

THE WITT RING KERNEL FOR A FOURTH DEGREE FIELD EXTENSION

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ABSTRACT. We compute the Witt ring kernel for an arbitrary field extension of degree 4 and characteristic different from 2 in terms of the coefficients of a polynomial determining the extension. In the case where the lower field is not formally real we prove that the intersection of any power n of its fundamental ideal and the Witt ring kernel is generated by n -fold Pfister forms.

Let F be a field of characteristic different from 2. As usual denote by $W(F)$ the Witt ring of F , i.e. the ring of equivalence classes of quadratic forms over F . In this note we compute the kernel $W(L/F)$ of the restriction map $W(F) \rightarrow W(L)$, where L/F is an arbitrary field extension of degree 4. This kernel was computed in [2] in the case where L/F is a tower of two quadratic extensions, and our result can be considered as its generalization. The main step is to show that $W(L/F)$ is generated as an ideal by 1-fold and 2-fold Pfister forms. The last statement is a particular case of a stronger result (see Corollary 2), which as far as we know, has not been known even for biquadratic extensions.

Our notation is standard, and the main reference is the book [4]. Notice only that all the fields considered below are of characteristic different from 2. We write $\varphi \simeq \psi$ if quadratic forms φ and ψ are isomorphic, and $\varphi = \psi$ if their classes in the Witt ring are equal. The anisotropic part of a form φ is denoted by φ_{an} . We use the sign \perp for the direct sum of two forms and the signs $+$ and $-$ for the sum and the difference of elements of the Witt ring. For a form φ over a field F we denote by $D_F(\varphi)$ the set of nonzero values of F represented by φ . By the n -fold Pfister form $\langle\langle a_1, \dots, a_n \rangle\rangle$ we mean the product $\langle 1, -a_1 \rangle \otimes \dots \otimes \langle 1, -a_n \rangle$. If V is a linear space over a field F and L/F is a field extension, then by definition $V_L = L \otimes_F V$.

We begin by the following statement, which plays the main role in the sequel.

Key words and phrases. Witt ring, Pfister form, polynomial.

The work under this publication was partially supported by Royal society Joint Project "Quadratic forms and central simple algebras under field extensions" and Emory University.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$ - $\text{T}\mathcal{E}\mathcal{X}$

Proposition 1. *Suppose $p(t)$ is an irreducible even degree polynomial over a field F , and θ is its root. Let k be a nonnegative integer, φ an anisotropic quadratic form over F such that $\dim \varphi \geq \frac{k}{2} \deg p + 1$ and $\dim(\varphi_{F(\theta)})_{an} = k$. Then there exists a subform $\varphi_0 \subset \varphi$ and $x \in F^*$ such that $\dim \varphi_0 \leq \frac{1}{2} \deg p + 1$ and $xp(t) \in D(\varphi_{0F(t)})$. In particular, the form $\varphi_{0F(\theta)}$ is isotropic.*

Proof. Let U be the linear space associated to φ , $U = \langle e_1, \dots, e_{\dim \varphi} \rangle$. Let further $V = \langle v_1, \dots, v_{\frac{1}{2}(\dim \varphi - k)} \rangle$ be a totally isotropic subspace of $U_{F(\theta)}$. Put $W = \sum_{i=0}^{\frac{1}{2} \deg p} \theta^i U$. We can consider V and W as F -linear subspaces of the F -linear space $U_{F(\theta)}$. Obviously,

$$\dim_F U_{F(\theta)} = \dim \varphi \deg p, \dim_F V = \frac{1}{2}(\dim \varphi - k) \deg p, \dim_F W = \left(\frac{1}{2} \deg p + 1\right) \dim \varphi.$$

Since $\dim \varphi > \frac{1}{2}k \deg p$, we have

$$\dim_F V + \dim_F W > \dim_F U_{F(\theta)}.$$

Hence $V \cap W \neq 0$. Let $0 \neq v = \sum_{i=0}^{\frac{1}{2} \deg p} \theta^i u_i \in V$, where $u_i \in U$. Denote by φ_0 the subform of φ determined by the F -linear subspace $U_0 = \langle u_0, \dots, u_{\frac{1}{2} \deg p} \rangle \subset U$.

Obviously, $\dim \varphi_0 \leq \frac{1}{2} \deg p + 1$. Put $\tilde{v} = \sum_{i=0}^{\frac{1}{2} \deg p} t^i u_i \in F[t] \otimes_F U_0$. Since $\varphi(v) = 0$, we have $p|\varphi_0(\tilde{v})$. Since $\deg \varphi_0(\tilde{v}) \leq \deg p$, we get that $\varphi_0(\tilde{v}) = xp$ for some $x \in F$. Finally, $x \neq 0$, because $v \neq 0$, and φ is anisotropic over F . \square

We apply Proposition 1 to field extensions of degree 4.

Corollary 2. *Under hypothesis of Proposition 1 suppose $\deg p = 4$. Then $\varphi = \psi + \sum \pi_i$, where $\dim \psi \leq 2k$, each π_i is similar to either a 2-fold or a 1-fold Pfister form and $\pi_{iF(\theta)} = 0$. Moreover, we can assume that each π_i is similar to a 2-fold Pfister form except the case where $k = 0$, the extension $F(\theta)/F$ contains a quadratic subextension $F(\sqrt{d})/F$ such that $\varphi_{F(\sqrt{d})} = 0$, and $\text{disc}(\varphi) = d$. In this case we can assume that exactly one π_i is similar to a 1-fold Pfister form (obviously, then $\text{disc}(\pi_i) = d$).*

In particular, if $k = 0$, i.e. $\varphi \in W(F(\theta)/F)$, we have $\psi = 0$, and $\varphi = \sum \pi_i$.

Proof. Assume first that $k \geq 1$. By Proposition 1 we get $\varphi \simeq \varphi_0 \perp \varphi_1$, where $\dim \varphi_0 = 3$ and $\varphi_{0F(\theta)}$ is isotropic. Let $\pi \simeq \varphi_0 \perp \det(\varphi_0)$. Then

$$\varphi = \varphi_1 + \langle -\det(\varphi_0) \rangle + \pi,$$

π is similar to a 2-fold Pfister form, $\pi_{F(\theta)} = 0$, and

$$\dim(\varphi_1 + \langle -\det(\varphi_0) \rangle)_{an} \leq \dim \varphi - 2.$$

Therefore, we can go on by induction on $\dim \varphi$. In the case $k = 0$ and $\dim \varphi \geq 3$ the same argument works. Finally, if $k = 0$ and $\dim \varphi = 2$, then φ is similar to a

1-fold Pfister form. It is obvious that in this case the field $F(\theta)$ contains the field $F(\sqrt{\text{disc } \varphi})$. \square

Remark. Let us call a form φ minimal with respect to an extension $F(\theta)/F$, if $\dim \psi \geq \dim \varphi$ for any form ψ such that $\varphi - \psi \in W(F(\theta)/F)$. Corollary 2 claims that if $\deg p = 4$, $\dim \varphi > 2k$ and $\dim(\varphi_{F(\theta)})_{an} = k$, then φ is not minimal. On the other hand, it is easy to see that if forms φ_1 and φ_2 over F are minimal with respect to the extension $F(\theta)/F$, then the form $\varphi_1 \perp x\varphi_2$ over $F((x))$ is minimal with respect to the extension $F((x))(\theta)/F((x))$. Let τ be a 4-dimensional form over F such that the form $\tau_{F(\theta)}$ is isotropic, but for any subform $\tau' \subset \tau$ of codimension 1 the form $\tau'_{F(\theta)}$ is anisotropic. (The existence of such an extension $F(\theta)/F$ of degree 4 and a form τ has been established in [1] and [5]). In particular, τ is minimal. Therefore, the forms $\psi_1 \simeq \tau \otimes \langle x_1, \dots, x_m \rangle$ and $\psi_2 \simeq \tau \otimes \langle x_1, \dots, x_m \rangle \perp \langle x_{m+1} \rangle$ over the field $K = F((x_1))((x_2)) \dots ((x_{m+1}))$ are minimal, which shows that the inequality $\dim \psi \leq 2k$ in Corollary 2 can not be improved. Therefore, in the case $\deg p = 4$ the inequality $\dim \varphi \geq \frac{k}{2} \deg p + 1$ in Proposition 1 can not be improved as well.

It remains to determine 2-fold Pfister forms in $W(F(\theta)/F)$. This can be done in terms of the coefficients of p .

Lemma 3. *Let $p(t) = t^4 + at^2 + bt + c$ be an irreducible polynomial over F , θ its root, π an anisotropic 2-fold Pfister form over F . Suppose that $\pi_{F(\sqrt{d})} \neq 0$ for any $d \in F^*$ such that $F(\sqrt{d}) \subset F(\theta)$. Then the following two conditions are equivalent:*

1) $\pi_{F(\theta)} = 0$.

2) there exist $x, y \in F^*$ such that $\det \begin{pmatrix} -x & 0 & \frac{1}{2}(y - ax) \\ 0 & -y & -\frac{1}{2}bx \\ \frac{1}{2}(y - ax) & -\frac{1}{2}bx & -cx \end{pmatrix} \in F^{*2}$

and $\pi \simeq \langle\langle x, y \rangle\rangle$.

Proof. 1) \implies 2). Assume that $\pi_{F(\theta)} = 0$. Let (V, π') be the quadratic space of the pure subform of π , i.e. $\pi \simeq \langle 1 \rangle \perp \pi'$. All 3-dimensional subforms of π are similar to each other. Therefore, by Proposition 1 there are $x \in F^*$ and $v_0, v_1, v_2 \in V$ such that

$$\pi'(t^2v_2 + tv_1 + v_0) = -xp.$$

We have $\varphi_{F(\theta)} \neq 0$ for any 2-dimensional subform $\varphi \subset \pi'$. Indeed, otherwise we would have $F(\sqrt{\text{disc } \varphi}) \subset F(\theta)$, hence $\pi_{F(\sqrt{\text{disc } \varphi})} = 0$, a contradiction to the hypothesis of the proposition. Therefore, the vectors v_0, v_1, v_2 are linearly independent. Comparing the coefficients at the powers of t on both sides of the last equality we get

$$\begin{cases} \pi'(v_2, v_2) = -x, \\ \pi'(v_1, v_2) = 0, \\ \pi'(v_1, v_1) + 2\pi'(v_0, v_2) = -ax, \\ 2\pi'(v_0, v_1) = -bx, \\ \pi'(v_0, v_0) = -cx. \end{cases}$$

Let $\pi'(v_0, v_2) = \frac{1}{2}(y - ax)$. Then the matrix of the form π' with respect to the

basis (v_2, v_1, v_0) is $\begin{pmatrix} -x & 0 & \frac{1}{2}(y-ax) \\ 0 & -y & -\frac{1}{2}bx \\ \frac{1}{2}(y-ax) & -\frac{1}{2}bx & -cx \end{pmatrix}$. Since $\det \pi' = 1$, we get $\pi' \simeq \langle -x, -y, xy \rangle$, hence $\pi \simeq \langle \langle x, y \rangle \rangle$.

2) \implies 1). We have $\pi'(t^2v_2 + tv_1 + v_0) = -xp$ for some vectors $v_0, v_1, v_2 \in V$. In particular, $\pi'_{F(\theta)}$ is isotropic, which means that $\pi_{F(\theta)} = 0$. \square

Corollary 4. *Suppose a polynomial $p(t) = t^4 + at^2 + bt + c$ is irreducible over a field F , and π is an anisotropic 2-fold Pfister form over F such that $\pi_{F(\theta)} = 0$. Then at least one of the following two conditions hold:*

- 1) $\pi \simeq \langle \langle d, e \rangle \rangle$, where $F(\sqrt{d}) \subset F(\theta)$, $e \in F^*$.
 - 2) $\pi \simeq \langle \langle \alpha(\alpha - a)^2 - 4c\alpha + b^2, -\alpha \rangle \rangle$ where $\alpha \in F$, and $(\alpha(\alpha - a)^2 - 4c\alpha + b^2)\alpha \neq 0$.
- Conversely, if π is a 2-fold Pfister form of one of two types above, then $\pi_{F(\theta)} = 0$.

Proof. Assume that $\pi_{F(\theta)} = 0$, and condition 1) does not hold. Then, as we have noticed already, the vectors v_0, v_1, v_2 introduced in Lemma 3 are linearly independent. Put $\alpha = \frac{y}{x}$. Since

$$\det \begin{pmatrix} -x & 0 & \frac{1}{2}(y-ax) \\ 0 & -y & -\frac{1}{2}bx \\ \frac{1}{2}(y-ax) & -\frac{1}{2}bx & -cx \end{pmatrix} = x \left(\frac{1}{4}\alpha(\alpha - a)^2 - \alpha c + \frac{1}{4}b^2 \right) \in F^{*2},$$

and $\langle \langle x, -x \rangle \rangle = 0$, we have

$$\pi \simeq \langle \langle x, y \rangle \rangle \simeq \langle \langle x, \alpha x \rangle \rangle \simeq \langle \langle x, -\alpha \rangle \rangle = \langle \langle \alpha(\alpha - a)^2 - 4c\alpha + b^2, -\alpha \rangle \rangle.$$

Conversely, if $(\alpha(\alpha - a)^2 - 4c\alpha + b^2)\alpha \neq 0$, then put $x = \alpha(\alpha - a)^2 - 4c\alpha + b^2$, $y = x\alpha$. The implication 2) \implies 1) of Lemma 3 shows that

$$\langle \langle \alpha(\alpha - a)^2 - 4c\alpha + b^2, -\alpha \rangle \rangle_{F(\theta)} = 0.$$

\square

Remark. The polynomial $f(\alpha) = \alpha(\alpha - a)^2 - 4c\alpha + b^2$ is a cubic resolvent of the polynomial p .

Example. Suppose $a \in F^* \setminus F^{*2}$, $b, c \in F$, $b + 2c\sqrt{a} \notin F(\sqrt{a})^2$. Consider the quartic extension $F(\sqrt{b + 2c\sqrt{a}})/F$. If $\theta = \sqrt{b + 2c\sqrt{a}}$, then

$$p(\theta) = \theta^4 - 2b\theta^2 + (b^2 - 4ac^2) = 0.$$

Let π be a 2-fold Pfister form such that $\pi_{F(\theta)} = 0$, $\pi_{F(\sqrt{a})} \neq 0$. By Corollary 4 we get that

$$\pi \simeq \langle \langle \alpha(\alpha + 2b)^2 - 4(b^2 - 4ac^2)\alpha, -\alpha \rangle \rangle \simeq \langle \langle (\alpha + 2b)^2 - 4(b^2 - 4ac^2), -\alpha \rangle \rangle \simeq \langle \langle ae^2 + be + c^2, -e \rangle \rangle,$$

where $e = \frac{4c^2}{\alpha}$. In fact we have got the result of theorem 3.9 from [1].

Proposition 5. *Let L/F be a field extension of degree 4, and F is not formally real. Then for any n*

$$W(L/F) \cap I^n(F) = (W(L/F) \cap I^2(F))I^{n-2}(F).$$

Hence in terminology of [1] the ideal $W(L/F) \cap I^n(F)$ is an n -Pfister ideal.

Proof. For any field K denote by G_K its absolute Galois group. Unfortunately the proof depends on a very deep result of Voevodsky, which claims that there exists an isomorphism $I^n/I^{n+1}(K) \simeq H^n(G_K, \mathbb{Z}/2\mathbb{Z})$, which takes an n -fold Pfister form $\langle\langle a_1, \dots, a_n \rangle\rangle$ to the cup product $(a_1) \cup \dots \cup (a_n)$ ([6], [7]). First we need a couple of lemmas.

Lemma 6. *Let F be a field. The following two conditions are equivalent:*

- 1) F is not formally real.
- 2) There exists a subfield $F_1 \subset F$ and a positive integer N such that $I^N(F_1) = 0$.

Proof. 2) \implies 1). If F is formally real, then so is any subfield F_1 of F , hence $I^N(F_1) \neq 0$ for any N .

1) \implies 2). Assume that F is not formally real, and $s(F) = m = 2^k$ is its level, i.e. the minimal number m such that there exist $x_1, \dots, x_m \in F$ with the equality $x_1^2 + \dots + x_m^2 = -1$. Let F_0 be the prime subfield of F , i.e. F_0 is either \mathbb{Q} or $\mathbb{Z}/p\mathbb{Z}$. Put $F_1 = F_0(x_1, \dots, x_m) \subset F$, and $N = m + k + 3$. Let us show that $I^N(F_1) = 0$. Since $\text{cd}_2(F_0(\sqrt{-1})) \leq 2$, we have $\text{cd}_2(F_1(\sqrt{-1})) \leq m + 2$, hence $I^{m+3}(F_1(\sqrt{-1})) = 0$. Therefore, for any $a_1, \dots, a_{m+3} \in F_1^*$ we get

$$\langle\langle a_1, \dots, a_{m+3} \rangle\rangle \simeq \langle\langle -1, b_1, \dots, b_{m+2} \rangle\rangle$$

for some $b_1, \dots, b_{m+2} \in F_1^*$. From this it follows at once that for any $a_1, \dots, a_N \in F_1^*$

$$\langle\langle a_1, \dots, a_N \rangle\rangle \simeq \langle\langle -1 \rangle\rangle^{\otimes(k+1)} \otimes \langle\langle b_1, \dots, b_{N-k-1} \rangle\rangle$$

for some $b_1, \dots, b_{N-k-1} \in F_1^*$. On the other hand, since $x_1, \dots, x_m \in F_1$, we have $\langle\langle -1 \rangle\rangle^{\otimes(k+1)} = 0 \in W(F_1)$. This implies that $\langle\langle a_1, \dots, a_N \rangle\rangle = 0 \in I^N(F_1)$, which finishes the proof. \square

Lemma 7. *The kernel of the restriction map $I^2/I^3(F) \rightarrow I^2/I^3(L)$ is generated by 2-fold Pfister forms belonging to $W(L/F)$.*

Proof. Assume that $\alpha \in \ker(I^2/I^3(F) \rightarrow I^2/I^3(L)) = \ker({}_2\text{Br}(F) \rightarrow {}_2\text{Br}(L))$. Obviously, $\text{ind } \alpha \leq 4$. Let φ be an Albert form associated to α . Then $\varphi_L \in I^3(L)$. Since $\dim \varphi = 6$, we conclude that $\varphi \in W(L/F)$. In view of Corollary 2 the lemma is proved. \square

We return to the proof of Proposition 5. It is proven in [3] (cor.18 and the remark after conjecture 20) that the kernel of the restriction map of the graded rings $H^*(G_F, \mathbb{Z}/2\mathbb{Z}) \rightarrow H^*(G_L, \mathbb{Z}/2\mathbb{Z})$ is generated in degrees 1 and 2 as an ideal in $H^*(G_F, \mathbb{Z}/2\mathbb{Z})$. Let $\alpha \in W(L/F) \cap I^n(F)$. Assume that the conditions of Lemma 6 are fulfilled. Then it is easy to see that there is an intermediate subfield $F_1 \subset \tilde{F} \subset F$ such that $\alpha \in W(L\tilde{F}/\tilde{F}) \cap I^n(\tilde{F})$ and $\text{cd}_2(\tilde{F}) < \infty$. Interpreting the graded

ring $H^*(G_{\tilde{F}}, \mathbb{Z}/2\mathbb{Z})$ (resp. $H^*(G_{L\tilde{F}}, \mathbb{Z}/2\mathbb{Z})$) as the graded ring $I^*/I^{*+1}(\tilde{F})$ (resp. $I^*/I^{*+1}(L\tilde{F})$) and using Lemma 7 we can prove Proposition 5 straightforwardly by decreasing induction on n .

Open question. Does Proposition 5 remain valid for a formally real field F ?

Acknowledgements. This research has been carried out at Emory University, Atlanta and the University of Nottingham in the framework of Royal society Joint Project "Quadratic forms and central simple algebras under field extensions". The author expresses his thanks to Detlev Hoffmann, Raman Parimala and Alexander Vishik for very fruitful discussions.

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