}_t = \mathfrak{g}(I)$ by (18). For $z = t + x \in \mathcal{D}(I, x)$, we have $t = z_s$, $x = z_n$. This yields

$$\mathfrak{g}_z = (\mathfrak{g}_t)_x = (\mathfrak{t}(I) \oplus \mathfrak{s}(I))_x = \mathfrak{t}(I) \oplus \mathfrak{s}(I)_x. \tag{20}$$

¿From (17) and (20) we then deduce that $\dim \mathcal{D}(I, x) = \dim \mathfrak{t}(I) + m - \dim \mathfrak{g}_z = m - \dim \mathfrak{s}(I)_x$. This proves (i).

It is well known [K$_2$] that

$$\mathfrak{g}^{\rm irr} = \{z \in \mathfrak{g} \mid \dim \mathfrak{g}_z > r\}. \tag{21}$$

By (18) we have $\dim \mathfrak{t}(I) = r - |I|$, so (20) yields $\dim \mathfrak{g}_z = r - |I| + \dim \mathfrak{s}(I)_x$; whence by (21) the equivalence (a)$\Leftrightarrow$(b) in (ii). Since $\mathfrak{s}(I)$ is a semisimple Lie algebra of rank $|I|$, the equivalence (b)$\Leftrightarrow$(c) in (ii) follows from the description of irregular loci in reductive Lie algebras given by (21). □

Corollary 3.4. *Let $\alpha \in \Delta$. Then $\mathcal{D}(\alpha, x) \subseteq \mathfrak{g}^{\rm irr}$ if and only if $x = 0$.*

Proof. Since $\mathfrak{s}(\alpha)$ is isomorphic to $\mathfrak{sl}_2$ and $(\mathfrak{sl}_2)^{\rm irr} = \{0\}$, see Example 1.10, the claim follows from the equivalence (a)$\Leftrightarrow$(c) in Lemma 3.3(ii). □

3.5. The following statement should be known to the experts, but I failed to find a proper reference and shall give a short proof (for the conjugating action of G on G, the counterpart of this statement is proved in [St$_1$] (see also [St$_2$])).

Lemma 3.6.

(i) *For every* $\alpha \in \Delta$, *the variety* $\overline{G(\operatorname{Ker}\alpha)} = \overline{\mathcal{D}(\alpha,0)}$ *is an irreducible component of* $\mathfrak{g}^{\mathrm{irr}}$.
(ii) *Every irreducible components of* $\mathfrak{g}^{\mathrm{irr}}$ *is of this type.*
(iii) $\dim \mathcal{D}(\alpha,0) = m-3$ *for every* $\alpha \in \Phi$.

Proof. Since in every decomposition class all G-orbits are of the same dimension, (21) implies that $\mathfrak{g}^{\mathrm{irr}}$ is a union of decomposition classes. Take a decomposition class $\mathcal{D}(I,y) \subseteq \mathfrak{g}^{\mathrm{irr}}$. Then by Lemma (3.3)(ii) we have $\mathfrak{s}(I) \neq 0$ and $y \in \mathfrak{s}(I)^{\mathrm{irr}}$. Hence by [K$_1$, Theorem 5.3] there is a root $\alpha \in I$ such that the orbit of y under the action of the adjoint group of $\mathfrak{s}(I)$ intersects the subalgebra $\sum_{\gamma\in\Phi(I)^+\setminus\{\alpha\}} \mathfrak{g}^{\gamma}$. Therefore

$$\mathcal{D}(I,y) = \mathcal{D}(I,x) \quad \text{for some } x \in \textstyle\sum_{\gamma\in\Phi(I)^+\setminus\{\alpha\}} \mathfrak{g}^{\gamma}. \tag{22}$$

On the other hand, according to [BK, 5.4] (see also [TY, 39.2.2]),

$$\overline{\mathcal{D}(\alpha,0)} \supseteq \mathfrak{t}(\alpha) + \textstyle\sum_{\gamma\in\Phi^+\setminus\{\alpha\}} \mathfrak{g}^{\gamma}. \tag{23}$$

¿From (18), (19), (22), (23) we then deduce that

$$\mathcal{D}(I,y) \subseteq \overline{\mathcal{D}(\alpha,0)} = \overline{G(\operatorname{Ker}\alpha)}. \tag{24}$$

For every root $\gamma \in \Delta$, by Corollary 3.4 we have $\mathcal{D}(\gamma,0) \subseteq \mathfrak{g}^{\mathrm{irr}}$ and, since $\mathfrak{s}(\gamma)$ is isomorphic to $\mathfrak{sl}_2$, Lemma 3.3(i) yields that $\dim \mathcal{D}(\gamma,0) = m-3$. By virtue of (24) this completes the proof. $\square$

3.7. *Remarks.* (a) $\mathcal{D}(\alpha,0) = \mathcal{D}(\beta,0)$ if and only if $W(\alpha) = W(\beta)$ where W is the Weyl group. Hence the number of irreducible components of $\mathfrak{g}^{\mathrm{irr}}$ is equal to $|\Phi/W|$.

(b) Since $\mathcal{D}(\alpha,0) = G(\mathfrak{t}(\alpha))$ and $\mathfrak{t}(\alpha)$ is a reductive subalgebra of $\mathfrak{g}$, Lemma 3.6(iii) is a special case of the following more general

Lemma 3.8. *Let* $\mathfrak{l}$ *be a reductive subalgebra of* $\mathfrak{g}$ *and let* $\mathfrak{c}$ *be its maximal torus. Assume that* $\mathfrak{c} \subseteq \mathfrak{t}$. *Then*

$$\dim \overline{G(\mathfrak{l})} = \dim \mathfrak{c} + |\{\alpha \in \Phi \mid \mathfrak{c} \nsubseteq \operatorname{Ker}\alpha\}|.$$

Proof. Taking into account that the image of morphism $G\times\mathfrak{l} \to \mathfrak{g}$, $(g,x) \mapsto g(x)$, contains an open subset of $\overline{G(\mathfrak{l})}$, and the union of maximal tori of $\mathfrak{l}$ contains an open subset of $\mathfrak{l}$, we coclude that points x in general position in $\mathfrak{l}$ are nonsingular points of $\overline{G(\mathfrak{l})}$ and $\mathrm{T}_x(\overline{G(\mathfrak{l})}) = \mathfrak{l} + \mathfrak{g}(x)$. Hence $\dim \overline{G(\mathfrak{l})} = \dim(\mathfrak{l} + \mathfrak{g}(x))$. The root decomposition of $\mathfrak{g}$ with respect to $\mathfrak{t}$ yields

$$\mathfrak{g}(x) = \textstyle\bigoplus_{\{\alpha\in\Phi \,\mid\, \mathfrak{c}\nsubseteq \operatorname{Ker}\alpha\}} \mathfrak{g}^{\alpha}. \tag{25}$$

The right-hand side of (25) is the sum of all weight subspaces of $\mathfrak{g}$ with respect to $\mathfrak{c}$ with the nonzero weights. Since every root space of $\mathfrak{l}$ with respect to $\mathfrak{c}$ lies in this sum, we obtain

$$\mathfrak{l} + \mathfrak{g}(x) = \mathfrak{c} + \textstyle\bigoplus_{\{\alpha\in\Phi \,\mid\, \mathfrak{c}\nsubseteq \operatorname{Ker}\alpha\}} \mathfrak{g}^{\alpha};$$

whence the claim. $\square$

3.9. It is convenient to introduce the following notation.

Let $\mathfrak{a}$ be a reductive Lie algebra and let x be its element. Put

$$\mathcal{I}(\mathfrak{a}_x) := \{y \in \mathfrak{a}_x \mid \dim(\mathfrak{a}_x)_y > \operatorname{rk}\mathfrak{a}\}. \tag{26}$$

Clearly, $\mathcal{I}(\mathfrak{a}_x)$ is empty if and only if $x \in \mathfrak{a}^{\mathrm{reg}}$. If x is semisimple, then $\mathfrak{a}_x$ is a reductive algebra of rank $\operatorname{rk}\mathfrak{a}$, whence $\mathcal{I}(\mathfrak{a}_x) = \mathfrak{a}_x^{\mathrm{irr}}$. For nonsemisimple x, this equality, in general, does not hold (see below Subsection 4.5).

Lemma 3.10. *Let* $x \in [\mathfrak{a},\mathfrak{a}]$. *Then* $\mathcal{I}(a_x)$ *is isomorphic to* $\mathfrak{z}(\mathfrak{a}) \times \mathcal{I}([\mathfrak{a},\mathfrak{a}])_x)$.

Proof. Since $\mathfrak{a} = \mathfrak{z}(\mathfrak{a}) \oplus [\mathfrak{a}, \mathfrak{a}]$, we have $\mathfrak{a}_x = \mathfrak{z}(\mathfrak{a}) \oplus [\mathfrak{a}, \mathfrak{a}]_x$. Hence, for $z \in \mathfrak{z}(\mathfrak{a})$ and $y \in [\mathfrak{a}, \mathfrak{a}]_x$, we have $(\mathfrak{a}_x)_{z+y} = \mathfrak{z}(\mathfrak{a}) \oplus ([\mathfrak{a}, \mathfrak{a}]_x)_y$. Since $\operatorname{rk} \mathfrak{a} = \dim \mathfrak{z}(\mathfrak{a}) + \operatorname{rk} [\mathfrak{a}, \mathfrak{a}]$, this and (26) show that $z + y \in \mathcal{I}(\mathfrak{a}_x)$ if an only if $y \in \mathcal{I}([\mathfrak{a}, \mathfrak{a}])_x$; whence the claim. □

By (18) this yields

Corollary 3.11. *Let I be a subset of Δ and let x be an element of $\mathfrak{s}(I)$. Then*

$$\dim \mathcal{I}(\mathfrak{g}(I)_x) = r - |I| + \dim \mathcal{I}(\mathfrak{s}(I)_x). \tag{27}$$

3.12. It follows from (7), (8), (21) that the restriction to $\mathcal{C}^{\mathrm{irr}}$ of the projection π_1 (see (9)) is a surjective morphism

$$\pi := \pi_1|_{\mathcal{C}^{\mathrm{irr}}} : \mathcal{C}^{\mathrm{irr}} \to \mathfrak{g}^{\mathrm{irr}}. \tag{28}$$

Let y be a point of $\mathfrak{g}^{\mathrm{irr}}$. Then by (8) and (26) we have

$$\pi^{-1}(y) = \{(y, z) \in \mathfrak{g} \times \mathfrak{g} \mid z \in \mathcal{I}(\mathfrak{g}_y)\}.$$

This shows that the second projection yields an isomorphism

$$\pi^{-1}(y) \xrightarrow{\simeq} \mathcal{I}(\mathfrak{g}_y). \tag{29}$$

3.13. Now we shall prove lower bound (2) for $\operatorname{codim}_{\mathcal{C}} \mathcal{C}^{\mathrm{irr}}$.

Theorem 3.14. $\operatorname{codim}_{\mathcal{C}} \mathcal{C}^{\mathrm{irr}} \geqslant 2$.

Proof. Let X be an irreducible component of $\mathcal{C}^{\mathrm{irr}}$. By (10) we have to show that

$$\dim X \leqslant m + r - 2. \tag{30}$$

By theorem on dimension of fibres, for every point $y \in \overline{\pi(X)}$, we have the inequality

$$\dim X \leqslant \dim \overline{\pi(X)} + \dim(\pi^{-1}(y) \cap X) \tag{31}$$

Since $\overline{\pi(X)}$ is an irreducible variety and $\mathfrak{g}$ is the union of decomposition classes, there is a decomposition class $\mathcal{D} = \mathcal{D}(I, x)$ such that

$$\overline{\pi(X) \cap \mathcal{D}} = \overline{\pi(X)}. \tag{32}$$

By (32) we have $\pi(X) \cap \mathcal{D} \neq \varnothing$. Take a point $y \in \pi(X) \cap \mathcal{D}$. Since $\pi^{-1}(y)$ is isomorphic to $\mathcal{I}(\mathfrak{g}_y)$, we have

$$\dim(\pi^{-1}(y) \cap X) \leqslant \dim \mathcal{I}(\mathfrak{g}_y). \tag{33}$$

¿From (17) we obtain that

$$\dim \mathcal{D} = \dim \mathfrak{t}(I) + \dim G(y). \tag{34}$$

It follows from (32) that $\overline{\pi(X)} \subseteq \overline{\mathcal{D}}$. This and (31), (33), (34) then imply that

$$\begin{aligned} \dim X &\leqslant \dim \mathfrak{t}(I) + \dim G(y) + \dim \mathcal{I}(\mathfrak{g}_y) \\ &= \dim \mathfrak{t}(I) + \dim G(y) + \dim \mathfrak{g}_y - \operatorname{codim}_{\mathfrak{g}_y} \mathcal{I}(\mathfrak{g}_y) \\ &= \dim \mathfrak{t}(I) + m - \operatorname{codim}_{\mathfrak{g}_y} \mathcal{I}(\mathfrak{g}, y) \leqslant \dim \mathfrak{t}(I) + m. \end{aligned} \tag{35}$$

As $\pi(X) \subseteq \mathfrak{g}^{\mathrm{irr}}$ and all G-orbits in $\mathcal{D}$ are of the same dimension, $\pi(X) \cap \mathcal{D} \neq \varnothing$ implies that $\mathcal{D} \subseteq \mathfrak{g}^{\mathrm{irr}}$. By Lemma 3.3 this yields $I \neq \varnothing$, hence $\dim \mathfrak{t}(I) \leqslant r - 1$.

If $\dim \mathfrak{t}(I) \leqslant r - 2$, then (35) implies (30).

So it remains to consider the case where $\dim \mathfrak{t}(I) = r - 1$, i.e., $\mathcal{D} = \mathcal{D}(\alpha, x)$ for some root α. Since $\mathcal{D} \subseteq \mathfrak{g}^{\mathrm{irr}}$, we deduce from Corollary 3.4 that $x = 0$. By (19) this means that y is a semisimple element. Hence $\mathcal{I}(\mathfrak{g}_y) = (\mathfrak{g}_y)^{\mathrm{irr}}$. By Lemma 3.6 this implies that $\operatorname{codim}_{\mathfrak{g}_y} \mathcal{I}(\mathfrak{g}_y) = 3$. Plugging this in (35), we obtain $\dim X \leqslant m + r - 4$; whence (30). This completes the proof. □

3.15. The following statement is used in the proof of upper bound (2) for $\operatorname{codim}_{\mathcal{C}} \mathcal{C}^{\mathrm{irr}}$.

Lemma 3.16. *Let $\varphi\colon X \to Y$ be a dominant morphism of algebraic varieties. Assume that Y is irreducible (but X may be not). Then*

(i) *There are an integer $c \geqslant 0$ and an irreducible component Z of X such that*
  (a) $\dim \varphi^{-1}(y) = c$ *for points y in general position in Y;*
  (b) $\dim Z = c + \dim Y$.
  (c) $\overline{\varphi(Z)} = Y$;

(ii) *If the fibers of φ over points in general position in Y are irreducible, then Z is the unique irreducible component of Y whose image under φ is dense in Y and there is an open subset U of Y such that $Z = \overline{\varphi^{-1}(U)}$.*

Proof. Since φ is dominant and Y is irreducible, there is an irreducible component of X whose image under φ is dense in Y. Let $Z_1, \ldots, Z_n$ be all such components. Put $\psi_i := \varphi|_{Z_i}\colon Z_i \to Y$. By theorem of dimension of fibers applied to ψ_i there is an integer $c_i \geqslant 0$ such that $\dim \psi_i^{-1}(y) = c_i$ for points y in general position in Y and

$$\dim Z_i = c_i + \dim Y. \tag{36}$$

Put $c = \max\limits_i c_i$ and let $c = c_{i_0}$. By construction, for points y in general position in Y, we have

$$\varphi^{-1}(y) = \textstyle\bigcup_{i=1}^n \psi_i^{-1}(y). \tag{37}$$

Hence $\dim \varphi^{-1}(y) = c$. This and (36) show that we can take $Z := Z_{i_0}$. This proves (i).

If $\varphi^{-1}(y)$ in (37) is irreducible, then $\varphi^{-1}(y) = \psi_{i_0}^{-1}(y)$ because of the dimension reason. Hence there is an open subset U in Y such that for every i we have $\psi_i^{-1}(U) \subseteq Z$. Since Z_i is irreducible, we obtain $Z_i = \overline{\psi_i^{-1}(U)} \subseteq Z$, i.e., $i = i_0$. This proves (ii). $\square$

3.17. *Proof of Theorem* 1.5. Let Y be an irreducible component of $\mathfrak{g}^{\rm irr}$, let $X := \pi^{-1}(Y)$ (see (28)), and let $\varphi := \pi|_X\colon X \to Y$. By Lemma 3.16 there is an irreducible component Z of X and an integer $c \geqslant 0$ such that properties (a), (b), (c) in the formulation of this lemma hold.

The variety Z is an irreducible component of $\mathcal{C}^{\rm irr}$. Indeed, since Z is irreducible, there is an irreducible component Z' of $\mathcal{C}^{\rm irr}$ containing Z. By (c) we have $Y \subseteq \overline{\pi(Z')}$. Since Y is an irreducible component of $\mathfrak{g}^{\rm irr}$ and $\overline{\pi(Z')}$ is an irreducible subvariety of $\mathfrak{g}^{\rm irr}$, this implies $Y = \overline{\pi(Z')}$. Hence $Z' \subseteq X$. Since Z is a maximal irreducible closed subset of X and $Z \subseteq Z'$, we have $Z = Z'$.

By Lemma 3.6 there is $\alpha \in \Delta$ such that

$$Y = \overline{G(\text{Ker } \alpha)}. \tag{38}$$

For every $y \in (\text{Ker } \alpha)^{\rm reg}$, we have $\mathfrak{g}_y = \mathfrak{g}(\alpha)$. Since $\mathfrak{s}(\alpha)$ is isomorphic to $\mathfrak{sl}_2$, this yields $\mathfrak{g}_y^{\rm irr} = \text{Ker } \alpha$. As y is semisimple, the discussion in Subsection 3.9 then implies that

$$\varphi^{-1}(y) = \{(y, z) \in \mathcal{C} \mid z \in \text{Ker } \alpha\}. \tag{39}$$

This, in particular, shows that $\varphi^{-1}(y)$ is irreducible and

$$c = \dim \varphi^{-1}(y) = r - 1. \tag{40}$$

¿From (38), (39), and Lemma 3.16 it clearly follows that $Z = \overline{G(\text{Ker } \alpha \times \text{Ker } \alpha)}$. Since by Lemma 3.6 we have

$$\dim Y = m - 3,$$

Lemma 3.16 and (40) yield that $\dim Z = m + r - 4$. By virtue of (10) this completes the proof. $\square$

3.18. Theorems 1.5 and 3.14 reduce computing $\text{codim}_{\mathcal{C}} \mathcal{C}^{\rm irr}$ to finding out whether there are irreducible components of $\mathcal{C}^{\rm irr}$ of codimensions 2 and 3 or not. We now turn to solving this problem.

Lemma 3.19. *Let C be an irreducible component of $\mathcal{C}^{\rm irr}$. Then there is a decomposition class $\mathcal{D} \subseteq \mathfrak{g}^{\rm irr}$ such that C is the closure of one of the irreducible components of $\pi^{-1}(\mathcal{D})$.*

Proof. Since $\mathfrak{g}^{\rm irr}$ is the union of decomposition classes and π is surjective, we have

$$\mathcal{C}^{\rm irr} = \textstyle\bigcup_{i=1}^{n} C_i,$$

where $C_1, \ldots, C_n$ is the set of all irreducible components of all varieties $\pi^{-1}(\mathcal{D})$ where $\mathcal{D}$ runs through the set of all decomposition classes contained in $\mathfrak{g}^{\rm irr}$. Hence

$$C = \textstyle\bigcup_{i=1}^{n} \overline{C \cap C_i}.$$

Since C is irreducible, this implies that $C = \overline{C \cap C_j}$ for some j. Hence $C \subseteq \overline{C_j} \subseteq \mathcal{C}^{\rm irr}$. As $\overline{C_j}$ is irreducible and C is a maximal irreducible closed subset of $\mathcal{C}^{\rm irr}$, from this we deduce that $C = \overline{C_j}$. $\square$

Lemma 3.20.

(i) *There is an irreducible component Z of $\pi^{-1}(\mathcal{D}(I, x))$ such that*

$$\operatorname{codim}_{\mathcal{C}} Z = \operatorname{codim}_{\mathfrak{s}(I)_x} \mathcal{I}(\mathfrak{s}(I)_x) + |I| \tag{41}$$

and $\pi(Z)$ is dense in $\mathcal{D}(I, x)$.

(ii) $\operatorname{codim}_{\mathcal{C}} Z' \geqslant \operatorname{codim}_{\mathcal{C}} Z$ *for every irreducible component Z' of $\pi^{-1}(\mathcal{D}(I, x))$.*

Proof. Let z be a point of $\mathfrak{t}(I)^{\rm reg}$ and let $y = z + x$. We have $\mathfrak{g}_z = \mathfrak{g}(I)$; whence $\mathfrak{g}_y = (\mathfrak{g}_z)_x = \mathfrak{g}(I)_x$. From this and (29) we deduce that $\pi^{-1}(y)$ is isomorphic to $\mathcal{I}\big(\mathfrak{g}(I)_x\big)$. Therefore by (27) we have

$$\dim \pi^{-1}(y) = r - |I| + \dim \mathcal{I}(\mathfrak{s}(I)_x). \tag{42}$$

It follows from (19) and (42) that dimension of fiber of π over every point of $\mathcal{D}(I, x)$ is equal to $r - |I| + \dim \mathcal{I}(\mathfrak{s}(I)_x)$. By theorem on dimension of fibers this, Lemma 3.3(i), and Lemma 3.16 imply that there is an irreducible component Z of $\pi^{-1}(\mathcal{D}(I, x))$ such that $\pi(Z)$ is dense in $\mathcal{D}(I, x)$,

$$\dim Z = m - \dim \mathfrak{s}(I)_x + r - |I| + \dim \mathcal{I}(\mathfrak{s}(I)_x), \tag{43}$$

and $\dim Z \geqslant \dim Z'$ for every irreducible component Z' of $\pi^{-1}(\mathcal{D}(I, x))$. Since (41) follows from (43) and (10), this completes the proof. $\square$

3.21. Theorems 1.5, 3.14 and Lemmas 3.19, 3.20, 3.3(ii) reduce our problem to finding the numbers

$$c(I, x) := \operatorname{codim}_{\mathfrak{s}(I)_x} \mathcal{I}(\mathfrak{s}(I)_x) + |I| \tag{44}$$

for all the cases where

$$1 \leqslant |I| \leqslant 3 \ \text{ and } x \text{ is a nilpotent element of } \mathfrak{s}(I)^{\rm irr}. \tag{45}$$

Namely, $\operatorname{codim}_{\mathcal{C}} \mathcal{C}^{\rm irr} = 2$ if and only if there is a subset I of Δ and a nilpotent element $x \in \mathfrak{s}(I)^{\rm irr}$ such that $|I| \leqslant 2$ and $c(I, x) = 2$. If $\operatorname{codim}_{\mathcal{C}} \mathcal{C}^{\rm irr} \neq 2$, then $\operatorname{codim}_{\mathcal{C}} \mathcal{C}^{\rm irr} = 3$ if and only if there is a subset I of Δ and a nilpotent element $x \in \mathfrak{s}(I)^{\rm irr}$ such that $|I| \leqslant 3$ and $c(I, x) = 3$. If $\operatorname{codim}_{\mathcal{C}} \mathcal{C}^{\rm irr} \neq 2$ and 3, then $\operatorname{codim}_{\mathcal{C}} \mathcal{C}^{\rm irr} = 4$ by Theorems 3.14 and 1.5.

3.22. If $x = 0$, then $\mathfrak{s}(I)_x = \mathfrak{s}(I)$ and $\mathcal{I}(\mathfrak{s}(I)_x) = \mathfrak{s}(I)^{\rm irr}$. So by Lemma 3.6 we have

$$c(I, 0) = 3 + |I|. \tag{46}$$

3.23. This covers the cases where $|I|$=1. Indeed, then $\mathfrak{s}(I)$ is isomorphic to $\mathfrak{sl}_2$, hence $x = 0$ by (45) and therefore by (46) in this case we have

$$c(I, x) = 4.$$

3.24. To explore the cases $|I| = 2$ and 3, in the next section we obtain a necessary information on $\operatorname{codim}_{\mathfrak{a}_x}\mathcal{I}(\mathfrak{a}_x)$ for some semisimple Lie algebras $\mathfrak{a}$ of rank $\leqslant 3$ and nonzero nilpotent elements $x \in \mathfrak{a}^{\mathrm{irr}}$.

Below, for such $\mathfrak{a}$, we denote by Ψ the root system of $\mathfrak{a}$ with respect to a fixed Cartan subalgebra $\mathfrak{c}$ and by $\{\alpha_i\}$ a system of simple roots of Ψ. We fix a Chevalley system $(X_\alpha)_{\alpha\in\Psi}$ of $(\mathfrak{a}, \mathfrak{c})$ and put $H_\alpha = [X_{-\alpha}, X_\alpha]$, cf. [Bo$_2$, §2, 4]. For classical $\mathfrak{a}$, we take X_α and H_α as in [Bo$_2$, §13]. The integers $N_{\alpha,\beta}$ for $\alpha, \beta, \alpha+\beta \in \Psi$ are defined by the equality $[X_\alpha, X_\beta] = N_{\alpha,\beta}X_{\alpha+\beta}$, cf. [Bo$_2$, §2, 4].

4. $\mathcal{I}(\mathfrak{a}_x)$ for some algebras $\mathfrak{a}$ of rank $\leqslant 3$

4.1. *Case* $\mathfrak{a} = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$.

Up to an outer automorphism, $x = (y, 0)$ for a nonzero nilpotent element $y \in \mathfrak{sl}_2$. Then $\mathfrak{a}_x = \langle y\rangle \oplus \mathfrak{sl}_2$ and so $\mathcal{I}(\mathfrak{a}_x) = \langle y\rangle \oplus \{0\}$. Therefore

$$\operatorname{codim}_{\mathfrak{a}_x}\mathcal{I}(\mathfrak{a}_x) = 3.$$

4.2. *Case* $\mathfrak{a} = \mathfrak{sl}_3$.

In this case, the subregular orbit is the unique nonzero nilpotent orbit of the adjoint group of $\mathfrak{a}$ in $\mathfrak{a}^{\mathrm{irr}}$. It contains X_{α_1}. Since $\operatorname{Ker}\alpha_1 = \langle H_{\alpha_1} + 2H_{\alpha_2}\rangle$ and, for every $\alpha \in \Psi$, the subalgebra $\mathfrak{a}_{X_\alpha}$ is the linear span of $\operatorname{Ker}\alpha$ and all the X_β's such that $\alpha + \beta \notin \Psi$, this yields

$$\mathfrak{a}_{X_{\alpha_1}} = \langle\, X_{\alpha_1}, X_{\alpha_1+\alpha_2}, X_{-\alpha_2}, H_{\alpha_1} + 2H_{\alpha_2}\rangle. \tag{47}$$

As $N_{\alpha_1+\alpha_2,-\alpha_2} = -1$, we obtain, for $a, b, c, d \in k$ and $y = aX_{\alpha_1} + bX_{\alpha_1+\alpha_2} + cX_{-\alpha_2} + d(H_{\alpha_1} + 2H_{\alpha_2}) \in \mathfrak{a}_{x_{\alpha_1}}$, that

$$\begin{aligned} [y, X_{\alpha_1+\alpha_2}] &= cX_{\alpha_1} + 3dX_{\alpha_1+\alpha_2},\\ [y, X_{-\alpha_2}] &= -bX_{\alpha_1} - 3dX_{-\alpha_2},\\ [y, H_{\alpha_1} + 2H_{\alpha_2}] &= 3bX_{\alpha_1+\alpha_2} - 3cX_{-\alpha_2}. \end{aligned} \tag{48}$$

¿From (47) and (48) we deduce that

$$\begin{aligned} \dim(\mathfrak{a}_{X_{\alpha_1}})_y &= \dim \mathfrak{a}_{X_{\alpha_1}} - \dim[y, \mathfrak{a}_{X_{\alpha_1}}]\\ &= 4 - \mathrm{rk}A, \end{aligned}$$

where

$$A = \begin{bmatrix} c & 3d & 0\\ -b & 0 & -3d\\ 0 & 3b & -3c \end{bmatrix}.$$

Since $\operatorname{rk} A \leqslant 2$ for all b, c, d, and $\operatorname{rk} A < 2$ only for $b = c = d = 0$, this implies that $\mathcal{I}(\mathfrak{a}_{X_{\alpha_1}})$ is the center of $\mathfrak{a}_{X_{\alpha_1}}$,

$$\mathcal{I}(\mathfrak{a}_{X_{\alpha_1}}) = \langle\, X_{\alpha_1}\,\rangle. \tag{49}$$

Therefore by (47) and (49) we have

$$\operatorname{codim}_{\mathfrak{a}_{X_{\alpha_1}}}\mathcal{I}(\mathfrak{a}_{X_{\alpha_1}}) = 3.$$

4.3. *Case* $\mathfrak{a} = \mathfrak{so}_5$.

Like in the previous case we obtain

$$\mathfrak{a}_{X_{\alpha_2}} = \langle\, X_{\alpha_2}, X_{\alpha_1+2\alpha_2}, X_{-\alpha_1}, H_{\alpha_1+\alpha_2}\,\rangle. \tag{50}$$

By the dimension reason (50) implies that X_{α_2} is a subregular element of $\mathfrak{a}$. If $a, b, c, d \in k$, then, for the element $y = aX_{\alpha_2} + bX_{\alpha_1+2\alpha_2} + cX_{-\alpha_1} + dH_{\alpha_1+\alpha_2} \in \mathfrak{a}_{X_{\alpha_2}}$, we have

$$\begin{aligned} [y, X_{\alpha_1+2\alpha_2}] &= 2dX_{\alpha_1+2\alpha_2},\\ [y, X_{-\alpha_1}] &= -2dX_{-\alpha_1},\\ [y, H_{\alpha_1+\alpha_2}] &= 2bX_{\alpha_1+2\alpha_2} - 2cX_{-\alpha_1}. \end{aligned} \tag{51}$$

¿From (50) and (51) we deduce that

$$\begin{aligned}\dim(\mathfrak{a}_{X_{\alpha_2}})_y &= \dim \mathfrak{a}_{X_{\alpha_2}} - \dim[y, \mathfrak{a}_{X_{\alpha_2}}]\\ &= 4 - \operatorname{rk} A,\end{aligned}$$

where

$$A = \begin{bmatrix} 2d & 0 \\ 0 & -2d \\ 2b & -2c \end{bmatrix}.$$

Since $\operatorname{rk} A = 2$ if $d \neq 0$, and $\operatorname{rk} A < 2$ otherwise, this implies that

$$\mathcal{I}\big(\mathfrak{a}_{X_{\alpha_2}}\big) = \langle\, X_{\alpha_1}, X_{2\alpha_1+\alpha_2}, X_{-\alpha_2}\,\rangle. \tag{52}$$

Therefore by (50) and (52) we have

$$\operatorname{codim}_{\mathfrak{a}_{X_{\alpha_2}}} \mathcal{I}(\mathfrak{a}_{X_{\alpha_2}}) = 1.$$

4.4. There is a unique nonzero nilpotent orbit $\mathcal{O}$ of the adjoint group of $\mathfrak{a}$ distinct from the subregular one, see, e.g., [CM]. Its dimension is 4. Since

$$\mathfrak{a}_{X_{\alpha_1}} = \langle X_{\alpha_1}, X_{\alpha_1+\alpha_2}, X_{\alpha_1+2\alpha_2}, X_{-\alpha_2}, X_{-\alpha_1-2\alpha_2}, H_{\alpha_1+2\alpha_2}\rangle, \tag{53}$$

by the dimension reason we have $X_{\alpha_1} \in \mathcal{O}$. As $N_{\alpha_1+\alpha_2,-\alpha_2} = 2$, $N_{\alpha_1+2\alpha_2,-\alpha_2} = -1$, $N_{\alpha_1+\alpha_2,-\alpha_1-2\alpha_2} = 1$, we obtain, for $a, b, c, d, e, f \in k$ and $y = aX_{\alpha_1} + bX_{\alpha_1+\alpha_2} + cX_{\alpha_1+2\alpha_2} + dX_{-\alpha_2} + eX_{-\alpha_1-2\alpha_2} + fH_{\alpha_1+2\alpha_2} \in \mathfrak{s}(I)_{X_{\alpha_1}}$, that

$$\begin{aligned}
[y, X_{\alpha_1+\alpha_2}] &= -2dX_{\alpha_1} - eX_{-\alpha_2} + fX_{\alpha_1+\alpha_2},\\
[y, X_{\alpha_1+2\alpha_2}] &= dX_{\alpha_1+\alpha_2} + eH_{\alpha_1+2\alpha_2} + 2fX_{\alpha_1+2\alpha_2},\\
[y, X_{-\alpha_2}] &= 2bX_{\alpha_1} - cX_{\alpha_1+\alpha_2} - fX_{-\alpha_2},\\
[y, X_{-\alpha_1-2\alpha_2}] &= bX_{-\alpha_2} - cH_{\alpha_1+2\alpha_2} - 2fX_{-\alpha_1-2\alpha_2},\\
[y, H_{\alpha_2+2\alpha_1}] &= -bX_{\alpha_1+\alpha_2} - 2cX_{\alpha_1+2\alpha_2} + dX_{-\alpha_2} + 2eX_{-\alpha_1-2\alpha_2}.
\end{aligned} \tag{54}$$

It follows from (53) and (54) that

$$\begin{aligned}\dim(\mathfrak{a}_{X_{\alpha_1}})_y &= \dim \mathfrak{a}_{X_{\alpha_1}} - \dim[y, \mathfrak{a}_{X_{\alpha_1}}]\\ &= 6 - \operatorname{rk} A,\end{aligned}$$

where

$$A = \begin{bmatrix} -2d & f & 0 & -e & 0 & 0 \\ 0 & d & 0 & 0 & 2f & e \\ 2b & -c & 0 & -f & 0 & 0 \\ 0 & 0 & 0 & b & -2f & 0 \\ 0 & b & -2c & d & 2e & 0 \end{bmatrix}.$$

An elementary exploration of $\operatorname{rk} A$ as function of b, c, d, e, f yields that $\mathcal{I}(\mathfrak{a}_{x_{\alpha_1}})$ is the union of six 3-dimensional linear subspaces of $\mathfrak{a}_{X_{\alpha_1}}$:

$$\begin{aligned}
\mathcal{I}(\mathfrak{a}_{X_{\alpha_1}}) = {} & \langle X_{\alpha_1}, X_{\alpha_1+2\alpha_2}, X_{-\alpha_2}\rangle \cup \langle X_{\alpha_1}, X_{\alpha_1+\alpha_2}, X_{\alpha_1+2\alpha_2}\rangle\\
& \cup \langle X_{\alpha_1}, X_{\alpha_1+\alpha_2}, X_{-\alpha_2}\rangle \cup \langle X_{\alpha_1}, X_{-\alpha_2}, X_{-\alpha_1-2\alpha_2}\rangle\\
& \cup \langle X_{\alpha_1}, X_{-\alpha_2} + \sqrt{-1}X_{\alpha_1+\alpha_2}, X_{\alpha_1+2\alpha_2}\rangle\\
& \cup \langle X_{\alpha_1}, X_{-\alpha_2} - \sqrt{-1}X_{\alpha_1+\alpha_2}, X_{\alpha_1+2\alpha_2}\rangle.
\end{aligned} \tag{55}$$

Therefore by (53) and (55) we have

$$\operatorname{codim}_{\mathfrak{a}_{X_{\alpha_1}}} \mathcal{I}(\mathfrak{a}_{X_{\alpha_1}}) = 3.$$

4.5. *Case* $\mathfrak{a} = \operatorname{Lie} \mathbf{G}_2$.

It is easy to verify (see also [GQT, p. 10]) that

$$e = X_{\alpha_2} + X_{3\alpha_1+\alpha_2},\ f = -2X_{-\alpha_2} - 2X_{-3\alpha_1-\alpha_2},\ h = 2H_{\alpha_2} + 2H_{3\alpha_1+\alpha_2}$$

is an $\mathfrak{sl}_2$-triple. Since $\alpha_1(h) = 0$, $\alpha_2(h) = 2$, the classification of nilpotent orbits in $\mathfrak{a}$, see, e.g., [CM, 8.4], implies that

$$x = e = X_{\alpha_2} + X_{3\alpha_1+\alpha_2} \tag{56}$$

is a subregular nilpotent element of $\mathfrak{a}$ and

$$\dim \mathfrak{a}_x = 4. \tag{57}$$

Since $[X_\alpha, X_\beta] = 0$ if $\alpha + \beta \notin \Phi$, we have $X_{\alpha_1+\alpha_2}, X_{3\alpha_1+2\alpha_2}, X_{2\alpha_1+\alpha_2} \in \mathfrak{a}_x$. Hence it follows from (56), (57) that

$$\mathfrak{a}_x = \langle X_{\alpha_2} + X_{3\alpha_1+\alpha_2}, X_{\alpha_1+\alpha_2}, X_{3\alpha_1+2\alpha_2}, X_{2\alpha_1+\alpha_2}\rangle. \tag{58}$$

¿From (56), (58) we deduce that, for every $y \in \mathfrak{a}_x$, we have $[y, \mathfrak{a}_x] \in \langle X_{3\alpha_1+2\alpha_2}\rangle$; whence

$$\dim(\mathfrak{a}_x)_y = \dim \mathfrak{a}_x - \dim[y, \mathfrak{a}_{X_{\alpha_1}}] \geqslant 4 - 1 = 3.$$

This proves that $\mathcal{I}(\mathfrak{a}_x) = \mathfrak{a}_x$, i.e.,

$$\operatorname{codim}_{\mathfrak{a}_x} \mathcal{I}(\mathfrak{a}_x) = 0.$$

4.6. As for the algebras $\mathfrak{a}$ of rank 3, it will be sufficient for our purposes to consider only those $\mathfrak{a}$ whose simple ideals are of type A, and to prove that for such $\mathfrak{a}$ the inequality $\operatorname{codim}_{\mathfrak{a}_x} \mathcal{I}(\mathfrak{a}_x) \geqslant 1$ always holds. We deduce this statement from the following general lemma.

Lemma 4.7. *Let x be a nilpotent element of $\mathfrak{m} := \mathfrak{sl}_{n_1} \oplus \ldots \oplus \mathfrak{sl}_{n_q}$. Then there is semisimple element $h \in \mathfrak{m}$ such that x is a regular element of $\mathfrak{m}_h$.*

Proof. Clearly, it suffices to prove this statement for $q = 1$. Therefore we now assume that $\mathfrak{m} = \mathfrak{sl}_n$. If $x \in \mathfrak{m}^{\rm reg}$, then $h = 0$. Now let $x \in \mathfrak{m}^{\rm irr}$. Then by the Jordan normal form theory we may (and shall) assume that

$$x = \operatorname{diag}(J_{d_1}, \ldots, J_{d_s}), \quad \text{where } J_i = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix} \in \operatorname{Mat}_{i\times i} \text{ and } s \geqslant 2. \tag{59}$$

Since $s \geqslant 2$, there are $a_1, \ldots, a_s \in \mathbb{Z}$ such that

$$a_1 > \ldots > a_s, \tag{60}$$

$$d_1a_1 + \cdots + d_sa_s = 0. \tag{61}$$

By (61) the semisimple matrix $\operatorname{diag}(a_1I_{d_1}, \ldots, a_sI_{d_s})$ lies in $\mathfrak{m}$. We claim that one can take

$$h := \operatorname{diag}(a_1I_{d_1}, \ldots, a_sI_{d_s}).$$

Indeed, from (60) we deduce that

$$[\mathfrak{m}_h, \mathfrak{m}_h] = \{\operatorname{diag}(A_1, \ldots, A_s) \mid A_i \in \mathfrak{sl}_{d_i} \text{ for every } i\}, \tag{62}$$

and the claim readily follows from (59) and (62). □

Corollary 4.8. *Maintain the notation of Lemma* 4.7. *Then*

$$\operatorname{codim}_{\mathfrak{m}_x} \mathcal{I}(\mathfrak{m}_x) \geqslant 1.$$

Proof. Since the element h from Lemma 4.7 is semisimple, the algebra $\mathfrak{m}_h$ is reductive and its rank is equal to that of $\mathfrak{m}$. As x is a regular element of $\mathfrak{m}_h$, this implies the equality $\dim(\mathfrak{m}_h)_x = \operatorname{rk} \mathfrak{m}$. But $(\mathfrak{m}_h)_x = (\mathfrak{m}_x)_h$. Therefore $y \notin \mathcal{I}(\mathfrak{m}_x)$; whence the claim. □

5. **Proofs of Theorems 1.6 and 1.12**

5.1. *Proof of Theorem* 1.6. Let I be a subset of Δ such that $1 \leqslant |I| \leqslant 3$ and let x be a nilpotent element of $\mathfrak{s}(I)^{\rm irr}$. If $|I|=1$, then according to Subsection 3.23 we have $c(I,x)$=4. Now consider the cases $|I|=2$ and 3.

(a) Let $\mathfrak{g}$ be of type A_r, D_r, E_6, E_7, or E_8. Then every simple ideal of $\mathfrak{s}(I)$ is of type A. Therefore, if $|I|=2$, then $\mathfrak{s}(I)$ is isomorphic to $\mathfrak{sl}_2\oplus\mathfrak{sl}_2$ or $\mathfrak{sl}_3$; whence by (46) and Subsections 4.1, 4.2 we have $c(I,x)=5$. If $|I|=3$, then $\mathfrak{s}(I)$ is isomorphic to $\mathfrak{sl}_2\oplus\mathfrak{sl}_2\oplus\mathfrak{sl}_2$, $\mathfrak{sl}_2\oplus\mathfrak{sl}_3$, or $\mathfrak{sl}_4$; whence by Corollary 4.8 we have $c(I,x)\geqslant 4$. As is explained in Subsection 3.21, this information implies that $\operatorname{codim}_{\mathcal{C}}\mathcal{C}^{\rm irr}=4$.

(b) Let $\mathfrak{g}$ be of type B_r, C_r, or F_4. If $|I|=2$, then $\mathfrak{s}(I)$ is isomorphic to $\mathfrak{so}_5$, $\mathfrak{sl}_2\oplus\mathfrak{sl}_2$, or $\mathfrak{sl}_3$, and there is I such that $\mathfrak{sl}(I)$ is isomorphic to $\mathfrak{so}_5$. By (46) and Subsections 4.3, 4.4, if $\mathfrak{s}(I)$ is isomorphic to $\mathfrak{so}_5$, then $c(I,x)\geqslant 3$ and there is x such that $c(I,x)=3$. On the other hand, as we have seen in (a), if $\mathfrak{s}(I)$ is isomorphic to $\mathfrak{sl}_2\oplus\mathfrak{sl}_2$ or $\mathfrak{sl}_3$, then $c(I,x)=5$. According to Subsection 3.21, this information implies that $\operatorname{codim}_{\mathcal{C}}\mathcal{C}^{\rm irr}=3$.

(c) Let $\mathfrak{g}$ be of type G_2. Then $I=\Delta$ and $\mathfrak{s}(I)=\mathfrak{g}$. According to Subsection 4.5, there is x such that $c(I,x)=2$. As is explained in Subsection 3.21, this implies that $\operatorname{codim}_{\mathcal{C}}\mathcal{C}^{\rm irr}=2$. $\square$

5.2. *Proof of Theorem* 1.12. Let N be the normalizer of $\mathfrak{t}$ in G. We endow $\mathfrak{t}\oplus\mathfrak{t}$ with the natural N-module structure. Let $G\times^N(\mathfrak{t}\oplus\mathfrak{t})$ be the algebraic homogeneous vector G-bundle over G/N with fiber $\mathfrak{t}\oplus\mathfrak{t}$, see [Se, §2], [PV$_2$, 4.8], [LPR, 2.17]. Denote by $g*t$ the image of point $(g,t)\in G\times(\mathfrak{t}\oplus\mathfrak{t})$ under the natural projection $G\times(\mathfrak{t}\oplus\mathfrak{t})\to G\times^N(\mathfrak{t}\oplus\mathfrak{t})$.

Since $\mathcal{C}=\overline{G(\mathfrak{t}\times\mathfrak{t})}$, the natural G-equivariant morphism

$$\varphi\colon G\times^N(\mathfrak{t}\oplus\mathfrak{t})\to\mathcal{C},\quad g*t\mapsto g(t), \tag{63}$$

is dominant. We claim that φ is a birational isomorphism. As $\operatorname{char}k=0$, proving this claim is equivalent to showing that $\varphi^{-1}(x)$ is a single point for points x in general position in $\mathcal{C}$. To show that the latter property holds, notice that there is a nonempty open G-stable subset U of $\mathcal{C}$ laying in $G(\mathfrak{t}^{\rm reg}\times\mathfrak{t}^{\rm reg})$. Let $x\in U$. Since φ is G-equivariant, the fibers $\varphi^{-1}(x)$ and $\varphi^{-1}(g(x))$ for $g\in G$ are isomorphic. Hence it would be sufficient to prove that $\varphi^{-1}(x)$ is a single point for $x\in\mathfrak{t}^{\rm reg}\times\mathfrak{t}^{\rm reg}$.

To do this, take a point $x=(x_1,x_2)\in\mathfrak{t}^{\rm reg}\times\mathfrak{t}^{\rm reg}$. Since $x_i\in\mathfrak{t}^{\rm reg}$, we have

$$\mathfrak{g}_{x_i}=\mathfrak{t}. \tag{64}$$

Let $g*t\in\varphi^{-1}(x)$ where $t=(t_1,t_2)\in\mathfrak{t}\oplus\mathfrak{t}$. By (63) we have $g(t_i)=x_i$. This and (64) yield

$$g(\mathfrak{g}_{t_i})=\mathfrak{g}_{g(t_i)}=\mathfrak{g}_{x_i}=\mathfrak{t}. \tag{65}$$

It follows from (65) that $\dim\mathfrak{g}_{t_i}=\dim\mathfrak{t}$. Since $\mathfrak{t}\subseteq\mathfrak{g}_{t_i}$, this yields $\mathfrak{g}_{t_i}=\mathfrak{t}$. By virtue of (65) the latter equality implies that $g\in N$. From this and the definition of $g*t$ we now deduce that $g*t=e*g(t)=e*x$. Thus, $\varphi^{-1}(x)=e*x$. This proves the claim.

Since algebraic homogeneous vector bundles are locally trivial in Zariski topology, see [Se], the varieties $G\times^N(\mathfrak{t}\oplus\mathfrak{t})$ and $G/N\times(\mathfrak{t}\oplus\mathfrak{t})$ are birationally isomorphic. Hence the varieties $\mathcal{C}$ and $G/N\times(\mathfrak{t}\oplus\mathfrak{t})$ are birationally isomorphic as well. Given this, the proof comes to a close because the variety G/N of maximal tori of G is rational, see [C], [G, 6.1], [BS, 7.9]. $\square$

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