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Specialization of Quadratic and Symmetric Bilinear Forms

Translated from German by Thomas Unger

Preliminary Version:

- **contains Chapters I and II;**
- **Chapters III and IV are in preparation.**

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Dedicated to the memory of my teachers

Emil Artin 1898 – 1962

Hel Braun 1914 – 1986

Ernst Witt 1911 – 1991

Preface

A Mathematician Said Who
Can Quote Me a Theorem that's True?
For the ones that I Know
Are Simply not So,
When the Characteristic is Two!

This pretty limerick first came to my ears in May 1998 during a talk by T.Y. Lam on field invariants from the theory of quadratic forms.[1] It is – poetic exaggeration allowed – a suitable motto for this monograph.

What is it about? In the beginning of the seventies I drew up a specialization theory of quadratic and symmetric bilinear forms over fields [K_4]. Let $\lambda\colon K \to L \cup \infty$ be a place. Then one can assign a form $\lambda_*(\varphi)$ to a form φ over K in a meaningful way if φ has "good reduction" with respect to λ (see §1). The basic idea is to simply apply the place λ to the coefficients of φ which therefore of course have to be in the valuation ring of λ.

The specialization theory of that time was satisfactory as long as the field L, and therefore also K, had characteristic $\neq 2$. It served me in the first place as the foundation for a theory of generic splitting of quadratic forms [K_5], [K_6]. After a very modest beginning, this theory is now in full bloom. It became important for the understanding of quadratic forms over fields, as can be seen from the book [IKKV] of Izhboldin-Kahn-Karpenko-Vishik for instance. One should note that there exists a theory of (partial) generic splitting of central simple algebras and reductive algebraic groups, parallel to the theory of generic splitting of quadratic forms (see [Ke R] and the literature cited there).

In this book I would like to present a specialization theory of quadratic and symmetric bilinear forms with respect to a place $\lambda\colon K \to L \cup \infty$, without the assumption that $\operatorname{char} L \neq 2$. This is where complications arise. We have to make a distinction between bilinear and quadratic forms and study them both over fields and valuation rings. From the viewpoint of reductive algebraic groups, the so-called regular quadratic forms (see below) are the natural objects. But, even if we are only interested in such forms, we have to know a bit about specialization of nondegenerate symmetric bilinear forms, since

[1] "Some reflections on quadratic invariants of fields", 3 May 1998 in Notre Dame (Indiana) on the occasion of O.T. O'Meara's 70th birthday.

they occur as "multipliers" of quadratic forms: if φ is such a bilinear form and ψ is a regular quadratic form, then we can form a tensor product $\varphi \otimes \psi$, see §5. This is a quadratic form, which is again regular when ψ has even dimension ($\dim \psi =$ number of variables occurring in ψ). However – and here already we run into trouble – when $\dim \psi$ is odd, $\varphi \otimes \psi$ is not necessarily regular.

Even if we only want to understand quadratic forms over a field K of characteristic zero, it might be necessary to look at specializations with respect to places from K to fields of characteristic 2, especially in arithmetic investigations. When K itself has characteristic 2, an often more complicated situation may occur, for which we are not prepared by the available literature. Surely, fields of characteristic 2 were already allowed in my work on specializations in 1973 [K_4], but from today's point of view satisfactory results were only obtained for symmetric bilinear forms. For quadratic forms there are gaping holes. We have to study quadratic forms over a valuation ring in which 2 is not a unit. Even the beautiful and extensive book of Ricardo Baeza [Ba] doesn't give us enough for the theory of specializations, although Baeza even allows semilocal rings instead of valuation rings. He only studies quadratic forms whose associated bilinear forms are nondegenerate. This forces those forms to have even dimension.

Let me now discuss the contents of this book. After an introduction to the problem in §1, which can be understood without any previous knowledge of quadratic and bilinear forms, the specialization theory of symmetric bilinear forms is presented in §2 - §3. There are good, generally accessible sources available for the foundations of the algebraic theory of symmetric bilinear forms. Therefore many results are presented without a proof, but with a reference to the literature instead. As an important application, the outlines of the theory of generic splitting in characteristic $\neq 2$ are sketched in §4, nearly without proofs.

From §5 onwards we address the theory of quadratic forms. In characteristic 2 fewer results can be found in the literature for such forms than for bilinear forms, even at the basic level. Therefore we present most of the proofs. We also concern ourselves with the so-called "weak specialization" (see §1) and get into areas which may seem strange even to specialists in the theory of quadratic forms. In particular we have to require a quadratic form over K to be "obedient" in order to weakly specialize it with respect to a place $\lambda\colon K \to L \cup \infty$ (see §7). I have never encountered such a thing anywhere in the literature.

At the end of Chapter I we reach a level in the specialization theory of quadratic forms that facilitates a generic splitting theory, useful for many applications. In the first two sections (§9, §10) of Chapter II we produce such a generic splitting theory in two versions, both of which deserve interest in their own right.

We call a quadratic form φ over a field k *nondegenerate* when its quasilinear part (cf. Arf [A]), which we denote by $QL(\varphi)$, is anisotropic. We further call – deviating from Arf [A] – φ *regular* when $QL(\varphi)$ is at most one-dimensional and *strictly regular* when $QL(\varphi) = 0$ (cf. §6, Definition 3). When

k has characteristic $\neq 2$, every nondegenerate form is strictly regular, but in characteristic 2 the quasilinear part causes complications. For in this case φ can become degenerate under a field extension $L \supset k$. Only in the regular case is this impossible.

In §9 we study the splitting behaviour of a regular quadratic form φ over k under field extensions, while in §10 any nondegenerate form φ, but only separable extensions of k are allowed. The theory of §9 incorporates the theory of §4, and so the missing proofs of §4 are subsequently filled in.

Until the end of §10 our specialization theory is based on an obvious "canonical" concept of *good reduction* of a form φ over a field K (quadratic or symmetric bilinear) to a valuation ring $\mathfrak{o}$ of K, similar to what is known under this name in other areas of mathematics (e.g. abelian varieties). There is nothing wrong with this theory, however for many applications it is too limited.

This is particularly clear when studying specializations with respect to a place $\lambda\colon K \to L \cup \infty$ with $\operatorname{char} K = 0$, $\operatorname{char} L = 2$. If φ is a nondegenerate quadratic form over K with good reduction with respect to λ, then the specialization $\lambda_*(\varphi)$ is automatically strictly regular. However, we would like to have a more general specialization concept, in which forms with quasilinear part $\neq 0$ can arise over L. Conversely, if the place λ is surjective, i.e. $\lambda(K) = L \cup \infty$, we would like to "lift" *every* nondegenerate quadratic form ψ over L with respect to λ to a form φ over K, i.e. to find a form φ over K which specializes to ψ with respect to λ. Then we could use the theory of forms over K to make statements about ψ.

We present such a general specialization theory in §11. It is based on the concept of "*fair reduction*", which is less orthodox than good reduction, but which possesses quite satisfying properties.

Next, in §12, we present a theory of generic splitting, which unites the theories of §4, §9 and §10 under one roof and which incorporates fair reduction. This theory is deepened in §13 and §14 through the study of generic splitting towers and so we reach the end of Chapter II.

Chapter III (§15 - §27) is a long chapter in which we present a panorama of results about quadratic forms over fields for which specialization and generic splitting of forms play an important role. This only scratches the surface of applications of the specialization theory of Chapters I and II. Certainly many more results can be unearthed.

We return to the foundations of specialization theory in the final short Chapter IV (§28 - §32). Quadratic and bilinear forms over a field can be specialized with respect to a more general "quadratic place" $\Lambda\colon K \to L \cup \infty$ (defined in §28) instead of a usual place $\lambda\colon K \to L \cup \infty$. This represents a considerable broadening of the specialization theory of Chapters I and II. Of course we require again "obedience" from a quadratic form q over K in order for its specialization $\Lambda_*(q)$ to reasonably exist. It then turns out that the generic splitting behaviour of $\Lambda_*(q)$ is governed by the splitting behaviour of q and Λ, in so far good or fair reduction is present in a weak sense, as elucidated for ordinary places in Chapter II.

Why are quadratic places of interest, compared to ordinary places? To answer this question we observe the following. If a form q over K has bad reduction with respect to a place $\lambda\colon K \to L\cup\infty$, it often happens that λ can be "enlarged" to a quadratic place $\Lambda\colon K \to L\cup\infty$ such that q has good or fair reduction with respect to Λ in a weak sense, and the splitting properties of q are handed down to $\Lambda_*(q)$ while there is no form $\lambda_*(q)$ available for which this would be the case. The details of such a notion of reduction are much more tricky compared to what happens in Chapters I and II. The central term which renders possible a unified theory of generic splitting of quadratic forms is called "stably conservative reduction", see §31.

One has to get used to the fact that for bilinear forms there is in general no Witt cancellation rule, in contrast to quadratic forms. Nevertheless the specialization theory is in many respects easier for bilinear forms than for quadratic forms.

On the other hand we do not have any theory of generic splitting for symmetric bilinear forms over fields of characteristic 2. Such a theory might not even be possible in a meaningful way. This may well be connected to the fact that the automorphism groups of such forms can be very far from being reductive groups (which may also account for the absence of a good cancellation rule).

This book is intended for audiences with different interests. For a mathematician with perhaps only a little knowledge of quadratic or symmetric bilinear forms, who just wants to get an impression of specialization theory, it suffices to read §1 - §4. The theory of generic splitting in characteristic $\neq 2$ will acquaint him with an important application area.

From §5 onwards the book is intended for scholars, working in the algebraic theory of quadratic forms and also for specialists in the area of algebraic groups, for they have always been given something to look at by the theory of quadratic forms.

When a reader has reached §10 of the book, he can lean back in his chair and take a well-deserved break. He has then learned about the specialization theory which is based on the concept of good reduction and has gained a certain perspective on specific phenomena in characteristic 2. Furthermore he has been introduced to the foundations of generic splitting and so has seen the specialization theory in action. Admittedly he has not yet seen independent applications of the weak specialization theory (§3, §7), for this theory has only appeared up to then as an auxiliary one.

The remaining sections §11 - §14 of Chapter II develop the specialization theory sufficiently far to allow an understanding of the classical algebraic theory of quadratic forms (as presented in the books of Lam [L], [L′] and Scharlau [S]) without the usual restriction that the characteristic should be different from 2. Precisely this happens in Chapter III where the reader will also obtain sufficient illustrations, enabling him to relieve other classical theorems from the characteristic $\neq 2$ restriction, although this is often a nontrivial task.

The final Chapter IV is ultimately intended for the mathematician who wants to embark on a more daring expedition in the realm of quadratic forms over fields. It cannot be a mere coincidence that the specialization theory for quadratic places is just as satisfying as the specialization theory for ordinary places. It is therefore a safe prediction that quadratic places will turn out to be generally useful and important in a future theory of quadratic forms over fields.

Manfred Knebusch
Regensburg, June 2007

Contents

Chapter I

Fundamentals of Specialization Theory

§1 Introduction: on the Problem of Specialization of Quadratic and Bilinear Forms

Let φ be a nondegenerate symmetric bilinear form over a field K, in other words

$$\varphi(x,y) = \sum_{i,j=1}^{n} a_{ij} x_i y_j,$$

where $x = (x_1, \ldots, x_n) \in K^n$ and $y = (y_1, \ldots, y_n) \in K^n$ are vectors, (a_{ij}) is a symmetric $(n \times n)$-matrix with coefficients $a_{ij} = a_{ji} \in K$ and $\det(a_{ij}) \neq 0$. We like to write $\varphi = (a_{ij})$. The number of variables n is called the *dimension of* φ, $n = \dim \varphi$.

Let also $\lambda\colon K \to L \cup \infty$ be a place, $\mathfrak{o} = \mathfrak{o}_\lambda$ the valuation ring associated to K and $\mathfrak{m}$ the maximal ideal of $\mathfrak{o}$. We denote the group of units of $\mathfrak{o}$ by $\mathfrak{o}^*$, $\mathfrak{o}^* = \mathfrak{o} \setminus \mathfrak{m}$.

We would like λ to "specialize" φ to a bilinear form $\lambda_*(\varphi)$ over L. When is this possible in a reasonable way? If all $a_{ij} \in \mathfrak{o}$ and if $\det(a_{ij}) \in \mathfrak{o}^*$, then one can associate the nondegenerate form $(\lambda(a_{ij}))$ over L to φ. This naive idea leads us to the following

Definition. We say that φ has *good reduction with respect to* λ when φ is isometric to a form (c_{ij}) over K with $c_{ij} \in \mathfrak{o}$, $\det(c_{ij}) \in \mathfrak{o}^*$. We then call the form $(\lambda(c_{ij}))$ "the" *specialization of* φ with respect to λ. We denote this specialization by $\lambda_*(\varphi)$.

Note. $\varphi = (a_{ij})$ is isometric to (c_{ij}) if and only if there exists a matrix $S \in \mathrm{GL}(n,K)$ with $(c_{ij}) = {}^tS(a_{ij})S$. In this case we write $\varphi \cong (c_{ij})$.

We also allow the case $\dim \varphi = 0$, standing for the unique bilinear form on the zero vector space, the form $\varphi = 0$. We agree that the form $\varphi = 0$ has good reduction and set $\lambda_*(\varphi) = 0$.

Problem 1. Is this definition meaningful? Up to isometry $\lambda_*(\varphi)$ should be independent of the choice of the matrix (c_{ij}).

We will later see that this indeed the case, provided $2 \notin \mathfrak{m}$, so that L has characteristic $\neq 2$. If L has characteristic 2, then $\lambda_*(\varphi)$ is well-defined up to "stable isometry" (see §3).

Problem 2. Is there a meaningful way in which one can associate a symmetric bilinear form over L to φ, when φ has bad reduction?

With regards to this problem we would like to recall a classical result of T.A. Springer, which lets us suspect that finding a solution to the problem is not completely beyond hope. Let $v: K \to \mathbb{Z} \cup \infty$ be a discrete valuation of a field K with associated valuation ring $\mathfrak{o}$. Let π be a generator of the maximal ideal $\mathfrak{m}$ of $\mathfrak{o}$, so that $\mathfrak{m} = \pi\mathfrak{o}$. Finally, let $k = \mathfrak{o}/\mathfrak{m}$ be the residue class field of $\mathfrak{o}$ and $\lambda: K \to k \cup \infty$ the canonical place with valuation ring $\mathfrak{o}$. We suppose that $2 \notin \mathfrak{m}$, so that $\operatorname{char} k \neq 2$ is.

Let φ be a nondegenerate symmetric bilinear form over K. Then there exists a decomposition $\varphi \cong \varphi_0 \perp \pi\varphi_1$, where φ_0 and φ_1 have good reduction with respect to λ. Indeed, we can choose a diagonalisation $\varphi \cong \langle a_1, \dots, a_n \rangle$. {As usual $\langle a_1, \dots, a_n \rangle$ denotes the diagonal matrix $\begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix}$.} Then we can make all a_i square-free, so that $v(a_i) = 0$ or 1 for each i, and renumber indices to get $a_i \in \mathfrak{o}^*$ for $1 \leq i \leq t$ and $a_i = \pi\varepsilon_i$, where $\varepsilon_i \in \mathfrak{o}^*$ for $t < i \leq n$. {Possibly $t = 0$, so that $\varphi_0 = 0$, or $t = n$, so that $\varphi_1 = 0$.}

Theorem (Springer 1955 [Sp]). *Let K be complete with respect to the discrete valuation v. If φ is anisotropic (i.e. there is no vector $x \neq 0$ in K^n with $\varphi(x, x) = 0$), then the forms $\lambda_*(\varphi_0)$ and $\lambda_*(\varphi_1)$ are anisotropic and up to isometry independent of the choice of decomposition $\varphi \cong \varphi_0 \perp \pi\varphi_1$.*

Conversely, if ψ_0 and ψ_1 are anisotropic forms over k, then there exists up to isometry a unique anisotropic form φ over K with $\lambda_*(\varphi_0) \cong \psi_0$ and $\lambda_*(\varphi_1) \cong \psi_1$.

Given any place $\lambda: K \to L \cup \infty$ and any form φ over K, Springer's theorem suggests to look for a "weak specialization" $\lambda_W(\varphi)$ by orthogonally decomposing φ in a form φ_0 with good reduction and a form φ_1 with "extremely bad" reduction, subsequently forgetting φ_1 and setting $\lambda_W(\varphi) = \lambda_*(\varphi_0)$.

Given an arbitrary valuation ring $\mathfrak{o}$, this sounds like a daring idea. Nonetheless we will see in §3 that a weak specialization can be defined in a meaningful way. Admittedly $\lambda_W(\varphi)$ is not uniquely determined by φ and λ up to isometry, but up to so-called Witt equivalence. In the situation of Springer's theorem, $\lambda_W(\varphi)$ is then the Witt class of φ_0 and $\lambda_W(\pi\varphi)$ the Witt class of φ_1.

A *quadratic form* q of dimension n over K is a function $q: K^n \to K$, defined by a homogeneous polynomial of degree 2,

$$q(x) = \sum_{1 \leq i \leq j \leq n} a_{ij} x_i x_j$$

$(x = (x_1, \dots, x_n) \in K^n)$. We can associate (a possibly degenerate) symmetric bilinear form

$$B_q(x, y) = q(x + y) - q(x) - q(y) = \sum_{i=1}^{n} 2a_{ii} x_i y_i + \sum_{i<j} a_{ij}(x_i y_j + x_j y_i)$$

to q. It is clear that $B_q(x, x) = 2q(x)$ for all $x \in K^n$.

If $\operatorname{char} K \neq 2$, then any symmetric bilinear form φ over K corresponds to just one quadratic form q over K with $B_q = \varphi$, namely $q(x) = \frac{1}{2}\varphi(x,x)$. In this way we can interpret a quadratic form as a symmetric bilinear form and vice versa. In characteristic 2 however, quadratic forms and symmetric bilinear forms are very different objects.

Problem 3. Let $\lambda: K \to L \cup \infty$ be a place.
(a) To which quadratic forms q over K can we associate "specialized" quadratic forms $\lambda_*(q)$ over L in a meaningful way?
(b) Let $\operatorname{char} L = 2$ and $\operatorname{char} K \neq 2$, hence $\operatorname{char} K = 0$. Should one specialize a quadratic form q over K with respect to λ as a quadratic form, or rather as a symmetric bilinear form?

In what follows we will present a specialization theory for arbitrary nondegenerate symmetric bilinear forms (§3), but only for a rather small class of quadratic forms, the so-called "obedient" quadratic forms (§7). Problem 3(b) will be answered unequivocally. If q is obedient, B_q will determine a really boring bilinear form $\lambda_*(B_q)$ (namely a hyperbolic form) which gives almost no information about q. However, $\lambda_*(q)$ can give important information about q. If possible, a specialization in the quadratic sense is thus to be preferred over a specialization in the bilinear sense.

§2 An Elementary Treatise on Symmetric Bilinear Forms

In this section a "form" will always be understood to be a *nondegenerate* symmetric bilinear form over a field. So let K be a field.

Theorem 1 ("Witt decomposition").
(a) *Any form* φ *over* K *has a decomposition*

$$\varphi \cong \varphi_0 \perp \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \perp \ldots \perp \begin{pmatrix} a_r & 1 \\ 1 & 0 \end{pmatrix}$$

with φ_0 *anisotropic and* $a_1, \ldots, a_r \in K \quad (r \geq 0)$.
(b) *The isometry class of* φ_0 *is uniquely determined by* φ. (*Therefore* $\dim \varphi_0$ *and the number* r *are uniquely determined.*)

To clarify these statements, let us recall the following:

(1) A form φ_0 over K is called *anisotropic* if $\varphi_0(x,x) \neq 0$ for all vectors $x \neq 0$.

(2) If $\operatorname{char} K \neq 2$, then we have for every $a \in K^*$ that

$$\begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} \cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cong \langle 1, -1 \rangle \cong \langle a, -a \rangle.$$

Is $\operatorname{char} K = 2$ however and $a \neq 0$, then $\left(\begin{smallmatrix} a & 1 \\ 1 & 0 \end{smallmatrix}\right) \not\cong \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Indeed if $\varphi = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ we have $\varphi(x,x) = 0$ for every vector $x \in K^2$, while this is not the case for $\varphi = \left(\begin{smallmatrix} a & 1 \\ 1 & 0 \end{smallmatrix}\right)$. In characteristic 2 we still have $\left(\begin{smallmatrix} a & 1 \\ 1 & 0 \end{smallmatrix}\right) \cong \langle a, -a \rangle \quad (a \in K^*)$, but $\left(\begin{smallmatrix} a & 1 \\ 1 & 0 \end{smallmatrix}\right)$ *need not* be isometric to $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right) \cong \langle 1, -1 \rangle$.

(3) The form $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ is given the name "hyperbolic plane" (even in characteristic 2), and every form φ, isometric to an orthogonal sum $r \times \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ of r copies of $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, is called "*hyperbolic*" $(r \geq 0)$.

(4) Forms which are isometric to an orthogonal sum $\left(\begin{smallmatrix} a_1 & 1 \\ 1 & 0 \end{smallmatrix}\right) \perp \ldots \perp \left(\begin{smallmatrix} a_r & 1 \\ 1 & 0 \end{smallmatrix}\right)$ are called *metabolic* $(r \geq 0)$. If $\operatorname{char} K \neq 2$, then every metabolic form is hyperbolic. This is not the case if $\operatorname{char} K = 2$.

(5) If $\operatorname{char} K = 2$, then φ is hyperbolic exactly when *every* vector x of the underlying vector space K^n is isotropic, i.e. $\varphi(x,x) = 0$. If φ is not hyperbolic, we can always find an orthogonal basis such that $\varphi \cong \langle a_1, \ldots, a_n \rangle$ for suitable $a_i \in K^*$.

One can find a proof of Theorem 1 in any book about quadratic forms when $\operatorname{char} K \neq 2$ (see in particular [Bo$_1$], [L], [S]). Part (b) of the theorem is then an immediate consequence of Witt's cancellation theorem. There is no general cancellation theorem in characteristic 2, as the following example shows:

$$(*) \qquad \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} \perp \langle -a \rangle \cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \perp \langle -a \rangle$$

for all $a \in K^*$. If e, f, g is a basis of K^3 which has the left-hand side of $(*)$ as value matrix, then $e+g, f, g$ will be a basis which has the right-hand side of $(*)$ as value matrix. For characteristic 2 one can find proofs of Theorem 1 and the other statements we made in [MH, Chap.I and Chap.III, §1], [K$_1$, §8], [M, §4]. The following is clear from formula $(*)$:

Lemma 1. *If a form φ with $\dim \varphi = 2r$ is metabolic, then there exists a form ψ such that $\varphi \perp \psi \cong r \times \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \perp \psi$.*

Definition 1. (a) In the situation of Theorem 1, we call the form φ_0 the *kernel form* of φ and r the (*Witt*) *index* of φ. We write $\varphi_0 = \ker(\varphi)$, $r = \operatorname{ind}(\varphi)$. {In the literature one frequently sees the notation $\varphi_0 = \varphi_{anisotropic}$ ("anisotropic part" of φ).}

(b) Two forms φ, ψ over K are called *Witt equivalent*, denoted by $\varphi \sim \psi$, if $\ker \varphi \cong \ker \psi$. We write $\varphi \approx \psi$ when $\ker \varphi \cong \ker \psi$ and $\dim \varphi = \dim \psi$. On the basis of the next theorem, we then call φ and ψ *stably isometric*.

Theorem 2. *$\varphi \approx \psi$ exactly when there exists a form χ such that $\varphi \perp \chi \cong \psi \perp \chi$.*

We omit the proof. It is easy when one uses Theorem 1, Lemma 1 and the following lemma.

Lemma 2. *The form $\chi \perp (-\chi)$ is metabolic for every form χ.*

Proof. From Theorem 1(a) we may suppose that χ is anisotropic. If χ is different from the zero form, then $\chi \cong \langle a_1, \ldots, a_n \rangle$ with elements $a_i \in K^*$ $(n \geq 1)$. Finally, $\langle a_i \rangle \perp \langle -a_i \rangle \cong \left(\begin{smallmatrix} a_i & 1 \\ 1 & 0 \end{smallmatrix}\right)$. $\square$

As is well-known, Witt's cancellation theorem (already mentioned above) is valid if $\operatorname{char} K \neq 2$. It says that two stably isometric forms are already isometric: $\varphi \approx \psi \Rightarrow \varphi \cong \psi$.

Let φ be a form over K. We call the equivalence class of φ with respect to the relation $\sim$, introduced above, the *Witt class* of φ and denote it by $\{\varphi\}$. We can add Witt classes together as follows:

$$\{\varphi\} + \{\psi\} := \{\varphi \perp \psi\}.$$

The class $\{0\}$ of the zero form, whose members are exactly the metabolic forms, is the neutral element of this addition. From Lemma 2 it follows that

$\{\varphi\} + \{-\varphi\} = 0$. In this way, the Witt classes of forms over K form an abelian group, which we denote by $W(K)$. We can also multiply Witt classes together:

$$\{\varphi\} \cdot \{\psi\} := \{\varphi \otimes \psi\}.$$

Remark. The definition of the tensor product $\varphi \otimes \psi$ of two forms φ, ψ belongs to the domain of linear algebra [Bo_1, §1, No. 9]. For diagonalizable forms we have

$$\langle a_1, \ldots, a_n\rangle \otimes \langle b_1, \ldots, b_m\rangle \cong \langle a_1b_1, \ldots, a_1b_m, a_2b_1, \ldots, a_nb_m\rangle.$$

We also have $\langle a_1, \ldots, a_n\rangle \otimes \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \cong n \times \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Finally, for a form $\left(\begin{smallmatrix} b & 1 \\ 1 & 0 \end{smallmatrix}\right)$ with $b \neq 0$ we have

$$\langle a\rangle \otimes \begin{pmatrix} b & 1 \\ 1 & 0 \end{pmatrix} \cong \langle a\rangle \otimes \langle b, -b\rangle \cong \langle ab, -ab\rangle \cong \begin{pmatrix} ab & 1 \\ 1 & 0 \end{pmatrix}.$$

Now it is clear that the tensor product of any given form and a metabolic form is again metabolic. {For a conceptual proof of this see [K_1, §3], [MH, Chap. I].} Therefore the Witt class $\{\varphi \otimes \psi\}$ is completely determined by the classes $\{\varphi\}$, $\{\psi\}$, independent of the choice of representatives φ, ψ.

With this multiplication, $W(K)$ becomes a commutative ring. The identity element is $\{\langle 1\rangle\}$. We call $W(K)$ the *Witt ring* of K. For $\operatorname{char} K \neq 2$ this ring was already introduced by Ernst Witt in 1937 [W].

We would like to describe the ring $W(K)$ by generators and relations. In characteristic $\neq 2$ this was already known by Witt [oral communication] and is implicitly contained in his work [W, Satz 7].

First we must recall the notion of determinant of a form. For $a \in K^*$, the isometry class of a one-dimensional form $\langle a\rangle$ will again be denoted by $\langle a\rangle$. The tensor product $\langle a\rangle \otimes \langle b\rangle$ will be abbreviated by $\langle a\rangle\langle b\rangle$. We have $\langle a\rangle\langle b\rangle = \langle ab\rangle$ and $\langle a\rangle\langle a\rangle = \langle 1\rangle$. In this way the isometry classes form an abelian group of exponent 2, which we denote by $Q(K)$. Given $a, b \in K^*$, it is clear that $\langle a\rangle = \langle b\rangle$ exactly when $b = ac^2$ for a $c \in K^*$. So $Q(K)$ is just the *group of square classes* K^*/K^{*2} in disguise. We identify $Q(K) = K^*/K^{*2}$.

It is well-known that for a given form $\varphi = (a_{ij})$ the square class of the determinant of the symmetric matrix (a_{ij}) only depends on the isometry class of φ. We denote this square class by $\det(\varphi)$, so $\det(\varphi) = \langle \det(a_{ij})\rangle$, and call it the *determinant of* φ. A slight complication arises from the fact that the determinant is not compatible with Witt equivalence. To remedy this, we introduce the *signed determinant*

$$d(\varphi) := \langle -1\rangle^{\frac{n(n-1)}{2}} \cdot \det(\varphi)$$

($n := \dim \varphi$). One can easily check that $d(\varphi \perp \left(\begin{smallmatrix} a & 1 \\ 1 & 0 \end{smallmatrix}\right)) = d(\varphi)$, for any $a \in K$. Hence $d(\varphi)$ depends only on the Witt class $\{\varphi\}$. The signed determinant $d(\varphi)$ also has a disadvantage though, in contrast with $\det(\varphi)$, $d(\varphi)$ does not behave well with respect to the orthogonal sum. Let $\nu(\varphi)$ denote the *dimension index* of φ, $\nu(\varphi) = \dim \varphi + 2\mathbb{Z} \in \mathbb{Z}/2\mathbb{Z}$. Then we have (cf. [S, I §2])

$$d(\varphi \perp \psi) = \langle -1 \rangle^{\nu(\varphi)\nu(\psi)} d(\varphi) d(\psi).$$

Let us now describe $W(K)$ by means of generators and relations. Every one-dimensional form $\langle a \rangle$ satisfies $d(\langle a \rangle) = \langle a \rangle$. This innocent remark shows that the map from $Q(K)$ to $W(K)$, which sends every isometry class $\langle a \rangle$ to its Witt class $\{\langle a \rangle\}$, is injective. We can thus interpret $Q(K)$ as a subgroup of the group of units of the ring $W(K)$, $Q(K) \subset W(K)^*$.

$W(K)$ is additively generated by the subset $Q(K)$, since every non-hyperbolic form can be written as $\langle a_1, \ldots, a_n \rangle = \langle a_1 \rangle \perp \ldots \perp \langle a_n \rangle$. Hence $Q(K)$ is a system of generators of $W(K)$. There is an obviously surjective ring homomorphism

$$\Phi\colon \mathbb{Z}[Q(K)] \twoheadrightarrow W(K)$$

from the group ring $\mathbb{Z}[Q(K)]$ to $W(K)$. Recall that $\mathbb{Z}[Q(K)]$ is the ring of formal sums $\sum_g n_g g$ with $g \in Q(K)$, $n_g \in \mathbb{Z}$, and almost all $n_g = 0$. Φ associates to such a sum an in $W(K)$ constructed sum $\sum_g n_g g$.

The elements of the kernel of Φ are the relations on $Q(K)$ we are looking for. We can write down some of those relations immediately: for every $a \in K^*$ is $\langle a \rangle + \langle -a \rangle$ clearly a relation. For $a, b \in K^*$ and given $\lambda, \mu \in K^*$, the form $\langle a, b \rangle$ represents the element $c := \lambda^2 a + \mu^2 b$. If $c \neq 0$, then we can find another element $d \in K^*$ with $\langle a, b \rangle \cong \langle c, d \rangle$ Comparing determinants shows that $\langle d \rangle = \langle abc \rangle$. Hence $\langle a \rangle + \langle b \rangle - \langle c \rangle - \langle abc \rangle = (\langle a \rangle + \langle b \rangle)(\langle 1 \rangle - \langle c \rangle)$ is also a relation. We have the technically important

Theorem 3. *The ideal* $\operatorname{Ker} \Phi$ *of the ring* $\mathbb{Z}[Q(K)]$ *is additively generated (i.e. as abelian group) by the elements* $\langle a \rangle + \langle -a \rangle$, $a \in K^*$ *and the elements* $\langle a \rangle + \langle b \rangle - \langle c \rangle - \langle abc \rangle$ *with* $a, b \in K^*$, $\langle b \rangle \neq \langle -a \rangle$, $c = \lambda^2 a + \mu^2 b$ *with* $\lambda, \mu \in K^*$.

Remark. $\operatorname{Ker} \Phi$ is therefore generated as an ideal by the element $\langle 1 \rangle + \langle -1 \rangle$ and the elements $(\langle 1 \rangle + \langle a \rangle)(1 - \langle c \rangle)$ with $\langle a \rangle \neq \langle -1 \rangle$, $c = 1 + \lambda^2 a$ with $\lambda \in K^*$. For application in the next section, the additive description of $\operatorname{Ker} \Phi$ above is more favourable though.

A proof of Theorem 3, which also works in characteristic 2, can be found in [K_1, §5], [KRW, §1], [K_2 II, §4] (even over semi-local rings instead of over fields[2]), [MH, p.85]. For characteristic $\neq 2$ the proof is a bit simpler, since every form has an orthogonal basis in this case, see [S, I § 9].

[2] The case where K has only 2 elements, $K = \mathbb{F}_2$, is not covered by the more general theorems there. The statement of Theorem 3 for $K = \mathbb{F}_2$ is trivial however, since K has only one square class $\langle 1 \rangle$ and $\langle 1, 1 \rangle \sim 0$.

§3 Specialization of Symmetric Bilinear Forms

In this section, a "form" will again be understood to be a nondegenerate symmetric bilinear form. Let $\lambda\colon K \to L \cup \infty$ be a place from the field K to a field L. Let $\mathfrak{o} = \mathfrak{o}_\lambda$ be the valuation ring associated to λ and $\mathfrak{m}$ its maximal ideal. As usual for rings, $\mathfrak{o}^*$ stands for the group of units of $\mathfrak{o}$, so that $\mathfrak{o}^* = \mathfrak{o} \setminus \mathfrak{m}$. This is the set of all $x \in K$ with $\lambda(x) \neq 0, \infty$.

We will denote the Witt class of a one-dimensional form $\langle a\rangle$ over K (or L) by $\{a\}$ this time. The group of square classes $Q(\mathfrak{o}) = \mathfrak{o}^*/\mathfrak{o}^{*2}$ can be embedded in $Q(K) = K^*/K^{*2}$ in a natural way via $a\mathfrak{o}^{*2} \mapsto aK^{*2}$. We interpret $Q(\mathfrak{o})$ as a subgroup of $Q(K)$, so $Q(\mathfrak{o}) = \{\langle a\rangle \mid a \in \mathfrak{o}^*\} \subset Q(K)$. Our specialization theory is based on the following

Theorem 1. *There exists a well-defined additive map $\lambda_W\colon W(K) \to W(L)$, given by $\lambda_W(\{a\}) = \{\lambda(a)\}$ if $a \in \mathfrak{o}^*$, and $\lambda_W(\{a\}) = 0$ if $\langle a\rangle \notin Q(\mathfrak{o})$ (i.e. $(aK^{*2}) \cap \mathfrak{o}^* = \emptyset$).*[3]

Proof. (Copied from [K_4, §3].) Our place λ is a combination of the canonical place $K \to (\mathfrak{o}/\mathfrak{m}) \cup \infty$ with respect to $\mathfrak{o}$, and a field extension $\overline{\lambda}\colon \mathfrak{o}/\mathfrak{m} \hookrightarrow L$. Thus it suffices to prove the theorem for the canonical place. So let $L = \mathfrak{o}/\mathfrak{m}$ and $\lambda(a) = \overline{a} := a + \mathfrak{m}$ for $a \in \mathfrak{o}$.

We have a well-defined additive map $\Lambda\colon \mathbb{Z}[Q(K)] \to W(L)$ such that $\Lambda(\langle a\rangle) = \{\overline{a}\}$ if $a \in \mathfrak{o}^*$, and $\Lambda(\langle a\rangle) = 0$ if $\langle a\rangle \notin Q(\mathfrak{o})$. Clearly Λ vanishes on all elements $\langle a\rangle + \langle -a\rangle$ with $a \in K^*$. According to §2, Theorem 3 we will be finished if we can show that Λ also disappears on every element

$$Z = \langle a_1\rangle + \langle a_2\rangle - \langle a_3\rangle - \langle a_4\rangle$$

with $a_i \in K^*$ and $\langle a_1, a_2\rangle \cong \langle a_3, a_4\rangle$. This will be the case when the four square classes $\langle a_i\rangle$ are not all in $Q(\mathfrak{o})$.

Suppose from now on, without loss of generality, that $a_1 \in \mathfrak{o}^*$. Then we have $Z = \langle a_1\rangle y$, where

$$y = 1 + \langle c\rangle - \langle b\rangle - \langle bc\rangle$$

is an element such that $\langle 1, c\rangle \cong \langle b, bc\rangle$. So $b = u^2 + w^2 c$ for elements $u, w \in K$. Clearly the equation $\Lambda(\langle a\rangle x) = \{\overline{a}\}\Lambda(x)$ is satisfied for any $a \in \mathfrak{o}^*$, $x \in \mathbb{Z}[Q(K)]$. Therefore it is enough to verify that $\Lambda(y) = 0$. We suppose without

[3] The letter W in the notation λ_W refers to "Witt" or "weak", see §1 and §7.

loss of generality that u and w are not both zero, otherwise we already have that $y = 0$.

Let us first treat the case $\langle c\rangle \in Q(\mathfrak{o})$, so without loss of generality $c \in \mathfrak{o}^*$. Then we have

$$\Lambda(y) = (1 + \{\overline{c}\})\Lambda(1 - \langle b\rangle).$$

If $\{\overline{c}\} = \{-1\}$, we are done. So suppose from now on that $\{\overline{c}\} \neq \{-1\}$. Then the form $\langle 1, \overline{c}\rangle$ is anisotropic over L. Since we are allowed to replace u and v by gu and gv for some $g \in K^*$, we may additionally assume that u and v are both in $\mathfrak{o}$, but not both in $\mathfrak{m}$. Since $\langle 1, \overline{c}\rangle$ is anisotropic, we have $\overline{b} = \overline{u}^2 + \overline{cw}^2 \neq 0$ and

$$\Lambda(y) = (1 + \{\overline{c}\})(1 - \{\overline{u}^2 + \overline{cw}^2\}) = 0.$$

The case which remains to be tackled is when the square class cK^{*2} doesn't contain a unit from $\mathfrak{o}$. Then $u^{-2}w^2c$ is definitely not a unit and either $b = u^2(1 + d)$ or $b = w^2c(1 + d)$ with $d \in \mathfrak{m}$. Hence $\Lambda(1 - \langle\overline{b}\rangle)$ is 0 or $1 - \{\overline{c}\}$, and both times $\Lambda(y) = 0$. □

Scholium. The map $\lambda_W\colon W(K) \to W(L)$ can be described very conveniently as follows: Let φ be a form over K. If φ is hyperbolic (or, more generally metabolic), then $\lambda_W(\{\varphi\}) = 0$. If φ is not hyperbolic, then consider a diagonalisation $\varphi \cong \langle a_1, a_2, \ldots, a_n\rangle$. Multiply each coefficient a_i, for which it is possible, by a square so that it becomes a unit in $\mathfrak{o}$, and leave the other coefficients as they are. Let for example $a_i \in \mathfrak{o}^*$ for $1 \leq i \leq r$ and $\langle a_i\rangle \notin Q(\mathfrak{o})$ for $r < i \leq n$ (possibly $r = 0$ or $r = n$). Then $\lambda_W(\{\varphi\}) = \{\langle\lambda(a_1), \ldots, \lambda(a_r)\rangle\}$.

Let us now recall a definition from the Introduction §1.

Definition 1. We say that a form φ over K has *good reduction with respect to* λ, or that φ is λ-*unimodular* if φ is isometric to a form (a_{ij}) with $a_{ij} \in \mathfrak{o}$ and $\det(a_{ij}) \in \mathfrak{o}^*$. We call such a representation $\varphi \cong (a_{ij})$ a λ-*unimodular representión of* φ (or a unimodular representation with respect to the valuation ring $\mathfrak{o}$).

This definition can be interpreted geometrically as follows. We associate to φ a couple (E, B), consisting of an n-dimensional K-vector space E ($n = \dim\varphi$) and a symmetric bilinear form $B\colon E \times E \to K$ such that B represents the form φ after a choice of basis of E. We denote this by $\varphi \mathrel{\widehat{=}} (E, B)$. Since φ has good reduction with respect to λ, E contains a free $\mathfrak{o}$-submodule M of rank n with $E = KM$, i.e. $E = K \otimes_{\mathfrak{o}} M$, and with $B(M \times M) \subset \mathfrak{o}$, such that the restriction $B|M \times M\colon M \times M \to \mathfrak{o}$ is a *nondegenerate bilinear form over* $\mathfrak{o}$, i.e. gives rise to an isomorphism $x \mapsto B(x, -)$ from the $\mathfrak{o}$-module M to the dual $\mathfrak{o}$-module $\check{M} = \mathrm{Hom}_{\mathfrak{o}}(M, \mathfrak{o})$.

By means of Theorem 1 we can now quite easily find a solution of the first problem posed in §1.

Theorem 2. *Suppose that the form φ over K has good reduction with respect to λ. Let $\varphi \cong (a_{ij})$ be a unimodular representation of φ. Then the Witt* *class $\lambda_W(\{\varphi\})$ is represented by the (nondegenerate!) form $(\lambda(a_{ij}))$ over L. Consequently the form $(\lambda(a_{ij}))$ is up to stable isometry independent of the choice of unimodular representation.* (Recall that if two forms ψ and ψ' are Witt equivalent and $\dim\psi = \dim\psi'$, then $\psi \approx \psi'$.)

To prove this theorem, we need the following easy lemma about lifting orthogonal bases.

Lemma. *Let M be a finitely generated free $\mathfrak{o}$-module, equipped with a nondegenerate symmetric bilinear form $B\colon M \times M \to \mathfrak{o}$. Let $k := \mathfrak{o}/\mathfrak{m}$ and let $\pi\colon M \to M/\mathfrak{m}M$ be the natural epimorphism from M to the k-vector space $M/\mathfrak{m}M$. Further, let $\overline{B}$ be the (again nondegenerate) bilinear form induced by B on $M/\mathfrak{m}M$, $\overline{B}(\pi(x), \pi(y)) := B(x, y) + \mathfrak{m}$. Suppose that the vector space $M/\mathfrak{m}M$ has a basis $\overline{e}_1, \dots, \overline{e}_n$, orthogonal with respect to $\overline{B}$. Then M has a basis $e_1, \dots, e_n$, orthogonal with respect to B, with $\pi(e_i) = \overline{e}_i \quad (1 \le i \le n)$.*

Proof. By induction on n, which obviously is the rank of the free $\mathfrak{o}$-module M. For $n = 1$ nothing has to be shown. So suppose that $n > 1$. We choose an element $e_1 \in M$ with $\pi(e_1) = \overline{e}_1$. Then $B(e_1, e_1) \in \mathfrak{o}^*$ since $\overline{B}(\overline{e}_1, \overline{e}_1) \neq 0$. Hence the restriction of B to the module $\mathfrak{o}e_1$ is a nondegenerate bilinear form on $\mathfrak{o}e_1$. Invoking a very simple theorem (e.g. [MH, p.5, Th.3.2], §5, Lemma 1 below) yields $M = (\mathfrak{o}e_1) \perp N$ with $N = (\mathfrak{o}e_1)^\perp = \{x \in M \mid B(x, e_1) = 0\}$. The restriction $\pi|N\colon N \to M/\mathfrak{m}M$ is then a homomorphism from N to $(k\overline{e}_1)^\perp = \bigoplus_{i=2}^{n} k\overline{e}_i$ with kernel $\mathfrak{m}N$. By our induction hypothesis, N contains an orthogonal basis $e_2, \dots, e_n$ with $\pi(e_i) = \overline{e}_i \quad (2 \le i \le n)$ which can be completed by e_1 to form an orthogonal basis of M which has the required property. $\square$

Remark. Clearly the lemma and its proof remain valid when $\mathfrak{o}$ is an arbitrary local ring with maximal ideal $\mathfrak{m}$, instead of a valuation ring.

We also need the following

Definition 2. A *bilinear $\mathfrak{o}$-module* is a couple (M, B) consisting of an $\mathfrak{o}$-module M and a symmetric bilinear form $B\colon M \times M \to \mathfrak{o}$. A bilinear module is called *free* when the $\mathfrak{o}$-module M is free of finite rank. If $e_1, \dots, e_n$ is a basis of M, we write $(M, B) \cong (a_{ij})$ with $a_{ij} := B(e_i, e_j)$. If $e_1, \dots, e_n$ is an orthogonal basis ($B(e_i, e_j) = 0$ for $i \neq j$), then we also write $(M, B) \cong \langle a_1, \dots, a_n\rangle$ with $a_i := B(e_i, e_i)$.

Note. The form B is nondegenerate exactly when $\det(a_{ij})$ is a unit in $\mathfrak{o}$, respectively when all a_i are units in $\mathfrak{o}$.

All this makes sense and remains correct when $\mathfrak{o}$ is an arbitrary commutative ring (with 1), instead of a valuation ring. As before, "$\cong$" stands for "isometric", also for bilinear modules.

Proof of Theorem 2. For $a \in \mathfrak{o}$, let $\overline{a}$ denote the image of a in $\mathfrak{o}/\mathfrak{m}$. We suppose for the moment that the bilinear space $(\overline{a}_{ij})$ over $\mathfrak{o}/\mathfrak{m}$ is not hyperbolic. Then it has an orthogonal basis. By the lemma, the bilinear module (a_{ij}) over $\mathfrak{o}$ also has an orthogonal basis. Hence over $\mathfrak{o}$,

$$(*) \qquad (a_{ij}) \cong \langle a_1, \ldots, a_n \rangle$$

for certain $a_i \in \mathfrak{o}^*$. The isometry $(*)$ is then also valid over K, and so we have in $W(L)$

$$\lambda_W(\{\varphi\}) = \{\langle \lambda(a_1), \ldots, \lambda(a_n) \rangle\}.$$

On the other hand $(*)$ implies that

$$(\overline{a}_{ij}) \cong \langle \overline{a}_1, \ldots, \overline{a}_n \rangle \quad \text{over } \mathfrak{o}/\mathfrak{m}.$$

If we now apply the (injective) homomorphism $\overline{\lambda}\colon \mathfrak{o}/\mathfrak{m} \to L$ induced by λ (thus we tensor with the field extension given by $\overline{\lambda}$), we obtain

$$(\lambda(a_{ij})) \cong \langle \lambda(a_1), \ldots, \lambda(a_n) \rangle$$

over L. Consequently the Witt class $\lambda_W(\{\varphi\})$ is represented by the form $(\lambda(a_{ij}))$.

Let us now tackle the remaining case, where the form $(\overline{a}_{ij})$ over $\mathfrak{o}/\mathfrak{m}$ is hyperbolic. We can apply what we just have proved to the form $\psi := \varphi \perp \langle 1 \rangle$. This gives us

$$\begin{aligned} \lambda_W(\{\psi\}) &= \{(\lambda(a_{ij})) \perp \langle 1 \rangle\} \\ &= \{(\lambda(a_{ij}))\} + \{\langle 1 \rangle\} \end{aligned}$$

in $W(L)$. On the other hand we have $\lambda_W(\{\psi\}) = \lambda_W(\{\varphi\}) + \{\langle 1 \rangle\}$, and we find again that $\lambda_W(\{\varphi\}) = \{(\lambda(a_{ij}))\}$. □

Remark. If $\operatorname{char} L \neq 2$, a hyperbolic form over $\mathfrak{o}/\mathfrak{m}$ also has an orthogonal basis, so that the distinction between the two cases above is unnecessary.

Definition 3. If φ has good reduction with respect to λ, $\varphi \cong (a_{ij})$ with $a_{ij} \in \mathfrak{o}$, $\det(a_{ij}) \in \mathfrak{o}^*$, we denote the form $(\lambda(a_{ij}))$ over L by $\lambda_*(\varphi)$ and call it "the" *specialization of φ with respect to λ.*

If $\operatorname{char} L = 2$ we run into trouble with this definition, since $\lambda_*(\varphi)$ is only up to *stable isometry* uniquely determined by φ . We nevertheless use it, since it is so convenient. If $\operatorname{char} L \neq 2$, $\lambda_*(\varphi)$ is up to isometry uniquely determined by φ.

Example. Every metabolic form φ over K has good reduction with respect to λ. Of course is $\lambda_*(\varphi) \sim 0$.

Proof. It suffices to prove this in the case $\dim \varphi = 2$, so $\varphi = \left(\begin{smallmatrix} a & 1 \\ 1 & 0 \end{smallmatrix}\right)$ with $a \in K$. Let $\varphi \widehat{=} (E, B)$ and let e, f be a basis of E with value matrix $\left(\begin{smallmatrix} a & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Choose an element $c \in K^*$ with $ac^2 \in \mathfrak{o}$. Then $ce, c^{-1}f$ is a basis of E with value matrix $\left(\begin{smallmatrix} ac^2 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. □

Theorem 3. *Let φ and ψ be forms over K, having good reduction with respect to λ. Then $\varphi \perp \psi$ also has good reduction with respect to λ and*

$$\lambda_*(\varphi \perp \psi) \approx \lambda_*(\varphi) \perp \lambda_*(\psi).$$

Proof. This is clear. □

Until now we got on with our specialization theory almost without any knowledge of bilinear forms over $\mathfrak{o}$. Except for the lemma above about the existence of orthogonal bases, we needed hardly anything from this area. We could even have avoided using this little bit of information if we would only have considered diagonalised forms over fields.

We are still missing one important theorem of specialization theory (especially for applications later on), Theorem 4 below. For a proof of this theorem we need the basics of the theory of forms over valuation rings, which we will present next using a "geometric" point of view. In other words, we interpret a form φ over a field as an "inner product" on a vector space and use more generally "inner products" on modules over rings, while until now a form was usually interpreted as a polynomial in two sets of variables $x_1, \ldots, x_n, y_1, \ldots, y_n$.

For the moment we allow local rings instead of the valuation ring $\mathfrak{o}$ since this won't cost us anything extra. So let A be a local ring.

Definition 4. A *bilinear space* M over A is a free A-module M of finite rank, equipped with a symmetric bilinear form $B\colon M \times M \to A$ which is *nondegenerate*, i.e. which determines an isomorphism $x \mapsto B(x, -)$ from M on the dual module $\check{M} = \mathrm{Hom}_A(M, A)$.

Remark. We usually denote a bilinear space by the letter M. If confusion is possible, we write (M, B) or even (M, B_M).

In what follows, M denotes a bilinear space over A, with associated bilinear form B.

Definition 5. A *subspace* V of M is a submodule V of M which is a direct summand of M, i.e. for which there exists another submodule W of M with $M = V \oplus W$.

To a subspace V we can associate the orthogonal submodule

$$V^\perp = \{x \in M \mid B(x, V) = 0\},$$

and we have an exact sequence

$$0 \longrightarrow V^\perp \longrightarrow M \xrightarrow{\varphi} \check{V} \longrightarrow 0.$$

Here $\check{V} = \mathrm{Hom}_A(V, A)$ and φ maps $x \in M$ to the linear form $y \mapsto B(x, y)$ on V. The sequence splits since $\check{V}$ is free. Thus $V^\perp$ is again a subspace of M.

Definition 6. A subspace V of M is called *totally isotropic* when $B(V, V) = \{0\}$, i.e. when $V \subset V^\perp$. V is called a *Lagrangian subspace* of M when $V = V^\perp$. If M contains a Lagrangian subspace, M is called *metabolic*. M is called *anisotropic* if it *doesn't* contain any totally isotropic subspace $V \neq \{0\}$.

Lemma 1.
(a) *Every bilinear space M over A has a decomposition*

$$M \cong M_0 \perp M_1$$

with M_0 anisotropic and M_1 metabolic.

(b) *Every metabolic space N over A is the orthogonal sum of spaces of the form* $\left(\begin{smallmatrix} a & 1 \\ 1 & 0 \end{smallmatrix}\right)$,

$$N \cong \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \perp \ldots \perp \begin{pmatrix} a_r & 1 \\ 1 & 0 \end{pmatrix}$$

with $a_1, \ldots, a_r \in A$.

These statements can be inferred from more general theorems, which can be found in e.g. [Ba, §1], [K_1, §3], [K_2, I §3], [KRW, §1].

Remark. If 2 is a unit in A, Witt's cancellation theorem ([K_3], [R]) holds for bilinear spaces over A and every metabolic space over A is even hyperbolic, i.e. is an orthogonal sum $r \times \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ of copies of the "hyperbolic plane" $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ over A. Now the anisotropic space M_0 in Lemma 1(a) is up to isometry uniquely determined by M. If $2 \notin A^*$ this is false in general.

If $\alpha\colon A \to C$ is a homomorphism from A to another local ring C, we can associate to a bilinear space $(M, B) = M$ over A a bilinear space $(C \otimes_A M, B') = C \otimes_A M$ over C as follows: the underlying free C-module is the tensor product $C \otimes_A M$ determined by α, and the C-bilinear form B' on this module is obtained from B by means of a basis extension, so

$$B'(c \otimes x, d \otimes y) = cd\,\alpha(B(x, y))$$

($x, y \in M; c, d \in C$). The form B' is again nondegenerate. If (a_{ij}) is the value matrix of B with respect to a basis $e_1, \ldots, e_n$ of M, then $(\alpha(a_{ij}))$ is the value matrix of B' with respect to the basis $1 \otimes e_1, \ldots, 1 \otimes e_n$ of $C \otimes_A M$.

If A doesn't contain any zero divisors and if K is the quotient field of A, we can in particular use the inclusion $A \hookrightarrow K$ to associate a bilinear space $K \otimes_A M$ to the bilinear space M over A. Now we can interpret M as an A-submodule of the K-vector space $K \otimes_A M$ ($x = 1 \otimes x$ for $x \in M$) and reconstruct B from B' by restriction, $B = B'|M \times M\colon\ M \times M \to A$.

Let us return to our place $\lambda\colon K \to L \cup \infty$ and the valuation ring $\mathfrak{o}$.

Lemma 2. *Let M be a bilinear space over $\mathfrak{o}$.*
(a) *If $K \otimes_{\mathfrak{o}} M$ is isotropic, then M is isotropic.*
(b) *If $K \otimes_{\mathfrak{o}} M$ is metabolic, then M is metabolic.*

Proof. Let $E := K \otimes_{\mathfrak{o}} M$. We interpret M as an $\mathfrak{o}$-submodule of E and have $E = KM$.

(a) If E is isotropic, there exists a subspace $W \neq \{0\}$ in E with $W \subset W^{\perp}$. The $\mathfrak{o}$-submodule $V := W \cap M$ of M satisfies $KV = W$ and so $V \neq \{0\}$. Furthermore $V \subset V^{\perp}$. The $\mathfrak{o}$-module M/V is torsion free and finitely generated, hence free. This is because every finitely generated ideal in $\mathfrak{o}$ is principal, cf. [CE, VII, §4]. Therefore V is a totally isotropic subspace of E.

(b) If $W = W^{\perp}$, then $V = V^{\perp}$. Hence M is metabolic. □

Now we are fully equipped to prove the following important theorem [K_4, Prop.2.2].

Theorem 4. *Let φ and ψ be forms over K. If φ and $\varphi \perp \psi$ have good reduction with respect to λ, then ψ also has good reduction with respect to λ.*

Proof. Adopting geometric language, the statement says: Let F and G be bilinear spaces over K and $E := F \perp G$. If F and E have good reduction, i.e. $F \cong K \otimes_{\mathfrak{o}} N$, $E \cong K \otimes_{\mathfrak{o}} M$ for bilinear spaces N and M over $\mathfrak{o}$, then G has good reduction as well.

By §2, Theorem 1 there is a decomposition $G = G_0 \perp G_1$ with G_0 anisotropic and G_1 metabolic. From above (cf. the example following Definition 3), G_1 has good reduction. Hence it suffices to show that G_0 has good reduction.

Now $E \perp (-F) \cong F \perp (-F) \perp G_0 \perp G_1$.[4] Since $F \perp (-F) \perp G_0$ is metabolic, but G_1 anisotropic, G_1 is the kernel space of $E \perp (-F)$. ("Kernel space" is the pendant of the word "kernel form" (= anisotropic part) in the geometric language.) We decompose $M \perp (-N)$ following Lemma 1(a), $M \perp (-N) \cong R \perp S$ where R is anisotropic and S metabolic. Tensoring with K gives $E \perp (-F) \cong K \otimes_{\mathfrak{o}} R \perp K \otimes_{\mathfrak{o}} S$. Now $K \otimes_{\mathfrak{o}} S$ is metabolic and, according to Lemma 2, $K \otimes_{\mathfrak{o}} R$ is anisotropic. Hence $K \otimes_{\mathfrak{o}} R$ is also a kernel space of $E \perp (-F)$. Applying §2, Theorem 1 gives $K \otimes_{\mathfrak{o}} R \cong G_0$, and we are finished. □

Corollary. *Let φ and ψ be forms over K with $\varphi \sim \psi$. If φ has good reduction with respect to λ, ψ also has good reduction with respect to λ and $\lambda_*(\varphi) \sim \lambda_*(\psi)$. If furthermore $\varphi \approx \psi$, then $\lambda_*(\varphi) \approx \lambda_*(\psi)$.*

Proof. There are Witt decompositions $\varphi \cong \varphi_0 \perp \mu$, $\psi \cong \psi_0 \perp \nu$ with φ_0, ψ_0 anisotropic and μ, ν metabolic. As established above, μ and ν have good reduction with respect to λ and $\lambda_*(\mu)$, $\lambda_*(\nu)$ are metabolic. By assumption φ has good reduction with respect to λ and φ_0 is isometric to ψ_0. Theorem 4 implies that φ_0 has good reduction with respect to λ. Therefore ψ_0, and hence ψ, has good reduction with respect to λ, and (according to Theorem 3)

$$\lambda_*(\varphi) \approx \lambda_*(\varphi_0) \perp \lambda_*(\mu) \sim \lambda_*(\varphi_0),$$
$$\lambda_*(\psi) \approx \lambda_*(\psi_0) \perp \lambda_*(\nu) \sim \lambda_*(\psi_0).$$

[4] If $E = (E, B)$ is a bilinear space, then $-E$ denotes the space $(E, -B)$.

Naturally $\lambda_*(\varphi_0) \approx \lambda_*(\psi_0)$, so $\lambda_*(\varphi) \sim \lambda_*(\psi)$. If $\varphi \approx \psi$, then φ and ψ have the same dimension and so $\lambda_*(\varphi) \approx \lambda_*(\psi)$. $\square$

Let us give a small illustration of Theorem 4.

Definition 7. Let φ and ψ be forms over a field K. If there exists a form χ over K with $\varphi \cong \psi \perp \chi$, we say that φ *represents* the form ψ and write $\psi < \varphi$.

For example, the one-dimensional forms represented by φ are exactly the square classes $\langle \varphi(x, x) \rangle$, where x runs through the anisotropic vectors of the space belonging to φ.

Theorem 5 (Substitution Principle). *Let k be a field and $K = k(t)$, where $t = (t_1, \ldots, t_r)$ is a set of indeterminates. Let $(f_{ij}(t))$ be a symmetric $(n \times n)$-matrix and $(g_{kl}(t))$ a symmetric $(m \times m)$-matrix, for polynomials $f_{ij}(t) \in k[t]$, and $g_{kl}(t) \in k[t]$. Let further be given a field extension $k \subset L$ and a point $c \in L^r$ with $\det(f_{ij}(c)) \neq 0$ and $\det(g_{kl}(c)) \neq 0$. If $\operatorname{char} k = 2$, also suppose that the form $(f_{ij}(c))$ is anisotropic over L.*

Claim: if $(g_{kl}(t)) < (f_{ij}(t))$ (as forms over K), then $(g_{kl}(c)) < (f_{ij}(c))$ (as forms over L).

Proof. Going from $k[t]$ to $L[t]$, we suppose without loss of generality that $L = k$. For every $s \in \{1, \ldots, r\}$ there is exactly one corresponding place $\lambda_s : k(t_1, \ldots, t_s) \to k(t_1, \ldots, t_{s-1}) \cup \infty$ with $\lambda_s(u) = u$ for all $u \in k(t_1, \ldots, t_{s-1})$ and $\lambda_s(t_s) = c_s$. {Read $k(t_1, \ldots, t_{s-1}) = k$ when $s = 1$.} The composition $\lambda_1 \circ \lambda_2 \circ \cdots \circ \lambda_s$ of these places is a place $\lambda : K \to k \cup \infty$ with $\lambda(a) = a$ for all $a \in k$ and $\lambda(t_i) = c_i$ for $i = 1, \ldots, r$. Let φ denote the form $(f_{ij}(t))$ over K and ψ the form $(g_{kl}(t))$ over K. {Note that obviously $\det(f_{ij}(t)) \neq 0$, $\det(g_{kl}(t)) \neq 0$.} The forms φ and ψ both have good reduction with respect to λ and $\lambda_*(\varphi) \approx (f_{ij}(c))$, $\lambda_*(\psi) \approx (g_{kl}(c))$.

Now let $\psi < \varphi$. Then there exists a form χ over K with $\psi \perp \chi \cong \varphi$. According to Theorem 4, χ has good reduction with respect to λ and according to Theorem 3, $\lambda_*(\psi) \perp \lambda_*(\chi) \approx \lambda_*(\varphi)$. Hence $\lambda_*(\psi) \perp \lambda_*(\chi) \cong \lambda_*(\varphi)$ if $\operatorname{char} k \neq 2$. If $\operatorname{char} k = 2$ and $\lambda_*(\varphi)$ is anisotropic, this remains true, since $\lambda_*(\varphi)$ is up to isometry the unique anisotropic form in the Witt class $\lambda_W(\varphi)$, and $\lambda_*\ (\psi) \perp \lambda_*\ (\chi)$ has the same dimension as $\lambda_*(\varphi)$. $\square$

Let us now return to our arbitrary place $\lambda : K \to L \cup \infty$ and to the conventions made at the beginning of the paragraph. The Lemmas 1 and 2 allow us to give an easier proof of Theorem 2, which is interesting in its own right.

Second proof of Theorem 2. We adopt the geometric language. Let E be a bilinear space over K, having good reduction with respect to λ and let M and N be bilinear spaces over $\mathfrak{o}$ with $E \cong K \otimes_{\mathfrak{o}} M \cong K \otimes_{\mathfrak{o}} N$. We have to show that $L \otimes_\lambda M \approx L \otimes_\lambda N$, where the tensor product is taken over $\mathfrak{o}$, and L is regarded as an $\mathfrak{o}$-algebra via the homomorphism $\lambda|\mathfrak{o} : \mathfrak{o} \to L$. The space $K \otimes_{\mathfrak{o}} (M \perp (-N))$ is metabolic. According to Lemma 2, $M \perp (-N)$ is metabolic. Hence $L \otimes_\lambda (M \perp (-N)) = L \otimes_\lambda M \perp (-L \otimes_\lambda N)$ is also metabolic. Therefore $L \otimes_\lambda M \approx L \otimes_\lambda N$. □

We can now describe the property "good reduction" and the specialization of a form by means of diagonal forms as follows.

Theorem 6. *Let φ be a form over K, $\dim\varphi = n$.*

(*a*) *The form φ has good reduction with respect to λ if and only if φ is Witt equivalent to a diagonal form $\langle a_1, \ldots, a_r\rangle$ with units $a_i \in \mathfrak{o}^*$. In this case $\dim\lambda_*(\varphi) = n$ and $\lambda_*(\varphi) \sim \langle\lambda(a_1), \ldots, \lambda(a_r)\rangle$. Furthermore one can choose $r = n + 2$.*

(*b*) *Let $2 \in \mathfrak{o}^*$, i.e. $\operatorname{char} L \neq 2$. The form φ has good reduction with respect to λ if and only if φ is isometric to a diagonal form $\langle a_1, \ldots, a_n\rangle$ with $a_i \in \mathfrak{o}^*$. In this case $\lambda_*(\varphi) \cong \langle\lambda(a_1), \ldots, \lambda(a_n)\rangle$.*

Proof of part (*a*). If $\varphi \sim \langle a_1, \ldots, a_r\rangle$ with units $a_i \in \mathfrak{o}^*$, then by the Corollary φ has good reduction with respect to λ and $\lambda_*(\varphi) \sim \langle\lambda(a_1), \ldots, \lambda(a_r)\rangle$. Suppose now that φ has good reduction with respect to λ. The form φ corresponds to a bilinear space $K \otimes_{\mathfrak{o}} M$ over K, which comes from a bilinear space M over $\mathfrak{o}$. If $M/\mathfrak{m}M$ is not hyperbolic, then M has an orthogonal basis by the Lemma following Theorem 2. Therefore $\varphi \cong \langle a_1, \ldots, a_n\rangle$ with units $a_i \in \mathfrak{o}^*$. In general we consider the space $M' := M \perp \langle 1, -1\rangle$ over $\mathfrak{o}$. Then $M'/\mathfrak{m}M'$ is definitely *not* hyperbolic. Hence $\varphi \perp \langle 1, -1\rangle \cong \langle b_1, \ldots, b_{n+2}\rangle$ with units b_i. □

The proof of part (b) is similar, but simpler since now $M/\mathfrak{m}M$ always has an orthogonal basis. We don't need the Corollary here.

Finally, we consider the specialization of tensor products of forms.

Theorem 7. *Let φ and ψ be two forms over K, which have good reduction with respect to λ. Then $\varphi \otimes \psi$ also has good reduction with respect to λ, and $\lambda_*(\varphi \otimes \psi) \approx \lambda_*(\varphi) \otimes \lambda_*(\psi)$.*

Proof. According to Theorem 6 we have the following Witt equivalences, $\varphi \sim \langle a_1, \ldots, a_m\rangle$, $\psi \sim \langle b_1, \ldots, b_n\rangle$, with units $a_i, b_j \in \mathfrak{o}^*$. Then $\varphi \otimes \psi \sim \langle a_1b_1, a_1b_2, \ldots, a_1b_n, \ldots, a_mb_n\rangle$. Again according to Theorem 6, $\varphi \otimes \psi$ has good reduction and

$$\begin{aligned}\lambda_*(\varphi \otimes \psi) &\sim \langle\lambda(a_1)\lambda(b_1), \ldots, \lambda(a_m)\lambda(b_n)\rangle \\ &\cong \langle\lambda(a_1), \ldots, \lambda(a_m)\rangle \otimes \langle\lambda(b_1), \ldots, \lambda(b_n)\rangle \\ &\sim \lambda_*(\varphi) \otimes \lambda_*(\psi).\end{aligned}$$

Now the forms $\lambda_*(\varphi \otimes \psi)$ and $\lambda_*(\varphi) \otimes \lambda_*(\psi)$ both have the same dimension as $\varphi \otimes \psi$. Therefore $\lambda_*(\varphi \otimes \psi) \approx \lambda_*(\varphi) \otimes \lambda_*(\psi)$. □

If we use a bit more multilinear algebra, we can come up with the following conceptually more pleasing proof of Theorem 7.

Second proof of Theorem 7. Adopting geometric language, φ corresponds to a bilinear space $K \otimes_{\mathfrak{o}} M$ and ψ corresponds to a space $K \otimes_{\mathfrak{o}} N$ with bilinear spaces M and N over $\mathfrak{o}$. Hence $\varphi \otimes \psi$ corresponds to the space

$$(K \otimes_{\mathfrak{o}} M) \otimes_K (K \otimes_{\mathfrak{o}} N) \;\cong\; K \otimes_{\mathfrak{o}} (M \otimes_{\mathfrak{o}} N)$$

over K. Now the free bilinear module $M \otimes_{\mathfrak{o}} N$ is again nondegenerate (cf. e.g. [MH, I §5]). Consequently $\varphi \otimes \psi$ has good reduction with respect to λ, and $\lambda_*(\varphi \otimes \psi)$ can be represented by the space $L \otimes_\lambda (M \otimes_{\mathfrak{o}} N)$, obtained from the space $M \otimes_{\mathfrak{o}} N$ by base extension to L using the homomorphism $\lambda|\mathfrak{o} : \mathfrak{o} \to L$. Now $L \otimes_\lambda (M \otimes_{\mathfrak{o}} N) \;\cong\; (L \otimes_\lambda M) \otimes_L (L \otimes_\lambda N)$. In other words, $\lambda_*(\varphi \otimes \psi) \cong \lambda_*(\varphi) \otimes \lambda_*(\psi)$, since $\lambda_*(\varphi) \mathrel{\widehat{=}} L \otimes_\lambda M$, $\lambda_*(\psi) \mathrel{\widehat{=}} L \otimes_\lambda N$. $\square$

§4 Generic Splitting in Characteristic $\neq 2$

In this section we outline an important application area of the specialization theory developed in §3, namely the theory of generic splitting of bilinear forms. Many proofs will only be given in §7, after also having developed a specialization theory of quadratic forms.

Let k be a field and let φ be a *form* over k, which is just as before understood to be a nondegenerate symmetric bilinear form over k. Our starting point is the following simple

Observation. Let K and L be fields, containing k, and let $\lambda: K \to L \cup \infty$ be a place *over* k, i.e. with $\lambda(c) = c$ for all $c \in k$.

(a) Then $\varphi \otimes K$ has good reduction with respect to λ and $\lambda_*(\varphi \otimes K) \approx \varphi \otimes L$.[5] Indeed, if $\varphi \cong (a_{ij})$ with $a_{ij} \in k$, $\det(a_{ij}) \neq 0$, then also $\varphi \otimes K \cong (a_{ij})$, and this is a unimodular representation of φ with respect to λ, since k is contained in the valuation ring $\mathfrak{o}$ of λ. So $\lambda_*(\varphi \otimes K) \approx (\lambda(a_{ij})) = (a_{ij})$ and since this symmetric matrix is now considered over L, we conclude that $\lambda_*(\varphi \otimes K) \approx \varphi \otimes L$.

(b) Suppose that $\varphi \otimes K$ has kernel form φ_1 and Witt index r_1, i.e.

$$\varphi \otimes K \;\approx\; \varphi_1 \perp r_1 \times H,$$

where H denotes from now on the hyperbolic “plane”[6] $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. According to the corollary of Theorem 4 in §3, the form φ_1 has good reduction with respect to λ. Therefore, applying λ_* yields

$$\varphi \otimes L \;\approx\; \lambda_*(\varphi_1) \perp r_1 \times H.$$

Hence we conclude that $\operatorname{ind}(\varphi \otimes L) \geq \operatorname{ind}(\varphi \otimes K)$ and that

$$\lambda_*(\ker(\varphi \otimes K)) \;\sim\; \ker(\varphi \otimes L).$$

(Recall the terminology of §2, Definition 1.)

Definition 1. We call two fields $K \supset k$, $L \supset k$ over k *specialization equivalent*, or just *equivalent*, if there exist places $\lambda: K \to L \cup \infty$ and $\mu: L \to K \cup \infty$ over k. We then write $K \sim_k L$.

[5] $\varphi \otimes K$ is the form φ, considered over K instead of over k. If E is a bilinear space over k corresponding to φ, then $K \otimes_k E$ – as described in §3 – is a bilinear space corresponding to $\varphi \otimes K$.

[6] We do not make a notational distinction between the forms $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ over the different fields occurring here.

The following conclusions can be drawn immediately from our observation:

Remark 1. If $K \sim_k L$, then every form φ over k satisfies:

(1) $\operatorname{ind}(\varphi \otimes K) = \operatorname{ind}(\varphi \otimes L)$.

(2) $\ker(\varphi \otimes K)$ has good reduction with respect to every place λ from K to L and $\lambda_*(\ker(\varphi \otimes K)) \cong \ker(\varphi \otimes L)$. (Note that "$\cong$" holds, not just "$\approx$"!)

From a technical point of view, it is a good idea to treat the following special case of Definition 1 separately:

Definition 2. We call a field extension $K \supset k$ *inessential* if there exists a place $\lambda: K \to k \cup \infty$ over k.

Obviously this just means that K is equivalent to k over k. In this case, Remark 1 becomes:

Remark 2. If K is an inessential extension of k, then every form φ over k satisfies:

(1) $\operatorname{ind}(\varphi \otimes K) = \operatorname{ind}(\varphi)$,

(2) $\ker(\varphi \otimes K) = \ker(\varphi) \otimes K$.

We will see that the forms φ and $\varphi \otimes K$ exhibit the "same" splitting behaviour with respect to an inessential extension K/k in a broader sense (see Scholium 4 below).

Let us return to the general observation above. It gives rise to the following

Problem. Let φ be a form over k, $\dim \varphi = n \geq 2$, and let t be an integer with $1 \leq t \leq \frac{n}{2}$. Does there exist a field extension $K \supset k$ which is "generic with respect to splitting off t hyperbolic planes"?, i.e. with the following properties:

(a) $\operatorname{ind}(\varphi \otimes K) \geq t$.

(b) If L is a field extension of k with $\operatorname{ind}(\varphi \otimes L) \geq t$, then there exists a place from K to L over k.

We first address this problem for the case $t = 1$. As before, let φ be a form over a field k.

Definition 3. An extension field $K \supset k$ is called a *generic zero field of* φ if the following conditions hold:

(a) $\varphi \otimes K$ is isotropic.

(b) For every field extension $k \hookrightarrow L$ with $\varphi \otimes L$ isotropic, there exists a place $\lambda: K \to L \cup \infty$ over k.

Note that if φ is isotropic, then the field k is of course itself a generic zero field of φ.

In the rest of this section, we assume that $\operatorname{char} k \neq 2$. Now φ can also be viewed as a quadratic form,[7] $\varphi(x) := \varphi(x,x)$. We define a field extension $k(\varphi)$ of k, which is a priori suspected to be a generic zero field of φ.

Definition 4. Let $\dim \varphi \geq 3$ or let $\dim \varphi = 2$ and $\varphi \ncong H$. Let $k(\varphi)$ denote the function field of the affine quadric $\varphi(X_1, \ldots, X_n) = 0$ (where $n = \dim \varphi$), i.e. the quotient field of the ring $A := k[X_1, \ldots, X_n]/(\varphi(X_1, \ldots, X_n))$ with indeterminates $X_1, \ldots, X_n$.

Observe that the polynomial $\varphi(X_1, \ldots, X_n)$ is irreducible. To prove this we may suppose that φ is diagonalised, $\varphi = \langle a_1, a_2, \ldots, a_n \rangle$. The polynomial $a_1X_1^2 + a_2X_2^2 + \ldots + a_nX_n^2$ is clearly not a product of two linear forms (since $\operatorname{char} k \neq 2$).

If $\varphi = H$, then $\varphi(X_1, X_2) = X_1X_2$. On formal grounds we then set $k(\varphi) = k(t)$ for an indeterminate t.

Let $x_1, \ldots, x_n$ be the images of the indeterminates $X_1, \ldots, X_n$ in A. Then $\varphi(x_1, \ldots, x_n) = 0$. Hence $\varphi \otimes k(\varphi)$ is isotropic. (This is also true for $\varphi \cong H$.) On top of that we have the following important

Theorem 1. *Let* $\dim \varphi \geq 2$. *Then* $k(\varphi)$ *is a generic zero field of* φ.

Note that the assumption $\dim \varphi \geq 2$ is necessary for the existence of a generic zero field, since forms of dimension ≤ 1 are never isotropic.

Theorem 1 can already be found in [K$_5$]. We will wait until §9 to prove it in the framework of a generic splitting theory of quadratic forms.

It is clear from above that every other generic zero field K of φ over k is equivalent to $k(\varphi)$. It is unknown which bilinear forms possess generic zero fields in characteristic 2.

Given an arbitrary form φ over k, we now construct a tower of fields $(K_r \mid 0 \leq r \leq h)$ together with anisotropic forms φ_r over K_r and numbers $i_r \in \mathbb{N}_0$ as follows: choose K_0 to be any inessential extension of the field k.[8] Let φ_0 be the kernel form of $\varphi \otimes K_0$ and i_0 the Witt index of φ. Then

$$\varphi \otimes K_0 \cong \varphi_0 \perp i_0 \times H.$$

If $\dim \varphi_0 \leq 1$, then Stop! Otherwise choose a generic zero field $K_1 \supset K_0$ of φ_0. Let φ_1 be the kernel form of $\varphi_0 \otimes K_1$ and i_1 the Witt index of $\varphi_0 \otimes K_1$. Then

$$\varphi_0 \otimes K_1 \cong \varphi_1 \perp i_1 \times H.$$

If $\dim \varphi_1 \leq 1$, then Stop! Otherwise choose a generic zero field $K_2 \supset K_1$ of φ_1. Let φ_2 be the kernel form of $\varphi_1 \otimes K_2$ and i_2 the Witt index of $\varphi_1 \otimes K_2$. Then

$$\varphi_1 \otimes K_2 \cong \varphi_2 \perp i_2 \times H,$$

etc.

[7] In keeping with our earlier conventions, it would perhaps be better to write $\varphi(x) = \frac{1}{2}\varphi(x,x)$. However, the factor $\frac{1}{2}$ is not important for now.

[8] In earlier works (especially [K$_5$]) K_0 was always chosen to be k. From a technical point of view it is favourable to allow K_0 to be an inessential extension of k, just as in [KR].

The construction halts after $h \le \left[\frac{\dim \varphi}{2}\right]$ steps with a Witt decomposition

$$\varphi_{h-1} \otimes K_h \cong \varphi_h \perp i_h \times H,$$

$\dim \varphi_h \le 1$.

Definition 5. $(K_r \mid 0 \le r \le h)$ is called a *generic splitting tower of* φ. The number h is called the *height* of φ, denoted $h = h(\varphi)$. The number i_r is called the r-th *higher index* of φ and the form φ_r the r-th *higher kernel form* of φ.

Remark. Obviously φ_r is the kernel form of $\varphi \otimes K_r$ and $\operatorname{ind}(\varphi \otimes K_r) = i_0 + \ldots + i_r$. Note that $h = 0$ iff the form φ "splits", i.e. iff its kernel form is zero or one-dimensional.

We will see that the number h and the sequence $(i_0, \ldots, i_h)$ are independent of the choice of generic splitting tower and also that the forms φ_r are determined uniquely by φ in a precise way. For this the following theorem is useful.

Theorem 2. *Let ψ be a form over a field K. Let $\lambda: K \to L \cup \infty$ be a place, such that ψ has good reduction with respect to λ. Suppose that L (and therefore K) has characteristic $\ne 2$. Then $\lambda_*(\psi)$ is isotropic if and only if λ can be extended to a place $\mu: K(\psi) \to L \cup \infty$.*

If λ is a trivial place, i.e. a field extension, then this theorem says once more that $K(\psi)$ is a generic zero field of ψ (Theorem 1).

One direction of the assertion is trivial, just as for Theorem 1: if λ can be extended to a place $\mu: K(\psi) \to L \cup \infty$, then $\psi \otimes K(\psi)$ has good reduction with respect to μ and $\mu_*(\psi \otimes K(\psi)) \cong \lambda_*(\psi)$. Since $\psi \otimes K(\psi)$ is isotropic, $\lambda_*(\psi)$ is also isotropic.

The other direction will be established in §9. For a shorter and simpler proof, see [K_5] and the books [S], [KS].

Theorem 3 (Corollary of Theorem 2). *Let φ be a form over a field k. Let $(K_r \mid 0 \le r \le h)$ be a generic splitting tower of φ with associated higher kernel forms φ_r and indices i_r. Suppose that φ has good reduction with respect to some place $\gamma: k \to L \cup \infty$. Suppose that L (and hence k) has characteristic $\ne 2$. Finally let $\lambda: K_m \to L \cup \infty$ be a place for some m, $0 \le m \le h$ which extends γ and which <u>cannot</u> be extended to K_{m+1} if $m < h$. Then φ_m has good reduction with respect to λ. The form $\gamma_*(\varphi)$ has kernel form $\lambda_*(\varphi_m)$ and Witt index $i_0 + \ldots + i_m$.*

Proof. There is an isometry

(1) $$\varphi \otimes K_m \cong \varphi_m \perp (i_0 + \ldots + i_m) \times H.$$

The form $\varphi \otimes K_m$ has good reduction with respect to λ and $\lambda_*(\varphi \otimes K_m) = \gamma_*(\varphi)$. Using Theorem 4 and Theorem 3 from §3, (1) implies that φ_m has good reduction with respect to λ and

(2) $$\gamma_*(\varphi) \cong \lambda_*(\varphi_m) \perp (i_0 + \ldots + i_m) \times H$$

(cf. the observation at the beginning of this section). If $\lambda_*(\varphi_m)$ were isotropic, then we would have $m < h$, since $\dim \varphi_h \leq 1$. It would then follow from Theorem 2 that λ can be extended to a place from K_{m+1} to L, contradicting our assumption. Therefore $\lambda_*(\varphi_m)$ is anisotropic and so (2) is the Witt decomposition of $\gamma_*(\varphi)$. □

Note that this theorem implies in particular, that any attempt to successively extend the given place $\lambda: k \to L \cup \infty$ to a "storey" K_m of the generic splitting tower, which is as high as possible, always ends at the same number m.

Theorem 3 gives rise to a number of interesting statements about the splitting behaviour of forms and the extensibility of places.

Scholium 1. Let φ be a form over k and $(K_r \mid 0 \leq r \leq h)$ a generic splitting tower of φ with kernel forms φ_r and indices i_r. Furthermore, let $k \subset L$ be a field extension. If we apply Theorem 3 to the trivial place $\gamma: k \hookrightarrow L$, we get:

(1) Let $\lambda: K_m \to L \cup \infty$ be a place over k $\quad (0 \leq m \leq h)$, which *cannot* be extended to K_{m+1} in case $m < h$. Then φ_m has good reduction with respect to λ and $\lambda_*(\varphi_m)$ is the kernel form of $\varphi \otimes L$. The index of $\varphi \otimes L$ is $i_0 + \ldots + i_m$.

(2) If $\lambda': K_r \to L \cup \infty$ is a place over k, then $r \leq m$ and λ' can be extended to a place $\mu: K_m \to L \cup \infty$.

(3) Given a number t with $1 \leq t \leq \left[\frac{\dim \varphi}{2}\right] = i_0 + \ldots + i_h$, let $m \in \mathbb{N}_0$ be minimal with $t \leq i_0 + \ldots + i_m$. Then K_m is a generic field extension of k with respect to splitting off t hyperbolic planes of φ (in the context of our problem above).

Definition 6. Let φ be a form over k. We call the set of indices $\operatorname{ind}(\varphi \otimes L)$, where L traverses all field extensions of k, the *splitting pattern of* φ, denoted by $\mathrm{SP}(\varphi)$.

This definition also makes sense in characteristic 2, and it is a priori clear that $\mathrm{SP}(\varphi)$ consists of at most $\left[\frac{\dim \varphi}{2}\right] + 1$ elements. Usually the elements of $\mathrm{SP}(\varphi)$ are listed in ascending order. If $\operatorname{char} k \neq 2$ and $(i_r \mid 0 \leq r \leq h)$ is the sequence of higher indices of a generic splitting tower $(K_r \mid 0 \leq r \leq h)$ of φ, then Scholium 1 shows:

$$\mathrm{SP}(\varphi) = (i_0, i_0 + i_1, i_0 + i_1 + i_2, \ldots, i_0 + i_1 + \ldots + i_h).$$

Since the numbers i_r with $r > 0$ are all positive, it is evident that the height h and the higher indices i_r $(0 \leq r \leq h)$ are independent of the choice of generic splitting tower of φ. Obviously is $i_0 + \ldots + i_h = \left[\frac{\dim \varphi}{2}\right]$.

Scholium 2. Let $(K_r \mid 0 \leq r \leq h)$ and $(K'_r \mid 0 \leq r \leq h)$ be two generic splitting towers of the form φ over k. Applying Scholium 1 to the field extensions

$k \subset K_s$ and $k \subset K'_r$, yields: if $\lambda: K_r \to K'_s \cup \infty$ is a place over k, then $r \leq s$ and λ can be extended to a place $\mu: K_s \to K'_s \cup \infty$. The fields K_s and K'_s are equivalent over k. For every place $\mu: K_s \to K'_s \cup \infty$, the kernel form φ_s of $\varphi \otimes K_s$ has good reduction with respect to μ and $\mu_*(\varphi_s)$ is the kernel form φ'_s of $\varphi \otimes K'_s$.

Scholium 3. Let φ be a form over k and $\gamma: k \to L \cup \infty$ a place such that φ has good reduction with respect to γ. Applying Theorem 3 to the places $j \circ \gamma: k \to L \cup \infty$, being the composites of γ and trivial places $j: L \hookrightarrow L'$, yields:

(1) $\mathrm{SP}(\gamma_*(\varphi)) \subset \mathrm{SP}(\varphi)$.

(2) The higher kernel forms of $\gamma_*(\varphi)$ arise from certain higher kernel forms of φ by means of specialization. More precisely: if $(K_r \mid 0 \leq r \leq h)$ is a generic splitting tower of φ and $(K'_s \mid 0 \leq s \leq h')$ is a generic splitting tower of $\gamma_*(\varphi)$, then $h' \leq h$ and for every s with $0 \leq s \leq h'$ we have

$$\mathrm{ind}(\gamma_*(\varphi) \otimes K'_s) = i_0 + \ldots + i_m$$

for some $m \in \{0, \ldots, h\}$. The number m is the biggest integer such that γ can be extended to $\lambda: K_m \to K'_s \cup \infty$. The kernel form φ_m of $\varphi \otimes K_m$ has good reduction with respect to every extension λ of this kind, and $\lambda_*(\varphi_m)$ is the kernel form of $\gamma_*(\varphi) \otimes K'_s$.

(3) If $\rho: K_r \to K'_s \cup \infty$ is a place, which extends $\gamma: k \to L \cup \infty$, then $r \leq m$ and ρ can be further extended to a place from K_m to K'_s.

Scholium 4. Let L/k be an inessential field extension (see Definition 2 above) and φ again a form over k. Let $(L_i \mid 0 \leq i \leq h)$ be a generic splitting tower of $\varphi \otimes L$. It is then immediately clear from Definition 5 that this is also a generic splitting tower of φ. Hence $\mathrm{SP}(\varphi \otimes L) = \mathrm{SP}(\varphi)$, and the higher kernel forms of $\varphi \otimes L$ can also be interpreted as higher kernel forms of φ.

Which strictly increasing sequences $(0, j_1, j_2, \ldots, j_n)$ occur as splitting patterns of anisotropic forms? What do anisotropic forms of given height h look like? These questions are difficult and at the moment the object of intense research. The first efforts towards answering them can be found in [K$_4$], [K$_5$], while more recent ones can be found in [H$_2$], [HR$_1$], [HR$_2$], [Ka] amongst others.

A complete answer is only known in the case $h = 1$. A form $\langle 1, a_1 \rangle \otimes \ldots \otimes \langle 1, a_d \rangle$ $(d \geq 1, a_i \in k)$ is called a *d-fold Pfister form* over k.[9] If τ is a Pfister form of degree d, then the form τ', satisfying $\langle 1 \rangle \perp \tau' = \tau$, is called the *pure part* of τ.

Theorem 4. *An anisotropic form φ over k has height 1 if $\varphi \cong a\tau$ $(\dim \varphi$ even) or $\varphi \cong a\tau'$ $(\dim \varphi$ odd), with $a \in k^*$ and τ an anisotropic Pfister form of degree $d \geq 1$ in the first case and $d \geq 2$ in the second case.*

[9] The form $\langle 1 \rangle$ also counts as a Pfister form, more precisely a 0-fold Pfister form.

Note that therefore $\mathrm{SP}(\varphi) = (0, 2^{d-1})$ when $\dim\varphi$ is even and $\mathrm{SP}(\varphi) = (0, 2^{d-1} - 1)$ when $\dim\varphi$ is odd. Is $k = \mathbb{R}$ for example, then all $d \geq 1$, resp. $d \geq 2$ occur. One can take for instance $\varphi = 2^d \times \langle 1 \rangle$, resp. $\varphi = (2^d - 1) \times \langle 1 \rangle$.

A proof of Theorem 4 can be found in [K_5] and the books [S], [KS]. In §20 and §21 we will prove two theorems for fields of arbitrary characteristic, from which Theorem 4 can be obtained in characteristic $\neq 2$ (§20, Theorem 5, §21, Theorem ■).

Little is still known about forms of height 2, but the known results are interesting and partly deep, see e.g. [K_6], [F], [Ka], [H_1], [H_3].

§5 An Elementary Treatise on Quadratic Modules

We want to construct a specialization theory for *quadratic* forms, similar to the theory in §3 for symmetric bilinear forms. In order to do this we need some definitions and theorems about quadratic modules over rings, and in particular over valuation rings.

Let A be any ring (always commutative, with 1). We recall some fundamental definitions and facts about quadratic forms over A, cf. [MH, Appendix 1].

Definition 1. Let M be an A-module. A *quadratic form on* M is a function $q\colon M \to A$, satisfying the following conditions:

(1) $q(\lambda x) = \lambda^2 q(x)$ for $\lambda \in A$, $x \in M$.

(2) The function $B_q\colon M \times M \to A$, $B_q(x,y) := q(x+y) - q(x) - q(y)$ is a bilinear form on M.

The pair (M,q) is then called a *quadratic module* over A.

Note that the bilinear form B_q is symmetric. Furthermore $B_q(x,x) = 2q(x)$ for all $x \in M$. If 2 is a unit in A, $2 \in A^*$, we can retrieve q from B_q. Also, every symmetric bilinear form B on M comes from a quadratic form q in this case, namely $q(x) = \frac{1}{2}B(x,x)$. So, if $2 \in A^*$, bilinear modules (see §3, Def. 2) and quadratic modules over A are really the same objects.

If $2 \notin A^*$, and 2 is not a zero-divisor in A, we can still identify quadratic forms on an A-module M with special symmetric bilinear forms, namely the forms B with $B(x,x) \in 2A$ for all $x \in M$ ("even" bilinear forms). However, if 2 is a zero-divisor in A, then quadratic and bilinear modules over A are very different objects.

In what follows, primarily *free quadratic modules* will play a rôle, i.e. quadratic modules (M,q) for which the A-module is free and always of finite rank. If M is a free A-module with basis $e_1, \dots, e_n$ and (a_{ij}) a symmetric $(n \times n)$-matrix, then there exists exactly one quadratic form q on M with $q(e_i) = a_{ii}$ and $B_q(e_i, e_j) = a_{ij}$ for $i \neq j$ $\quad (1 \leq i, j \leq n)$, namely

$$q\left(\sum_{i=1}^{n} x_i e_i\right) = \sum_{i=1}^{n} a_{ii} x_i^2 + \sum_{1 \leq i < j \leq n} a_{ij} x_i x_j .$$

We denote this quadratic module (M,q) by a symmetric matrix in square brackets, $(M,q) = [a_{ij}]$, and call $[a_{ij}]$ the *value matrix* of q with respect to the basis $e_1, \dots, e_n$. If (a_{ij}) is a diagonal matrix with coefficients $a_1, \dots, a_n$, then we write $(M, q) = [a_1, \dots, a_n]$.

Definition 2. Let (M_1, q_1) and (M_2, q_2) be quadratic A-modules. The *orthogonal sum* $(M_1, q_1) \perp (M_2, q_2)$ is the quadratic A-module

$$(M_1 \oplus M_2, q_1 \perp q_2),$$

consisting of the direct sum $M_1 \oplus M_2$ and the form

$$(q_1 \perp q_2)(x_1 + x_2) := q_1(x_1) + q_2(x_2)$$

for $x_1 \in M_1$, $x_2 \in M_2$.

If $(M_1, q_1) = [A_1]$ and $(M_2, q_2) = [A_2]$ are free quadratic modules with corresponding symmetric matrices A_1, A_2, then

$$(M_1, q_1) \perp (M_2, q_2) = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}.$$

Now it is also clear how to construct a multiple orthogonal sum $(M_1, q_1) \perp \dots \perp (M_r, q_r)$. In particular we have for elements $a_1, \dots, a_r \in A$ that

$$[a_1] \perp \dots \perp [a_r] = [a_1, \dots, a_r].$$

Let (M, q) be a quadratic $\mathfrak{o}$-module and suppose that M_1 and M_2 are submodules of the $\mathfrak{o}$-module M, then we write $M = M_1 \perp M_2$, when $B_q(M_1, M_2) = 0$ ("internal" orthogonal sum, in contrast with the "external" orthogonal sum of Definition 2). Clearly $(M, q) \cong (M_1, q|M_1) \perp (M_2, q|M_2)$ in this case.

If $\beta: M \times M \to A$ is a – *not necessarily symmetric* – bilinear form, then $q(x) := \beta(x, x)$ is a quadratic form on M. If M is free, one can easily verify that every quadratic form on M is of this form. Furthermore, two bilinear forms β, β' give rise to the same quadratic form q exactly when $\beta - \beta' = \gamma$ is an *alternating* bilinear form: $\gamma(x, x) = 0$ for all $x \in M$, and hence $\gamma(x, y) = -\gamma(y, x)$ for all $x, y \in M$.

Suppose now that (M_1, B_1) is a free bilinear module (cf. §3) and that (M_2, q_2) is a free quadratic module. We equip the free module $M := M_1 \otimes_A M_2$ with a quadratic form q as follows: first we choose a bilinear form β_2 on M_2 with $q_2(x) = \beta_2(x, x)$ for all $x \in M_2$. Next we form the tensor product $\beta := B_1 \otimes \beta_2: M \times M \to A$ of the bilinear forms B_1, β_2. This bilinear form β is characterized by

$$\beta(x_1 \otimes x_2, y_1 \otimes y_2) = B_1(x_1, y_1)\beta_2(x_2, y_2)$$

for $x_1, y_1 \in M_1$ and $x_2, y_2 \in M_2$ (cf. [Bo$_1$, §1, No.9]). Finally we let $q(x) := \beta(x, x)$ for $x \in M_1 \otimes M_2$. This quadratic form q is *independent* of the choice of the bilinear form β_2, since if γ_2 is an alternating bilinear form on M_2, then $B_1 \otimes \gamma_2$ is an alternating bilinear form on M.

Definition 3. We call q the *tensor product* of the symmetric bilinear form B_1 and the quadratic form q_2, denoted $q = B_1 \otimes q_2$, and call the quadratic module (M, q) the tensor product of the bilinear module (M_1, B_1) and the quadratic module (M_2, q_2), $(M, q) = (M_1, B_1) \otimes (M_2, q_2)$.

The quadratic form $q = B_1 \otimes q_2$ is characterized by $B_q = B_1 \otimes B_{q_2}$ and

$$q(x_1 \otimes x_2) = B_1(x_1, x_1)\, q_2(x_2)$$

for $x_1 \in M_1$, $x_2 \in M_2$. For a one-dimensional bilinear module $\langle c \rangle$ and a quadratic free module (M, q) there is a natural isometry $\langle c \rangle \otimes (M, q) \cong (M, cq)$. In particular is for a symmetric $(n \times n)$-matrix A

$$\langle c \rangle \otimes [A] \;\cong\; [cA].$$

Later on we will often denote a quadratic module (M, q) by the single letter M and an (always symmetric) bilinear module (E, B) by the single letter E. The tensor product $E \otimes M$ is clearly additive in both arguments,

$$\begin{aligned} (E_1 \perp E_2) \otimes M \;&\cong\; (E_1 \otimes M) \perp (E_2 \otimes M) \\ E \otimes (M_1 \perp M_2) \;&\cong\; (E \otimes M_1) \perp (E \otimes M_2). \end{aligned}$$

Consequently we have for example $(a_i, b_j \in A)$:

$$\langle a_1, \dots, a_r \rangle \otimes [b_1, \dots, b_s] \;\cong\; [a_1 b_1, a_1 b_2, \dots, a_r b_s].$$

Let $M = (M, q)$ be a free quadratic A-module and $\alpha\colon A \to C$ a ring homomorphism. We associate to M a quadratic C-module $M' = (M', q')$ as follows: The C-module M' is the tensor product $C \otimes_A M$, formed by means of α. Choose a bilinear module $\beta\colon M \times M \to A$ with $q(x) = \beta(x, x)$ for all $x \in M$. Let $\beta'\colon M' \times M' \to C$ be the bilinear form over C associated to β, in other words

$$(*) \qquad \beta'(c \otimes x, d \otimes y) = cd\, \alpha(\beta(x, y))$$

for $x, y \in M$, $c, d \in C$. Let $q'(u) := \beta'(u, u)$ for $u \in M'$. This quadratic form q' is independent of the choice of β', since if γ is an alternating bilinear form on M, then the associated C-bilinear form γ' on M' is again alternating. The form q' can be characterized as follows:

$$q'(c \otimes x) = c^2 \alpha(q(x)), \quad B_{q'}(c \otimes x, d \otimes y) = cd\, \alpha(B_q(x, y)),$$

for $x \in M$, $y \in M$, $c \in C$, $d \in C$.

Definition 4. We say that the quadratic C-module (M', q') arises from $M = (M, q)$ by means of a *base extension* determined by α, and denote M' by $C \otimes_A M$ or by $C \otimes_\alpha M$ or, even more precisely, by $C \otimes_{A,\alpha} M$. We also use the notation $q' = q_C$.

If M is given by a symmetric matrix, $M = [a_{ij}]$ then $C \otimes_A M = [\alpha(a_{ij})]$.

Given a bilinear A-module $M = (M, B)$, we similarly define a bilinear C-module $C \otimes_A M = (C \otimes_A M, B_C)$, where B_C is determined in the obvious way by B, cf. $(*)$ above.

If $M = (M, q)$ is a quadratic module over A and N a submodule of M, we also interpret N as a quadratic module, $N = (N, q|N)$ (quadratic submodule). Furthermore we denote by $N^\perp$ the submodule of M consisting of all elements $x \in M$ with $B_q(x, y) = 0$ for all $y \in N$. Of course, we also interpret $N^\perp$ as a quadratic module. In particular we can look at the quadratic submodule $M^\perp$ of M. If $M^\perp$ is free of finite rank r, then $M^\perp$ has the form $[a_1, \ldots, a_r]$ with elements $a_i \in A$.

Later on we will frequently use the following elementary lemma.

Lemma 1 [MH, p.5]**.** *Let $M = (M, B)$ be a bilinear A-module and let N be a submodule of M. Suppose that the bilinear form B is nondegenerate on N, i.e. the homomorphism $x \mapsto B(x, -)|N$ from N to the dual module $\check{N} = \mathrm{Hom}_A(N, A)$ is an isomorphism. Then $M = N \perp N^\perp$.*

Let $M = (M, q)$ be a quadratic module and let N be a submodule of M such that the bilinear form B_q is nondegenerate on N, then according to the lemma we also have that $M = N \perp N^\perp$.

To finish this section, we briefly examine *free hyperbolic* modules, being quadratic modules of the form $r \times \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ with $r \in \mathbb{N}_0$, in other words direct sums of r copies of the quadratic module $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. {When $r = 0$, the zero module is meant.}

Lemma 2. *Let (M, q) be a quadratic A-module whose associated bilinear form B_q is nondegenerate. Let U be a submodule of M with $q|U = 0$. Suppose that U is free of rank r and that it is a direct summand of the A-module M. Then there exists a submodule $N \supset U$ of M with $N \cong r \times \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ and $M = N \perp N^\perp$.*

Proof. (cf. [Ba, p.13 f]). We saw already in §3 (just after Definition 5) that there exists an exact sequence of A-modules

$$0 \longrightarrow U^\perp \longrightarrow M \xrightarrow{\alpha} \check{U} \longrightarrow 0,$$

with $\check{U} := \mathrm{Hom}_A(U, A)$, where α maps an element $z \in M$ to the linear form $B(z, -)|U$ on U. Since $\check{U}$ is free, this sequence splits. Choose a submodule V of M with $M = U^\perp \oplus V$. Then $\alpha|V \colon V \to \check{U}$ is an isomorphism. The modules U and V are therefore in perfect duality with respect to the pairing $U \times V \to A$, $(x, y) \mapsto B_q(x, y)$.

Now choose a – not necessarily symmetric – bilinear form $\beta \colon V \times V \to A$ with $\beta(x, x) = q(x)$ for $x \in V$. We then have an A-linear map $\varphi \colon V \to U$ such that $\beta(v, x) = B_q(v, \varphi(x))$ for $v \in V$, $x \in V$. Therefore

$$q(x - \varphi(x)) = q(x) - B_q(x, \varphi(x)) = q(x) - \beta(x, x) = 0$$

for every $x \in V$. Let $W := \{x - \varphi(x) \mid x \in V\}$. Since

$$B_q(u, x - \varphi(x)) = B_q(u, x)$$

for all $u \in U$ and $x \in V$, the modules W and U are also in perfect duality with respect to B_q. Furthermore is $q|W = 0$. Choose a basis $e_1, \ldots, e_r$ for U and let $f_1, \ldots, f_r$ be the dual basis of W with respect to B_q, i.e. $B_q(e_i, f_j) = \delta_{ij}$. Then $N := U \oplus W$ can be written as

$$N = \mathop{\perp}_{i=1}^{r} (Ae_i + Af_i) \;\cong\; r \times \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

An appeal to Lemma 1 yields $M = N \perp N^{\perp}$. □

Given a quadratic A-module $M = (M, q)$, we denote $(M, -q)$ by $-M$, as is already our practice for bilinear spaces.

Theorem. *Let $M = (M, q)$ be a quadratic A-module. Suppose that the bilinear form B_q is nondegenerate and that M is free of rank r. Then*

$$M \perp (-M) \;\cong\; r \times \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Proof. Let $(E, \tilde{q}) := (M, q) \perp (M, -q)$. We interpret the module $E = M \oplus M$ as the cartesian product $M \times M$. The diagonal $D := \{(x, x) \mid x \in M\}$ is a free E-submodule of rank r and D is a direct summand of E, for $E = D \oplus (M \times \{0\})$. Furthermore $q|D = 0$. The statement now follows from Lemma 2. □

§6 Quadratic Modules over Valuation Rings

In this section we let $\mathfrak{o}$ be a valuation ring with maximal ideal $\mathfrak{m}$, quotient field K and residue class field $k = \mathfrak{o}/\mathfrak{m}$. The case $\mathfrak{m} = 0$, i.e. $K = k = \mathfrak{o}$ is explicitly allowed. We shall present the theorems about quadratic modules over $\mathfrak{o}$ necessary for our specialization theory, and prove most of them.

Lemma 1. *Let (M, B) be a free bilinear module over $\mathfrak{o}$, and let N be a submodule of M. The submodule $N^\perp$ of M is free with finite basis and is a direct summand of M. Every submodule L of M with $M = N^\perp \oplus L$ is also free with finite basis.*

Proof. $M/N^\perp$ is torsion free and finitely generated, hence free with finite basis. This follows from the fact that every finitely generated ideal of $\mathfrak{o}$ is principal, cf. [CE, VII, §4]. (We used this already in the proof of Lemma 2, §3.) If $\overline{x}_1, \dots, \overline{x}_r$ is a basis of $M/N^\perp$ and $x_1, \dots, x_r$ are the pre-images of the $\overline{x}_i$ in M, then $L_0 := \mathfrak{o}x_1 + \ldots + \mathfrak{o}x_r$ is free with basis $x_1, \dots, x_r$ and $M = N^\perp \oplus L_0$. If $M = N^\perp \oplus L$ as well, then $L \cong M/N^\perp \cong L_0$, so that L is also free with finite basis. Finally, $N^\perp$ is torsion free and finitely generated, hence free with finite basis. □

In particular, if (M, B) is a free bilinear module over $\mathfrak{o}$, there is an orthogonal decomposition $M = M_0 \perp M^\perp$ with $B|M^\perp \times M^\perp = 0$.

Definition 1. $M^\perp$ is called the *radical* of the bilinear module (M, B). If (M, q) is a free quadratic $\mathfrak{o}$-module, then the radical $M^\perp$ of (M, B_q), equipped with the quadratic form $q|M^\perp$, is called the *quasilinear part* of M, and is denoted by $QL(M)$. If $M = M^\perp$, the quadratic module (M, q) is called *quasilinear.*

Definition 2. Let M be a free $\mathfrak{o}$-module with basis $e_1, \dots, e_n$. We call a vector $x \in M$ *primitive in* M, if for the decomposition $x = \lambda_1 e_1 + \ldots + \lambda_n e_n$ ($\lambda_i \in \mathfrak{o}$) the ideal $\sum\limits_1^m \lambda_i \mathfrak{o}$ is equal to $\mathfrak{o}$, i.e. at least one λ_i is a unit.

This property is independent of the choice of basis: x is primitive when $M/\mathfrak{o}x$ is torsion free, hence when $\mathfrak{o}x$ is a direct summand of the module M.

Definition 3. We call a quadratic module (M, q) over $\mathfrak{o}$ *nondegenerate* if it satisfies the following conditions:

(Q0) M is free of finite rank.
(Q1) The bilinear form $\overline{B}_q$, induced by B_q on $M/M^\perp$ in the obvious way, is nondegenerate.
(Q2) $q(x) \in \mathfrak{o}^*$ for every vector x in $M^\perp$, which is primitive in $M^\perp$ (and hence in M).

If instead of (Q2), the following condition is satisfied

(Q2′) $QL(M) = 0$ or $QL(M) \cong [\varepsilon]$ with $\varepsilon \in \mathfrak{o}^*$,

then we call (M, q) *regular*. In the special case $M^\perp = 0$, we call (M, q) *strictly regular*.

Remark. The strictly regular quadratic modules are identical with the "quadratic spaces" over $\mathfrak{o}$ in [K$_4$]. In case $\mathfrak{o} = K$, the term "nondegenerate" has the same meaning as in [K$_4$]. "Strictly regular" is a neologism, constructed with the purpose of avoiding collision with the confusingly many overlapping terms in the literature.

From a technical point of view, Definition 3 is the core definition of this book. The idea behind it is, that the term "nondegenerate" captures a possibly large class of quadratic $\mathfrak{o}$-modules, which work well in the specialization theory (see §7 and following). {We will settle upon the agreement that a quadratic K-module E has "good reduction with respect to $\mathfrak{o}$" if $E \cong K \otimes_{\mathfrak{o}} M$, where M is a nondegenerate quadratic $\mathfrak{o}$-module, see §7, Def.1.} The requirements (Q0) and (Q1) are obvious, but (Q2) and (Q2′) deserve some explanation.

If $\operatorname{char} K \neq 2$, i.e. $2 \neq 0$ in $\mathfrak{o}$, then $q|M^\perp = 0$ and the requirement (Q2) implies that $M^\perp = 0$, hence implies – in conjunction with (Q0) and (Q1) – strict regularity. If $2 \in \mathfrak{o}^*$, then the nondegenerate quadratic $\mathfrak{o}$-modules are the same objects as the nondegenerate bilinear $\mathfrak{o}$-modules in the sense of our earlier definition.

Suppose now that $\operatorname{char} K = 2$. The condition $M^\perp = 0$, in other words strict regularity, is very natural, but too limited for applications. Indeed, if $M^\perp = 0$, then the bilinear module (M, B_q) is nondegenerate and we have $B_q(x, x) = 2q(x) = 0$ for every $x \in M$. This implies that M has even dimension, as is well-known. (To prove this, consider the vector space $K \otimes_{\mathfrak{o}} M$.) So if we insist on using strict regularity, we can only deal with quadratic forms of even dimension.

On the other hand, property (Q2) has an annoying defect: it is not always preserved under a basis extension. If $\mathfrak{o}' \supset \mathfrak{o}$ is another valuation ring, whose maximal ideal $\mathfrak{m}'$ lies over $\mathfrak{m}$, i.e. $\mathfrak{m}' \cap \mathfrak{o} = \mathfrak{m}$, and if M is nondegenerate, then $\mathfrak{o}' \otimes_{\mathfrak{o}} M$ can be degenerate.

However, if M satisfies (Q2′), this clearly cannot happen. Therefore we will have to limit ourselves later (from §9 onwards) in some important cases to regular modules.

The arguments will be clearer however, when we allow arbitrary nondegenerate quadratic modules as long as possible, and a lot of results in this generality are definitely important.

Our use of the terms "regular" and "nondegenerate" finds its justification in the requirements of the specialization theory presented in this book.

Regular quadratic modules over valuation rings and fields (in the sense of Definition 3) are just those quadratic modules, for which the generic splitting theory in particular functions in a "regular way", as we are used to in the absence of characteristic 2 (cf. §4), see §9 below. The nondegenerate quadratic modules are those ones, for which the generic splitting theory still can get somewhere in a sensible way, see §10 and §12 below.

The author is aware of the fact that in other areas (number theory in particular) a different terminology is used. For example, Martin Kneser calls our regular modules "half regular" and our strictly regular modules "regular" in his lecture notes [Kn_3], and has a good reason to do so. In the literature, the word "nondegenerate" is almost exclusively used for the more restrictive class of quadratic modules, which we call "strictly regular", see Bourbaki [Bo_1, §3, No. 4] in particular. In our context, however, it would be a little bit silly to give the label "degenerate" to every quadratic module, which is not strictly regular.

We need some formulas, related to quadratic modules of rank 2.

Lemma 2. *Let $\alpha \in \mathfrak{o}$, $\beta \in \mathfrak{o}$, $\lambda \in \mathfrak{o}^*$. Then*

$$\begin{bmatrix} \alpha & \lambda \\ \lambda & \beta \end{bmatrix} \cong \begin{bmatrix} \alpha & 1 \\ 1 & \lambda^{-2}\beta \end{bmatrix}, \tag{1}$$

and furthermore

$$\langle \lambda \rangle \otimes \begin{bmatrix} \alpha & 1 \\ 1 & \beta \end{bmatrix} \cong \begin{bmatrix} \lambda\alpha & 1 \\ 1 & \lambda^{-1}\beta \end{bmatrix}. \tag{2}$$

Finally, if $\alpha \in \mathfrak{o}^*$, $\beta \in \mathfrak{o}$ then

$$\begin{bmatrix} \alpha & 1 \\ 1 & \beta \end{bmatrix} \cong \langle \alpha \rangle \otimes \begin{bmatrix} 1 & 1 \\ 1 & \alpha\beta \end{bmatrix}. \tag{3}$$

Proof. Let (M, q) be a free quadratic module with basis e, f and associated value matrix $\left[\begin{smallmatrix} \alpha & \lambda \\ \lambda & \beta \end{smallmatrix}\right]$. Then $e, \lambda^{-1}f$ is also a basis of M, whose associated value matrix equals $\left[\begin{smallmatrix} \alpha & 1 \\ 1 & \lambda^{-2}\beta \end{smallmatrix}\right]$. This settles (1). Furthermore $\langle \lambda \rangle \otimes \left[\begin{smallmatrix} \alpha & 1 \\ 1 & \beta \end{smallmatrix}\right]$ is determined by the quadratic module $(M, \lambda q)$, and so $\langle \lambda \rangle \otimes \left[\begin{smallmatrix} \alpha & 1 \\ 1 & \beta \end{smallmatrix}\right] \cong \left[\begin{smallmatrix} \lambda\alpha & \lambda \\ \lambda & \lambda\beta \end{smallmatrix}\right]$. Applying (1) to this, results in (2). Clearly (3) is a special case of (2). □

Theorem 1. *Let $M = (M, q)$ be a nondegenerate quadratic module over $\mathfrak{o}$.*

(*a*) *Then M is an orthogonal sum of modules $\left[\begin{smallmatrix} \alpha & 1 \\ 1 & \beta \end{smallmatrix}\right]$ with $1 - 4\alpha\beta \in \mathfrak{o}^*$ and modules $[\varepsilon]$ with $\varepsilon \in \mathfrak{o}^*$.*

(*b*) *If M is regular and* $\dim M$ *even* (*recall that* $\dim M$ *denotes the rank of the free module M*), *then M is strictly regular and equal to an orthogonal sum of modules $\left[\begin{smallmatrix} \alpha & 1 \\ 1 & \beta \end{smallmatrix}\right]$ with $1 - 4\alpha\beta \in \mathfrak{o}^*$.*

Proof. First suppose that $2 \in \mathfrak{o}^*$. Then (M, B_q) is a bilinear space. Hence (M, B_q) has an orthogonal basis, as is well-known (cf. [MH, I, Cor.3.4]), and so

$$(M, B_q) \;\cong\; \langle \varepsilon_1, \ldots, \varepsilon_n \rangle.$$

Therefore $(M, q) \;\cong\; \left[\frac{\varepsilon_1}{2}, \ldots, \frac{\varepsilon_n}{2}\right]$. For a binary quadratic module $N = [a, b]$ with units $a, b \in \mathfrak{o}^*$, we have

$$[a, b] \;\cong\; \begin{bmatrix} a & 2a \\ 2a & a+b \end{bmatrix},$$

since, if e, f is a basis of the module N with value matrix $\left[\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right]$, then $e, e+f$ is a basis with value matrix $\left[\begin{smallmatrix} a & 2a \\ 2a & a+b \end{smallmatrix}\right]$. Lemma (2), formula (1) then yields

$$[a, b] \;\cong\; \begin{bmatrix} a & 1 \\ 1 & (2a)^{-1}(a+b) \end{bmatrix}.$$

Hence all the assertions of the theorem are clear when $2 \in \mathfrak{o}^*$.

Next suppose that $2 \in \mathfrak{m}$. The quasilinear part $M^\perp$ is of the form $[\varepsilon_1, \ldots, \varepsilon_r]$ with $\varepsilon_i \in \mathfrak{o}^*$. Let N be a module complement of $M^\perp$ in M. Then $(N, q|N)$ is strictly regular. We will show, by induction on $\dim N$, that N is the orthogonal sum of binary modules $\left[\begin{smallmatrix} \alpha & 1 \\ 1 & \beta \end{smallmatrix}\right]$.

If $N = 0$, nothing has to be done. So suppose that $N \neq 0$. We choose a primitive vector e in N. Since the bilinear form $B_q|N \times N$ is nondegenerate, there exists a vector $f \in N$ with $B_q(e, f) = 1$. Let $\alpha = q(e)$, $\beta = q(f)$. The determinant of the matrix $\left(\begin{smallmatrix} 2\alpha & 1 \\ 1 & 2\beta \end{smallmatrix}\right)$ is the unit $1 - 4\alpha\beta$. Hence the module $\mathfrak{o}e + \mathfrak{o}f$ is free with basis e, f. Using Lemma 1 from §5, we get an orthogonal decomposition $N = (\mathfrak{o}e + \mathfrak{o}f) \perp N_1$. Now $\mathfrak{o}e + \mathfrak{o}f \;\cong\; \left[\begin{smallmatrix} \alpha & 1 \\ 1 & \beta \end{smallmatrix}\right]$, and by our induction hypothesis, N_1 is also an orthogonal sum of binary quadratic modules of this form.

This establishes part (a) of the proof. In particular, N has even dimension. If M is regular, then $r \leq 1$. Is in addition $\dim M$ even, then $r = 0$, in other words, M has to be strictly regular. □

If M is a free $\mathfrak{o}$-module, we interpret M as a subset of the K-vecor space $K \otimes_{\mathfrak{o}} M$, by identifying $x \in M$ with $1 \otimes x$. Then $K \otimes_{\mathfrak{o}} M = KM$.

Definition 4. A quadratic $\mathfrak{o}$-module (M, q) is called *isotropic* if there exists a vector $x \neq 0$ in M with $q(x) = 0$. Otherwise (M, q) is called *anisotropic*.

Remark. In this definition of "isotropic", we proceed in a different way, compared to §3, Definition 6, since the bilinear form B_q can be degenerate. This would cause trouble over a local ring instead of over $\mathfrak{o}$.

Lemma 3. *Suppose that (M, q) is a quadratic $\mathfrak{o}$-module, and that M is free of finite rank over $\mathfrak{o}$. The following assertions are equivalent:*

(*a*) *(M, q) is isotropic.*

(*b*) $(K \otimes_{\mathfrak{o}} M, q_K)$ *is isotropic.*

(*c*) *The module M has a direct summand $V \neq 0$ with $q|V = 0$.*

Proof. The implications (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (a) are trivial. (b) $\Rightarrow$ (c): Let $E = K \otimes_{\mathfrak{o}} M = KM$. Now $q = q_K|M$. The K-vector space E contains a subspace $W \neq 0$ with $q_K|W = 0$. Let $V := W \cap M$. The $\mathfrak{o}$-module V is a direct summand of M, since M/V is torsion free and finitely generated, hence free. We have $KV = W$. Therefore certainly $V \neq 0$ and $q|V = 0$. $\square$

Definition 5. Let (M, q) be a quadratic $\mathfrak{o}$-module. A pair of vectors e, f in M is called *hyperbolic* if $q(e) = q(f) = 0$ and $B_q(e, f) = 1$.

Note that according to §5, Lemma 1, we then have that

$$M = (\mathfrak{o}e + \mathfrak{o}f) \perp N \cong \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \perp N$$

where N is a quadratic submodule of M, namely $N = (\mathfrak{o}e + \mathfrak{o}f)^{\perp}$.

Lemma 4. *Let (M, q) be a nondegenerate quadratic $\mathfrak{o}$-module, and let e be a primitive vector in M with $q(e) = 0$. Then e can be completed to a hyperbolic vector pair e, f.*

Proof. We choose a decomposition $M = N \perp M^{\perp}$ and write $e = x + y$ with $x \in N$, $y \in M^{\perp}$. Suppose for the sake of contradiction that the vector x is not primitive in N, hence not primitive in M. Then y is primitive in M. According to condition (Q2), we then have $q(y) \in \mathfrak{o}^*$. Hence also $q(x) = -q(y) \in \mathfrak{o}^*$. Therefore x has to be primitive, contradiction.

Hence x is primitive in N. Since B_q is nondegenerate on N, there exists an element $z \in N$ with $B_q(x, z) = 1$. We also have $B_q(e, z) = 1$. Clearly $f := z - q(z)e$ completes the vector e to a hyperbolic pair. $\square$

Theorem 2 (Cancellation Theorem). *Let M and N be free quadratic $\mathfrak{o}$-modules. Let G be a* strictly regular *quadratic $\mathfrak{o}$-module, and suppose that $G \perp M \cong G \perp N$. Then $M \cong N$.*

For the proof we refer to [K$_3$], in which such a cancellation theorem is proved over local rings. An even more general theorem can be found in Kneser's works [Kn$_2$], [Kn$_3$, Ergänzung zu Kap. I]. A very accessible source for many aspects of the theory of quadratic forms over local rings is the book [Ba] of R. Baeza. Admittedly, Baeza only treats strictly regular quadratic modules, which he calls "quadratic spaces".

In the field case $\mathfrak{o} = K$, Theorem 2 was already demonstrated for $2 = 0$ by Arf [A] and for $2 \neq 0$ – as is well-known – by Witt [W]. A very nice discussion of the cancellation problem over fields, with a view towards the theory over local rings, can again be found in Kneser's work [Kn$_1$].

Theorem 3. *Let M be a nondegenerate quadratic module over $\mathfrak{o}$. There exists a decomposition*

$$(*) \qquad M \cong M_0 \perp r \times \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

with M_0 nondegenerate and anisotropic, $r \geq 0$. The number $r \in \mathbb{N}_0$ and the isometry class of M_0 are uniquely determined by M.

Proof. Lemma 3 and Lemma 4 immediately imply the existence of a decomposition as in $(*)$. Suppose now that

$$M \cong M_0' \perp r' \times \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

is another decomposition. Suppose without loss of generality that $r \leq r'$. Then Theorem 2 implies that

$$M_0 \cong M_0' \perp (r' - r) \times \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Since M_0 is anisotropic, we must have $r = r'$. □

In the field case $\mathfrak{o} = K$, this theorem can again be found back in Arf [A] for $2 = 0$ and in Witt [W] for $2 \neq 0$.

Definition 6. We call a decomposition $(*)$, as in Theorem 3, a *Witt decomposition* of M. We call M_0 the *kernel module* of the nondegenerate quadratic module M and r the *Witt index* of M and write $M_0 = \ker(M)$, $r = \operatorname{ind}(M)$.

Furthermore, we call two nondegenerate quadratic modules M and N over $\mathfrak{o}$ *Witt equivalent*, denoted by $M \sim N$, when $\ker(M) \cong \ker(N)$. We denote the Witt class of M, i.e. the equivalence class of M with respect to $\sim$, by $\{M\}$.

It is now tempting to define an addition of Witt classes $\{M\}$, $\{N\}$ of two nondegenerate quadratic modules M, N over $\mathfrak{o}$ in the same way as we did this for nondegenerate bilinear modules over fields in §2, namely $\{M\} + \{N\} := \{M \perp N\}$.

Although the orthogonal sum of two nondegenerate quadratic modules could be degenerate, the addition makes sense if one of the modules M, N is strictly regular, as we will show now.

Proposition. *Let M, M' be strictly regular quadratic $\mathfrak{o}$-modules with $M \sim M'$, and let N, N' be nondegenerate quadratic $\mathfrak{o}$-modules with $N \sim N'$. Then the modules $M \perp N$ and $M' \perp N'$ are nondegenerate and $M \perp N \sim M' \perp N'$.*

Proof. It follows immediately from Definition 3 that $M \perp N$ and $M' \perp N'$ are nondegenerate. Suppose without loss of generality that $\dim M' \geq \dim M$. Then $M' \cong M \perp s \times \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ for an $s \geq 0$. Therefore $M' \perp N \cong (M \perp N) \perp s \times \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, and so $M' \perp N \sim M \perp N$. A similar argument shows that $M' \perp N \sim M' \perp N'$. □

Definition 7a. We denote the set of Witt classes of nondegenerate quadratic $\mathfrak{o}$-modules by $\widetilde{Wq}(\mathfrak{o})$. We denote the subset of Witt classes of regular, resp. strictly regular quadratic $\mathfrak{o}$-modules by $Wqr(\mathfrak{o})$, resp. $Wq(\mathfrak{o})$.

Because of the proposition above, we now have a well-defined "addition" of classes $\{M\} \in Wq(\mathfrak{o})$ with classes $\{N\} \in \widetilde{Wq}(\mathfrak{o})$,

$$\{M\} + \{N\} := \{M \perp N\}.$$

The zero module $M = 0$ gives rise to a neutral element for this addition, $\{0\} + \{N\} = \{N\}$, $\{M\} + \{0\} = \{M\}$. If M and N are strictly regular, then $\{M\} + \{N\} = \{N\} + \{M\} = \{M \perp N\} \in Wq(\mathfrak{o})$. Therefore $Wq(\mathfrak{o})$ is an abelian semigroup with respect to this sum. The theorem at the end of §5 shows that $Wq(\mathfrak{o})$ is in fact an abelian group, since if (M, q) is a strictly regular quadratic $\mathfrak{o}$-module, then $\{(M, q)\} + \{(M, -q)\} = 0$.

Definition 7b. We call the abelian group $Wq(\mathfrak{o})$ the *quadratic Witt group* of $\mathfrak{o}$, the set $\widetilde{Wq}(\mathfrak{o})$ the *quadratic Witt set* of $\mathfrak{o}$ and $Wqr(\mathfrak{o})$ the *regular quadratic Witt set* of $\mathfrak{o}$.

The group $Wq(\mathfrak{o})$ acts on the set $\widetilde{Wq}(\mathfrak{o})$ by means of the addition, introduced above. One can show easily that the isomorphism classes of nondegenerate quasilinear quadratic $\mathfrak{o}$-modules (see Definition 1) form a system of representatives of the orbits of this action. $Wqr(\mathfrak{o})$ is a union of orbits.

If $\mathfrak{m} = 0$, hence $\mathfrak{o} = K$, then Witt classes of *degenerate* quadratic modules can be defined without difficulties. We will explain this next.

Definition 8. Let $M = (M, q)$ be a finite dimensional quadratic module over K. The *defect* $\delta(M)$ of M is the set of all $x \in M^{\perp}$ with $q(x) = 0$.

The defect $\delta(M)$ is clearly a subspace of M, and M is nondegenerate if and only if $\delta(M) = 0$. The form q induces a quadratic form $\overline{q}\colon M/\delta(M) \to K$ on the vector space $M/\delta(M)$ in the obvious way: $\overline{q}(\overline{x}) := q(x)$ for $x \in M$, where $\overline{x}$ is the image of x in $M/\delta(M)$. The quadratic K-module $(M/\delta(M), \overline{q})$ is clearly nondegenerate.

Definition 9. If $\delta(M) = 0$, i.e. if M is nondegenerate, then we call $M = (M, q)$ a *quadratic space* over K. In general we denote the quadratic space $(M/\delta(M), \overline{q})$ over K by $\hat{M}$, and call $\hat{M}$ the *quadratic space associated to* M.

We have

$$M \cong \hat{M} \perp \delta(M) \cong \hat{M} \perp s \times [0]$$

with $s := \dim \delta(M)$. Furthermore, given a quadratic space E over K, we obviously have

$$\delta(E \perp M) = \delta(M) \quad \text{and} \quad (E \perp M)^{\wedge} \cong E \perp \hat{M}.$$

Now it is clear, that in the field case, Theorem 3 can be expanded as follows:

Theorem 3′. *Let M be a finite dimensional quadratic K-module. There exists a decomposition*

$$M \;\cong\; M_0 \perp r \times \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

with $\hat{M}_0$ anisotropic and $r \geq 0$. The number $r \in \mathbb{N}_0$ and the isometry class of M_0 are uniquely determined by M.

Definition 10. We call M_0 the *kernel module* of M and r the *Witt index* of M, and write $M_0 = \ker M$, $r = \operatorname{ind}(M)$. Just as before, we call two finite dimensional quadratic K-modules M and N *Witt equivalent*, and write $M \sim N$, when $\ker(M) \;\cong\; \ker(N)$. We denote the Witt equivalence class of M by $\{M\}$.

Remark. It may seem more natural to call M and N equivalent when their "kernel *spaces*" $(\ker M)^\wedge$, $(\ker N)^\wedge$ are isometric. Our results in §7 and §11 are a bit stronger if we use the finer equivalence defined above.

We denote the set of Witt classes $\{M\}$ of finite dimensional quadratic K-modules M by $\widehat{Wq}(K)$ and call this the *defective quadratic Witt set* of K. The abelian group $Wq(K)$ acts on $\widehat{Wq}(K)$ in the usual way: $\{E\} + \{M\} := \{E \perp M\}$ for $\{E\} \in Wq(K)$, $\{M\} \in \widehat{Wq}(K)$. It even makes sense to add two arbitrary Witt classes together:

$$\{M\} + \{N\} := \{M \perp N\}.$$

The result is independent of the choice of representatives M, N, as can be shown by an argument, analogous to the proof of the proposition, preceding Definition 7a. Therefore $\widehat{Wq}(K)$ is an abelian semigroup, having $Wq(K)$ as a subgroup.

Look out! If $\operatorname{char} K = 2$, the semigroup $\widehat{Wq}(K)$ does not satisfy a cancellation rule. Is for example $a \in K^*$, then $[a] \perp [a] \;\cong\; [0] \perp [a]$, but it is *not* so that $[a] \sim [0]$.

To a field homomorphism $\alpha\colon K \to K'$ (i.e. a field extension), we can associate a well-defined map $\alpha_*\colon \widehat{Wq}(K) \to \widehat{Wq}(K')$, which sends the Witt class $\{M\}$ to the class $\{K' \otimes_\alpha M\}$. This map is a semigroup homomorphism. By restricting α_*, we can find maps from $Wq(K)$ to $Wq(K')$ and from $Wqr(K)$ to $Wqr(K')$, but (if $\operatorname{char} K = 2$) in general *no* map from $\widetilde{Wq}(K)$ to $\widetilde{Wq}(K')$. This is the reason why we sometimes will extend the Witt set $\widetilde{Wq}(K)$ to the semigroup $\widehat{Wq}(K)$.

Let us now return to quadratic modules over arbitrary valuation rings.

At the end of §5 we introduced free hyperbolic modules. From now on, we call those modules simply "hyperbolic". { In this way, we stay in harmony with the terminology, used in the literature, for valuation rings (or local rings in general), cf. [B$_1$, V §2], [B$_2$].} Hence

Definition 11. A quadratic $\mathfrak{o}$-module M is called *hyperbolic* if

$$M \cong r \times \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

for some $r \in \mathbb{N}_0$.

So, if M is not the zero module, this means that M contains a basis $e_1, f_1, e_2, f_2, \ldots, e_r, f_r$, where e_i, f_i are hyperbolic vector pairs, such that $\mathfrak{o}e_i + \mathfrak{o}f_i$ is orthogonal with $\mathfrak{o}e_j + \mathfrak{o}f_j$ for $i \neq j$. Clearly a hyperbolic module is strictly regular.

Lemma 5. *Let $M = (M, q)$ be a strictly regular quadratic $\mathfrak{o}$-module. M is hyperbolic if and only if M contains a submodule V with $q|V = 0$ and $V = V^\perp$.*

Proof. If M is hyperbolic and $e_1, f_1, \ldots, e_r, f_r$ is a basis, consisting of pairwise orthogonal hyperbolic vector pairs, then the module $V\colon= \mathfrak{o}e_1 + \mathfrak{o}e_2 + \ldots + \mathfrak{o}e_r$ clearly has the properties $q|V = 0$ and $V = V^\perp$. Suppose now that M is strictly regular and that V is a submodule satisfying these properties. Since $M/V^\perp = M/V$ is torsion free and finitely generated, it is free. Therefore V is a direct summand of the $\mathfrak{o}$-module M. V is torsion free and finitely generated as well, and so also free. The form B_q gives rise to an exact sequence

$$0 \longrightarrow V^\perp \longrightarrow M \longrightarrow \overset{\vee}{V} \longrightarrow 0$$

(cf. the proof of §5, Lemma 2), from which we deduce that $\dim M = \dim V + \dim V^\perp = 2 \dim V$.

According to the theorem at the end of §5, there is an orthogonal decomposition $M = N \perp N^\perp$ with $V \subset N$ and N hyperbolic. Using V to form $V^\perp$ in N, we see that $V = V^\perp$. We conclude that $\dim N = 2 \dim V$ as well, so that $N = M$. □

Theorem 4. *Let M be a strictly regular quadratic $\mathfrak{o}$-module with $K \otimes_{\mathfrak{o}} M$ hyperbolic. Then M itself is hyperbolic.*

Proof. We apply the criterion, given by Lemma 5, two times. As before we interpret M as an $\mathfrak{o}$-submodule of $E\colon= K \otimes_{\mathfrak{o}} M$, i.e. $M \subset E$, $E = KM$. Now E contains a subspace W with $q|W = 0$, $W^\perp = W$. The intersection $V\colon= W \cap M$ is an $\mathfrak{o}$-submodule of M with $q|V = 0$ and $V^\perp = V$. (Here $V^\perp$ is considered in M.) □

Definition 12. The valuation ring $\mathfrak{o}$ is called *quadratically henselian* if for every $\gamma \in \mathfrak{m}$, there exists a $\lambda \in \mathfrak{o}$ such that $\lambda^2 - \lambda = \gamma$.

Remark. One can convince oneself that this means that "Hensel's Lemma" is satisfied by monic polynomials of degree 2. Every henselian valuation ring is of course quadratically henselian (cf. [Ri, Chap. F], [E, §16]). In Definition 12 we isolated that part of the property "henselian", which is important for the theory of quadratic forms. Note that, if $\mathfrak{o}$ is quadratically henselian, then every $\varepsilon \in 1+4\mathfrak{m}$ is a square in $\mathfrak{o}$, since $\varepsilon = 1+4\gamma$ and $\gamma = \lambda^2 - \lambda$ imply that $\varepsilon = (1-2\gamma)^2$.

Lemma 6. *Let $\mathfrak{o}$ be quadratically henselian. Let $\alpha, \beta \in K$ and $\alpha\beta \in \mathfrak{m}$. Then we have that over K:*

$$\begin{bmatrix} \alpha & 1 \\ 1 & \beta \end{bmatrix} \cong \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Proof. According to Lemma 2, formula (3), we have

$$\begin{bmatrix} \alpha & 1 \\ 1 & \beta \end{bmatrix} \cong \langle \alpha \rangle \otimes \begin{bmatrix} 1 & 1 \\ 1 & \alpha\beta \end{bmatrix}.$$

There exists a $\lambda \in \mathfrak{o}$ with $\lambda^2 + \lambda + \alpha\beta = 0$. If e, f is a basis of the module $\left[\begin{smallmatrix} 1 & 1 \\ 1 & \alpha\beta \end{smallmatrix}\right]$, having the indicated value matrix, then $q(\lambda e + f) = \lambda^2 + \lambda + \alpha\beta = 0$. Therefore $\left[\begin{smallmatrix} \alpha & 1 \\ 1 & \beta \end{smallmatrix}\right]$ is isotropic. Applying Lemma 4, with $\mathfrak{o} = K$, gives $\left[\begin{smallmatrix} \alpha & 1 \\ 1 & \beta \end{smallmatrix}\right] \cong \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. □

Theorem 5. *Let $\mathfrak{o}$ be quadratically henselian. Let (M, q) be a nondegenerate anisotropic quadratic module over $\mathfrak{o}$. Then $q(e)$ is a unit for every vector $e \in M$ which is primitive in M.*

Proof. We choose a decomposition $M = N \perp M^\perp$. Let $e \in M$ be primitive. Suppose for the sake of contradiction that $q(e) \in \mathfrak{m}$. We write $e = x + y$ with $x \in N$, $y \in M^\perp$, and have $q(e) = q(x) + q(y)$.

Case 1: x is primitive in M, hence also in N. Then there exists an $f \in N$ with $B_q(x, f) = 1$, since B_q is nondegenerate on N. We have $B_q(e, f) = 1$. Therefore

$$\mathfrak{o}e + \mathfrak{o}f \cong \begin{bmatrix} \alpha & 1 \\ 1 & \beta \end{bmatrix}$$

with $\alpha = q(e) \in \mathfrak{m}$, $\beta = q(f) \in \mathfrak{o}$. According to Lemma 6, $\mathfrak{o}e + \mathfrak{o}f$ is isotropic. This is a contradiction, since M is anisotropic.

Case 2: x is not primitive in M. Now $x = \lambda x_0$ with x_0 primitive in M, $\lambda \in \mathfrak{m}$. The vector y has to be primitive in M. Since q is nondegenerate, it follows that $q(y) \in \mathfrak{o}^*$. We have

$$q(e) = \lambda^2 q(x_0) + q(y) \in \mathfrak{o}^*,$$

contradicting the assumption that $q(e) \in \mathfrak{m}$.

We conclude that $q(e) \in \mathfrak{o}^*$. □

Later on, we will use Lemma 6 and Theorem 5 for a nondegenerate quadratic module M over an arbitrary valuation ring $\mathfrak{o}$, by going from M to the quadratic module $M^h := \mathfrak{o}^h \otimes_{\mathfrak{o}} M$ over the henselisation $\mathfrak{o}^h$ of $\mathfrak{o}$ (see [Ri, Chap.F], [E, §17]). {Remark: One can also form a "quadratic henselisation" $\mathfrak{o}^{qh}$ in the obvious way. It would suffice to take $\mathfrak{o}^{qh}$ instead of $\mathfrak{o}^h$.} In order to do so, the following lemma is important.

Lemma 7. *Let M be a nondegenerate quadratic module over $\mathfrak{o}$. Then the quadratic module M^h over $\mathfrak{o}^h$ is also nondegenerate.*

Proof. The properties (Q0) and (Q1) from Definition 3 stay valid for every basis extension. Hence we suppose that M satisfies (Q0) and (Q1), so that we only have to consider (Q2). We have $(M^h)^\perp = (\mathfrak{o}^h \otimes_{\mathfrak{o}} M)^\perp = \mathfrak{o}^h \otimes_{\mathfrak{o}} M^\perp = (M^\perp)^h$.

The property (Q2) for M says that the quasilinear space $k \otimes_{\mathfrak{o}} M^\perp = k \otimes_{\mathfrak{o}} QL(M)$ is anisotropic over k. To start with, we can identify $k \otimes_{\mathfrak{o}} M$ with the quadratic module $M/\mathfrak{m}M$ over $k = \mathfrak{o}/\mathfrak{m}$, whose quadratic form $\overline{q}\colon M/\mathfrak{m}M \to k$ is induced by the quadratic form $q\colon M \to \mathfrak{o}$ in the obvious way. In the same way we then have $k \otimes_{\mathfrak{o}} M^\perp = M^\perp/\mathfrak{m}M^\perp$. A vector $x \in M^\perp$ is primitive if and only if its image $\overline{x} \in M^\perp/\mathfrak{m}M^\perp$ is nonzero. (Q2) implies then that $\overline{q}(\overline{x}) \neq 0$.

Let $\mathfrak{m}^h$ denote the maximal ideal of $\mathfrak{o}^h$. It is well-known that $\mathfrak{m}^h \cap \mathfrak{o} = \mathfrak{m}$ and that the residue class field $\mathfrak{o}^h/\mathfrak{m}^h$ is canonically isomorphic to $\mathfrak{o}/\mathfrak{m} = k$. We identify $\mathfrak{o}^h/\mathfrak{m}^h$ with k. Then $k \otimes_{\mathfrak{o}^h} M^h = k \otimes_{\mathfrak{o}} M$ and also $k \otimes_{\mathfrak{o}^h} QL(M^h) = k \otimes_{\mathfrak{o}} QL(M)$. Therefore (Q2) is satisfied for M if and only if (Q2) is satisfied for M^h. $\square$

§7 Weak Specialization

In this section $\lambda: K \to L \cup \infty$ is a place, $\mathfrak{o} = \mathfrak{o}_\lambda$ is the valuation ring associated to λ, with maximal ideal $\mathfrak{m}$ and residue class field $k = \mathfrak{o}/\mathfrak{m}$. Given a quadratic module $E = (E, q)$ over K, we want to associate a quadratic module $\lambda_*(E)$ over L to it, insofar this is possible in a meaningful way.

In order to do this, we limit ourselves to the case where E is nondegenerate (see §6, Definition 3, but with K instead of $\mathfrak{o}$). This is not a real limitation, however. Since K is a field, every quadratic module $E = (E, q)$ of finite dimension over K has the form $E \cong F \perp [0, \dots, 0] = F \perp s \times [0]$ with F nondegenerate (see §6). So, when we have associated a specialization $\lambda_*(F)$ to F in a satisfactory way, it is natural to put $\lambda_*(E) = \lambda_*(F) \perp s \times [0]$. We stay with nondegenerate quadratic K-modules, however, and now call them **quadratic spaces** over K, as agreed in §6, Definition 9.

Definition 1. Let E be a quadratic space over K. We say that E has *good reduction with respect to* λ when $E \cong K \otimes_{\mathfrak{o}} M$, where M is a nondegenerate quadratic $\mathfrak{o}$-module.

This is the "good case". We would like to put $\lambda_*(E) := L \otimes_\lambda M$, where the tensor product is formed by means of the homomorphism $\lambda|\mathfrak{o}: \mathfrak{o} \to M$, see §5, Definition 4. This tensor product can more easily be described as follows: First we go from the quadratic $\mathfrak{o}$-module $M = (M, q)$ to the quadratic space $M/\mathfrak{m}M = (M/\mathfrak{m}M, \overline{q})$ over $k = \mathfrak{o}/\mathfrak{m}$, where $\overline{q}$ is obtained from $q: M \to \mathfrak{o}$ in the obvious way. Next we extend scalars by means of the field extension $\overline{\lambda}: k \hookrightarrow L$, determined by λ, thus obtaining $\lambda_*(E) = L \otimes_k (M/\mathfrak{m}M)$.

Is this meaningful? With a similar argument as towards the end of §3 (second proof of Theorem 2), we now show that the answer is affirmative when E is *strictly regular* (which means that $E^\perp = 0$, cf. §6, Definition 3).

Theorem 1. *Let E be strictly regular and let M, M' be nondegenerate quadratic $\mathfrak{o}$-modules with $E \cong K \otimes_{\mathfrak{o}} M \cong K \otimes_{\mathfrak{o}} M'$. Then $M \cong M'$, and hence $M/\mathfrak{m}M \cong M'/\mathfrak{m}M'$.*

Proof. Clearly $M^\perp = 0$ and $(M')^\perp = 0$. Hence M and M' are also strictly regular. By the theorem at the end of §5, the space $K \otimes_{\mathfrak{o}} [M' \perp (-M)] \cong E \perp (-E)$ is hyperbolic. So $M' \perp (-M)$ is itself hyperbolic, by §6, Theorem 4. $M \perp (-M)$ is also hyperbolic, again by the theorem at the end of §5. The modules $M' \perp (-M)$ and $M \perp (-M)$ have the same rank. Hence

$M' \perp (-M) \cong M \perp (-M)$, and so $M' \cong M$ by the Cancellation Theorem (§6, Theorem 2). $\square$

The problem remains to see that $\lambda_*(E)$ does not depend on the choice of the module M, when E has good reduction with respect to λ, but only is non-degenerate. Furthermore, in case of bad reduction (i.e. not good reduction), we would nevertheless like to associate a Witt class $\lambda_W(\{E\})$ over L to E in a meaningful way, just as for bilinear forms in §3 ("weak specialization"). Neither problem can be dealt with as in §3, in the first place because the Witt classes do not constitute an additive group this time. Thus we have to seek out a new path.

Lemma 1. *Let* $\mathfrak{o}$ *be quadratically henselian* (*see* §6, *Definition* 12) *and let* $V = (V, q)$ *be an anisotropic quadratic module over* K.

(*a*) *The sets*

$$\mu(V) := \{x \in V \mid q(x) \in \mathfrak{o}\} \quad \textit{and} \quad \mu_+(V) := \{x \in V \mid q(x) \in \mathfrak{m}\}$$

are $\mathfrak{o}$*-submodules of* V.

(*b*) *For any* $x \in \mu(V)$ *and* $y \in \mu_+(V)$, *we have* $q(x+y) - q(x) \in \mathfrak{m}$ *and* $B_q(x, y) \in \mathfrak{m}$.

Proof. (a) Let $x \in \mu(V)$ (resp. $x \in \mu_+(V)$) and $\lambda \in \mathfrak{o}$, then clearly $\lambda x \in \mu(V)$ (resp. $\mu_+(V)$). So we only have to show that for any $x, y \in \mu(V)$ (resp. $\mu_+(V)$), we have $x + y \in \mu(V)$ (resp. $\mu_+(V)$). Let $B := B_q$.

Let $x \in \mu(V)$, $y \in \mu(V)$. Suppose for the sake of contradiction that $x + y \notin \mu(V)$, i.e. $q(x + y) = q(x) + q(y) + B(x, y) \notin \mathfrak{o}$. Since $q(x) \in \mathfrak{o}$, $q(y) \in \mathfrak{o}$, we must have that $B(x, y) \notin \mathfrak{o}$. Hence $\lambda := B(x, y)^{-1} \in \mathfrak{m}$. The vectors $x, \lambda y$ give rise to the value matrix $\left[\begin{smallmatrix} q(x) & 1 \\ 1 & \lambda^2 q(y) \end{smallmatrix}\right]$. According to §6, Lemma 6, the space $Kx + Ky = Kx + K\lambda y$ is isotropic. Contradiction, since V is anisotropic. Therefore $x + y \in \mu(V)$.

Next, let $x \in \mu_+(V)$, $y \in \mu_+(V)$. Suppose again for the sake of contradiction that $x + y \notin \mu_+(V)$. We just showed that $q(x + y) \in \mathfrak{o}$. By our assumption $q(x + y) \notin \mathfrak{m}$, so that $q(x + y) \in \mathfrak{o}^*$. Since $q(x) \in \mathfrak{m}$, $q(y) \in \mathfrak{m}$, we have $B(x, y) \in \mathfrak{o}^*$. Write $B(x, y) = \lambda^{-1}$ with $\lambda \in \mathfrak{o}^*$. Again, the vectors $x, \lambda y$ give rise to the value matrix $\left[\begin{smallmatrix} q(x) & 1 \\ 1 & \lambda^2 q(y) \end{smallmatrix}\right]$. Just as before, the space $Kx + Ky$ is isotropic, according to §6, Lemma 6. Contradiction! We conclude that $x + y \in \mu_+(V)$.

(b) Suppose for the sake of contradiction that there exist vectors $x \in \mu(V)$, $y \in \mu_+(V)$ with $q(x+y) - q(x) \notin \mathfrak{m}$. We showed above that $q(x+y) \in \mathfrak{o}$. Hence $q(x + y) - q(x) = q(y) + B(x, y) \in \mathfrak{o}$. By assumption, this element doesn't live in $\mathfrak{m}$ and is thus a unit. Since $q(y) \in \mathfrak{m}$, we also have $B(x, y) \in \mathfrak{o}^*$. As before, we write $\lambda = B(x, y)^{-1}$ and observe that $Kx + Ky$ is isotropic, giving a contradiction. Therefore $q(x+y) - q(x) \in \mathfrak{m}$ and $B(x, y) = q(x+y) - q(x) - q(y) \in \mathfrak{m}$ as well. $\square$

We remain in the situation of Lemma 1. For $x \in \mu(V)$, $\lambda \in \mathfrak{m}$, we have $\lambda x \in \mu_+(V)$. Therefore

$$\rho(V) := \mu(V)/\mu_+(V)$$

is a k-vector space in a natural sense ($k = \mathfrak{o}/\mathfrak{m}$). We define a function $\overline{q}: \rho(V) \to k$ as follows:

$$\overline{q}(\overline{x}) := \overline{q(x)} \qquad (x \in \mu(V)),$$

where $\overline{x}$ denotes the image of $x \in \mu(V)$ in $\rho(V)$ and $\overline{a}$ denotes the image of $a \in \mathfrak{o}$ in k. Lemma 1 tells us that the map $\overline{q}$ is well-defined.

For $x \in \mu(V)$, $a \in \mathfrak{o}$, we have $\overline{q}(\overline{a}\,\overline{x}) = \overline{q}(\overline{ax}) = \overline{q(ax)} = \overline{a}^2\overline{q(x)} = \overline{a}^2\overline{q}(\overline{x})$. According to Lemma 1, the bilinear form $B := B_q$ has values in $\mathfrak{o}$ on $\mu(V) \times \mu(V)$ and values in $\mathfrak{m}$ on $\mu(V) \times \mu_+(V)$. Therefore, it induces on $\rho(V)$ a symmetric bilinear form $\overline{B}$ over k with $\overline{B}(\overline{x}, \overline{y}) = \overline{B(x, y)}$ for $x, y \in \mu(V)$. A very simple calculation now shows that

$$\overline{q}(\overline{x} + \overline{y}) - \overline{q}(\overline{x}) - \overline{q}(\overline{y}) = \overline{B}(\overline{x}, \overline{y})$$

for $x, y \in \mu(V)$. This furnishes the proof that $\overline{q}$ is a quadratic form on the k-vector space $\rho(V)$ with $B_{\overline{q}} = \overline{B}$. The quadratic k-module $(\rho(V), \overline{q})$ is clearly anisotropic.

Definition 2. ($\mathfrak{o}$ quadratically henselian.) We call the quadratic k-module $\rho(V) = (\rho(V), \overline{q})$ the *reduction* of the anisotropic quadratic K-module V with respect to the valuation ring $\mathfrak{o}$.

In order to associate to a quadratic space E over K, by means of the place $\lambda: K \to L \cup \infty$, a Witt class $\lambda_W\{E\}$ over L, the following path presents itself now: Let $\mathfrak{o}^h$ be the henselization of the valuation ring $\mathfrak{o} = \mathfrak{o}_\lambda$, $\mathfrak{m}^h$ the maximal ideal of $\mathfrak{o}^h$ and K^h the quotient field of $\mathfrak{o}^h$. The residue class field $\mathfrak{o}^h/\mathfrak{m}^h$ is canonically isomorphic to $k = \mathfrak{o}/\mathfrak{m}$ and will be identified with k. Let

$$\lambda_W\{E\} = \{L \otimes_{\overline{\lambda}} \rho(\mathrm{Ker}\ (K^h \otimes E))\},$$

where, as before, $\overline{\lambda}: k \hookrightarrow L$ is the field embedding, determined by λ. The space over L on the righthand side can then be considered to be a "weak specialization" of E with respect to λ.

All good and well, if only we knew whether the vector space

$$\rho(\mathrm{Ker}\ (K^h \otimes E))$$

has finite dimension! To guarantee this, we have to confine the class of allowed quadratic modules E.

In the following, $v: K \to \Gamma \cup \infty$ is a surjective valuation, associated to the valuation ring $\mathfrak{o}$, thus with Γ the value group of v, $\Gamma \cong K^*/\mathfrak{o}^*$. We use additive notation for Γ (so $v(xy) = v(x) + v(y)$ for $x, y \in K$).

Already in §3, we agreed to view the square class group $Q(\mathfrak{o})$ as a subgroup of $Q(K)$, and also to regard the elements of $Q(K)$ as one-dimensional bilinear spaces[10] over K.

Now we choose a complement Σ of $Q(\mathfrak{o})$ in $Q(K)$, in other words, a subgroup Σ of $Q(K)$ with $Q(K) = Q(\mathfrak{o}) \times \Sigma$. This is possible, since $Q(K)$ is elementary abelian of exponent 2, i.e. a vector space over the field with 2 elements. Further, we choose, for every square class $\sigma \in \Sigma$, an element $s \in \mathfrak{o}$ with $\sigma = \langle s \rangle$. For $\sigma = 1$ we choose the representative $s = 1$. Let S be the set of all elements s. For every $a \in K^*$ there exists exactly one $s \in S$ and elements $\varepsilon \in \mathfrak{o}^*$, $b \in K^*$ with $a = s\varepsilon b^2$. Since $K^*/\mathfrak{o}^* \;\cong\; \Gamma$, it is clear that S (resp. Σ) is a system of representatives of $\Gamma/2\Gamma$ in K^* (resp. $Q(K)$) for the homomorphism from K^* to $\Gamma/2\Gamma$ (resp. from $Q(K)$ to $\Gamma/2\Gamma$), determined by $v\colon K^* \to \Gamma$.

Definition 3. A quadratic space E over K is called *obedient with respect to* λ (or *obedient with respect to* $\mathfrak{o}$) if there exists a decomposition

$$E \;\cong\; \mathop{\perp}_{s\in S} \langle s \rangle \otimes (K \otimes_{\mathfrak{o}} M_s), \qquad (*)$$

where M_s is a nondegenerate quadratic $\mathfrak{o}$-module. (Of course $M_s \neq 0$ for finitely many s only.)

Clearly this property does not depend on the choice of system of representatives S.

Remarks.

(1) Obedience is much weaker than the property "good reduction" (Definition 1).

(2) Let E and F be quadratic spaces over K, obedient with respect to $\mathfrak{o}$. If at least one of them is strictly regular, then $E \perp F$ is obedient.

(3) If $\operatorname{char} K \neq 2$, then every quadratic space over K is strictly regular. Hence the orthogonal sum of two obedient quadratic spaces over K is again obedient.

(4) If $2 \in \mathfrak{o}^*$, i.e. $\operatorname{char} L \neq 2$, then *every* quadratic space E over K is obedient with respect to λ, since K also has characteristic $\neq 2$ in this case. E has a decomposition in one-dimensional spaces, which are clearly obedient.

(5) If $\operatorname{char} K = 0$, but $\operatorname{char} L = 2$, then every quadratic space of odd dimension over K is *disobedient* (= not obedient) with respect to λ. To see this, let E be an obedient quadratic space with orthogonal decomposition $(*)$, as in Definition 3. Then the space $K \otimes_{\mathfrak{o}} M_i$ has quasilinear part $(K \otimes_{\mathfrak{o}} M_i)^{\perp} = 0$ for every $i \in \{1, \dots, r\}$. Therefore M_i also has quasilinear part $M_i^{\perp} = 0$, and is thus strictly regular. Hence the space $M_i/\mathfrak{m}M_i$ over $k = \mathfrak{o}/\mathfrak{m}$ is strictly regular, and so must have even dimension. We conclude that E must have even dimension.

[10] Strictly speaking, square classes are *isomorphism classes* of one-dimensional spaces.

Let us now give an example of a two-dimensional disobedient space over a field K of characteristic 2.

First, recall the definition of the *Arf-invariant* $\mathrm{Arf}(\varphi)$ of a strictly regular form φ over a field k of characteristic 2. For $x \in k$, let $\wp(x) := x^2 + x$. Further, let $\wp(k)$ denote the set of all $\wp(x)$, $x \in k$. This is a subgroup of k^+, i.e. k regarded as an additive group. We choose a decomposition

$$(*) \qquad \varphi \cong \begin{bmatrix} a_1 & 1 \\ 1 & b_1 \end{bmatrix} \perp \ldots \perp \begin{bmatrix} a_m & 1 \\ 1 & b_m \end{bmatrix}$$

and set

$$\mathrm{Arf}(\varphi) := a_1 b_1 + \cdots + a_m b_m + \wp(k) \in k^+/\wp(k).$$

It is well-known that $\mathrm{Arf}(\varphi)$ is independent of the choice of the decomposition $(*)$ ([A], [S, Chap.IX. §4]).

Example. Let k be a field of characteristic 2 and $K = k(t)$ the rational function field in one variable t over k. Further, let $\mathfrak{o}$ be the discrete valuation ring of K with respect to the prime polynomial t in $k[t]$, i.e. $\mathfrak{o} = k[t]_{(t)}$.

Claim. The quadratic space $\left[\begin{smallmatrix} 1 & 1 \\ 1 & t^{-1} \end{smallmatrix}\right]$ over K is disobedient with respect to $\mathfrak{o}$.

Proof. Suppose for the sake of contradiction that this space is obedient. Then there exist a strictly regular space M over $\mathfrak{o}$ and an element $u \in K^*$ with $\left[\begin{smallmatrix} 1 & 1 \\ 1 & t^{-1} \end{smallmatrix}\right] \cong \langle u \rangle \otimes (K \otimes_{\mathfrak{o}} M)$. By §6, Theorem 1, we have $M \cong \left[\begin{smallmatrix} \alpha & 1 \\ 1 & \beta \end{smallmatrix}\right]$ with $\alpha, \beta \in \mathfrak{o}$. Hence, by §6, Lemma 2, (3), the following holds over K:

$$(1) \qquad \begin{bmatrix} 1 & 1 \\ 1 & t^{-1} \end{bmatrix} \cong \langle \alpha u \rangle \begin{bmatrix} 1 & 1 \\ 1 & \alpha\beta \end{bmatrix}.$$

Comparing Arf-invariants of both sides, shows that there exists an element $a \in K$ with

$$(2) \qquad t^{-1} = \alpha\beta + a^2 + a.$$

Hence there exists an element $a \in K$ with $t^{-1} + a^2 + a \in \mathfrak{o}$.

If we now move to the formal power series field $k((t)) \supset K$, we easily see that such an element a does not exist: For if $a = \sum\limits_{r \geq d} c_r t^r$ with $d \in \mathbb{Z}$ and coefficients $c_r \in k$, $c_d \neq 0$, then

$$\sum_{r \geq d} c_r^2 t^{2r} + \sum_{r \geq d} c_r t^r + t^{-1} \in k[[t]],$$

and so $d < 0$. The term of lowest degree, in the first sum on the left, is $c_d^2 t^{2d}$. It cannot be compensated by other summands on the left. Therefore $t^{-1} + a^2 + a \notin \mathfrak{o}$, and the space $\left[\begin{smallmatrix} 1 & 1 \\ 1 & t^{-1} \end{smallmatrix}\right]$ is disobedient. $\square$

In this proof we used the Arf-invariant. We remark that we can use easier aids: Anisotropic quadratic forms like $\left[\begin{smallmatrix} 1 & 1 \\ 1 & c \end{smallmatrix}\right]$ are norm forms of separable quadratic field extensions, and are as such "multiplicative". Therefore, we can deduce immediately from (1) that $\begin{bmatrix} 1 & 1 \\ 1 & t^{-1} \end{bmatrix} \cong \begin{bmatrix} 1 & 1 \\ 1 & \alpha\beta \end{bmatrix}$ and then, that there exists a relation (2).

Remark. Hitherto the quasilinear parts of free quadratic modules played a predominantly negative role. This will continue to be the case. Rather often they cause a lot of complications, compared to the theory of nondegenerate bilinear modules. But sometimes quasilinear quadratic modules can do good things. Suppose again that $K = k(t)$, $\mathfrak{o} = k[t]_{(t)}$ as in the example above. The space $\begin{bmatrix} 1 & 1 \\ 1 & t^{-1} \end{bmatrix}$ is disobedient with respect to $\mathfrak{o}$. However, $\begin{bmatrix} 1 & 1 \\ 1 & t^{-1} \end{bmatrix} \perp [t^{-1}]$ is obedient, since this space over K is isometric to $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \perp [t^{-1}]$, and hence to $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \perp [t]$.

In connection with the definition of obedience (Definition 3), we agree upon some more terminology. If $E = (E, q)$ is a quadratic space, obedient with respect to $\mathfrak{o}$, then we have an (internal) orthogonal decomposition

$$E = \perp_{s \in S} E_s$$

of the following kind: Every vector space E_s contains a free $\mathfrak{o}$-submodule M_s with $E_s = KM_s$, such that $q|E_s = sq_s$ for a quadratic form $q_s\colon E_s \to K$, which takes values in $\mathfrak{o}$ on M_s and for which $(M_s, q_s|M_s)$ is a nondegenerate quadratic module. (Of course $E_s \neq 0$ for finitely many $s \in S$ only.)

Definition 4. We call such a decomposition $E = \perp_{s \in S} E_s$, together with a choice of modules $M_s \subset E_s$, a λ-*modular*, or also, an $\mathfrak{o}$-*modular representation* of the obedient quadratic space E.

Lemma 2. *Suppose again that $\mathfrak{o}$ is quadratically henselian and $E = (E, q)$ is a quadratic space over K, obedient with respect to $\mathfrak{o}$. Let $E = \perp_{s \in S} E_s$, $E_s = KM_s$ be an $\mathfrak{o}$-modular representation of E. Suppose furthermore that E is anisotropic. Then we have, with the notation of Lemma 1,*[11]

$$\mu(E) = M_1 \perp \perp_{s \neq 1} \mu_+(E_s),$$
$$\mu_+(E) = \mathfrak{m}M_1 \perp \perp_{s \neq 1} \mu_+(E_s).$$

Proof. As above, we set $q|E_s = sq_s$. Let x be a vector of E with $x \neq 0$. Then there exist finitely many pairwise distinct elements $s_1, \dots, s_r$ in S, as well as *primitive* vectors $x_i \in M_i := M_{s_i}$ and scalars $a_i \in K^*$ $(1 \leq i \leq r)$, such that $x = \sum_{i=1}^{r} a_i x_i$. We have

[11]In accordance with our earlier agreement, every E_s is considered as a quadratic subspace of E, $E_s = (E_s, q|E_s)$.

$$q(x) = \sum_{i=1}^{r} a_i^2 s_i q_i(x_i),$$

where we used the abbreviation $q_i := q_{s_i}$. According to §6, Theorem 5, we have $q_i(x_i) \in \mathfrak{o}^*$ for every $i \in \{1, \ldots, r\}$. Hence, for $i \neq j$, the values $v(a_i^2 s_i q_i(x_i)) = v(a_i^2 s_i)$ and $v(a_j^2 s_j q_j(x_j)) = v(a_j^2 s_j)$ are different. Let $v(a_k^2 s_k)$ be the smallest value among the $v(a_i^2 s_i)$, $1 \leq i \leq r$. Then $v(q(x)) = v(a_k^2 s_k) = v(q(a_k x_k))$. This shows that

$$\mu(E) = \mathop{\perp}_{s \in S} \mu(E_s), \quad \mu_+(E) = \mathop{\perp}_{s \in S} \mu_+(E_s).$$

For $s \neq 1$, $a \in K^*$ and primitive $y \in E_s$, we have $q(ay) = sa^2 q_s(y) \notin \mathfrak{o}^*$. Therefore, $\mu(E_s) = \mu_+(E_s)$ for every $s \in S \setminus \{1\}$. We still have to determine the modules $\mu(E_1)$ and $\mu_+(E_1)$. So, let $x \in E_1$, $x \neq 0$. We write $x = ay$ with $a \in K^*$ and primitive $y \in M_1$. Then $q(x) = a^2 q(y)$ and $q(y) \in \mathfrak{o}^*$. Hence $q(x) \in \mathfrak{o}$ exactly when $a \in \mathfrak{o}$, and $q(x) \in \mathfrak{m}$ exactly when $a \in \mathfrak{m}$. Therefore we conclude that $\mu(E_1) = M_1$ and $\mu_+(E_1) = \mathfrak{m} M_1$. $\square$

An immediate consequence of the lemma is

Theorem 2. *Let $\mathfrak{o}$ be quadratically henselian, and let $E = (E, q)$ be an anisotropic quadratic space over K, obedient with respect to $\mathfrak{o}$. Let $E = \mathop{\perp}_{s \in S} E_s$, $E_s = KM_s$ be an $\mathfrak{o}$-modular representation of E. Then we have for the reduction $\rho(E) = (\rho(E), \overline{q})$ of E with respect to $\mathfrak{o}$:*

$$(\rho(E), \overline{q}) \;\cong\; (M_1/\mathfrak{m}M_1, \overline{q}_1),$$

where $\overline{q}_1 \colon M_1/\mathfrak{m}M_1 \to k$ is the quadratic form over k, determined by $q|M_1 \colon M_1 \to \mathfrak{o}$ in the obvious way. $\square$

Furthermore, the lemma tells us that $M_1 = \mu(E_1)$ and $\mathfrak{m}M_1 = \mu_+(E_1)$. In particular we have

Theorem 3. *Let $\mathfrak{o}$ be quadratically henselian and let $E = (E, q)$ be an anisotropic quadratic space over K, having good reduction with respect to $\mathfrak{o}$. Then $\mu(E) = (\mu(E), q|\mu(E))$ is a nondegenerate quadratic $\mathfrak{o}$-module with $E = K \otimes_{\mathfrak{o}} \mu(E)$, and $\mu_+(E) = \mathfrak{m}\mu(E)$. Note that $\mu(E)$ is the only nondegenerate quadratic $\mathfrak{o}$-module M in E, with $E = KM$.* $\square$

We tone down Definition 4 as follows:

Definition 5. Let $E = (E, q)$ be a quadratic space over K, obedient with respect to λ. A *λ-modular* (or *$\mathfrak{o}$-modular*) *decomposition of* E is an orthogonal decomposition $E = \mathop{\perp}_{s \in S} E_s$, in which every space $\langle s \rangle \otimes E_s = (E_s, s \cdot (q|E_s))$ has good reduction with respect to λ.

If $\mathfrak{o}$ is quadratically henselian and E is anisotropic then, according to Theorem 3, every λ-modular decomposition corresponds to *exactly one* λ-modular representation of E.

Lemma 3. *Let $\mathfrak{o}$ be quadratically henselian. Let $s_1, \dots, s_r$ be different elements of S and $M_1, \dots, M_r$ anisotropic nondegenerate quadratic modules over $\mathfrak{o}$. Then*

$$E := \perp_{i=1}^{r} \langle s_i \rangle \otimes (K \otimes_{\mathfrak{o}} M_i)$$

is an anisotropic quadratic space over K.

Proof. Let a primitive vector $x_i \in M_i$ be given for every $i \in \{1, \dots, r\}$. Also, let $a_1, \dots, a_r \in K$ be scalars with

$$(\dagger) \qquad \sum_{i=1}^{r} s_i a_i^2 q_i(x_i) = 0,$$

where q_i denotes the quadratic form on M_i. We must show that all $a_i = 0$.

Suppose for the sake of contradiction that this is not so. After renumbering the M_i, we can assume without loss of generality that for a certain $t \in \{1, \dots, r\}$, $a_i \neq 0$ for $1 \leq i \leq t$ and $a_i = 0$ for $t < i \leq r$. By §6, Theorem 5, $q(x_i) \in \mathfrak{o}^*$ for every $i \in \{1, \dots, t\}$. The values $v(s_i a_i^2 q_i(x)) = v(s_i a_i^2)$ with $1 \leq i \leq t$ are pairwise different. Since this contradicts equation (†), all $a_i = 0$. □

We arrive at the main theorem of this chapter.

Theorem 4. *Let E be a quadratic space over K, obedient with respect to $\mathfrak{o}$. Let*

$$E = \perp_{s \in S} E_s = \perp_{s \in S} F_s$$

be two $\mathfrak{o}$-modular decompositions of E, and also let M_1, N_1 be nondegenerate quadratic modules with $E_1 \cong K \otimes_{\mathfrak{o}} M_1$, $F_1 \cong K \otimes_{\mathfrak{o}} N_1$. Then the quadratic spaces $M_1/\mathfrak{m}M_1$ and $N_1/\mathfrak{m}N_1$ over $k = \mathfrak{o}/\mathfrak{m}$ are Witt equivalent.

Proof. (a) For every $s \in S \setminus \{1\}$ we choose nondegenerate quadratic $\mathfrak{o}$-modules M_s, N_s with $E_s \cong \langle s \rangle \otimes (K \otimes_{\mathfrak{o}} M_s)$, $F_s \cong \langle s \rangle \otimes (K \otimes_{\mathfrak{o}} N_s)$. Suppose first that $\mathfrak{o}$ is quadratically henselian. In accordance with §6, Theorem 3, we choose Witt decompositions

$$M_s = M_s^\circ \perp i_s \times \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad N_s = N_s^\circ \perp j_s \times \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

with M_s°, N_s° anisotropic nondegenerate quadratic $\mathfrak{o}$-modules. By Lemma 3 above, the nondegenerate quadratic K-modules

$$U := \perp_{s \in S} \langle s \rangle \otimes (K \otimes_{\mathfrak{o}} M_s^\circ), \quad V := \perp_{s \in S} \langle s \rangle \otimes (K \otimes_{\mathfrak{o}} N_s^\circ)$$

are anisotropic. Now,

$$E \cong U \perp \left(\sum_{s\in S} i_s\right) \times \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] \cong V \perp \left(\sum_{s\in S} j_s\right) \times \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right].$$

Hence $U \cong V$. According to Theorem 2, we have $\rho(U) \cong M_1^\circ/\mathfrak{m}M_1^\circ$, $\rho(V) \cong N_1^\circ/\mathfrak{m}N_1^\circ$. Therefore, $M_1^\circ/\mathfrak{m}M_1^\circ \cong N_1^\circ/\mathfrak{m}N_1^\circ$. Furthermore,

$$M_1/\mathfrak{m}M_1 \cong M_1^\circ/\mathfrak{m}M_1^\circ \perp i_1 \times \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right],\ N_1/\mathfrak{m}N_1 \cong N_1^\circ/\mathfrak{m}N_1^\circ \perp j_1 \times \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right].$$

We conclude that $M_1/\mathfrak{m}M_1 \sim N_1/\mathfrak{m}N_1$.

(b) Suppose next that $\mathfrak{o}$ is arbitrary. We go from $(K,\mathfrak{o})$ to the henselization $(K^h,\mathfrak{o}^h)$. (By definition, K^h is the quotient field of the henselian valuation ring $\mathfrak{o}^h$.) As is well-known, the valuation $v\colon K \to \Gamma \cup \infty$ extends uniquely to a valuation $v^h\colon K^h \to \Gamma \cup \infty$, which therefore again has Γ as value group. Also, $\mathfrak{o}^h$ is the valuation ring of v^h. Consequently, S is also a system of representatives of $Q(K^h)/Q(\mathfrak{o}^h) \cong \Gamma/2\Gamma$.

Given a free quadratic module M over $\mathfrak{o}$, we set – as before – $M^h := \mathfrak{o}^h \otimes_{\mathfrak{o}} M$ and for a free quadratic module U over K, we set $U^h := K^h \otimes_K U$. By §6, Lemma 7, the quadratic $\mathfrak{o}^h$-modules M_s^h, N_s^h are nondegenerate. Furthermore, $E^h = \perp_{s\in S} E_s^h = \perp_{s\in S} F_s^h$ and $E_s^h \cong K^h \otimes_{\mathfrak{o}^h} M_s^h$, $F_s^h \cong K^h \otimes_{\mathfrak{o}^h} N_s^h$ for every $s \in S$.

By above, the quadratic spaces $M_1^h/\mathfrak{m}^h M_1^h$ and $N_1^h/\mathfrak{m}^h N_1^h$ over k are equivalent. However, these spaces can be canonically identified with $M_1/\mathfrak{m}M_1$ and $N_1/\mathfrak{m}N_1$ (cf. the end of the proof of §6, Lemma 7). We conclude that $M_1/\mathfrak{m}M_1 \sim N_1/\mathfrak{m}N_1$. □

Definition 6. Let E be a quadratic space over K, obedient with respect to λ. If $E = \perp_{s\in S} E_s$ is a λ-modular decomposition of E and M_1 a nondegenerate quadratic $\mathfrak{o}$-module with $E_1 \cong K \otimes_{\mathfrak{o}} M_1$, then we call the quadratic space $L \otimes_\lambda M_1$ a *weak specialization of E with respect to* λ. (As before, $\otimes_\lambda$ denotes a base extension with respect to the homomorphism $\lambda|\mathfrak{o}\colon \mathfrak{o} \to L$.) By Theorem 4, the space $L \otimes_\lambda M_1$ is uniquely determined by E and λ, up to Witt equivalence. We denote its Witt class by $\lambda_W(E)$, i.e.

$$\lambda_W(E) := \{L \otimes_\lambda M_1\}.$$

("W" as in "Witt" or "weak".)

If $\operatorname{char} K = 2$, then $M_1/\mathfrak{m}M_1$ is a quadratic space over k, to be sure. Nevertheless $L \otimes_\lambda M_1 = L \otimes_{\overline{\lambda}} (M_1/\mathfrak{m}M_1)$ can be degenerate. In this case, we only have $\lambda_W(E) \in \widehat{Wq}(L)$ (see §6, from Definition 10 onwards). If $\operatorname{char} K \neq 2$, this cannot happen since $M_1/\mathfrak{m}M_1$ is strictly regular in this case. So now, $\lambda_W(E) \in Wq(L)$.

Theorem 5. *If E and E' are quadratic spaces over K, obedient with respect to λ, and if $E \sim E'$, then $\lambda_W(E) = \lambda_W(E')$.*

Proof. Suppose without loss of generality that $\dim E \le \dim E'$. Then $E' \cong E \perp r \times \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ for a certain $r \in \mathbb{N}_0$. If we choose a nondegenerate $\mathfrak{o}$-module

M_1 for E, as in Definition 6, then $M_1' := M_1 \perp r \times \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ is a possibe choice for E'. (Here we consider $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ over $\mathfrak{o}$ instead of over K.) Therefore, $L \otimes_\lambda M_1' \sim L \otimes_\lambda M_1$. □

Example. Let $K = k(t_1, \dots, t_n)$ be the rational function field in n variables $t_1, \dots, t_n$ over an arbitrary field k. For every multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$, let t^α denote the monomial $t_1^{\alpha_1} \dots t_n^{\alpha_n}$. We order the abelian group $\mathbb{Z}^n$ lexicographically and then have exactly one valuation $v: K \to \mathbb{Z}^n \cup \infty$ with $v(t^\alpha) = \alpha$ for every $\alpha \in \mathbb{N}_0^n$. Its valuation ring contains the field k, and the residue class field $\mathfrak{o}/\mathfrak{m}$ coincides with k. Let $\lambda: K \to k \cup \infty$ be the canonical place corresponding to $\mathfrak{o}$. Furthermore, let A be the set of all multi-indices $(\alpha_1, \dots, \alpha_n)$ with $\alpha_i \in \{0, 1\}$ for every i, in other words $A = \{0,1\}^n \subset \mathbb{N}_0^n$. As system of representatives S, in the above sense, we take the set of t^α with $\alpha \in A$.

Suppose now that we are given a family $(F_\alpha \mid \alpha \in A) = \mathcal{F}$ of 2^n quadratic spaces over k. We construct the space

$$E := \perp_{\alpha \in A} \langle t^\alpha \rangle \otimes (K \otimes_k F_\alpha)$$

over K. For every $\alpha \in A$, $\langle t^\alpha \rangle \otimes E$ is obedient with respect to λ and

$$\lambda_W(\langle t^\alpha \rangle \otimes E) = \{F_\alpha\}.$$

Hence the family $\mathcal{F}$ can be recovered from the space E, up to Witt equivalence. If all F_α are anisotropic, we even get the F_α back from E as kernel spaces of the Witt classes $\lambda_W(\langle t^\alpha \rangle \otimes E)$. In this case, E is anisotropic as well. This can easily be seen, using an argument similar to the one used in the proof of Lemma 3. One could say that the family $\mathcal{F}$ is "stored" in the space E over $k((t_1, \dots, t_n))$.

This simple example reminisces of Springer's Theorem in §1. Without our theory of weak specialization, we can use Springer's Theorem and induction on n to show that the F_α are uniquely determined by E (at least when $\operatorname{char} k \neq 2$). This is so, because the valuation v is "n-fold discrete". For more complicated value groups, we cannot fall back on Springer's Theorem for specialization arguments.

We return to the situation of arbitrary places $\lambda: K \to k \cup \infty$.

Definition 7. (a) We denote the sets of Witt classes $\{E\}$ of obedient quadratic spaces (with respect to $\mathfrak{o}$) and obedient strictly regular quadratic spaces (with respect to $\mathfrak{o}$) by $\widetilde{Wq}(K, \mathfrak{o})$ and $Wq(K, \mathfrak{o})$ respectively. These are thus subsets of the sets $\widetilde{Wq}(K)$ and $Wq(K)$, introduced in §6, Definition 7b.

(b) We define a map $\lambda_W: \widetilde{Wq}(K, \mathfrak{o}) \to \widehat{Wq}(L)$ by setting $\lambda_W(\{E\}) := \lambda_W(E)$. This map is well-defined by Theorem 5.

Clearly $Wq(K, \mathfrak{o})$ is a subgroup of the abelian group $Wq(K)$. The group $Wq(K, \mathfrak{o})$ acts on the set $\widetilde{Wq}(K, \mathfrak{o})$ by restriction of the action of $Wq(K)$ on $\widetilde{Wq}(K)$, explained in §6. If $\operatorname{char} K \neq 2$, then naturally $\widetilde{Wq}(K, \mathfrak{o}) = Wq(K, \mathfrak{o})$, and λ_W maps $Wq(K, \mathfrak{o})$ to $Wq(L)$.

Remark. Let E and F be quadratic spaces over K, obedient with respect to λ and suppose that E is strictly regular. Obviously we then have

$$\lambda_W(E \perp F) = \lambda_W(E) + \lambda_W(F).$$

Therefore, restricting the map $\lambda_W\colon \widetilde{Wq}(K,\mathfrak{o}) \to \widehat{Wq}(L)$ gives rise to a homomorphism from $Wq(K,\mathfrak{o})$ to $Wq(L)$, which we also denote by λ_W. The map $\lambda_W\colon \widetilde{Wq}(K,\mathfrak{o}) \to \widehat{Wq}(L)$ is equivariant with respect to the homomorphism $\lambda_W\colon Wq(K,\mathfrak{o}) \to Wq(L)$.

In §3, we got a homomorphism $\lambda_W\colon W(K) \to W(L)$, using a different method. Given a bilinear space $E = (E,B)$ over K, does there exist a description of the Witt class $\lambda_W(\{E\})$, analogous to our current Definition 5?

First of all, when $2 \in \mathfrak{o}^*$, i.e. $\operatorname{char} L \neq 2$, we find complete harmony between §3 and §6. In this case, bilinear spaces over K and L are the same objects as quadratic spaces, and every such space over K is obedient with respect to λ.

Theorem 6. *If* $\operatorname{char} L \neq 2$*, the homomorphism* $\lambda_W\colon W(K) \to W(L)$ *from* §3 *coincides with the homomorphism* $\lambda_W\colon Wq(K) \to Wq(L)$*, defined just now.*

Proof. It suffices to show that for a one-dimensional space $[a] = \langle 2a\rangle$, the element $\lambda_W(\{\langle 2a\rangle\})$ from §3 coincides with the currently defined Witt class $\lambda_W([a])$. If $a \in \mathfrak{o}^*$, then $\lambda_W(\{\langle 2a\rangle\}) = \{\langle 2\lambda(a)\rangle\} = \{[\lambda(a)]\}$, according to §3. However, if $\langle a\rangle \notin Q(\mathfrak{o})$, then $\langle 2a\rangle \notin Q(\mathfrak{o})$ and $\lambda_W(\{\langle 2a\rangle\}) = 0$. We obtain the same values for $\lambda_W([a])$, using Definition 6 above. $\square$

And so, when $\operatorname{char} L \neq 2$, the work performed hitherto gives us a new – more conceptual – proof of §3, Theorem 1.

What is the situation when $\operatorname{char} L = 2$? First, we encounter in all generality definitions for bilinear spaces, analogous to Definitions 1 and 5.

Definition 8. Let $E = (E,B)$ be a bilinear space over K.

(a) E has *good reduction with respect to* λ (or *with respect to* $\mathfrak{o}$) when $E \cong K \otimes_{\mathfrak{o}} M$ for a bilinear space M over $\mathfrak{o}$. {Note: this is just a translation of the original definition of "good reduction" of §1 (= §3, Definition 1) in geometric language.}

(b) A λ*-modular* (or $\mathfrak{o}$*-modular*) *decomposition of* E is an orthogonal decomposition $E = \perp_{s\in S} E_s$, in which every space $\langle s\rangle \otimes E_s$ has good reduction with respect to λ.

In contrast to the quadratic case, *every* bilinear space E over K has λ-modular decompositions. This is obvious, since E is the orthogonal sum of one-dimensional bilinear spaces and copies of the space $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, see §2.

Let $\lambda_W\colon W(K) \to W(L)$ be the map, introduced in §3. When E is a bilinear space over K, we set $\lambda_W(E) := \lambda_W(\{E\}) \in W(L)$.

Theorem 7. *Let E be a bilinear space over K and let $E = \perp_{s \in S} E_s$ be a λ-modular decomposition of E. Also, let M_1 be a bilinear space over $\mathfrak{o}$ with $E_1 \cong K \otimes_{\mathfrak{o}} M_1$. Then*

$$\lambda_W(E) = \{L \otimes_\lambda M_1\}.$$

Proof. We have $\lambda_W(E) = \sum_{s \in S} \lambda_W(E_s)$. By §3, Theorem 2, we know that $\lambda_W(E_1) = \{L \otimes_\lambda M_1\}$. It remains to show that $\lambda_W(E_s) = 0$ for every $s \in S$ with $s \neq 1$. Hence we have to show that if $\langle s \rangle \in Q(K)$ is a square class, not in $Q(\mathfrak{o})$, and if N is a bilinear space over $\mathfrak{o}$, then $\lambda_W(\langle s \rangle \otimes (K \otimes_{\mathfrak{o}} N)) = 0$.

By a lemma from §3, N has an orthogonal basis when the bilinear space $N/\mathfrak{m}N$ over k is not hyperbolic. In this case we have

$$\langle s \rangle \otimes (K \otimes_{\mathfrak{o}} N) \ \cong\ \langle s\varepsilon_1, \ldots, s\varepsilon_r \rangle$$

with units $\varepsilon_i \in \mathfrak{o}^*$. By definition of λ_W, we have $\lambda_W(\langle s\varepsilon_i \rangle) = 0$ for all $i \in \{1, \ldots, r\}$, hence $\lambda_W(\langle s \rangle \otimes (K \otimes_{\mathfrak{o}} N)) = 0$. If $N/\mathfrak{m}N$ is hyperbolic, we construct the space $N' := N \perp \langle 1 \rangle$ over $\mathfrak{o}$. Since N' has an orthogonal basis, we get $\lambda_W(\langle s \rangle \otimes (K \otimes_{\mathfrak{o}} N')) = 0$. Since $\lambda_W(\langle s \rangle \otimes \langle 1 \rangle) = 0$ as well, we conclude that $\lambda_W(\langle s \rangle \otimes (K \otimes N)) = 0$. □

Theorem 7 allows us to speak of "weak specialization" of bilinear spaces, in analogy with Definition 6 for quadratic spaces.

Definition 9. Let E be a bilinear space over K. Also, let $E = \perp_{s \in S} E_s$ be a λ-modular decomposition of E and M_1 a bilinear space over $\mathfrak{o}$ with $E_1 \cong K \otimes_{\mathfrak{o}} M_1$. Then we call the space $L \otimes_\lambda M_1$ a *weak specialization* of the space E with respect to λ.

According to Theorem 7, a weak specialization of E with respect to λ is completely determined by E and λ up to Witt equivalence. One could ask if this can be shown in a direct – geometric – way, comparable to how we have done this for obedient quadratic spaces above. Reversing direction, this would give a new proof of §3, Theorem 1, also when $\operatorname{char} L = 2$.

We leave this question open. After all, we have a proof of §3, Theorem 1, and with it a weak specialization theory for bilinear spaces. This theory is more satisfying than the corresponding theory for quadratic spaces, in that we do not have to demand obedience of the spaces.

We can now also make a statement about Problem 3b of §1. Let $\operatorname{char} L = 2$ and let $\operatorname{char} K = 0$. Let (E, q) be a quadratic space, obedient with respect to λ. Associated to it, we have the bilinear space (E, B_q). It may be that (E, q) and (E, B_q) are in principle the same object, but it does make a difference wether we weakly specialize E as a quadratic or a bilinear space. What is better?

Let (M, q) be a regular (= strictly regular) quadratic $\mathfrak{o}$-module. According to §6, Theorem 1, we have a decomposition

$$(M,q) \;\cong\; \begin{bmatrix} a_1 & 1 \\ 1 & b_1 \end{bmatrix} \perp \ldots \perp \begin{bmatrix} a_r & 1 \\ 1 & b_r \end{bmatrix}.$$

(Note: $(M/\mathfrak{m}M, \overline{q})$ is strictly regular, and thus has even dimension.) Then

$$(M,B_q) \;\cong\; \begin{pmatrix} 2a_1 & 1 \\ 1 & 2b_1 \end{pmatrix} \perp \ldots \perp \begin{pmatrix} 2a_r & 1 \\ 1 & 2b_r \end{pmatrix}.$$

Hence,

$$L \otimes_\lambda (M,q) \;\cong\; \begin{bmatrix} \lambda(a_1) & 1 \\ 1 & \lambda(b_1) \end{bmatrix} \perp \ldots \perp \begin{bmatrix} \lambda(a_r) & 1 \\ 1 & \lambda(b_r) \end{bmatrix}.$$

However, $L \otimes_\lambda (M,B_q) \;\cong\; r \times \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Therefore, a weak specialization of the space (E,B_q) is always hyperbolic and so gives hardly any information about (E,q). However, a weak specialization of (E,q) with respect to λ can give an interesting result.

Final Consideration. A last word about the central Definition 3 of an obedient quadratic space. It seems obvious to formally weaken it, by considering quadratic *modules* E instead of spaces E over K, subject only to the requirements that, as a K-vector space, E should have finite dimension and a decomposition $(*)$ as presented there. But then E is already nondegenerate. We namely have

$$QL(E) \;\cong\; \mathop{\perp}_{s\in S} \langle s\rangle \otimes (K \otimes_{\mathfrak{o}} QL(M_s)),$$

and as in the proof of Lemma 3, we see that $QL(E)$ is anisotropic (without the requirement that $\mathfrak{o}$ is quadratically henselian). Thus these "obedient quadratic modules" are the same objects as those given by Definition 3.

§8 Good Reduction

As before, we let $\lambda: K \to L \cup \infty$ be a place, $\mathfrak{o} = \mathfrak{o}_\lambda$ the valuation ring of K, $\mathfrak{m}$ its maximal ideal, $k = \mathfrak{o}/\mathfrak{m}$ its residue class field and v a valuation on K, associated to $\mathfrak{o}$, with value group Γ. Also, let $\Sigma \subset Q(K)$ be a complement of $Q(\mathfrak{o})$ in $Q(K)$ and S a system of representatives of Σ in K^* (with $1 \in S$), as introduced in §7.

From now on, we call a nondegenerate quadratic $\mathfrak{o}$-module a **quadratic space over** $\mathfrak{o}$. In case $\mathfrak{o} = K$, we already used this terminology in §7. Recall further the concept of a *bilinear space* over $\mathfrak{o}$ (§3, Definition 4).

If E is a quadratic (resp. bilinear) space over K then, according to Definitions 1 and 8 in §7, E has *good reduction with respect to* λ (or: with respect to $\mathfrak{o}$) if there exists a quadratic (resp. bilinear) space M over $\mathfrak{o}$ with $E \cong L \otimes_{\mathfrak{o}} M$.

Theorem 1. *Let E be a quadratic or bilinear space over K, which has good reduction with respect to λ, and let M, M' be quadratic (resp. bilinear) spaces over $\mathfrak{o}$ with $E \cong K \otimes_{\mathfrak{o}} M \cong K \otimes_{\mathfrak{o}} M'$. Then the k-spaces $M/\mathfrak{m}M$ and $M'/\mathfrak{m}M'$ are isometric in the quadratic case and stably isometric*[12] *in the bilinear case. Therefore we also have, $L \otimes_\lambda M \cong L \otimes_\lambda M'$ resp. $L \otimes_\lambda M \approx L \otimes_\lambda M'$.*

Proof. By the theory of §7 (Theorems 4 and 7), $M/\mathfrak{m}M$ and $M'/\mathfrak{m}M'$ are Witt equivalent. Since these spaces have the same dimension, namely $\dim E$, they are isometric resp. stably isometric. □

In the following, the words "good reduction" are used so often, that it is appropriate to introduce an abbreviation. *From now on, we mostly write* GR *instead of "good reduction".*

Definition 1. Let E be a quadratic or bilinear space over K, which has GR with respect to λ, and let M be a quadratic, resp. bilinear, space over $\mathfrak{o}$ with $E \cong K \otimes_{\mathfrak{o}} M$. Then we denote the quadratic, resp. bilinear, module $L \otimes_\lambda M$ by $\lambda_*(E)$ and call it *"the" specialization of E with respect to λ.*

In the bilinear case, $\lambda_*(E)$ is nondegenerate, and thus a space. This is true in the quadratic case as well, as long as $\operatorname{char} K \neq 2$. If $\operatorname{char} K = 2$

[12] See §2 for the term "stably isometric".

however, $\lambda_*(E)$ can be degenerate (cf. our discussion about $\lambda_W(E)$ in §7 after Definition 6).

This terminology is convenient, but sloppy. According to Theorem 1, $\lambda_*(E)$ is only determined by E up to isometry in the quadratic case, and in the bilinear case even only up to stable isometry. Anyway, in what follows, we are interested in quadratic spaces (or modules) only up to isometry, and in bilinear spaces almost always only up to stable isometry.

Note that in the bilinear case, Definition 1 is only a translation of §3, Definition 3 in geometric language.

Remark. Let F and G be bilinear spaces over K, which have GR with respect to λ. Then $F \perp G$ clearly also has GR with respect to λ, and

$$\lambda_*(F \perp G) \approx \lambda_*(F) \perp \lambda_*(G).$$

For quadratic spaces, we have to realize that (when $\operatorname{char} K = 2$) the orthogonal sum of two spaces over $\mathfrak{o}$ can possibly be degenerate and thus need not again be a space. However, if F and G are quadratic spaces, which have GR with respect to λ, *and if F is strictly regular*, then $F \perp G$ has GR with respect to λ and

$$\lambda_*(F \perp G) \cong \lambda_*(F) \perp \lambda_*(G). \qquad \square$$

For further applications, the following theorem is of the utmost importance.

Theorem 2. (*a*) *Let F and G be bilinear spaces over K. If the spaces F and $F \perp G$ have* GR *with respect to λ, then G also has* GR *with respect to λ.*

(*b*) *Let F and G be quadratic spaces over K. Suppose that F is strictly regular. If F and $F \perp G$ have* GR *with respect to λ, then G also has* GR *with respect to λ.*

Proof. Part (a) has already been proved in §3, Theorem 4. Upon replacing every occurrence of "metabolic" by "hyperbolic", the argument there also yields a proof of part (b). (Use Theorem 3 and Lemma 3 in §6.) $\square$

Corollary. *Let E and F be bilinear* (*resp. quadratic*) *spaces with $E \sim F$. If E has* GR *with respect to λ, then so has F, and $\lambda_*(E) \sim \lambda_*(F)$. Besides, if $E \approx F$ (resp. $E \cong F$), then $\lambda_*(E) \approx \lambda_*(F)$ (resp. $\lambda_*(E) \cong \lambda_*(F)$).*

This was already established in §3 for the bilinear case. The quadratic case can be proved similarly.

In part (b) of Theorem 2, the assumed strict regularity of F is essential.

Example. Let $\operatorname{char} K = 2$ and let E be a 3-dimensional quadratic space over K with basis e, f, g and corresponding value matrix $\left[\begin{smallmatrix} 1 & 1 \\ 1 & a \end{smallmatrix}\right] \perp [\varepsilon]$, where $a \in \mathfrak{o}$ and $\varepsilon \in \mathfrak{o}^*$. Let $F := E^\perp = Kg$ and $G := Ke + K(f + cg) \cong \left[\begin{smallmatrix} 1 & 1 \\ 1 & a+c^2\varepsilon \end{smallmatrix}\right]$ for some $c \in K$. We have $E = F \perp G$. The quadratic spaces E and F have good reduction, but G can have bad reduction because the possibility exists that the Arf-invariant $a + c^2\varepsilon + \wp K$, $\wp K := \{x^2 + x \mid x \in K\}$ does not contain an element of $\mathfrak{o}$.

For instance, let $K = k(t)$ where k is an imperfect field of characteristic 2 and t is an indeterminate. As in the example of a disobedient space, §7, let $\mathfrak{o} = k[t]_{(t)}$. We choose $\varepsilon \in k^*$ such that ε is not a square in k. Furthermore, we choose $a = 0$, $c = t^{-2}$. We have $\varepsilon t^{-2} + x^2 + x \notin \mathfrak{o}$ for *every* $x \in K$. This is immediately clear when we look at the power series expansions in $k((t))$: Let $x = \sum_{n \geq d} c_n t^n$ for some $d \in \mathbb{Z}$, all $c_n \in k$ and $c_d \neq 0$. If it was true that $\varepsilon t^{-2} + x^2 + x \in \mathfrak{o}$, then we would have

$$\varepsilon t^{-2} + \sum_{n \geq d} c_n^2 t^{2n} + \sum_{n \geq d} c_n t^n \in k[[t]].$$

But then $d < 0$, and so $d = -1$. It would follow that $\varepsilon = c_d^2$, an impossibility.

Theorem 3. *Let E be a quadratic space over K. E has* GR *with respect to λ if and only if $QL(E)$ has* GR *with respect to λ and there exists a decomposition $E = F \perp QL(E)$ such that F has* GR *with respect to λ. In this case $\lambda_*(E) \cong \lambda_*(F) \perp \lambda_*(QL(E))$.*

Proof. Suppose first that we have a decomposition $E = F \perp QL(E)$, in which both F and $QL(E)$ have GR with respect to λ. Then, by the remark preceding Theorem 2, E has GR with respect to λ and $\lambda_*(E) \cong \lambda_*(F) \perp \lambda_*(QL(E))$, because F is strictly regular.

Next, suppose that E has GR. We choose a quadratic space M over $\mathfrak{o}$ with $E \cong K \otimes_{\mathfrak{o}} M$ and a decomposition $M = N \perp QL(M)$. From the definition of spaces over $\mathfrak{o}$, in other words of nondegenerate quadratic $\mathfrak{o}$-modules (§6, Definition 3), follows immediately that $QL(M)$ is nondegenerate and even that N is strictly regular. Now $E \cong K \otimes_{\mathfrak{o}} N \perp K \otimes_{\mathfrak{o}} QL(M)$, so that $QL(E) \cong K \otimes_{\mathfrak{o}} QL(M)$. Therefore $QL(E)$ has GR. Furthermore $F := K \otimes_{\mathfrak{o}} N$ has GR and is strictly regular. □

If F and G are bilinear spaces over K, which have GR with respect to λ, then we know from §3 that $F \otimes G$ also has GR with respect to λ and that

$$\lambda_*(F \otimes G) \approx \lambda_*(F) \otimes \lambda_*(G),$$

see §3, Theorem 6. As explained in §5, given a bilinear space F and a quadratic space G over K (or $\mathfrak{o}$), we can also construct a quadratic module $F \otimes G$ over K (resp. $\mathfrak{o}$), which can however be degenerate when $\operatorname{char} K = 2$. For $F \otimes G$ to be again a space, we must require that G is strictly regular. Analogous to the above statement, we have

Theorem 4. *Let F be a bilinear space over K, which has* GR *with respect to λ and let G be a quadratic space, having* GR *with respect to λ. In case* $\operatorname{char} K = 2$, *suppose furthermore that G is strictly regular. Then $F \otimes G$ als has* GR *with respect to λ and*

$$\lambda_*(F \otimes G) \;\cong\; \lambda_*(F) \otimes \lambda_*(G).$$

Proof. We choose spaces M and N over $\mathfrak{o}$ with $F \;\cong\; K \otimes_{\mathfrak{o}} M$, $G \;\cong\; K \otimes_{\mathfrak{o}} N$. Then N is strictly regular. Therefore $M \otimes_{\mathfrak{o}} N$ is a strictly regular quadratic space over $\mathfrak{o}$. Furthermore, $F \otimes G \;\cong\; K \otimes_{\mathfrak{o}} (M \otimes_{\mathfrak{o}} N)$. Hence, $F \otimes G$ has GR and

$$\lambda_*(F \otimes G) \;\cong\; L \otimes_\lambda (M \otimes_{\mathfrak{o}} N) \;\cong\; (L \otimes_\lambda M) \otimes_L (L \otimes_\lambda N) \;\cong\; \lambda_*(F) \otimes \lambda_*(G).$$

□

Remark. $\lambda_*(F)$ is only determined up to stable isometry and $\lambda_*(F \otimes G)$ only up to isometry. The theorem is valid – as the proof shows – for every choice of $\lambda_*(F)$. More generally we have: if E and E' are bilinear space over K with $E \approx E'$, and if F is a strictly regular quadratic space over K, then $E \otimes F \;\cong\; E' \otimes F$. This follows from the fact that there exists a bilinear space U over K with $E \perp U \;\cong\; E' \perp U$, implying that $(E \otimes F) \perp (U \otimes F) \;\cong\; (E' \otimes F) \perp (U \otimes F)$, and so $E \otimes F \;\cong\; E' \otimes F$ by the Cancellation Theorem.

The currently developed notion of the specialization $\lambda_*(E)$ of a space E which has GR, allows us to complete our understanding of the results from §7 about obedience and weak specialization.

Theorem 5. *Let E be a quadratic module over K. E is nondegenerate and obedient with respect to λ if and only if*

$$(*) \qquad E \;\cong\; \perp_{s \in S} \langle s \rangle \otimes F_s,$$

where the F_s are spaces over K, which have GR *with respect to λ, only finitely many of them being nonzero. For every decomposition of the form $(*)$, we have $\lambda_W(E) = \{\lambda_*(F_1)\}$.*

Proof. If E is a space and is obedient with respect to λ, we clearly have a decomposition as above. The equality $\lambda_W(E) = \{\lambda_*(F_1)\}$ follows from the definition of $\lambda_W(E)$ in §7 and the definition of $\lambda_*(F_1)$.

Now let $(F_s \mid s \in S)$ be a family of spaces, having GR with respect to λ and $F_s = 0$ for almost all $s \in S$. Let $E := \perp_{s \in S} \langle s \rangle \otimes F_s$. Then, the quadratic module E over K has finite dimension and $E^\perp = \perp_{s \in S} \langle s \rangle \otimes F_s^\perp$. According to the Final Consideration of §7, $E^\perp$ is anisotropic and therefore nondegenerate. It follows immediately from §7, Definition 3 that E is obedient with respect to λ. □

We call every decomposition of the form $(*)$, having the properties of Theorem 5, a λ-*modular decomposition* of E. This terminology is a bit sloppier than in §7, in the sense that we no longer discriminate between internal and external orthogonal sums. We talk about λ-modular decompositions of bilinear spaces in a similar fashion.

Theorem 6. *Let F be a bilinear space over K, which has* GR *with respect to λ.*

(i) *If G is a bilinear space over K, then*

$$(\dagger) \qquad \lambda_W(F \otimes G) = \lambda_W(F)\lambda_W(G) = \{\lambda_*(F)\}\lambda_W(G).$$

(ii) *If G is a strictly regular quadratic space over K, obedient with respect to λ, then $F \otimes G$ is also obedient with respect to λ, and $(\dagger)$ holds again.*

Proof. (ii) Let $G \cong \perp_{s\in S} \langle s\rangle \otimes G_s$ be a λ-modular decomposition of G. Every G_s has GR with respect to λ and is strictly regular. By Theorem 4, $F \otimes G_s$ has GR with respect to λ and $\lambda_*(F \otimes G_s) \cong \lambda_*(F) \otimes \lambda_*(G_s)$. Hence,

$$F \otimes G \cong \perp_{s\in S} \langle s\rangle \otimes (F \otimes G_s)$$

is a λ-modular decomposition of $F \otimes G$ and $\lambda_W(F \otimes G) = \{\lambda_*(F \otimes G_1)\} = \{\lambda_*(F) \otimes \lambda_*(G_1)\} = \{\lambda_*(F)\}\{\lambda_*(G_1)\} = \lambda_W(F)\lambda_W(G)$.

The proof of (i) is analogous. Less care is needed here than for (ii). □

Similarly we can show,

Theorem 7. *Let F be a bilinear space and G a strictly regular quadratic space over K. Suppose that the space G has* GR *with respect to λ. Then $F \otimes G$ is obedient with respect to λ, and*

$$\lambda_W(F \otimes G) = \lambda_W(F)\lambda_W(G) = \lambda_W(F)\{\lambda_*(G)\}. \qquad \square$$

To conclude this section, we have a look at the reduction behaviour of quadratic spaces under basis extensions. The following – almost banal – theorem, together with Theorem 2(b) above, lays the foundations upon which we will build the generic splitting theory of regular quadratic spaces in the next section.

Theorem 8. *Let $K' \supset K$ be a field extension and $\mu: K' \to L \cup \infty$ an extension of the place $\lambda: K \to L \cup \infty$. Let E be a regular quadratic space over K, which has* GR *with respect to λ. The space $K' \otimes_K E$ has* GR *with respect to μ and $\mu_*(K' \otimes_K E) \cong \lambda_*(E)$.*

Proof. Let $\mathfrak{o}'$ be the valuation ring associated to μ. Furthermore, let M be a regular quadratic space over $\mathfrak{o}$ with $E \cong K \otimes_{\mathfrak{o}} M$. Then $\mathfrak{o}' \otimes_{\mathfrak{o}} M$ is a regular quadratic space over $\mathfrak{o}'$ (sic!), and $K' \otimes_K E \cong K' \otimes_{\mathfrak{o}} M = K' \otimes_{\mathfrak{o}'} (\mathfrak{o}' \otimes_{\mathfrak{o}} M)$. Therefore, $K' \otimes_K E$ has GR with respect to μ and

$$\mu_*(K' \otimes_K E) \;\cong\; L \otimes_\mu (\mathfrak{o}' \otimes_{\mathfrak{o}} M) = L \otimes_\lambda M \;\cong\; \lambda_*(E).$$

$\square$

If we only require that E is nondegenerate instead of regular, the statement of Theorem 8 becomes false. Looking for a counterexample, we can restrict ourselves to quasilinear spaces, by Theorems 3 and 8.

Example. Let k be an imperfect field of characteristic 2 and let a be an element of k which is not a square. Consider the power series field $K = k((t))$ in one indeterminate t. Let $\lambda_0 \colon K \to k \cup \infty$ be the place with valuation ring $\mathfrak{o} = k[[t]]$, which maps every power series $f(t) \in \mathfrak{o}$ to its constant term $f(0)$. The quasilinear quadratic $\mathfrak{o}$-module $M = [1, a+t]$ is nondegenerate. For, if $f(t) = \sum\limits_{i \geq 0} b_i t^i$ and $g(t) = \sum\limits_{i \geq 0} c_i t^i$ are elements of $\mathfrak{o}$, whose constant terms b_0, c_0 are not both zero, then $f(t)^2 + (a+t)g(t)^2$ has nonzero constant term $b_0^2 + ac_0^2$, and is thus a unit in $\mathfrak{o}$. Therefore, axiom (QM2) of §6, Definition 3 is satisfied.

Hence, the space $E \colon= K \otimes_{\mathfrak{o}} M$ has GR with respect to λ_0, and $(\lambda_0)_*(E) \cong [1, a]$. Consider now $L \colon= k(\sqrt{a})$ and the place $\lambda \colon= j \circ \lambda_0 \colon K \to L \cup \infty$, obtained by composing λ_0 with the inclusion $j \colon k \hookrightarrow L$. Then E also has GR with respect to λ, and over L we have,

$$\lambda_*(E) = L \otimes_k (\lambda_0)_*(E) \;\cong\; [1, a] \;\cong\; [1, 1] \;\cong\; [1, 0].$$

Finally, let $K' \colon= K(\sqrt{a}) = L((t))$ and let $\mu \colon K' \to L \cup \infty$ be the place with valuation ring $\mathfrak{o}' \colon= L[[t]]$, which again maps every power series $f(t) \in \mathfrak{o}'$ to its constant term $f(0)$. μ extends the place λ. Over K', we have

$$K' \otimes_K E = [1, (\sqrt{a})^2 + t] \;\cong\; [1, t].$$

Therefore, $K' \otimes_K E$ is obedient with respect to μ and $\mu_W(K' \otimes_K E) = \{[1]\}$. It is now clear that $K' \otimes_K E$ has bad reduction with respect to μ, since the Witt class $\mu_W(K' \otimes_K E)$ would contain a two-dimensional module otherwise.

Remark. This argumentation shows nicely that it is profitable to give such an elaborate definition of Witt equivalence of degenerate quadratic modules over fields, as done in §6, Definition 10.

Chapter II

Generic Splitting Theory

§9 Generic Splitting of Regular Quadratic Forms

From now on we leave the geometric arena behind, and mostly talk of quadratic and bilinear *forms*, instead of *spaces*, over fields. The importance of the geometric point of view was to bring quadratic and bilinear modules over valuation rings into the game. For our specialization theory, these modules were merely an aid however, and their rôle has now more or less ended.

We should indicate one problem though, which occurs when we make the transition from quadratic spaces to quadratic forms: If $\varphi(x_1, \dots, x_n)$ and $\psi(x_1, \dots, x_n)$ are two quadratic forms over a field K, they are isometric, $\psi \cong \varphi$, if the polynomial ψ emerges from φ through a linear coordinate transformation. Now, if φ – i.e. the space (E, q) associated to φ – is degenerate, then it is possible that not all of the coordinates $x_1, \dots, x_n$ $(n := \dim E)$ occur in the polynomial ψ (and possibly neither in φ). The dimension of ψ in the naive sense, i.e. the number of occurring variables, can be smaller than the dimension of E. In order to recover the space (E, q) from ψ, one would have to attach a "virtual dimension" n to ψ, pretty horrible!

As soon as one allows degenerate forms, the geometric language is more precise – and thus more preferred – than the "algebraic" language. But now we only consider *regular* quadratic forms ($\widehat{=}$ regular quadratic spaces) over fields, so that the forms are guaranteed to remain nondegenerate under base field extensions.

Many concept which we introduced for quadratic spaces over fields (§5 – §8), will now be used for quadratic forms, usually without any further comments. *In what follows, a "form" is always a regular quadratic form over a field.* We recall once more the concepts GR (= good reduction) and specialization of forms and try to be as down to earth as possible.

So, let φ be a form over a field K. If $\dim \varphi$ is even (resp. odd), then $\varphi = [a_{ij}]$ (resp. $\varphi \cong [a_{ij}] \perp [c]$), where (a_{ij}) is a symmetric $(2m) \times (2m)$-matrix, such that $(r := 2m)$

$$\det \begin{pmatrix} 2a_{11} & a_{12} & \dots & a_{1r} \\ a_{21} & 2a_{22} & \dots & a_{2r} \\ \vdots & \vdots & & \vdots \\ a_{r1} & a_{r2} & \dots & 2a_{rr} \end{pmatrix} \neq 0$$

and $c \neq 0$. (See the beginning of §5 for the notation used here.)

Let $\lambda: K \to L \cup \infty$ be a place and $\mathfrak{o} = \mathfrak{o}_\lambda$ its valuation ring. The form φ has GR with respect to λ when $\varphi \cong [a_{ij}]$, resp. $\varphi \cong [a_{ij}] \perp [c]$, such that all a_{ij} are in $\mathfrak{o}$ and c, as well as the determinant above, are units in $\mathfrak{o}$. We then have

$\lambda_*(\varphi) \cong [\lambda(a_{ij})]$ resp. $\lambda_*(\varphi) \cong [\lambda(a_{ij})] \perp [\lambda(c)]$. By §6, Theorem 1(b), $[a_{ij}]$ is the orthogonal sum of m binary forms $\begin{bmatrix} a_i & 1 \\ 1 & b_i \end{bmatrix}$ $(1 \leq i \leq m)$ with $a_i \in \mathfrak{o}$, $b_i \in \mathfrak{o}$, $1 - 4a_ib_i \in \mathfrak{o}^*$. Thus, after a coordinate transformation, we have

$$(*) \qquad \varphi(x_1, \ldots, x_n) = \sum_{i=1}^{m} (a_i x_{2i}^2 + x_{2i}x_{2i+1} + b_i x_{2i+1}^2) \qquad (\quad + c x_{2m+1}^2).$$

We obtain $(\lambda_*\varphi)(x_1, \ldots, x_n)$ from this form, upon replacing the coefficients a_i, b_i, c by $\lambda(a_i)$, $\lambda(b_i)$, $\lambda(c)$.

Let $K' \supset K$ be a field extension. If $\varphi = \varphi(x_1, \ldots, x_n)$ is a form over K, we write $\varphi \otimes K'$ or $\varphi \otimes_K K'$ for this form considered over K'. If (E, q) is the quadratic space associated to φ, i.e. $\varphi \mathrel{\widehat{=}} (E, q)$, then the basis extension $(K' \otimes_K E,\ q_{K'})$ of (E, q) is the quadratic space associated to $\varphi \otimes K'$.[13]

Let $\mu: K' \to L \cup \infty$ be an extension of the place $\lambda: K \to L \cup \infty$. If φ has GR with respect to λ, then $\varphi \otimes K'$ has GR with respect to μ and $\mu_*(\varphi \otimes K') \cong \lambda_*(\varphi)$. This is clear now and was already established in §8, Theorem 8 anyway.

We can extend the observation we made, at the start of our treatment of generic splitting in §4, to regular quadratic forms. Thus let k be a field and φ a form over k. Furthermore, let K and L be fields, containing k, and let $\lambda: K \to L \cup \infty$ be a place over k. Then $\varphi \otimes K$ has GR with respect to λ and $\lambda_*(\varphi \otimes K) \cong \varphi \otimes L$.

We use H to denote the quadratic form $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, regardless of the field we are working in. The bilinear form B_φ, associated to $\varphi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We denote this bilinear form henceforth by $\widetilde{H}$. (This notation differs from §4!) If

$$\varphi \otimes K \cong \varphi_1 \perp r_1 \times H$$

is the Witt decomposition of $\varphi \otimes K$, then the form φ_1 has GR with respect to λ by §8, Theorem 2(b) and we have

$$\varphi \otimes L \cong \lambda_*(\varphi_1) \perp r_1 \times H.$$

Therefore, $\operatorname{ind}(\varphi \otimes L) \geq \operatorname{ind}(\varphi \otimes K)$. If K and L are specialization equivalent over k (see §4, Definition 1), then

$$\operatorname{ind}(\varphi \otimes L) = \operatorname{ind}(\varphi \otimes K) \quad \text{and} \quad \ker(\varphi \otimes L) \cong \lambda_*(\varphi_1).$$

We transfer the definition of generic zero field (§4, Definition 2) literally to the current situation. Just as in §4 we define, for $n := \dim \varphi \geq 2$, a field extension $k(\varphi)$ of k as follows: If $n > 2$ or $n = 2$ and $\varphi \not\cong H$, let $k(\varphi)$ be the quotient field of the integral domain $k[X_1, \ldots, X_n]/(\varphi(X_1, \ldots, X_n))$. {It is easy to see that the polynomial $\varphi(X_1, \ldots, X_n)$ is irreducible.} If $\varphi \cong H$ however, i.e. $\varphi(X_1, X_2) \cong X_1X_2$, let $k(\varphi) = k(t)$ for an indeterminate t.

[13] We write $\varphi \otimes K'$ instead of $K' \otimes \varphi$ in order to be in harmony with the notation in §3 and in the literature. {Many authors more briefly write $\varphi_{K'}$. Starting with §16 we occasionally will do this too.}

We want to show that $k(\varphi)$ is a generic zero field of φ. This is very simple when φ is isotropic: Obviously k itself is a generic zero field of φ in this case. We thus have to show that $k(\varphi)$ is specialization equivalent to k over k, which now means that there exists a place from $k(\varphi)$ to k over k.

Lemma 1. *If φ is an isotropic form over k, then $k(\varphi)$ is a purely transcendental field extension of k.*

Proof. This is by definition clear when $\dim \varphi = 2$, i.e. when $\varphi \cong H$. Now let $n := \dim \varphi > 2$. We have a decomposition $\varphi \cong H \perp \psi$ and may suppose without loss of generality that $\varphi = H \perp \psi$. So, if $X_1, \ldots, X_n$ are indeterminates, we have

$$\varphi(X_1, \ldots, X_n) = X_1 X_2 + \psi(X_3, \ldots, X_n).$$

Therefore,[14]

$$k(\varphi) = \mathrm{Quot}(k[X_1, \ldots, X_n]/(X_1 X_2 + \psi(X_3, \ldots, X_n)) = k(x_1, \ldots, x_n),$$

where x_i of course denotes the image of X_i in $k(\varphi)$. The elements $x_2, x_3, \ldots, x_n$ are algebraically independent over k and $x_1 = -x_2^{-1}\psi(x_3, \ldots, x_n)$. Therefore, $k(\varphi) = k(x_2, \ldots, x_n)$ is purely transcendental over k. □

Since $k(\varphi)/k$ is purely transcendental, there are many places from $k(\varphi)$ to k over k, cf. [Bo$_2$, §10, Prop. 1].

We will move on to prove that also for anisotropic φ with $\dim \varphi \geq 2$, $k(\varphi)$ is a generic zero field of φ. De facto we will obtain a stronger result (Theorem 4 below), and we will need its full strength later on as well. We require a lemma about the extension of places to quadratic field extensions.

Lemma 2. *Let E be a field, K a quadratic extension of E and α a generator of K over E. Let $p(T) = T^2 - aT + b \in E[T]$ be the minimal polynomial of α over E. Furthermore, let $\rho: E \to L \cup \infty$ be a place with $\rho(b) \neq \infty$ and $\rho(a) \neq \infty$, $\rho(a) \neq 0$. Finally, let β be an element of L such that*

$$\beta^2 - \rho(a)\beta + \rho(b) = 0.$$

Then there exists a unique place $\lambda: K \to L \cup \infty$ with $\lambda(\alpha) = \beta$, which extends ρ.

Proof. Let $\mathfrak{o}$ denote the valuation ring of ρ, $\mathfrak{m}$ its maximal ideal and k the residue class field $\mathfrak{o}/\mathfrak{m}$. Let $\gamma: E \to k \cup \infty$ be the canonical place associated to $\mathfrak{o}$. Then $\rho = \overline{\rho} \circ \gamma$, where $\overline{\rho}: k \hookrightarrow L$ is a field embedding. We may assume without loss of generality that k is a subfield of L and that $\overline{\rho}$ is the inclusion map $k \hookrightarrow L$. Every place $\lambda: K \to L \cup \infty$ which extends ρ, has as image a field which is algebraic over k. Therefore we may replace L by the algebraic closure of k in L and thus assume without loss of generality that L is algebraic over k. If c is an element of $\mathfrak{o}$, then $\overline{c}$ will denote the image of c in $\mathfrak{o}/\mathfrak{m} = k$, i.e. $\overline{c} = \gamma(c)$. We have $\beta^2 - \overline{a}\beta + \overline{b} = 0$.

[14] If A is an integral domain, Quot$A$ denotes the quotient field of A.

By the general extension theorem for places [Bo_2, §2, Prop. 3], there exists a place from K to the algebraic closure $\tilde{k}$ of k, which extends γ. We choose such a place $\delta\colon K \to \tilde{k} \cup \infty$. Let $\mathfrak{O}$ be the valuation ring of K, associated to δ, $\mathfrak{M}$ its maximal ideal and $F := \mathfrak{O}/\mathfrak{M}$ its residue class field. Finally, let $\sigma\colon K \to F \cup \infty$ be the canonical place of $\mathfrak{O}$. Then σ extends the place $\gamma\colon E \to k \cup \infty$. By general valuation theory [Bo_2, §8, Th. 1], we have $[F\colon k] \leq 2$. We may envisage F and L as subfields of $\tilde{k}$, which both contain k. One of the things we will show, is that F is equal to the subfield $L' := k(\beta)$ of L.

The field extension K/L is separable since the coefficient a in the minimal polynomial $p(T)$ is different from zero. Let j be the involution of K over E, i.e. the automorphism of K with fixed field E.

Case 1: $L' \neq k$. So $\beta \notin k$ and $T^2 - \overline{a}T + \overline{b}$ is the minimal polynomial of β over k. Now $\alpha^2 - \alpha a + b = 0$ implies $\sigma(\alpha)^2 - \sigma(\alpha)\overline{a} + \overline{b} = 0$. Thus $\sigma(\alpha) = \beta$ or $\sigma(\alpha) = \overline{a} - \beta$. In particular, $L' = k(\sigma(\alpha)) \subset F$. Since $[L'\colon k] = 2$ and $[F\colon k] \leq 2$, we have $L' = F$ and $F \subset L$.

We have $j(\alpha) = a - \alpha$, and thus $\sigma j(\alpha) = \overline{a} - \sigma(\alpha)$. Since $\overline{a} \neq 0$ this implies $\sigma j(\alpha) \neq \sigma(\alpha)$. Therefore the places σ and $\sigma \circ j$ from K to L are different. They both extend the place ρ. By general valuation theory [Bo_2, §8, Th. 1], there can be no other places from K to $\tilde{k}$ which extend ρ. Now take $\lambda = \sigma$ in case $\sigma(\alpha) = \beta$, and $\lambda = \sigma \circ j$ in case $\sigma(\alpha) = \overline{a} - \beta$. Then $\lambda\colon K \to L \cup \infty$ is the only place which extends ρ and maps α to β.

Case 2: $L' = k$. This time $\beta \in k$. Since β is a root of $T^2 - \overline{a}T + \overline{b}$, we have $T^2 - \overline{a}T + \overline{b} = (T - \beta)(T - \overline{a} + \beta)$ and $\alpha^2 - \alpha\overline{a} + \overline{b} = 0$ implies again that $\sigma(\alpha)^2 - \sigma(\alpha)\overline{a} + \overline{b} = 0$. Therefore,

$$(\sigma(\alpha) - \beta)(\sigma(\alpha) - \overline{a} + \beta) = 0.$$

Hence $\sigma(\alpha) = \beta$ or $\sigma(\alpha) = \overline{a} - \beta$. In particular we have $\sigma(\alpha) \in k$.

Again the places σ and $\sigma \circ j$ are different. By general valuation theory, they are exactly all the places from K to $\tilde{k}$ which extend ρ and $[F\colon k] = 1$, i.e. $F = k$. As in Case 1, we take $\lambda = \sigma$ in case $\sigma(\alpha) = \beta$, and $\lambda = \sigma \circ j$ in case $\sigma(\alpha) = \overline{a} - \beta$. Again λ is the only place from K to L (even the only place from K to $\tilde{k}$) which extends ρ and maps α to β. □

Theorem 3. *Let $\lambda\colon K \to L \cup \infty$ be a place and φ a form over K with* GR *with respect to λ and* $\dim\varphi \geq 2$. *Let $\overline{\varphi} := \lambda_*(\varphi)$. Then λ can be extended to a place $\mu\colon K(\varphi) \to L(\overline{\varphi}) \cup \infty$.*

Proof. If φ is isotropic, then $K(\varphi)/K$ is purely transcendental by Theorem 1. In this case λ can be extended in many ways to a place from $K(\varphi)$ to L. So suppose that φ is anisotropic.

Let $n := \dim\varphi$ and let $\mathfrak{o}$ be the valuation ring of λ. If $n = 2$ we assume in addition that $\overline{\varphi} \not\cong H$, deferring the case $\overline{\varphi} \cong H$ to the end of the proof. After a linear transformation of K^n we may assume without loss of generality that φ is of the form $(*)$ as in the beginning of this section (thus with a_i, $b_i \in \mathfrak{o}$, $c \in \mathfrak{o}^*$ etc.). We denote by $\overline{\varphi}$ the form over L, obtained from φ by replacing the coefficients a_i, b_i and – if n is odd – c by $\overline{a}_i := \lambda(a_i)$, $\overline{b}_i := \lambda(b_i)$,

$\overline{c} := \lambda(c)$. Let $X_1, \ldots, X_n$, resp. $U_1, \ldots, U_n$ be indeterminates, then we also write

$$\varphi(X_1, \ldots, X_n) = a_1 X_1^2 + X_1 X_2 + b_1 X_2^2 + \psi(X_3, \ldots, X_n),$$

and accordingly

$$\overline{\varphi}(U_1, \ldots, U_n) = \overline{a}_1 U_1^2 + U_1 U_2 + \overline{b}_1 U_2^2 + \overline{\psi}(U_3, \ldots, U_n).$$

Furthermore we write

$$K(\varphi) = \mathrm{Quot}\frac{K[X_1, \ldots, X_n]}{(\varphi(X_1, \ldots, X_n))} = K(x_1, \ldots, x_n),$$
$$L(\overline{\varphi}) = \mathrm{Quot}\frac{L(U_1, \ldots, U_n)}{(\overline{\varphi}(U_1, \ldots, U_n))} = L(u_1, \ldots, u_n),$$

where x_i denotes of course the image of X_i in $K(\varphi)$ and u_i the image of U_i in $L(\overline{\varphi})$. We then have the relations

$$a_1 x_1^2 + x_1 x_2 + b_1 x_2^2 + \psi(x_3, \ldots, x_n) = 0,$$
$$\overline{a}_1 u_1^2 + u_1 u_2 + \overline{b}_1 u_2^2 + \overline{\psi}(u_3, \ldots, u_n) = 0.$$

{If $n = 2$, the last summands on the left should be read as zero.} It is easy to see that the space $N := \begin{bmatrix} a_1 & 1 \\ 1 & b_1 \end{bmatrix}$ over $\mathfrak{o}$ is isometric to a space $\begin{bmatrix} a_1' & 1 \\ 1 & b_1' \end{bmatrix}$ with $a_1' \in \mathfrak{o}^*$. {Lift a suitable basis of $N/\mathfrak{m}N$ to a basis of N.} Therefore we additionally assume, without loss of generality, that $a_1 \in \mathfrak{o}^*$, i.e. $\overline{a}_1 \neq 0$.

The elements $x_2, \ldots, x_n$ are algebraically independent over K and likewise the elements $u_2, \ldots, u_n$ are algebraically independent over L. Let

$$E := K(x_2, \ldots, x_n) \subset K(\varphi)$$

and

$$F := L(u_2, \ldots, u_n) \subset L(\overline{\varphi}).$$

Our place λ has exactly one extension $\tilde{\lambda}: E \to F \cup \infty$ with $\tilde{\lambda}(x_i) = u_i$ $(2 \leq i \leq n)$ (cf. [Bo$_2$, §10, Prop. 2]).

Let $\tilde{\mathfrak{o}}$ be the valuation ring of $\tilde{\lambda}$. We have $K(\varphi) = E(x_1)$, $L(\overline{\varphi}) = F(u_1)$ with relations

$$x_1^2 + a x_1 + b = 0, \quad u_1^2 + \overline{a} u_1 + \overline{b} = 0,$$

where $a := a_1^{-1} x_2$, $b := a_1^{-1}(b_1 x_2^2 + \psi(x_3, \ldots, x_n))$ are in $\tilde{\mathfrak{o}}$ and $\overline{a} = \tilde{\lambda}(a)$, $\overline{b} = \tilde{\lambda}(b)$.

Surely $K(\varphi) \neq E$, because $\varphi \otimes K(\varphi)$ is isotropic, but $\varphi \otimes E$ is anisotropic since the extension E/K is purely transcendental. Therefore $K(\varphi)$ is a quadratic field extension of E. Furthermore $\overline{a} = \overline{a}_1^{-1} u_2 \neq 0$. Hence we can apply Lemma 2 to the place $\tilde{\lambda}: E \to F \cup \infty$ and the field extensions $K(\varphi)/E$, $L(\overline{\varphi})/F$. According to this lemma, there exists a (unique) place $\mu: K(\varphi) \to L(\overline{\varphi})$ which extends $\tilde{\lambda}$ and maps x_1 to u_1. This proves the theorem in case $\overline{\varphi} \not\cong H$.

Finally, let $n = 2$, φ anisotropic, but $\overline{\varphi}$ isotropic. We can still use the description of $K(\varphi)$ given above (this time with $\psi = 0$), i.e. $\varphi = \begin{bmatrix} a_1 & 1 \\ 1 & b_1 \end{bmatrix}$ with $a_1 \in \mathfrak{o}^*$, $b_1 \in \mathfrak{o}$, $K(\varphi) = K(x_1, x_2)$ with x_2 transcendental over K and

$$(1) \qquad a_1x_1^2 + x_1x_2 + b_1x_2^2 = 0.$$

We want to extend λ to a place $\mu: K(\varphi) \to L \cup \infty$. In order to do this, we choose a nontrivial zero $(c_1, c_2) \in L^2$ of $\overline{\varphi}$. Then we choose an extension $\tilde{\lambda}: K(x_2) \to L \cup \infty$ of λ with $\tilde{\lambda}(x_2) = c_2$, cf. [Bo$_2$, §10, Prop. 1]. {Take the "variable" $x_2 - c_2$ there.} We have

$$(2) \qquad \overline{a}_1c_1^2 + c_1c_2 + \overline{b}_1c_2^2 = 0.$$

If $c_2 = 0$, then $c_1 = 0$ by (2). Our zero is nontrivial however, so $c_2 \neq 0$.

Equation (1) shows that $x_1^2 + ax_1 + b = 0$ with $a := a_1^{-1}x_2$, $b := a_1^{-1}b_1x_2^2$. We have $\tilde{\lambda}(a) = \overline{a}_1^{-1}c_2 \neq 0$, $\tilde{\lambda}(b) = \overline{a}_1^{-1}\overline{b}_1c_2^2$. Equation (2) shows that $c_1^2 + \tilde{\lambda}(a)c_1 + \tilde{\lambda}(b) = 0$. Since φ is anisotropic, $K(\varphi) \neq K(x_1)$, thus $[K(\varphi): K(x_1)] = 2$. Lemma 2 tells us that there exists a (unique) place $\mu: K(\varphi) \to L \cup \infty$ which extends $\tilde{\lambda}$. □

Remark. In the proof we did not need the uniqueness statement of Lemma 2. Nonetheless, it deserves some attention. For example, one can use it to deduce from our proof that, given an anisotropic $\overline{\varphi}$, there is *exactly one* place $\mu: K(\varphi) \to L(\overline{\varphi}) \cup \infty$ which extends λ and maps x_i to u_i $(1 \leq i \leq n)$. Now, this holds for the generators $x_1, \dots, x_n$ of $K(\varphi)$ and $u_1, \dots, u_n$ of $L(\overline{\varphi})$, associated to the special representation $(*)$ of φ above (still with $a_1 \in \mathfrak{o}^*$) and can, by means of a coordinate transformation (with coefficients in $\mathfrak{o}$, etc.), be transferred to the case where $\varphi = [a_{ij}]$ for an arbitrary symmetric matrix (a_{ij}) over $\mathfrak{o}$ with

$$\det \begin{pmatrix} 2a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & 2a_{nn} \end{pmatrix} \in \mathfrak{o}^*.$$

Theorem 4. *Let $\lambda: K \to L \cup \infty$ be a place. Let φ be a form over k with* $\dim \varphi \geq 2$*, which has* GR *with respect to λ. Then $\lambda_*(\varphi)$ is isotropic if and only if λ can be extended to a place $\mu: K(\varphi) \to L \cup \infty$.*

Proof. If λ can be extended to a place μ from $K(\varphi)$ to L, then $\lambda_*(\varphi)$ has to be isotropic by the standard argument, which we already used in §4, just after Theorem 2.

Conversely, suppose that $\overline{\varphi} := \lambda_*(\varphi)$ is isotropic. By Theorem 3, there exists a place

$$\mu: K(\varphi) \to L(\overline{\varphi}) \cup \infty$$

which extends λ. By Lemma 1, $L(\overline{\varphi})/L$ is a purely transcendental extension. Therefore there exists a place

$$\rho: L(\overline{\varphi}) \to L \cup \infty$$

over L. Now $\rho \circ \mu: K(\varphi) \to L \cup \infty$ is a place which extends λ. $\square$

Remark. We could have adapted the proof of Theorem 3 in such a way, that we would have obtained Theorem 4 immediately (see in particular our argument there in the case $\overline{\varphi} \cong H$). On the other hand, we can obtain Theorem 3 from Theorem 4 by applying the latter to the place

$$j \circ \lambda: K(\varphi) \to L(\overline{\varphi}) \cup \infty,$$

where j is the inclusion $L \hookrightarrow L(\overline{\varphi})$. For our further investigations (§10, §12, §13) it is better however, to isolate Theorem 3 and its proof as a stopover to Theorem 4.

Corollary. *Let φ be a form over a field k with $\dim \varphi \geq 2$. Then $k(\varphi)$ is a generic zero field of φ.*

Proof. Clearly $\varphi \otimes k(\varphi)$ is isotropic. Now let $L \supset k$ be a field extension with $\varphi \otimes L$ isotropic. Applying Theorem 4 to the trivial place $k \hookrightarrow L$, we see that there exists a place $\mu: k(\varphi) \to L \cup \infty$ over k. $\square$

Since Theorems 1 and 2 of §4 are subcases of our current Theorem 4, they are now proved. The definitions and theorems following these two theorems remain valid. Later on they will be used without further comments in renewed generality, namely for arbitrary characteristic instead of characteristic $\neq 2$, regular quadratic forms instead of the nondegenerate symmetric bilinear forms which occur there. In particular, for every regular quadratic form φ over k, we have a generic splitting tower $(K_i \mid 0 \leq i \leq h)$ with higher indices i_r and higher kernel forms φ_r.

Comments. It should be noted that I. Kersten and U. Rehmann have used a different route in [Ke R, §6] to construct, for every form φ over k and every $r \leq \left[\frac{\dim \varphi}{2}\right]$, a field extension of k which is generic for the splitting off of r hyperbolic planes. In particular one can already find a generic splitting tower of φ in their work. In [KR] the generic splitting of regular quadratic forms in arbitrary characteristic is based on these foundations.

The results about forms of height 1 and 2, cited at the end of our §4, have so far only been established for characteristic $\neq 2$ in the literature.

§10 Separable Splitting

All fields occurring in this section are supposed to have characteristic 2. If φ is a nondegenerate quadratic form over such a field k, then its quasilinear part (cf. §6, Definition 1), which we denote by $QL(\varphi)$, is anisotropic. Now, if $K \supset k$ is a field extension, then it can happen that $QL(\varphi \otimes K) = QL(\varphi) \otimes K$ is isotropic, and thus that $\varphi \otimes K$ is degenerate. This possibility prompted us, when dealing with the theory of generic splitting in the previous sections, to only allow regular forms, i.e. quadratic forms φ with $\dim QL(\varphi) \leq 1$.

We will now make a course change and allow arbitrary nondegenerate quadratic forms. We will however only tolerate a restricted class of field extensions, the so-called separable field extensions. We will see that a reasonably satisfying theory of generic splitting is still possible in this case.

A field extension $K \supset k$ is called *separable* if every finitely generated subextension $E \supset k$ has a *separating transcendence basis*, i.e. a transcendence basis $t_1, \ldots, t_n$, such that E is separably algebraic over $k(t_1, \ldots, t_n)$, cf. [Bo$_3$], [Lg$_1$, X, §6], [J, IV, §5]. If K is already finitely generated over k, then it suffices to check if K itself contains a separating transcendence basis over k (loc. cit.).

In what follows, a "form" will always be understood to be a quadratic form. The foundation for the rest of this section is

Theorem 1. *Let φ be a nondegenerate form over k, and let $K \supset k$ be a separable field extension. Then $\varphi \otimes K$ is also nondegenerate.*

Proof. We work in the algebraic closure $\widetilde{K}$ of K. We have $QL(\varphi) = [a_1, \ldots, a_r]$ with $a_i \in K$. This form is anisotropic. We have to show that $QL(\varphi) \otimes K$ is also anisotropic. Each a_i has exactly one square root $\sqrt{a_i} \in \widetilde{K}$, which is already in the radical closure $k^{1/2^\infty} \subset K^{1/2^\infty} \subset \widetilde{K}$ of k. Now, since $QL(\varphi)$ is anisotropic, the elements $a_1, \ldots, a_n$ are linearly independent over the subfield $k^2 = \{x^2 \mid x \in k\}$ of k, or equivalently, the elements $\sqrt{a_1}, \ldots, \sqrt{a_n}$ are linearly independent over k. By an important theorem about separable field extensions ("MacLane's Criterion", loc. cit.), the fields K and $k^{1/2^\infty}$ are linearly disjoint over k. Therefore, the elements $\sqrt{a_1}, \ldots, \sqrt{a_r}$ are linearly independent over K. This shows that $QL(\varphi) \otimes K$ is anisotropic. □

If $\mathfrak{o}$ is a valuation ring with maximal ideal $\mathfrak{m}$, we denote the residue class field $\mathfrak{o}/\mathfrak{m}$ from now on by $\kappa(\mathfrak{o})$.

Theorem 2. *Let $\lambda: K \to L \cup \infty$ be a place and φ a form which has* GR *with respect to λ.*[15] *Suppose that the form $\lambda_*(\varphi)$ is nondegenerate. Suppose further that $K' \supset K$ is a field extension and that $\mu: K' \to L \cup \infty$ is an extension of the place λ. Then the form $\varphi \otimes K'$ has* GR *with respect to μ and $\mu_*(\varphi \otimes K') \cong \lambda_*(\varphi)$.*

Proof. Let $\mathfrak{o} := \mathfrak{o}_\lambda$, $\mathfrak{o}' := \mathfrak{o}_\mu$. The field extension $\overline{\lambda}: \kappa(\mathfrak{o}) \hookrightarrow L$ is a combination of the extensions $\kappa(\mathfrak{o}) \hookrightarrow \kappa(\mathfrak{o}')$ and $\overline{\mu}: \kappa(\mathfrak{o}') \hookrightarrow L$, where the first extension is induced by the inclusion $\mathfrak{o} \hookrightarrow \mathfrak{o}'$.

Let E be a quadratic space for φ and M a nondegenerate quadratic $\mathfrak{o}$-module with $E \cong K \otimes_{\mathfrak{o}} M$. Then $K' \otimes E = K' \otimes_{\mathfrak{o}'} M'$ with $M' := \mathfrak{o}' \otimes_{\mathfrak{o}} M$. The quasilinear quadratic $\kappa(\mathfrak{o})$-module $G := \kappa(\mathfrak{o}) \otimes_{\mathfrak{o}} QL(M)$ is anisotropic. By assumption, $L \otimes_{\overline{\lambda}} G = QL(L \otimes_\lambda M)$ is also anisotropic. Therefore

$$\kappa(\mathfrak{o}') \otimes_{\kappa(\mathfrak{o})} G = \kappa(\mathfrak{o}') \otimes_{\mathfrak{o}'} QL(M')$$

is anisotropic. This proves that M' is a nondegenerate quadratic $\mathfrak{o}'$-module. Hence $\varphi \otimes K'$ is nondegenerate and has GR with respect to μ. Furthermore, $\mu_*(\varphi \otimes K')$ corresponds to the quadratic space

$$L \otimes_\mu M' = L \otimes_\mu (\mathfrak{o}' \otimes_{\mathfrak{o}} M) = L \otimes_\lambda M.$$

Hence $\mu_*(\varphi \otimes K') \cong \lambda_*(\varphi)$. □

Now we can faithfully repeat the observation, made at start of our treatment of the theory of generic splitting in §4.

So, let φ be a nondegenerate form over a field k (of characteristic 2), and let $K \supset k$ and $L \supset k$ be field extensions of k with $L \supset k$ separable. Let $\lambda: K \to L \cup \infty$ be a place over k. On the basis of Theorem 1, we can apply Theorem 2 to the trivial place $k \hookrightarrow L$ and its extension λ. We see that $\varphi \otimes K$ is nondegenerate and has GR with respect to λ, and that $\lambda_*(\varphi \otimes K) \cong \varphi \otimes L$.

Let $\varphi \otimes K \cong \varphi_1 \perp r_1 \times H$ be the Witt decomposition of φ. By an established argument (§8, Theorem 2(b)), φ_1 has GR with respect to λ, and $\varphi \otimes L \cong \lambda_*(\varphi_1) \perp r_1 \times H$. Hence, $\operatorname{ind}(\varphi \otimes L) \geq \operatorname{ind}(\varphi \otimes K)$. If K is also separable over k and if K and L are specialization equivalent over k, it follows again that $\operatorname{ind}(\varphi \otimes L) = \operatorname{ind}(\varphi \otimes K)$ and $\ker(\varphi \otimes L) \cong \lambda_*(\varphi_1)$.

Definition. A *generic separable zero field of* φ is a separable field extension $K \supset k$ which has the following properties:

(a) $\varphi \otimes K$ is isotropic.

(b) If $L \supset k$ is a separable field extension with $\varphi \otimes L$ isotropic, then there exists a place $\lambda: K \to L \cup \infty$ over k.

According to Theorem 1, φ can become isotropic over a separable field extension L of k only if $\varphi \neq QL(\varphi)$, i.e. if $\dim \varphi - \dim QL(\varphi) \geq 2$. If φ

[15] This assumption presupposes that φ is nondegenerate (cf. §7, Definition 1).

is such a form with $\dim \varphi \neq 2$, then the polynomial $\varphi(X_1, \dots, X_n)$ is irreducible over the algebraic closure $\widetilde{k}$ of k. This can easily be seen by writing $\varphi(X_1, \dots, X_n) = X_1X_2 + \psi(X_3, \dots, X_n)$ over $\widetilde{k}$, after a coordinate transformation. Here ψ is a quadratic polynomial which is not the zero polynomial. Therefore, we can again construct the field

$$k(\varphi) = \mathrm{Quot}\frac{k[X_1, \dots, X_n]}{(\varphi(X_1, \dots, X_n))}.$$

This extends our definition of $k(\varphi)$ in §9 from regular forms φ to nondegenerate forms φ. {Obviously $k(\varphi)$ will have its original meaning when $\dim \varphi = 2$, $QL(\varphi) = 0$.}

It is now easy to see that $k(\varphi)$ is separable over k: Let $x_1, \dots, x_n$ be the images of $X_1, \dots, X_n$ in $k(\varphi)$, i.e. $k(\varphi) = k(x_1, \dots, x_n)$. After a coordinate transformation we may suppose, without loss of generality, that

$$\varphi(X_1, \dots, X_n) = a_1X_1^2 + X_1X_2 + b_1X_2^2 + \psi(X_3, \dots, X_n).$$

The elements $x_2, \dots, x_n$ form a transcendence basis of $k(\varphi)$ over k. If $x_1 \notin k(x_2, \dots, x_n)$, then $k(\varphi)$ is a separable quadratic extension of $k(x_2, \dots, x_n)$. If the form φ is isotropic, we can make it so that $a_1 = 0$. Then $k(\varphi) = k(x_2, \dots, x_n)$ is purely transcendental over k.

Theorem 3. *Let φ be a nondegenerate form over a field K with $\varphi \neq QL(\varphi)$ and let $\lambda\colon K \to L \cup \infty$ be a place such that φ has* GR *with respect to λ. Suppose that also the form $\lambda_*(\varphi)$ is nondegenerate. Then $\lambda_*(\varphi)$ is isotropic if and only if λ can be extended to a place $\mu\colon K(\varphi) \to L \cup \infty$.*

Proof. If there exists such a place μ, then $\lambda_*(\varphi)$ is isotropic by an established argument, using Theorem 2 above.

Suppose now that $\overline{\varphi} := \lambda_*(\varphi)$ is isotropic. By the theory in §9, we may suppose that $\dim QL(\varphi) \geq 2$, thus $\dim \varphi \geq 4$. Just as in the proof of Theorem 3 in §9, we see that λ can be extended to a place $\mu\colon K(\varphi) \to L(\overline{\varphi}) \cup \infty$. {Note that this is also true in case $\overline{\varphi}$ is anisotropic.} Since $\overline{\varphi}$ is isotropic, $L(\overline{\varphi})$ is a purely transcendental extension of L, as established above. Hence there exists a place ρ from $L(\overline{\varphi})$ to L over L. The place $\rho \circ \mu\colon K(\varphi) \to L \cup \infty$ extends λ. □

Corollary. *Let φ be a nondegenerate form over a field k with $\varphi \neq QL(\varphi)$. Then $k(\varphi)$ is a generic separable zero field of φ.* □

Since we have secured the existence of a generic separable zero field, we obtain for every form φ over k a *generic separable splitting tower*

$$(K_r \mid 0 \leq r \leq h)$$

with higher indices i_r and higher kernel forms φ_r $(0 \leq r \leq h)$, in complete analogy with the construction of generic splitting towers in §4 (just before §4, Definition 5). Thus K_0/k is a *separable* inessential field extension,

$\varphi_r = \ker(\varphi \otimes K_r)$, and K_{r+1} is a generic separable zero field of φ_r for $r < h$. The height h satisfies $h \leq \frac{1}{2}(\dim \varphi - \dim QL(\varphi))$.

In analogy with §4, Theorem 3, we have the following theorem, with mutatis mutandis the same proof.

Theorem 4. *Let φ be a nondegenerate form over k. Let $(K_r \mid 0 \leq r \leq h)$ be a generic separable splitting tower of φ with associated higher kernel forms φ_r and indices i_r. Let $\gamma \colon k \to L \cup \infty$ be a place such that φ has* GR *with respect to γ. Suppose that the form $\gamma_*(\varphi)$ is nondegenerate. Finally, for an m with $0 \leq m \leq h$, let a place $\lambda \colon K_m \to L \cup \infty$ be given, which extends γ and which cannot be extended to K_{m+1} in case $m < h$. Then φ_m has* GR *with respect to λ. The form $\gamma_*(\varphi)$ has kernel form $\lambda_*(\varphi_m)$ and Witt index $i_0 + \ldots + i_m$.* □

Now we can faithfully repeat Scholium 1 to 4 from §4, but this time with generic separable splitting towers and separable field extensions. We leave this to the reader. {In Scholium 3, on should of course assume – as in Theorem 4 above – that $\gamma_*(\varphi)$ is nondegenerate.}

In particular, the generic separable splitting tower $(K_r \mid 0 \leq r \leq h)$ regulates the splitting behaviour of φ with respect to *separable* field extensions $L \supset k$, as described in §4, Scholium 1.

§11 Fair Reduction and Weak Obedience

Our specialization theory of quadratic forms, developed in §6 – §8, gave a satisfying basis for understanding the splitting behaviour of quadratic forms under field extensions (if the forms were not regular, we had to limit ourselves to separable field extensions). There is one important point however, where the specialization theory is disappointing.

For instance, let $\lambda\colon K \to L \cup \infty$ be a place from a field K of characteristic 0 to a field L of characteristic 2. If φ is a quadratic form over K which has good reduction with respect to λ, then $\lambda_*(\varphi)$ is automatically a strictly regular form. Conversely, given a strictly regular form ψ over L, one can easily find a strictly regular (= nondegenerate) form φ over K which has good reduction with respect to λ and such that $\lambda_*(\varphi) \cong \psi$. One could then try to deduce properties of ψ from properties of the "lifting" φ of ψ, in the hope that φ is easier to deal with than ψ because $\operatorname{char} K = 0$. Good examples can be obtained from the generalization of §4, Scholium 3, at the end of §9, but we will not carry this out.

It is furthermore desirable to lift a nondegenerate form ψ over L, with quasilinear part $QL(\psi) \neq 0$, to a form φ over K. This is not possible with our specialization theory as it stands.

Let us review the foundations of the current theory! We return to the use of geometric language. Let $\lambda\colon K \to L \cup \infty$ be a place with associated valuation ring $\mathfrak{o}$. Let $\mathfrak{m}$ be the maximal ideal of $\mathfrak{o}$ and $k = \mathfrak{o}/\mathfrak{m}$ its residue class field. Finally, let $\overline{\lambda}\colon k \hookrightarrow L$ be the field extension determined by λ.

Given a suitable quadratic space E over K, the idea was to attach a space F over k to E and then to define the specialization $\lambda_*(E)$ of E with respect to λ as the space $L \otimes_{\overline{\lambda}} F$. We required E to have good reduction, i.e. $E \cong K \otimes_{\mathfrak{o}} M$ for a nondegenerate quadratic module M over $\mathfrak{o}$. Then we could choose our space F over k to be $F = M/\mathfrak{m}M$.

The preceding is analogous to the specialization theory for varieties in algebraic and arithmetic geometry (and related areas, such as rigid analysis): One defines a class of "nondegenerate" objects $\mathcal{X}$ over the valuation ring $\mathfrak{o}$ and associates to it, by means of a base extension, the objects $K \otimes_{\mathfrak{o}} \mathcal{X}$ and $k \otimes_{\mathfrak{o}} \mathcal{X}$ over K and k. Finally one decrees that $L \otimes_{\overline{\lambda}} (k \otimes_{\mathfrak{o}} \mathcal{X}) = L \otimes_{\lambda} \mathcal{X}$ is the specialization of $K \otimes_{\mathfrak{o}} \mathcal{X}$ with respect to λ.

Without any doubt, we have found in the nondegenerate quadratic spaces, as defined in §6, Definition 3, a respectable class $\mathcal{X}$ of such objects which easily comes to mind. Nevertheless we are a bit unfair to all the quadratic spaces over K, waiting to be specialized. It would be fairer to only require that $E \cong K \otimes_{\mathfrak{o}} M$ with M a free quadratic $\mathfrak{o}$-module such that $M/\mathfrak{m}M$ is nondegenerate, and then to define $\lambda_*(E) := L \otimes_\lambda M = L \otimes_{\overline{\lambda}} (M/\mathfrak{m}M)$. But then one should show, that up to isometry, $\lambda_*(E)$ only depends on λ and E and not on the choice of M.

This is possible, as we shall show now. In the following a *quadratic module* over a valuation ring (in particular a field) will always be understood to be a *free quadratic module of finite rank.*

Let $\mathfrak{o}$ be a valuation ring with maximal ideal $\mathfrak{m}$, quotient field K and residue class field $k = \mathfrak{o}/\mathfrak{m}$. If $2 \notin \mathfrak{m}$, then we already know all what follows. On formal grounds, we allow nevertheless the uninteresting case that char $k \neq 2$.

Definition 1.

(a) A quadratic module M over $\mathfrak{o}$ is called *reduced nondegenerate* if the quadratic module $M/\mathfrak{m}M$ over k is nondegenerate. (Note: If char $k \neq 2$, this implies that M itself is nondegenerate.)

(b) A quadratic module E over K has *fair reduction* (or: FR for short) *with respect to* λ (or: *with respect to* $\mathfrak{o}$), if there exists a reduced nondegenerate quadratic module M over $\mathfrak{o}$ such that $E \cong K \otimes_{\mathfrak{o}} M$.[16]

It is our task to prove that in the situation of Definition 1(b), the space $M/\mathfrak{m}M$ is independent of the choice of reduced nondegenerate module M up to isometry. We will however proceed in a more general direction than necessary, in order to develop at the same time the equipment necessary to obtain a generalization of the important Theorem 2(b) (§8) for fair reduction instead of good reduction. First a very general definition.

Definition 2. Let $M = (M, q)$ and $M' = (M', q')$ be quadratic modules over a ring A. We say that M *represents* the module M', and write $M' < M$, if the A-module M has a direct sum decomposition $M = M_1 \oplus M_2$ with $(M_1, q|M_1) \cong (M', q')$.

Now, if M is a quadratic $\mathfrak{o}$-module, we always regard M as an $\mathfrak{o}$-submodule of the K-vector space $K \otimes_{\mathfrak{o}} M$. So we have $K \otimes_{\mathfrak{o}} M = KM$. In the following, we will almost always denote the quadratic form on M by q, and its associated bilinear form B_q by B. We will also use q to denote the quadratic form on $K \otimes_{\mathfrak{o}} M$ with values in K, obtained from q. We will often denote the module $M/\mathfrak{m}M$ over k by $\overline{M}$, and the image of a vector $x \in M$ in $\overline{M}$ with $\overline{x}$. Likewise we denote the image of a scalar $a \in \mathfrak{o}$ in k by $\overline{a}$. Finally, we denote by $\overline{q}$ the quadratic form on $\overline{M}$ with values in k, induced by q, i.e. $\overline{q}(\overline{x}) = \overline{q(x)}$ for $x \in M$. Furthermore, $\overline{B}$ will stand for the associated bilinear form $B_{\overline{q}}$, i.e. $\overline{B}(\overline{x}, \overline{y}) = \overline{B(x, y)}$ for $x, y \in M$.

[16] In [K_4] the words "nearly good reduction" are used instead of "fair reduction". We avoid this terminology here, because we will talk about a different concept, "almost good reduction" later (Chapter IV).

As before, we call a nondegenerate quadratic module over a field or valuation ring a *quadratic space*, or just a *space*. If M is a reduced nondegenerate quadratic $\mathfrak{o}$-module, then $\overline{M} = M/\mathfrak{m}M$ is a space over k. It is easy to show that $K \otimes M$ is a space over K too (see the Remark below, after Definition 3), but that is not so important at the moment.

Lemma 1. *Let M be a reduced nondegenerate quadratic module over $\mathfrak{o}$. There exists a decomposition $M = M_1 \perp M_2$ with M_1 strictly regular and $B(M_2 \times M_2) \subset \mathfrak{m}$. If such a decomposition is given and x is a primitive vector of M_2, then $q(x) \in \mathfrak{o}^*$.*

Proof. We have a decomposition $\overline{M} = U \perp QL(\overline{M})$ with U strictly regular.[17] Let $x_1, \dots, x_r$ be vectors of M, such that the images $\overline{x}_1, \dots, \overline{x}_1$ in $\overline{M}$ form a basis of the k-vector space U. Then the determinant of the value matrix $(B(x_i, x_j))$ with $1 \le i,\ j \le r$ is a unit of $\mathfrak{o}$. Therefore, $M_1 := \sum_{i=1}^{r} \mathfrak{o}x_i$ is a strictly regular space. Let M_2 be the orthogonal complement $M_1^\perp$ of M_1 in M. Then $M = M_1 \perp M_2$ and $M_2/\mathfrak{m}M_2 = QL(\overline{M})$. Since $QL(\overline{M})$ is quasilinear, we have $B(M_2 \times M_2) \subset \mathfrak{m}$. Conversely, if such a decomposition $M = M_1 \perp M_2$ is given, with M_1 strictly regular and $B(M_2 \times M_2) \subset \mathfrak{m}$, then $\overline{M}_2 = QL(\overline{M})$. Since $QL(\overline{M})$ is anisotropic, every primitive vector x in M_2 has a value $q(x) \in \mathfrak{o}^*$. □

Lemma 2. *Suppose again that M is a reduced nondegenerate quadratic $\mathfrak{o}$-module. Then M is maximal among all finitely generated* (*thus free*) *$\mathfrak{o}$-modules $N \subset K \otimes_{\mathfrak{o}} M$ with $q(N) \subset \mathfrak{o}$.*

Proof. We work in the quadratic K-module $E := K \otimes_{\mathfrak{o}} M$. According to Lemma 1, there is a decomposition $M = M_1 \perp M_2$ with M_1 strictly regular and $B(M_2, M_2) \subset \mathfrak{m}$. Let $x \in E$ and $q(M + \mathfrak{o}x) \subset \mathfrak{o}$. We have to show that $x \in M$.

We write $x = x_1 + x_2$ with $x_1 \in K \otimes M_1$, $x_2 \in K \otimes M_2$. For every $y \in M_1$, we have $B(x_1, y) = B(x, y) = q(x + y) - q(x) - q(y) \in \mathfrak{o}$. Since the bilinear form $B = B_q$ is nondegenerate on M_1, it follows that $x_1 \in M_1$. Therefore, $M + \mathfrak{o}x = M + \mathfrak{o}x_2$ and so $q(x_2) \in \mathfrak{o}$. Now, $x_2 = az$ for a primitive vector $z \in M_2$ and $a \in K$. Since $q(z) \in \mathfrak{o}^*$, we get $a \in \mathfrak{o}$, and so $x_2 \in M$. We conclude that $x \in M$. □

Lemma 3. (Extension of §6, Lemma 4.) *Let M be a reduced nondegenerate quadratic $\mathfrak{o}$-module and let e be a primitive isotropic vector in M. Then e can be completed to a hyperbolic vector pair e, f in M.*

Proof. We work again in $E = K \otimes M$. The ideal $B(e, M)$ of $\mathfrak{o}$ is finitely generated. Therefore we have $B(e, M) = a\mathfrak{o}$ for an element a of $\mathfrak{o}$. If a were equal to zero, then the vector $\overline{e} \in \overline{M}$ would lie in the quasilinear part $QL(\overline{M})$ of $\overline{M}$. However, $\overline{e} \neq 0$ and $\overline{q}(\overline{e}) = 0$, so that $\overline{e} \notin QL(\overline{M})$ due to the anisotropy

[17] Recall that $QL(\overline{M})$ denotes the quasilinear part of $\overline{M}$, see the beginning of §6.

of $QL(\overline{M})$. Therefore, $a \neq 0$. Since $B(e, M) = a\mathfrak{o}$ and $q(M) \subset \mathfrak{o}$, we have $q(M + \mathfrak{o}(a^{-1}e)) \subset \mathfrak{o}$. Thus, by Lemma 2, $a^{-1}e \in M$. Since e is primitive, we conclude that $a^{-1} \in \mathfrak{o}$, i.e. $a \in \mathfrak{o}^*$. Hence, $B(e, M) = \mathfrak{o}$. As in the proof of §6, Lemma 4, we choose $z \in M$ with $B(e, z) = 1$ and complete e to a hyperbolic pair with the vector $f := z - q(z)e$. □

Theorem 1. *Let M be strictly regular, N a reduced nondegenerate quadratic $\mathfrak{o}$-module and $K \otimes M < K \otimes N$. Then $M < N$. Furthermore, $N \cong M \perp P$ where P is a reduced nondegenerate quadratic $\mathfrak{o}$-module.*

Proof. If Q is a strictly regular quadratic submodule of N, then $N = Q \perp P$ with $P := Q^\perp = \{x \in N \mid B(x, Q) = 0\}$, since Q is nondegenerate with respect to the bilinear form $B = B_q$ (§5, Lemma 1). We have $\overline{N} = \overline{Q} \perp \overline{P}$ and conclude that $\overline{P}$ is nondegenerate, i.e. P is reduced nondegenerate.

Because of this preliminary remark, it suffices to show that $M < N$. Suppose first of all that M is hyperbolic, i.e. $M \cong r \times H$ for some $r > 0$, where H denotes the quadratic module $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ over $\mathfrak{o}$. We proceed by induction on r.

Since $K \otimes M < K \otimes N$, $K \otimes N$ contains isotropic vectors. We can then choose a *primitive* isotropic vector e in N. This vector can be completed to a hyperbolic vector pair in N, by Lemma 3. Hence, $H < N$ and so $N \cong H \perp N'$ where N' is a reduced nondegenerate quadratic $\mathfrak{o}$-module. From $r \times (K \otimes H) < K \otimes N$, we get $K \otimes N \cong r \times (K \otimes H) \perp G$ for some space G over K. On the other hand, $K \otimes N \cong (K \otimes H) \perp (K \otimes N')$. By the decomposition theorem over K (§6, Theorem 2), we get $(r-1) \times (K \otimes H) \perp G \cong K \otimes N'$. Therefore, $(r-1) \times (K \otimes H) < K \otimes N'$. The induction hypothesis gives us $(r-1) \times H < N'$ and so $r \times H < N$.

In the general case, $K \otimes M < K \otimes N$ implies that $K \otimes (M \perp (-M)) < K \otimes (N \perp (-M))$. Since $M \perp (-M)$ is hyperbolic, it follows from above that $M \perp (-M) < N \perp (-M)$, and so, $N \perp (-M) \cong M \perp (-M) \perp P$ by the remark at the beginning of the proof, where P is another reduced nondegenerate quadratic $\mathfrak{o}$-module. By the decomposition theorem over $\mathfrak{o}$ (§6, Theorem 2), we may conclude that $N \cong M \perp P$, and so $M < N$. □

Theorem 2 [K_4, Lemma 2.8]**.** *Suppose that M and N are reduced nondegenerate quadratic $\mathfrak{o}$-modules with $K \otimes M < K \otimes N$. The spaces $\overline{M} = M/\mathfrak{m}M$ and $\overline{N} = N/\mathfrak{m}N$ over k have the following properties:*

(*a*) $\overline{M} < \overline{N}$.

(*b*) *More precisely: there exist quadratic subspaces S and T of $\overline{N}$ with $\overline{M} < S$, $\overline{N} = S \perp T$, T strictly regular, $\dim S \leq \dim QL(\overline{M}) + \dim QL(\overline{N}) + \dim M$.*

(*c*) *If $\overline{N}$ is anisotropic, then $M < N$.*

Remark. We will not need properties (*b*) and (*c*) later on. They can however be obtained in the following proof at no extra cost.

Proof of Theorem 2. (i) By Lemma 1, we can choose an orthogonal decomposition $M = M_1 \perp M_2$ with M_1 strictly regular and $B(M_2 \times M_2) \subset \mathfrak{m}$. We have $K \otimes M_1 < K \otimes N$. By Theorem 1, we have $N \cong M_1 \perp N_2$, where N_2 is a reduced nondegenerate quadratic $\mathfrak{o}$-module. Then $K \otimes M_1 \perp K \otimes M_2 < K \otimes M_1 \perp K \otimes N_2$ implies $K \otimes M_2 < K \otimes N_2$ by the decomposition theorem over K (§6, Theorem 2) as usual. If we could prove the claims of the theorem for M_2 and N_2 instead of M and N, then they would follow immediately for M and N as well. Hence we assume now, without loss of generality, that $B(M \times M) \subset \mathfrak{m}$.

(ii) If $M = \{0\}$, nothing has to be done. So suppose that $M \neq \{0\}$. Surely k has characteristic 2. We suppose, without loss of generality that $F := K \otimes M$ is a subspace of $E := K \otimes N$ (instead of only: F is isomorphic to a subspace of E). Since $\overline{M}$ is anisotropic, we have $M = \{x \in F \mid q(x) \in \mathfrak{o}\}$ (as already ascertained before). Therefore, $N_1 := N \cap F \subset M$. Moreover, N_1 is a direct summand of the $\mathfrak{o}$-module N, since N/N_1 is torsion free and finitely generated, and thus free. Hence N_1 itself is also free.

Let $m := \dim M$. By the Elementary Divisor Theorem for valuation rings, there exist bases $x_1, \dots, x_m$ and $y_1, \dots, y_m$ of the free $\mathfrak{o}$-modules M and N_1 with $y_i = c_i x_i$, $c_i \in \mathfrak{o}$ $(1 \leq i \leq m)$. We suppose without loss of generality that $c_i = 1$ for $1 \leq i \leq s$ and $c_i \in \mathfrak{m}$ for $s < i \leq m$. {It is allowed that $s = 0$ or $s = m$.}

If $\overline{N}$ is anisotropic, then $q(x) \in \mathfrak{o}^*$ for every primitive vector x of N, and thus for every vector in N_1 which is primitive in N_1. Hence $s = m$, i.e. $N_1 = M$ and $M < N$. This establishes property (c).

(iii) Let V be the image of N_1 in $\overline{N} = N/\mathfrak{m}N$. Since N_1 is a direct summand of N, we may make the identification $V = \overline{N_1}$. Since $B(M \times M) \subset \mathfrak{m}$, we have $B(N_1 \times N_1) \subset \mathfrak{m}$, thus $\overline{B}(V \times V) = 0$. Hence V is quasilinear. We have

$$\text{(1)} \qquad \overline{M} = k\overline{x}_1 \oplus \cdots \oplus k\overline{x}_m \cong [a_1, \dots, a_m], \text{ with } a_i := \overline{q}(\overline{x}_i) \in k^*.$$

$$\text{(2)} \qquad V = k\overline{y}_1 \oplus \cdots \oplus k\overline{y}_m \cong [a_1, \dots, a_s] \perp (m - s) \times [0].$$

Let $R := QL(\overline{N})$ be the quasilinear part of $\overline{N}$ and $V_0 := V \cap R$. Since R is anisotropic, V_0 is also isotropic. From (2) it follows that $V_0 < [a_1, \dots, a_s]$, and so

$$\text{(3)} \qquad [a_1, \dots, a_s] \cong V_0 \perp [b_1, \dots, b_{s-t}]$$

for elements $b_i \in k^*$ and $t := \dim V_0$. {If $t = s$, the righthand side of (3) should be read as V_0.} From (2) and (3) we get a decomposition $V = V_0 \perp U$, with

$$\text{(4)} \qquad U \cong [b_1, \dots, b_{s-t}] \perp (m - s) \times [0].$$

Since $V \cap R = V_0$, we have $U \cap R = \{0\}$. We choose a submodule W of $\overline{N}$ such that $\overline{N} = (R + U) \oplus W = R \oplus U \oplus W$. Since R is the quasilinear part of $\overline{N}$, $P := U \oplus W$ is strictly regular and $\overline{N} = R \perp P$. Let $u_1, \dots, u_{m-t}$ be a basis of U, associated with the representation (4). P is a symplectic vector space with respect to the bilinear form $\overline{B} = B_{\overline{q}}$. Hence we can complete

$u_1, \dots, u_{m-t}$ to a basis $u_1, \dots, u_{m-t}, z_1, \dots, z_{m-t}$ of a subspace P_1 of P with $\overline{B}(u_i, z_j) = \delta_{ij} \quad (1 \le i, j \le m-t)$. As a quadratic space, P_1 is of the form

$$P_1 = (ku_1 + kz_1) \perp \ldots \perp (ku_{m-t} + kz_{m-t})$$

and $P = P_1 \perp T$ for a strictly regular space T.

For $s - t < i \le m - t$, we have $ku_i + kz_i \cong \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Therefore, $[a_{t+i}] < ku_i + kz_i$ for this i. Furthermore we have $V_0 < R$ and $[b_i] < ku_i + kz_i$ for $1 \le i \le s - t$, since $b_i = \overline{q}(u_i)$. Putting everything together, (1) and (3) yield

$$\overline{M} \cong V_0 \perp [b_1, \dots, b_{s-t}] \perp [a_{s+1}, \dots, a_m] < R \perp P_1,$$

and $R \perp P = R \perp P_1 \perp T = \overline{N}$. Let $S := R \perp P_1$, then $\overline{M} < S$ and $\overline{N} = S \perp T$. Since $\overline{M} = QL(\overline{M})$ and $R = QL(\overline{N})$, we have furthermore

$$\begin{aligned} \dim S &= \dim QL(\overline{N}) + 2(m-t) \\ &= \dim QL(\overline{N}) + \dim QL(\overline{M}) + \dim \overline{M} - 2t \\ &\le \dim QL(\overline{N}) + \dim QL(\overline{M}) + \dim M. \end{aligned}$$

□

As a special case of Theorem 2, we obtain

Corollary 1. *Let M and M' be two reduced nondegenerate quadratic $\mathfrak{o}$-modules with $K \otimes M \cong K \otimes M'$. Then $M/\mathfrak{m}M \cong M'/\mathfrak{m}M'$.* □

Now let $\lambda: K \to L \cup \infty$ again be a place and $\mathfrak{o}$ its associated valuation ring. As before, we denote by $\overline{\lambda}$ the field embedding $k \hookrightarrow L$ determined by λ. By Corollary 1 it makes sense to make the following definition:

Definition 3. Let E be a space over K which has FR with respect to λ. Let M be a reduced nondegenerate quadratic $\mathfrak{o}$-module with $E \cong K \otimes M$. We call the quadratic L-module $L \otimes_\lambda M = L \otimes_{\overline{\lambda}} (M/\mathfrak{m}M)$ (which is uniquely determined by E up to isometry) the *specialization of E with respect to* λ, and denote it by $\lambda_*(E)$.

Note. If E has good reduction with respect to λ, then $\lambda_*(E)$ has the old meaning.

Look out! If $\operatorname{char} L = 2$, then $\lambda_*(E)$ can be a degenerate quadratic L-module, even if E is strictly regular. However, this does not happen when the field embedding $\overline{\lambda}$ is separable, see §10.

Remark. If M is a reduced nondegenerate quadratic $\mathfrak{o}$-module, then the quadratic K-module $K \otimes M$ is definitely not degenerate, for we know that $QL(K \otimes M) = K \otimes QL(M)$, and if $QL(K \otimes M)$ were isotropic, then $QL(M)$, and thus $QL(M/\mathfrak{m}M)$, would also be isotropic. Therefore our assumption in Definition 3, that the quadratic K-module E is nondegenerate, is a natural one and does not cause a loss of generality.

We illustrate the concept of fair reduction with a few examples.

Example 1. Let $\mathfrak{o}$ be a valuation ring with $\operatorname{char} K = 0$, $\operatorname{char} k = 2$, and let $\lambda: K \to k \cup \infty$ be the canonical place of $\mathfrak{o}$. We want to lift an arbitrary space S over k to a space over K by means of λ. We choose a decomposition

$$S \cong \begin{bmatrix} \alpha_1 & 1 \\ 1 & \beta_1 \end{bmatrix} \perp \ldots \perp \begin{bmatrix} \alpha_m & 1 \\ 1 & \beta_m \end{bmatrix} \perp [\gamma_1, \ldots, \gamma_r]$$

with α_i, β_i, $\gamma_j \in k$. {Since the quasilinear part of S is anisotropic, the elements $\gamma_1, \ldots, \gamma_r$ are linearly independent over k^2.} Next we choose pre-images a_i, b_i, c_j of the elements α_i, β_i, γ_j in $\mathfrak{o}$. Suppose that $r = 2s$ is even. We choose elements $t_1, \ldots, t_s \in \mathfrak{m}$. Then the K-space

$$E := \begin{bmatrix} a_1 & 1 \\ 1 & b_1 \end{bmatrix} \perp \ldots \perp \begin{bmatrix} a_m & 1 \\ 1 & b_m \end{bmatrix} \perp \begin{bmatrix} c_1 & t_1 \\ t_1 & c_2 \end{bmatrix} \perp \ldots \perp \begin{bmatrix} c_{2s-1} & t_s \\ t_s & c_{2s} \end{bmatrix}$$

clearly has FR with respect to λ and $\lambda_*(E) \cong S$. If $r = 2s + 1$ is odd, we again choose elements $t_1, \ldots, t_s \in \mathfrak{m}$. This time the K-space

$$E := \begin{bmatrix} a_1 & 1 \\ 1 & b_1 \end{bmatrix} \perp \ldots \perp \begin{bmatrix} a_m & 1 \\ 1 & b_m \end{bmatrix} \perp \begin{bmatrix} c_1 & t_1 \\ t_1 & c_2 \end{bmatrix} \perp \ldots \perp \begin{bmatrix} c_{2s-1} & t_s \\ t_s & c_{2s} \end{bmatrix} \perp [c_{2s+1}]$$

has FR with respect to λ and $\lambda_*(E) \cong S$. In the special case that all $t_i = 0$, we obtain a K-space

$$F := \begin{bmatrix} a_1 & 1 \\ 1 & b_1 \end{bmatrix} \perp \ldots \perp \begin{bmatrix} a_m & 1 \\ 1 & b_m \end{bmatrix} \perp [c_1, \ldots, c_r]$$

for every r. Since $\operatorname{char} K = 0$, we can interpret F (and more generally E) as a bilinear space,

$$F \cong \begin{pmatrix} 2a_1 & 1 \\ 1 & 2b_1 \end{pmatrix} \perp \ldots \perp \begin{pmatrix} 2a_m & 1 \\ 1 & 2b_m \end{pmatrix} \perp \langle 2c_1, \ldots, 2c_r \rangle.$$

The elements $a_1, \ldots, a_m$ can always be chosen to be $\neq 0$ and we obtain the diagonalization

$$F \cong \langle 2a_1, d_1, \ldots, 2a_m, d_m, 2c_1, \ldots, 2c_r \rangle$$

with

$$d_i := \frac{4a_i b_i - 1}{2a_i} \quad (1 \leq i \leq r).$$

Example 2. Let k be an imperfect field of characteristic 2, $\mathfrak{o}$ the power series ring $k[[t]]$ in one variable t, and so $K = k((t))$. Choose $c \in k \setminus k^2$ and let E be the space $\left[\begin{smallmatrix} 1 & 1 \\ 1 & ct^{-2} \end{smallmatrix}\right]$ over K.

Claim. E has FR, but does not have GR with respect to $\mathfrak{o}$.

Proof. We have $E \cong K \otimes M$, where M is the quadratic $\mathfrak{o}$-module $\left[\begin{smallmatrix} 1 & t \\ t & c \end{smallmatrix}\right]$. Its reduction $\overline{M} = M/\mathfrak{m}M$ is the space $\left[\begin{smallmatrix} 1 & 0 \\ 0 & c \end{smallmatrix}\right] = [1, c]$ over k, which is anisotropic. Therefore M is reduced nondegenerate and E has FR.

Suppose for the sake of contradiction that $E \cong K \otimes N$ where N is a quadratic space over $\mathfrak{o}$. Since E does not have a quasilinear part, the same is true for N. Hence, $N \cong \begin{bmatrix} \alpha & 1 \\ 1 & \beta \end{bmatrix}$ with $\alpha, \beta \in \mathfrak{o}$. We then have $\left[\begin{smallmatrix} 1 & 1 \\ 1 & ct^{-2} \end{smallmatrix}\right] \cong \begin{bmatrix} \alpha & 1 \\ 1 & \beta \end{bmatrix}$ over K. An inspection of the Arf-invariants shows that $ct^{-2} = \alpha\beta + x^2 + x$ for some $x \in k((t))$. We must have $x \neq 0$. We write $x = \sum_{i \geq d} a_i t^i$ for some $d \in \mathbb{Z}$ and $a_d \neq 0$. Since $\alpha\beta \in \mathfrak{o}$, we have $d = -1$ and $c = a_{-1}^2$. This is a contradiction since c is not a square in k. We conclude that E does not have GR. $\square$

Example 3. Again let $\mathfrak{o} = k[[t]]$ and k a field of characteristic 2. In §7 we determined that the space $E := \left[\begin{smallmatrix} 1 & 1 \\ 1 & t^{-1} \end{smallmatrix}\right]$ does not have GR with respect to $\mathfrak{o}$. Could it be that E has at least FR with respect to $\mathfrak{o}$?

Let us assume that this is so. Then there exists a reduced nondegenerate quadratic $\mathfrak{o}$-module M with $E \cong K \otimes M$. We choose a decomposition $M = M_1 \perp M_2$ with M_1 strictly regular and $B(M_2 \times M_2) \subset \mathfrak{m}$. We already know that M is degenerate, so $\dim M_2 > 0$. Since $\dim M = 2$ and $\dim M_1$ is even, we must have $M_1 = 0$. Hence $\overline{M}$ is quasilinear. But M is not quasilinear, since E is not quasilinear. Therefore we have a representation

$$M \cong \begin{bmatrix} a & t^n c \\ t^n c & b \end{bmatrix}$$

with $n \in \mathbb{N}$ and units $a, b, c \in \mathfrak{o}^*$. The space $\overline{M} \cong [\overline{a}, \overline{b}]$ is anisotropic and so the element $\overline{a}\overline{b}$ is not a square in k. Over K we have

$$\begin{bmatrix} 1 & 1 \\ 1 & t^{-1} \end{bmatrix} \cong K \otimes M \cong \langle a \rangle \otimes \begin{bmatrix} 1 & t^n ca^{-1} \\ t^n ca^{-1} & ba^{-1} \end{bmatrix} \cong \langle a \rangle \otimes \begin{bmatrix} 1 & 1 \\ 1 & abc^{-2}t^{-2n} \end{bmatrix}.$$

Comparing Arf-invariants shows

$$(*) \qquad abc^{-2}t^{-2n} = t^{-1} + x^2 + x$$

with $x \in K$. Let $v := K \to \mathbb{Z} \cup \infty$ be the valuation associated to $\mathfrak{o}$. From $(*)$ we get $v(x) < 0$ and thus $v(x) = -n$. Therefore $x = t^{-n} \sum_{i=0}^{\infty} x_i t^i$ with $x_i \in k$, $x_0 \neq 0$. Comparing the coefficients of t^{-2n} on the left and righthand side of $(*)$ gives

$$\overline{a}\overline{b}\overline{c}^{-2} = x_0^2.$$

So $\overline{a}\overline{b}$ is a square in k after all, a contradiction. Therefore E does not have FR.

We continue with the general theory. So $\lambda : K \to L \cup \infty$ is again an arbitrary place. We immediately obtain a consequence of Theorem 2 which is not contained in the results of §8 in the case of good reduction.

Corollary 2. *Let E and F be spaces over K which have* FR *with respect to λ and with $F < E$. Then $\lambda_* F < \lambda_* E$.* □

This corollary engenders a substitution principle for quadratic forms, modeled on §3, Theorem 5.

Theorem 3. *Let $(g_{kl}(t))_{1 \le k,l \le m}$ and $(f_{ij}(t))_{1 \le i,j \le n}$ be symmetric matrices whose coefficients are polynomials in variables $t = (t_1, \dots, t_r)$ over a field k. Let $L \supset k$ be a field extension of k and $c = (c_1, \dots, c_r)$ an r-tuple with coefficients in L. Suppose that the quadratic forms $[g_{kl}(t)]$ and $[f_{ij}(t)]$ over $k(t)$ satisfy $[g_{kl}(t)] < [f_{ij}(t)]$. Suppose also that the quadratic forms $[g_{kl}(c)]$ and $[f_{ij}(c)]$ over L are nondegenerate. Then $[g_{kl}(c)] < [f_{ij}(c)]$ (over L).* □

We return to an arbitrary place $\lambda\colon K \to L \cup \infty$.

Theorem 4. (Extension of §8, Lemma 2(b).) *Let G be a space and F a strictly regular space over K. Suppose that F has* GR *with respect to λ. Finally, let $E\colon= F \perp G$.*

Claim: E has FR *with respect to λ if and only if G has* FR *with respect to λ. In this case we have $\lambda_* E \cong \lambda_* F \perp \lambda_* G$.*

Proof. We have $F = K \otimes M$ where M is a strictly regular quadratic $\mathfrak{o}$-module. If G has FR with respect to λ, then $G = K \otimes P$ where P is a reduced nondegenerate quadratic $\mathfrak{o}$-module. The quadratic $\mathfrak{o}$-module $N\colon= M \perp P$ is then also reduced nondegenerate and

$$L \otimes_\lambda N = (L \otimes_\lambda M) \perp (L \otimes_\lambda P) = F \perp G = E.$$

Hence E has FR with respect to λ and $\lambda_* E \cong \lambda_* F \perp \lambda_* G$.

Suppose now that E has FR with respect to λ. We have $E \cong K \otimes N$ where N is a reduced nondegenerate quadratic $\mathfrak{o}$-module. By Theorem 1, $K \otimes M < K \otimes N$ implies that $N \cong M \perp P$ where P is a reduced nondegenerate quadratic $\mathfrak{o}$-module. Hence

$$F \perp (K \otimes P) \cong E \cong F \perp G.$$

Now $G \cong K \otimes P$ by the cancellation theorem over K (§6, Theorem 2). Therefore G has FR with respect to λ. □

To the currently developed theory of fair reduction, we can associate a variation of the weak specialization theory of §7, which we will briefly present in the following.

Theorem 5. (Extension of §6, Theorem 5.) *Let $\mathfrak{o}$ be quadratically henselian. Let (M, q) be a reduced nondegenerate and anisotropic quadratic $\mathfrak{o}$-module. Furthermore, let e be a primitive vector in M. Then $q(e) \in \mathfrak{o}^*$.*

Proof. We choose a decomposition $M = N \perp M'$ with N strictly regular and $B(M' \times M') \subset \mathfrak{m}$. Traversing the proof of §6, Theorem 5 and replacing $M^\perp$

by M' everywhere will give proof since all arguments will faithfully remain valid. $\square$

As before, $\mathfrak{o}^h$ denotes the henselization of the valuation ring $\mathfrak{o}$.

Lemma 4. *Let M be a quadratic $\mathfrak{o}$-module. Then $M^h := \mathfrak{o}^h \otimes_{\mathfrak{o}} M$ is reduced nondegenerate if and only if M is reduced nondegenerate.*

Proof. This is evident since $\mathfrak{o}^h/\mathfrak{m}^h = k$, so that $M^h/\mathfrak{m}^h M^h$ is canonically isomorphic to the quadratic k-module $M/\mathfrak{m}M$. $\square$

Now the road is clear to extend, in an appropriate way, the main result of §7 (Theorem 4), using fair reduction. As before, let $\lambda: K \to L \cup \infty$ be a place with associated valuation ring $\mathfrak{o}$. Let S be a system of representatives of $Q(K)/Q(\mathfrak{o})$ in $\mathfrak{o}$, as introduced in §7.

Definition 4. Let $E = (E, q)$ be a quadratic K-module (always free of finite rank). We say that E is *weakly obedient with respect to* λ (or: with respect to $\mathfrak{o}$) when E has a decomposition

$$(*) \qquad E = \perp_{s \in S} E_s$$

such that the quadratic K-module $(E_s, s^{-1}(q|E_s))$ has FR with respect to λ for every $s \in S$. Every decomposition of the form $(*)$ is called a *weakly λ-modular* (or: *weakly $\mathfrak{o}$-modular*) *decomposition* of E.

Remark. Let E be weakly obedient with respect to λ. Then

$$E \cong \perp_{s \in S} \langle s \rangle \otimes (K \otimes M_s)$$

with reduced nondegenerate quadratic $\mathfrak{o}$-modules M_s. For every $s \in S$ we may choose a decomposition $M_s = N_s \perp M'_s$ with strictly regular N_s and $B_s(M'_s \times M'_s) \subset \mathfrak{m}$, where B_s is the bilinear form associated to M_s. Just as indicated in §7, we see that

$$G := \perp_{s \in S} \langle s \rangle \otimes (K \otimes M'_s)$$

is anisotropic. Furthermore,

$$F := \perp_{s \in S} \langle s \rangle \otimes (K \otimes N_s)$$

is strictly regular. We have $E \cong F \perp G$. Therefore E is definitely not degenerate and is thus a space over K. $\square$

Theorem 6. (Extension of §7, Theorem 4.) *Let E be a space over K which is weakly obedient with respect to λ and let*

$$E = \perp_{s \in S} E_s = \perp_{s \in S} F_s$$

be two weakly λ*-modular decompositions of* E*. Then*[18] $\lambda_*(E_1) \sim \lambda_*(F_1)$.

Proof. The arguments in the proof of §7, Theorem 4, remain valid in the current more general situation. One should use Theorem 5 and Lemma 4 above. $\square$

Now the following definition makes sense:

Definition 5. Let E be a quadratic space, weakly obedient with respect to λ and let $E = \perp_{s \in S} E_s$ be a weakly λ-modular decomposition of E. Then we call $\lambda_*(E_1)$ a *weak specialization of* E *with respect to* λ. We denote the Witt class of $\lambda_*(E_1)$ by $\lambda_W(E)$. In other words, $\lambda_W(E) := \{\lambda_*(E_1)\} \in \widehat{W}q(L)$. By Theorem 6, $\lambda_W(E)$ is uniquely determined by E and λ.

Remark. Our proof of Theorem 6 is independent of the main result Theorem 2 of the specialization theory developed above. We could also have deduced the important Corollary 1 of Theorem 2 (specialization by FR is well-defined) from Theorem 6, analogous to the proof of §8, Theorem 1 in the quadratic case. In other words, we could have established the weak specialization theory first and then develop from this the basic idea of specialization by FR (Definition 3 above), just as before in §7 and §8.

In the theorems of §8 about the weak specialization of tensor products (Theorem 6(ii), Theorem 7) we may not simply replace the word GR by FR and the word "obedient" by "weakly obedient". This already shows us that in our theory, good reduction does not become superfluous in any way after the introduction of fair reduction.

More important even is the observation that in the theory of generic splitting in §9 and §10 good reduction appears center stage, and not fair reduction: For example, let φ be a regular quadratic form over a field k and let $L \supset k$ be a field extension. Let $(K_r \mid 0 \le r \le h)$ be a generic splitting tower of φ. If $\lambda\colon K_r \to L \cup \infty$ is a place over k for some $r \in \{1, \ldots, h\}$, then the kernel form φ_r of $\varphi \otimes K_r$ automatically has *good* reduction with respect to λ.

[18] See §6, Definition 10, for the definition of Witt equivalence $\sim$.

§12 Unified Theory of Generic Splitting

Just as in §9 and §10 we will not use geometric language and talk of quadratic forms instead of quadratic spaces in this section. The definitions concerning quadratic spaces over fields, encountered in §11, will be faithfully adopted in the language of forms. *In what follows, a "form" will always be understood to be a quadratic form.*

If $\gamma\colon k \to L \cup \infty$ is a place and φ a *regular* form over k which has GR with respect to γ, then the splitting behaviour of $\gamma_*(\varphi)$ under extensions of the field L is controlled by a given generic splitting tower $(K_r \mid 0 \le r \le h)$ of φ, as seen in §4 and §9 (§9, Theorem 4 and generalization of §4, Scholium 3). According to §10, something similar happens for a place γ and a form φ over k having GR with respect to γ for which $\gamma_*(\varphi)$ is nondegenerate, provided some care is exercised. Now we want to extend these results to the case where φ has just FR with respect to γ.

At the same time we want to unite the results of §9 and §10 under one roof, starting with the following definition.

Definition 1. Let φ be a nondegenerate form over a field K. We call a field extension $K \hookrightarrow L$ *φ-conservative* (or: *conservative for* φ) if the form $\varphi \otimes L$ is again nondegenerate, i.e. if the quasilinear part $QL(\varphi)$ of φ remains anisotropic under a field extension from K to L.

If φ is regular, then $\dim QL(\varphi) \le 1$ and so every field extension of K is φ-conservative. This is true in particular when $\operatorname{char} K \ne 2$. If $\operatorname{char} K = 2$, then every separable field extension $K \hookrightarrow L$ is φ-conservative by §10, Theorem 1.

Theorem 1. *Let $\lambda\colon K \to L \cup \infty$ be a place, $K' \supset K$ a field extension and $\mu\colon K' \to L \cup \infty$ an extension of λ. Let φ be a (nondegenerate) form over K, having* FR (*resp.* GR) *with respect to λ, and let $\lambda_*(\varphi)$ be nondegenerate.*

Claim: the extension $K \hookrightarrow K'$ is φ-conservative. The form $\varphi \otimes K'$ has FR (*resp.* GR) *with respect to μ and $\mu_*(\varphi \otimes K') \cong \lambda_*(\varphi)$.*

Proof. In the case of good reduction, this is Theorem 2 from §10. The proof in the case of fair reduction is similar and goes as follows.

Let $\mathfrak{o} := \mathfrak{o}_\lambda$ and $\mathfrak{o}' := \mathfrak{o}_\mu$. Let $E = (E, q)$ be a space corresponding to φ and M a reduced nondegenerate quadratic $\mathfrak{o}$-module with $E \cong K \otimes_{\mathfrak{o}} M$. Then $\lambda_*(\varphi)$ corresponds to the space $\lambda_*(E) \cong L \otimes_{\overline{\lambda}} \overline{M}$, where $\overline{\lambda}\colon \kappa(\mathfrak{o}) \hookrightarrow L$ is the field extension associated to λ. We have the factorization $\overline{\lambda} = \overline{\mu} \circ j$, featuring the inclusion $j\colon \kappa(\mathfrak{o}) \hookrightarrow \kappa(\mathfrak{o}')$ and the field homomorphism $\overline{\mu}\colon \kappa(\mathfrak{o}') \hookrightarrow L$, determined by μ. Let $\overline{\varphi}$ be the form over $\kappa(\mathfrak{o})$, associated to $\overline{M}$, i.e. $\lambda_*(\varphi) = \overline{\varphi} \otimes_{\overline{\lambda}} L$. Since $\lambda_*(\varphi)$ is nondegenerate by assumption, $\overline{\lambda}\colon \kappa(\mathfrak{o}) \hookrightarrow L$ is conservative for $\overline{\varphi}$. Therefore $j\colon \kappa(\mathfrak{o}) \hookrightarrow \kappa(\mathfrak{o}')$ is conservative for $\overline{\varphi}$ and $\overline{\mu}\colon \kappa(\mathfrak{o}') \hookrightarrow L$ is conservative for $\overline{\varphi} \otimes \kappa(\mathfrak{o}')$. Now $\overline{\varphi} \otimes \kappa(\mathfrak{o}')$ belongs to the quadratic $\kappa(\mathfrak{o}')$-module $\overline{M'}$ with $M' := \mathfrak{o}' \otimes_{\mathfrak{o}} M$. Therefore $\overline{M'}$ is nondegenerate, in other words M' is reduced nondegenerate. We have $\overline{M'} = \kappa(\mathfrak{o}') \otimes_{\kappa(\mathfrak{o})} \overline{M}$. Furthermore, $K' \otimes E \cong K' \otimes_{\mathfrak{o}'} M'$. Thus $\varphi \otimes K'$ has FR with respect to μ and

$$\mu_*(\varphi \otimes K') \cong (\overline{\varphi} \otimes \kappa(\mathfrak{o}')) \otimes_{\overline{\mu}} L = \overline{\varphi} \otimes_{\overline{\lambda}} L \cong \lambda_*(\varphi).$$

In particular, $\varphi \otimes K'$ is nondegenerate (cf. §11, the Remark following Definition 3). We conclude that $K \hookrightarrow K'$ is φ-conservative. $\square$

In the following let φ be a nondegenerate form over a field k. Over every field K we denote the form $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ by H.

Theorem 2. *Let K and L be extensions of the field k and let $\lambda\colon K \to L \cup \infty$ be a place over k. Suppose further that L is φ-conservative and let $\varphi \otimes K \cong r \times H \perp \psi$ be the Witt decomposition of φ.*

Claim: K is also a φ-conservative extension of k. The form ψ has GR *with respect to λ, and $\varphi \otimes L \cong r \times H \perp \lambda_*(\psi)$. Thus* $\operatorname{ind}(\varphi \otimes L) \geq \operatorname{ind}(\varphi \otimes K)$.

Proof. We apply Theorem 1 to the trivial place $\gamma\colon k \hookrightarrow L$ and its extension λ. Now φ has GR with respect to γ and $\gamma_*(\varphi) = \varphi \otimes L$. By Theorem 1, K is conservative for φ, in other words $\varphi \otimes K$ is nondegenerate. Again by this theorem, $\varphi \otimes K$ has GR with respect to λ and $\lambda_*(\varphi \otimes K) \cong \varphi \otimes L$. Since $r \times H$ also has GR with respect to λ, it follows that ψ has GR with respect to λ (§8, Theorem 2), and we have $\varphi \otimes L \cong r \times H \perp \lambda_*(\psi)$. $\square$

Corollary. *Let K and L be specialization equivalent extensions of the field k. Then K is φ-conservative if and only if L is φ-conservative. In this case we have* $\operatorname{ind}(\varphi \otimes K) = \operatorname{ind}(\varphi \otimes L)$. *Furthermore, if $\lambda\colon K \to L \cup \infty$ is a place over k, then* $\ker(\varphi \otimes K)$ *has* GR *with respect to λ and* $\lambda_*(\ker(\varphi \otimes K)) \cong \ker(\varphi \otimes L)$. $\square$

Next we define a *generic splitting tower* $(K_r \mid 0 \leq r \leq h)$ associated to φ just as in §4 for $\operatorname{char} k \neq 2$ and in §9 for regular forms, with *higher kernel forms* φ_r and *higher indices* i_r $(0 \leq r \leq h)$: K_0 is an inessential extension of k, $\varphi_r := \ker(\varphi \otimes K_r)$, $K_{r+1} \sim_k K_r(\varphi_r)$, $i_{r+1} := \operatorname{ind}(\varphi_r \otimes K_{r+1})$, in case $r < h$. The construction stops with step h if $\dim \varphi_h - \dim QL(\varphi_h) \leq 1$.

All this makes sense, since an inductive argument shows that $QL(\varphi_r) = QL(\varphi) \otimes K_r$ is anisotropic for every $r \in \{0, \dots h\}$. Indeed, the field extension $K_r(\varphi_r)/K_r$ is separable for $r < h$. Hence we can conclude by Theorem 2 that K_{r+1}/K_r is φ_r-conservative once we already know that $QL(\varphi_r)$ is anisotropic, i.e. that φ_r is nondegenerate. The extension K_h/k is therefore φ-conservative.

Note. If $\operatorname{char} k \neq 2$, then $\dim \varphi_h \leq 1$. If $\operatorname{char} k = 2$, then $\varphi_h = QL(\varphi_h)$.

We now have to convince ourselves that the field tower $(K_r \mid 0 \le r \le h)$ really accomplishes what we expect from a generic splitting tower. As before in more special situations, it suffices for this purpose, to prove a theorem of the following sort.

Theorem 3 (Extension of §9, Theorem 4 and §10, Theorem 3). *Let $\gamma\colon k \to L \cup \infty$ be a place. Suppose that the form φ has* FR *with respect to γ and that $\gamma_*(\varphi)$ is nondegenerate. Then $\gamma_*(\varphi)$ is isotropic if and only if γ can be extended to a place $\lambda\colon k(\varphi) \to L \cup \infty$.*

Proof. We have $\varphi \otimes k(\varphi) \cong H \perp \psi$ for a form ψ over $k(\varphi)$. If there exists a place $\lambda\colon k(\varphi) \to L \cup \infty$ then, by Theorem 1, $\varphi \otimes k(\varphi)$ has FR with respect to λ and $\lambda_*(\varphi \otimes k(\varphi)) \cong \gamma_*(\varphi)$. Then it follows from §11, Theorem 3, that ψ has FR with respect to λ and that $\gamma_*(\varphi) \cong H \perp \lambda_*(\psi)$. Hence $\gamma_*(\varphi)$ is isotropic.

Conversely, suppose that the nondegenerate form $\overline{\varphi} := \gamma_*(\varphi)$ is isotropic. Since $H < \overline{\varphi}$ we have $\dim \overline{\varphi} - \dim QL(\overline{\varphi}) \ge 2$. Now, by earlier work, we see that γ can be extended to a place $\tilde{\gamma}\colon k(\varphi) \to L(\overline{\varphi}) \cup \infty$ (look again at the proof of §9, Theorem 3). {Note: here we do not need the assumption that $\overline{\varphi}$ is isotropic, but only that $\dim \overline{\varphi} - \dim QL(\overline{\varphi}) \ge 2$.} Since $\overline{\varphi}$ is isotropic, $L(\overline{\varphi})$ is a purely transcendental extension of L (§9, Lemma 1). Thus there exists a place $\rho\colon L(\overline{\varphi}) \to L \cup \infty$ over L. The place $\lambda := \rho \circ \tilde{\gamma}\colon\ k(\varphi) \to L \cup \infty$ is an extension of γ. □

In what follows, let $(K_r \mid 0 \le r \le h)$ be a generic splitting tower associated to φ with higher kernel forms φ_r and indices i_r.

Theorem 4 (Extension of §4, Theorem 3 and §10, Theorem 4). *Let $\gamma\colon k \to L \cup \infty$ be a place with respect to which φ has* FR (*resp.* GR). *Suppose that the form $\gamma_*(\varphi)$ is nondegenerate. Finally, suppose that $\lambda\colon K_m \to L \cup \infty$ is a place for some $m \in \{0, \dots, h\}$, which extends γ and which cannot be extended to K_{m+1} in case $m < h$. Then φ_m has* FR (*resp.* GR) *with respect to λ. The form $\gamma_*(\varphi)$ has kernel form $\lambda_*(\varphi_m)$ and Witt index $i_0 + \dots + i_m$.*

Proof. We give the proof for FR. The case of GR can be treated in an analogous way. We have an isometry

$$\varphi \otimes K_m \cong \varphi_m \perp (i_0 + \dots + i_m) \times H. \tag{1}$$

By Theorem 1, $\varphi \otimes K_m$ has FR with respect to λ and $\lambda_*(\varphi \otimes K_m) \cong \gamma_*(\varphi)$. From (1) we get, according to §11, Theorem 4, that φ_m has FR with respect to λ and that

$$\gamma_*(\varphi) \cong \lambda_*(\varphi_m) \perp (i_0 + \dots + i_m) \times H. \tag{2}$$

In particular, $\lambda_*(\varphi_m)$ is nondegenerate. If $\lambda_*(\varphi_m)$ were isotropic, then we would surely have that $\dim \varphi_m \ge \dim QL(\varphi_m) + 2$, i.e. $m < h$. But then it follows from Theorem 3 that λ can be extended to a place $\mu\colon K_{m+1} \to$

$L \cup \infty$, contradiction! Therefore $\lambda_*(\varphi_m)$ is anisotropic and (2) is the Witt decomposition of $\gamma_*(\varphi_m)$. □

By applying this theorem to the trivial place γ, we obtain

Scholium 1 (Extension of §4, Scholium 1). Let $L \supset k$ be a φ-conservative field extension.

(1) Let $\lambda: K_m \to L \cup \infty$ be a place over k for some $m \in \{0, \dots, h\}$, which *cannot* be extended to a place from K_{m+1} to L in case $m < h$. Then φ_m has good reduction with respect to λ and $\lambda_*(\varphi_m)$ is the kernel form of $\varphi \otimes L$.

(2) If $\lambda': K_r \to L \cup \infty$ is a place over k, then $r \leq m$ and λ' can be extended to a place $\mu: K_m \to L \cup \infty$.

(3) For a given number t with

$$0 \leq t \leq \left[\frac{\dim \varphi - \dim QL(\varphi)}{2}\right] = i_0 + \dots + i_h$$

and $m \in \mathbb{N}_0$ minimal such that $t \leq i_0 + \dots + i_m$, K_m is a generic φ-conservative field extension of k for the splitting off of t hyperbolic planes of φ (cf. the problem posed at the beginning of §4). □

Thus, the field tower $(K_r \mid 0 \leq r \leq h)$ rightfully merits the name "generic splitting tower for φ" also in the current general situation.

Definition 2. As before we call h the *height* of the form φ and write $h = h(\varphi)$. Further, we define the *splitting pattern* $\mathrm{SP}(\varphi)$ as the set of all indices $\mathrm{ind}(\varphi \otimes L)$, where L runs through all φ-conservative field extensions of k. By the scholium we have

$$\mathrm{SP}(\varphi) = \{i_0 + \dots + i_r \mid 0 \leq r \leq h\}.$$

Definition 3. We call any field extension $E \supset k$ which is specialization equivalent to $k(\varphi)$ over k a *generic zero field of* φ. Further, for $r \in \{0, \dots, h\}$, we call any field extension $F \supset k$ which is specialization equivalent to K_r over k a *partially generic splitting field of* φ or, more precisely, a *generic splitting field of level* r. This signifies that F is generic for the splitting off of as many hyperbolic planes as the $(r+1)$-th number $i_0 + \dots + i_r$ in $\mathrm{SP}(\varphi)$ indicates. In case $r = h$, we speak of a *totally generic splitting field.*

This terminology generalizes notions from §4 and §9, but one has to exercise some caution; the extensions $E \supset k$ and $F \supset k$ have the mentioned generic properties – as far as we know – *only within the class of φ-conservative field extensions of k.*

Comment on the Theory so far. If the form φ over k is degenerate, but $\dim \varphi - \dim QL(\varphi) \geq 2$, then the function field $k(\varphi)$ can be constructed just as in the nondegenerate case. Thus we can formally define a "generic splitting tower" $(K_r \mid 0 \leq r \leq h)$ as above. We may also inquire about the splitting behaviour of φ under extensions $L \supset k$ of the field k, in other words about the possibilities for the index $\operatorname{ind}(\varphi \otimes L)$ and the kernel form $\ker(\varphi \otimes L)$ (see Definition 10 in §6). However, can we also extend the theory so far to degenerate forms?

We have a decomposition $\varphi \cong \widehat{\varphi} \perp \delta(\varphi)$ with $\delta(\varphi) \cong t \times [0]$ and $\widehat{\varphi}$ nondegenerate, cf. §6, Definition 9 ff. Clearly $k(\varphi)$ is a purely transcendental extension of $k(\widehat{\varphi})$ of transcendence degree t. Therefore $k(\varphi) \sim_k k(\widehat{\varphi})$. Consequently, the above tower $(K_r \mid 0 \leq r \leq h)$ is a generic splitting tower of $\widehat{\varphi}$. It is now clear that our theorems so far all remain valid for degenerate φ, as long as we complete Definition 1 above as follows: for degenerate φ, a field extension $K \hookrightarrow L$ is called conservative for φ if it is conservative for $\widehat{\varphi}$.

This, however, furnishes us with a fairly bland extension of the theory so far to degenerate forms. The obvious, difficult question seems to be: Let φ be a nondegenerate form over a field k and $L \supset k$ a field extension with $\varphi \otimes L$ degenerate. Is it so that the splitting behaviour of $\varphi \otimes L$ under field extensions of L is controlled by a given generic splitting tower $(K_r \mid 0 \leq r \leq h)$ of φ in a similar way as indicated in Scholium 1(1) above? For example, is $\mathrm{SP}(\varphi \otimes L) \subset \mathrm{SP}(\varphi)$?

Our theory does not give any information here. The main problem seems to occur in Theorem 1 above. If we do not know there that $\lambda_*(\varphi)$ is nondegenerate, we can not conclude – as far as I can see – that $(\varphi \otimes K')^\wedge$ has fair reduction with respect to μ. □

After this digression, we suppose again that φ is a *nondegenerate* form over k and that $(K_r \mid 0 \leq r \leq h)$ is a generic splitting tower of φ with higher kernel forms φ_r and indices i_r. From Theorem 4 above we immediately obtain a literal repetition of §4, Scholium 2 in the current, more general situation, if we bear in mind that for every generic splitting tower $(K'_s \mid 0 \leq s \leq h)$ of φ, all extensions K'_s of k are conservative for φ. Furthermore, we obtain from Theorem 4 an extension of §4, Scholium 3 in the same way as we obtained §4, Scholium 3 from §4, Theorem 3:

Scholium 2. Let $\gamma\colon k \to L \cup \infty$ be a place with respect to which φ has FR (resp. GR). Suppose that $\gamma_*(\varphi)$ is nondegenerate. Then:

(1) $\mathrm{SP}(\gamma_*(\varphi)) \subset \mathrm{SP}(\varphi)$.

(2) The higher kernel forms of $\gamma_*(\varphi)$ arise from certain higher kernel forms of φ by means of specialization. More precisely: if $(L_s \mid 0 \leq s \leq e)$ is a generic splitting tower of $\gamma_*(\varphi)$, then $e \leq h$ and, for every s with $0 \leq s \leq e$, we have

$$\operatorname{ind}(\gamma_*(\varphi) \otimes L_s) = i_0 + \ldots + i_m$$

with $m \in \{0, \ldots, h\}$. The number m is the biggest integer such that γ can be extended to $\lambda\colon K_m \to L_s \cup \infty$. The kernel form φ_m of $\varphi \otimes K_m$ has FR (resp. GR) with respect to every extension λ of this kind, and $\lambda_*(\varphi_m)$ is the kernel form of $\gamma_*(\varphi) \otimes L_s$.

(3) If $\rho: K_r \to L_s \cup \infty$ is a place, which extends $\gamma: k \to L \cup \infty$, then $r \leq m$ and ρ can be further extended to a place from K_m to L_s. □

§13 Regular Generic Splitting Towers and Base Extension

Before turning towards generic splitting towers, we will give two general definitions which will also serve us well in later sections.

As before, the word *form* over a field k will always be understood to mean a nondegenerate quadratic form over k. In the following let φ be a form over k.

Definition 1. Let $\dim\varphi$ be even, $\varphi \neq 0$ and $QL(\varphi) = 0$.

(a) The *discriminant algebra* $\Delta(\varphi)$ is defined as follows: If $\operatorname{char} k \neq 2$, we let $\Delta(\varphi) := k[X]/(X^2 - a)$, where a is a representative of the signed determinant $d(\tilde{\varphi}) = ak^{*2}$ of the bilinear form $\tilde{\varphi} = B_\varphi$ associated to φ (cf. §2). If $\operatorname{char} k = 2$, we let $\Delta(\varphi) := k[X]/(X^2 + X + c)$, where c is a representative of the Arf-invariant $\operatorname{Arf}(\varphi) \in k^+/\wp k$.

(b) We define the *discriminant* of φ to be the isomorphism class of $\Delta(\varphi)$ as k-algebra. We will denote it sloppily by $\Delta(\varphi)$ as well. The discriminant is independent of the choice of a resp. c above.

(c) We say that $\Delta(\varphi)$ *splits* when $\Delta(\varphi)$ is not a field, i.e. when $\Delta(\varphi) \cong k \times k$. We will symbolically write $\Delta(\varphi) = 1$ when φ splits and $\Delta(\varphi) \neq 1$ when φ doesn't split.

Remark 1. This notation is not completely groundless, for the isomorphism classes of quadratic separable k-algebras form a group in a natural way with unit element $k \times k$. In this group the equality $\Delta(\varphi \perp \psi) = \Delta(\varphi)\Delta(\psi)$ holds. We will not discuss the group of quadratic separable k-algebras any deeper since we will not make any serious use of it.

Remark 2. If K/k is a field extension,[19] then $\Delta(\varphi \otimes K) = \Delta(\varphi) \otimes_k K$.

Remark 3. If τ is the norm form of the k-algebra $\Delta(\varphi)$, then $\Delta(\varphi) = \Delta(\tau)$. Is further $\dim\varphi = 2$, then $\varphi \cong c\tau$ for some $c \in k^*$. If we write $\tau = \left[\begin{smallmatrix} 1 & 1 \\ 1 & \delta \end{smallmatrix}\right]$, then $\Delta(\varphi) = k[X]/(X^2 + X + \delta)$, also when $\operatorname{char} k \neq 2$.

Definition 2 (cf. [KR, Def.1.1]). Let $\dim\varphi > 1$. We say that "*φ is of outer type*" when $\dim\varphi$ is even, $QL(\varphi) = 0$ and $\Delta(\varphi) \neq 1$. In all other cases (thus in particular when $QL(\varphi) \neq 0$) we say that "*φ is of inner type*".

[19]From now on we will often denote a field extension $F \hookrightarrow E$ by E/F, as has been customary in algebra for a long time.

Remark 4. For those readers who are at home in the theory of reductive algebraic groups, we want to remark that Definition 2 leans on the concepts of inner/outer type in use there: is φ regular (i.e. $\dim QL(\varphi) \leq 1$) and $\dim \varphi \geq 3$, then the group $\mathrm{SO}(\varphi)$ is almost simple. This group is of inner/outer type if and only if this is the case for φ in the sense of Definition 2.

We want to construct "regular" generic splitting towers of φ, having particularly convenient properties with respect to extensions of the ground field k, which are nonetheless sufficiently general. Starting with such a regular generic splitting tower of φ we will then construct a regular generic splitting tower of $\varphi \otimes L$ for every φ-conservative field extension L/k.

For every form ψ with $\dim \psi \geq 2 + \dim QL(\psi)$ over a field K we introduced the field extension $K(\psi)$ of K earlier (§4; §9; §10, following Def.2). $K(\psi)$ was defined to be the function field of the affine quadric $\psi(X) = 0$ over K, except when $\psi \cong \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, in which case this quadric degenerates in two lines. In this situation we defined $K(\psi) = K(t)$ for some indeterminate t over K. Sometimes it is more natural to use the *projective* quadric $\psi(X) = 0$ over K instead of the affine quadric $\psi(X) = 0$. Thus we arrive at the following

Definition 3. Let ψ be a (nondegenerate quadratic) form over K with $\dim \psi \geq 2 + \dim QL(\psi)$. We define a field extension $K\{\psi\}$ of K as follows: If $\psi \cong \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, we let $K\{\psi\} = K$. Otherwise we let $K\{\psi\}$ be the subfield $K(\frac{x_1}{x_i}, \ldots, \frac{x_n}{x_i})$ of $K(\psi) = \mathrm{Quot} K[X_1, \ldots, X_n]/\psi(X_1, \ldots, X_n) = K(x_1, \ldots, x_n)$ for some $i \in [1, n]$.[20]

In the main case $\psi \not\cong \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ this field is obviously independent of the choice of $i \in [1, n]$. We have $K(\psi) = K\{\psi\}(x_i)$ and x_i is transcendental over $K\{\psi\}$. Also in the case $\psi \cong \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ we have $K(\psi) = K\{\psi\}(t)$ where t is an indeterminate.

$K(\psi)$ is an inessential extension of $K\{\psi\}$. Thus, all this time we could have used $K\{\psi\}$ instead of $K(\psi)$.

Back to our form φ over k! Let $h = h(\varphi)$ be the height of φ.

Definition 4. The *projective standard tower* of φ is the field tower $(K_r \mid 0 \leq r \leq h)$ with $K_0 = k$, $K_{r+1} = K_r\{\varphi_r\} \quad (0 \leq r \leq h-1)$, where φ_r denotes the kernel form of $\varphi \otimes K_r$. Similarly one obtains the *affine standard tower* $(K'_r \mid 0 \leq r \leq h)$ of φ by replacing the $K_r\{\psi_r\}$ with the function fields $K_r(\varphi_r)$ of the affine quadrics $\varphi_r = 0$.

Remark 5. From the definition of generic splitting tower (§12, just after Theorem 2) it follows immediately that $(K_r \mid 0 \leq r \leq h)$ and $(K'_r \mid 0 \leq r \leq h)$ are both generic splitting towers of φ. For every $r \in \{0, \ldots, h\}$ we have that K'_r is a purely transcendental field extension of K_r of transcendence degree r.

[20] In [K_5] this field is denoted by $K(\psi)_0$. $[1, n]$ denotes the set $\{1, 2, \ldots, n\}$.

Let us recall that a field extension L/K is called *regular* when it is separable and K is algebraically closed in L. This is synonymous with saying that L and the algebraic closure $\widetilde{K}$ of K in L are linearly disjoint, i.e. that $L \otimes_K \widetilde{K}$ is a field, cf. [Lg$_2$, Chap. 3]. If L/K is regular, then the K-algebra $L \otimes_K E$ does not contain zero divisors for any field extension E/K.

Theorem 1. *Let $(K_r|0 \le r \le h)$ be the projective standard tower of φ. Then the field extensions K_r/K_{r-1} with $0 < r < h$ are all regular. The same is true for K_h/K_{h-1} in case $h > 0$ and φ is of inner type. Furthermore, the kernel form φ_{h-1} of $\varphi \otimes K_{h-1}$ has dimension ≥ 3 in this situation. If φ is of outer type, then $h \ge 1$ and $K_h = K_{h-1} \otimes_k \Delta(\varphi)$. Furthermore, we then have $\varphi_{h-1} \cong c(\tau \otimes K_{h-1})$, where τ is the norm form of the field extension $\Delta(\varphi)$ of k and $c \in K_{h-1}^*$ is a constant. In particular we have* $\dim \varphi_{h-1} = 2$.

Proof. If ψ is a nondegenerate quadratic form over a field K and if $\dim \psi > 2$, then the extension $K\{\psi\}/K$ is clearly regular. This established the first statement of the theorem. If $h > 0$ and φ is of inner type, then $\varphi \otimes K_{h-1}$ is also of inner type and thus the kernel form φ_{h-1} of $\varphi \otimes K_{h-1}$ is of inner type. This shows that $\dim \varphi_{h-1} \ge 3$ and so $K_h = K_{h-1}\{\varphi_{h-1}\}$ over K_{h-1} is again regular. Suppose now that φ is of outer type. φ does not split, so that $h > 0$. The extension K_{h-1}/K_0 is regular and K_0/k is inessential. Therefore k is algebraically closed in K_{h-1} and $\Delta(\varphi_{h-1}) = \Delta(\varphi \otimes K_{h-1}) = \Delta(\varphi) \otimes_k K_{h-1}$ is a field. If it would be true that $\dim \varphi_{h-1} > 2$, then $K_h = K_{h-1}\{\varphi_{h-1}\}$ over K_{h-1} would be regular. However, $\Delta(\varphi_{h-1}) \otimes_{K_{h-1}} K_h = \Delta(\varphi \otimes K_h)$ is not a field since $\varphi \otimes K_h \sim 0$. Therefore the dimension of φ_{h-1} is 2. By Remarks 2 and 3 above, we have $\Delta(\varphi_{h-1}) = \Delta(\varphi \otimes K_{h-1}) = \Delta(\varphi) \otimes_k K_{h-1} = \Delta(\tau) \otimes_k K_{h-1} = \Delta(\tau \otimes K_{h-1})$ (and $\tau \otimes K_{h-1}$ is the norm form of K_h/K_{h-1}). We conclude that $\varphi_{h-1} \cong c(\tau \otimes K_{h-1})$ for some element c of K_{h-1}. $\square$

Theorem 2. *Let $r \in [0, h]$ and let E/k be a partially generic splitting field of level r of φ.*

(*a*) *If $r < h$, then k is algebraically closed in E.*

(*b*) *If $r = h$ and φ is of inner type, then k is likewise algebraically closed in E.*

(*c*) *If $r = h$ and φ is of outer type, then k has algebraic closure $\Delta(\varphi)$ in E.*

Proof. By Theorem 1 these statements hold when E is the r-field K_r in the projective standard tower $(K_i \mid 0 \le i \le h)$ of φ. The theorem now follows in all generality from the following simple lemma.

Lemma 1. *Let K and L be extensions of the field k which are specialization equivalent, $K \sim_k L$. Then every place $\lambda: K \to L \cup \infty$ over k maps the algebraic closure of k in K isomorphically to the algebraic closure of k in L.*

Proof of the lemma. Let K° be the algebraic closure of k in K and L° the algebraic closure of k in L. Let $\lambda: K \to L \cup \infty$ and $\mu: L \to K \cup \infty$ be places over k. Since K° and L° are algebraic over k, the places λ and μ are finite on K° resp. L°. Their restrictions to K° and L° are thus field homomorphisms $\lambda': K^\circ \to L^\circ$, $\mu': L^\circ \to K^\circ$, which are both the identity on k. Now $\mu' \circ \lambda'$ is an endomorphism of K°/k and thus automatically an automorphism of K°/k. Likewise, $\lambda' \circ \mu'$ is an automorphism of L°/k. The statement of the lemma is now clear. □

Now we can precisely formulate a desirable property of generic splitting towers.

Definition 5. We call a generic splitting tower $(K_r \mid 0 \le r \le h)$ of φ *regular* when the following holds:

(1) For every r with $0 \le r < h-1$ the extension K_{r+1}/K_r is regular.

(2) If φ is of inner type and $h > 0$, then K_h/K_{h-1} is also regular. On the other hand, if φ is of outer type, then K_h is regular over the composite $K_{h-1} \cdot \Delta(\varphi) = K_{h-1} \otimes_k \Delta(\varphi)$.

Examples. The projective standard tower of φ is regular by Theorem 1. The affine standard tower of φ is likewise regular. □

Now let $(K_r \mid 0 \le r \le h)$ be a regular generic splitting tower of φ and let L/k be a φ-conservative field extension of k. Using $(K_r \mid 0 \le r \le h)$ and L we want to construct a generic splitting tower of $\varphi \otimes L$.

Definition 6. For every $r \in \{0, 1, \dots, h\}$ we construct a field composite $K_r \cdot L$ of K_r and L over k as follows: if $r < h$, or if $r = h$ and φ is of inner type, then $K_r \otimes_k L$ is free of zero divisors and we let $K_r \cdot L$ be the quotient field of this ring, i.e. the uniquely determined free composite of K and L over k. If $r = h$ and φ is of outer type, we distinguish the cases where $\varphi \otimes L$ is of outer/inner type.

First, let $\varphi \otimes L$ be of outer type. Then $\Delta(\varphi) \otimes_k L$ is a field. Let $K_h \cdot L$ be the free composite of K_h with $\Delta(\varphi) \otimes_k L$ over $\Delta(\varphi)$, i.e. again the quotient field of $K_h \otimes_{\Delta(\varphi)} (\Delta(\varphi) \otimes_k L) = K_h \otimes_k L$. Finally, let $\varphi \otimes L$ be of inner type. Now $\Delta(\varphi)$ can be embedded over k in L in two ways. We choose one such embedding and set $K_h \cdot L = \mathrm{Quot}(K_h \otimes_{\Delta(\varphi)} L)$. □

If necessary, we write more precisely $K_r \cdot_k L$ $(0 \le r \le h)$ for the field composite $K_r \cdot L$. Later on we will call $K_r \cdot L$ sloppily *"the" free composite of K_r and L over k*, also in the case $r = h$, φ of outer type, $\varphi \otimes L$ of inner type. It will never matter which of the two embeddings $\Delta(\varphi) \hookrightarrow L$ we have chosen.

We use φ_r to denote – as before – the kernel form of $\varphi \otimes K_r$ (r-th higher kernel form) and i_r to denote the Witt index of $\varphi \otimes K_r$ (r-th higher index, $0 \le r \le h$).

Theorem 3. *Let J be the set of all $r \in [0, h]$ such that $\varphi_r \otimes K_r \cdot L$ is anisotropic, i.e. such that $\mathrm{ind}(\varphi \otimes K_r) = \mathrm{ind}(\varphi \otimes K_r \cdot L)$. Let $r_0 < r_1 < \cdots < r_e$ be the elements of J.*

Claim:
(*a*) $(K_{r_i} \cdot L \mid 0 \le i \le e)$ *is a regular generic splitting tower of* $\varphi \otimes L$.

(*b*) $K_{r+1} \cdot L / K_r \cdot L$ *is a regular inessential extension for every* $r \in [0, h] \setminus J$.

For the proof of this technically very important theorem, we need a general lemma about places.

Lemma 2. *Let* $K \supset k$ *and* $L \supset k$ *be arbitrary extensions of a field* k *and let* $\lambda\colon K \to L \cup \infty$ *be a place over* k. *Furthermore, let* $E \supset k$ *be a field extension which is linearly disjoint from* K *and* L *over* k. *Consider the free composites* $K \cdot E$ *and* $L \cdot E$ *of* K *resp.* L *with* E *over* k.

Claim: λ *has a unique extension* $\tilde{\lambda}$: $K \cdot E \to L \cdot E \cup \infty$ *to a place over* E.

Proof. $K \cdot E$ is the quotient field of $K \otimes_k E$ and $L \cdot E$ is the quotient field of $L \otimes_k E$. Let $\mathfrak{o} := \mathfrak{o}_\lambda$ and let α: $\mathfrak{o} \otimes_k E \to L \otimes_k E$ be the homomorphism of E-algebras induced by $\lambda | \mathfrak{o}\colon \mathfrak{o} \to L$. Following the general extension theorem for places [Bo$_2$, §2, Prop.3], we choose a place μ: $K \cdot E \to \widetilde{(L \cdot E)} \cup \infty$ in the algebraic closure of $L \cdot E$ which extends the homomorphism α. We will now show that μ is the only such extension of α and that $\mu(K \cdot E) \subset (L \cdot E) \cup \infty$.

For this purpose we choose a basis $(\omega_i \mid i \in I)$ of E over k (as k-vector space). Let $z \neq 0$ be an element in $K \cdot E$. We write $z = \frac{x}{y}$, where $x, y \in \mathfrak{o} \otimes E$ are both non-zero. We then have equations

$$x = u \cdot \sum_{i \in I} a_i \otimes \omega_i \ , \quad y = v \cdot \sum_{i \in I} b_i \otimes \omega_i$$

with $u, v \in K$ and families $(a_i \mid i \in I)$ and $(b_i \mid i \in I)$ in $\mathfrak{o}$, both of them not fully contained in the maximal ideal $\mathfrak{m}$ of $\mathfrak{o}$. Then

$$\mu\left(\sum_{i \in I} a_i \otimes \omega_i\right) = \sum_{i \in I} \lambda(a_i)\omega_i \neq 0, \quad \mu\left(\sum_{i \in I} b_i \otimes \omega_i\right) = \sum_{i \in I} \lambda(b_i)\omega_i \neq 0,$$

since $(\omega_i \mid i \in I)$ is also a family of elements of $L \cdot E$, linearly independent over L. We thus obtain

$$\mu(z) = \lambda\left(\frac{u}{v}\right) \cdot \left(\sum_{i \in I} \lambda(a_i)\omega_i\right) \cdot \left(\sum_{i \in I} \lambda(b_i)\omega_i\right)^{-1} \in (L \cdot E) \cup \infty.$$

□

Proof of Theorem 3. We assume without loss of generality that φ is anisotropic and use induction on the height h. Remark that $K_0 \sim_k k$ implies that $K_0 \cdot L \sim_L L$ by the lemma. Therefore the extension $K_0 \cdot L / L$ is inessential.

For $h = 0$ nothing more has to be done, so suppose that $h > 0$. First we assume that $\dim \varphi > 2$. We will deal with the (easier) case $\dim \varphi = 2$ at the end.

The form $\varphi_1 = \ker(\varphi \otimes K_1)$ has height $h-1$ and regular generic splitting tower $(K_r | 1 \le r \le h)$. We want to apply the induction hypothesis to φ_1, this tower and the field extension $K_1 \cdot L/K_1$.

Clearly the extension $K_1 \cdot L/K_0 \cdot L$ is regular (cf. e.g. [Lg$_2$, p.58]). Furthermore, $K_0 \cdot L/L$ is inessential and L/k φ-conservative. It follows that $K_1 \cdot L/k$ is φ-conservative and then that $K_1 \cdot L/K_1$ is φ_1-conservative. For every r with $1 \le r \le h-1$ we have

$$K_r \cdot_k L = \mathrm{Quot}(K_r \otimes_k L) = \mathrm{Quot}(K_r \otimes_{K_1} (K_1 \otimes_k L)) = K_r \cdot_{K_1} (K_1 \cdot_k L).$$

Also, $K_h \cdot_k L = K_h \cdot_{K_1} (K_1 \cdot_k L)$. This can be seen using the same calculation in case φ is of inner type, and also when φ is of outer type and $\Delta(\varphi)$ cannot be embedded in L. If φ is of outer type and $\Delta(\varphi) \subset L$, we perform the following calculation (bearing Definition 6 in mind).

$$\begin{aligned} K_h \cdot_k L := K_h \cdot_{\Delta(\varphi)} L &= \mathrm{Quot}(K_h \otimes_{\Delta(\varphi)} L) \\ &= \mathrm{Quot}(K_h \otimes_{K_1 \cdot \Delta(\varphi)} (K_1 \cdot \Delta(\varphi) \otimes_{\Delta(\varphi)} L)). \end{aligned}$$

We have $K_1 \cdot \Delta(\varphi) = K_1 \otimes_k \Delta(\varphi) = \Delta(\varphi_1)$ and do indeed obtain again

$$K_h \cdot_k L = \mathrm{Quot}(K_h \otimes_{\Delta(\varphi_1)} (K_1 \otimes_k L)) = K_h \cdot_{K_1} (K_1 \cdot_k L).$$

Thus we can apply the induction hypothesis, which tells us among other things that $K_{r+1} \cdot L/K_r \cdot L$ is regular and inessential for every $r \in [1, h] \setminus J$. We now distinguish the cases $r_0 \ge 1$ and $r_0 = 0$.

Suppose that $r_0 \ge 1$. By the induction hypothesis, $(K_{r_i} \cdot L | 0 \le i \le e)$ is a regular generic splitting tower of $\varphi \otimes K_1 \cdot L$. Let $F_0 := K_0 \cdot L$. The form $\varphi_0 \otimes F_0 = \varphi \otimes F_0$ is isotropic. Therefore the extension $F_0(\varphi_0 \otimes F_0) = K_0(\varphi_0) \cdot_{K_0} F_0$ of F_0 is purely transcendental, and thus inessential. By the lemma, $K_1 \cdot_{K_0} F_0 \sim_{F_0} K_0(\varphi_0) \cdot_{K_0} F_0$. Hence $K_1 \cdot_{K_0} F_0/F_0$ is also inessential. In analogy with the first calculation above, we find that

$$K_1 \cdot_{K_0} F_0 = K_1 \cdot_{K_0} (K_0 \cdot_k L) = K_1 \cdot_k L,$$

so that $K_1 \cdot_k L/K_0 \cdot_k L$ is inessential. This proves statement (b) for the current case.

We saw that $K_0 \cdot L/L$ is inessential. Thus $K_1 \cdot L/L$ is inessential. The generic splitting tower $(K_{r_i} \cdot L | 0 \le i \le e)$ of $\varphi \otimes K_1 \cdot L$ is thus also a generic splitting tower of $\varphi \otimes L$.

We come to the case $\dim \varphi > 2$, $r_0 = 0$. Statement (b) is covered by the induction hypothesis. $\varphi \otimes L$ is anisotropic. By the induction hypothesis, $\varphi \otimes K_1 \cdot L$ has regular generic splitting tower $(K_{r_i} \cdot L | 1 \le i \le e)$. The extension $K_{r_1} \cdot L/K_0 \cdot L$ is regular. For the proof of statement (a), it remains to be shown that $K_{r_1} \cdot L$ is the generic zero field of $\varphi \otimes K_0 \cdot L$. We already know that $K_{r_1} \cdot L/K_1 \cdot L$ is inessential. Thus it suffices to show that $K_1 \cdot L$ is the generic zero field of $\varphi \otimes (K_0 \cdot L)$.

Let $F_0 := K_0 \cdot L$. We have $\varphi_0 \otimes F_0 = \varphi \otimes F_0$, and thus $F_0(\varphi \otimes F_0) = K_0(\varphi_0) \cdot_{K_0} F_0$. Also $K_1 \sim_{K_0} K_0(\varphi_0)$. By the lemma, this shows that

$$K_1 \cdot_{K_0} F_0 \sim_{F_0} K_0(\varphi_0) \cdot_{K_0} F_0.$$

Furthermore, $K_1 \cdot_{K_0} F_0 = K_1 \cdot_k L$. Thus, $K_1 \cdot_k L \sim_{F_0} F_0(\varphi_0 \otimes F_0)$, what we wanted to show.

Finally, assume that $\dim \varphi = 2$. Now we have $h = 1$. The field $\Delta(\varphi_0) = \Delta(\varphi) \otimes_k K_0$ is a quadratic extension of K_0 and K_1 is a regular extension of $\Delta(\varphi_0)$. We have a place $\lambda: K_1 \to \Delta(\varphi_0) \cup \infty$ over K_0. After composing λ with the non-trivial automorphism of $\Delta(\varphi_0)/K_0$ (if necessary), we obtain a place $\mu: K_1 \to \Delta(\varphi_0) \cup \infty$ over $\Delta(\varphi_0)$. Therefore $K_1/\Delta(\varphi_0)$ is an inessential extension and regular.

Let $F_0 := K_0 \cdot L$, $F_1 := K_1 \cdot L$. We again make a distinction between the cases $\varphi \otimes L$ isotropic, resp. anisotropic.

Suppose first that $\varphi \otimes L$ is isotropic. Now $J = \{1\}$, and $\Delta(\varphi) \otimes_k L$ splits. We choose an embedding $\Delta(\varphi) \hookrightarrow L$ and obtain from this an embedding $K_0 \cdot \Delta(\varphi) = \Delta(\varphi_0) \hookrightarrow K_0 \cdot L = F_0$. By Definition 6, we now have

$$K_1 \cdot L := K_1 \cdot_{\Delta(\varphi)} L = \mathrm{Quot}(K_1 \otimes_{\Delta(\varphi)} L) = \mathrm{Quot}(K_1 \otimes_{\Delta(\varphi_0)} (\Delta(\varphi_0) \otimes_{\Delta(\varphi)} L)),$$

and furthermore $\Delta(\varphi_0) \otimes_{\Delta(\varphi)} L = K_0 \otimes_k L$. It follows that

$$F_1 = K_1 \cdot L = K_1 \cdot_{\Delta(\varphi_0)} (K_0 \cdot L) = K_1 \cdot_{\Delta(\varphi_0)} F_0.$$

It is now clear that the extension F_1/F_0 is regular. Since $K_1/\Delta(\varphi_0)$ is inessential, it follows furthermore from the lemma that F_1/F_0 is inessential. F_0/L is also inessential. Thus F_1/L is inessential. This establishes statements (a) and (b) in this case.

Finally, let $\dim \varphi = 2$ and assume that $\varphi \otimes L$ is anisotropic. Now $J = \{0, 1\}$. Statement (b) is vacuous. We already know that F_0/L is inessential. Furthermore, $K_1/\Delta(\varphi_0)$ is regular and inessential. Therefore

$$F_1 = K_1 \cdot_{\Delta(\varphi_0)} (\Delta(\varphi_0) \cdot_{K_0} F_0)$$

is regular and inessential over $\Delta(\varphi_0) \cdot_{K_0} F_0 = \Delta(\varphi_0 \otimes F_0)$. Furthermore, $\Delta(\varphi_0 \otimes F_0) = F_0\{\varphi_0 \otimes F_0\}$. Hence F_1 is the generic zero field of the form $\varphi_0 \otimes F_0$. Therefore (F_0, F_1) is a regular generic splitting tower of $\varphi \otimes L$. $\square$

Corollary. *In the situation of Theorem 3, $\varphi \otimes L$ has height e and generic splitting pattern*

$$\mathrm{SP}(\varphi \otimes L) = \{i_0 + \ldots + i_{r_j} \mid 0 \le j \le e\}.$$

For every $j \in [0, e]$ is $\varphi_{r_j} \otimes (K_{r_j} \cdot L)$ furthermore the j-th higher kernel form of $\varphi \otimes L$ with respect to the generic splitting tower constructed here. $\square$

§14 Generic Splitting Towers of a Specialized Form

In the following, let $\gamma: k \to L \cup \infty$ be a place and φ a quadratic form over k having FR with respect to γ. Assume that the form $\overline{\varphi} := \gamma_*(\varphi)$ over L is nondegenerate. Finally, let $(K_r \mid 0 \leq r \leq h)$ be a generic splitting tower of φ.

We have seen in §12 that the splitting behaviour of $\overline{\varphi}$ with respect to $\overline{\varphi}$-conservative extensions of the field L is controlled by the tower $(K_r \mid 0 \leq r \leq h)$, cf. §12, Theorem 4 and Scholium 2. Hence we may hope that it is actually possible to construct a generic splitting tower of $\overline{\varphi}$ in a natural way from $(K_r \mid 0 \leq r \leq h)$. We will devote ourselves to the task of constructing such a tower.

In the special case where the place γ is trivial and the tower $(K_r \mid 0 \leq r \leq h)$ is regular, we already dealt with this task in the previous section (§13, Theorem 3). In general we cannot expect to find a solution as nice as the solution in §13.

Let us continue by fixing some notation. Let i_r be the higher indices and φ_r the higher kernel forms of φ with respect to the given splitting tower $(K_r \mid 0 \leq r \leq h)$ of φ. Thus we have $\mathrm{SP}(\varphi) = \{i_0 + \ldots + i_r \mid 0 \leq r \leq h\}$. Further, let J be the subset of $\{0, 1, \ldots, h\}$ with $\mathrm{SP}(\overline{\varphi}) = \{i_0 + \ldots + i_r \mid r \in J\}$. The set J contains $e := h(\overline{\varphi})$ elements. Let us list them as

$$0 \leq t(0) < t(1) < \ldots < t(e) = h.$$

We formally set $t(-1) = -1$.

Let $(L_s \mid 0 \leq s \leq e)$ be a generic splitting tower of $\overline{\varphi}$. Following §12, Scholium 2 we choose places $\mu_s: K_{t(s)} \to L_s \cup \infty \quad (0 \leq s \leq e)$ such that μ_0 extends the place γ and such that every μ_s with $1 \leq s \leq e$ extends the place μ_{s-1}. For every $r \in \{0, \ldots, h\}$ we have an $s \in \{0, \ldots, e\}$ with $t(s-1) < r \leq t(s)$. Let $\lambda_r: K_r \to L_s \cup \infty$ be the restriction of the place μ_s to K_r. {In particular, $\mu_s = \lambda_{t(s)}$.} Let $\psi_s = \ker(\overline{\varphi} \otimes L_s)$ be the s-th higher kernel form of $\overline{\varphi}$. By §12 $\varphi_{t(s)}$ has FR with respect to μ_s and $(\mu_s)_*(\varphi_{t(s)}) \cong \psi_s$. Furthermore, for $t(s-1) < r < t(s)$, φ_r also has FR with respect to λ_r and

$$(\lambda_r)_*(\varphi_r) \cong \psi_s \perp (i_{r+1} + \ldots + i_{t(s)}) \times H.$$

Given a place $\alpha: E \to F \cup \infty$ and a subfield K of E, we denote the image of K under α in general sloppily with $\alpha(K)$, in other words,

$$\alpha(K) := \alpha(\mathfrak{o}_\alpha \cap K).$$

In the important case that the place γ is surjective, i.e. $\gamma(k) = L$, we would like to choose the places μ_s in such a way that $(\mu_s(K_{t(s)}) \mid 0 \leq s \leq e)$ is a generic splitting tower of $\overline{\varphi}$. For not necessarily surjective γ, one could try to achieve that $(L \cdot \mu_s(K_{t(s)}) \mid 0 \leq s \leq e)$ is a generic splitting tower of $\overline{\varphi}$, where $L \cdot \mu_s(K_{t(s)})$ denotes the subfield of L_s generated by L and $\mu_s(K_{t(s)})$ in L_s.[21] In any case, for arbitrary choice of μ_s – as above – the following theorem holds.

Theorem 1. *For $t(s-1) < r \leq t(s)$ the subfield $\mu_s(K_r) \cdot L$ of L_s, generated by $\mu_s(K_r)$ and L, is a generic splitting field of $\overline{\varphi}$ of level s.*

Proof. Let $F_r := \lambda_r(K_r) \cdot L$. By restricting the range, we obtain from λ_r a place from K_r to F_r which extends γ. Hence

$$\operatorname{ind}(\overline{\varphi} \otimes F_r) \geq \operatorname{ind}(\varphi \otimes K_r) = i_0 + \ldots + i_r.$$

Since $\operatorname{ind}(\overline{\varphi} \otimes F_r)$ is a number of $\operatorname{SP}(\overline{\varphi})$, we get

$$\operatorname{ind}(\overline{\varphi} \otimes F_r) \geq i_0 + \ldots + i_{t(s)}.$$

Thus there is a place from L_s to F_r over L. Since $F_r \subset L_s$, we get $F_r \sim_L L_s$. Thus F_r is a generic splitting field of $\overline{\varphi}$ of level s. $\square$

This theorem does *not* imply that $(K_{t(s)} \cdot L \mid 0 \leq s \leq e)$ is a generic splitting tower of $\overline{\varphi}$. However, in the following we will produce more special situations for which this *is* the case.

Theorem 2. *There exists a regular*[22] *generic splitting tower $(L_s \mid 0 \leq s \leq e)$ of $\overline{\varphi}$ and a place $\lambda\colon K_h \to L_e \cup \infty$ having the following properties:*

(i) *λ extends γ.*

(ii) *For every $s \in \{0, 1, \ldots, e\}$ and every r with $t(s-1) < r \leq t(s)$ we have $L_s = \lambda(K_r) \cdot L$.*

(iii) *$L_0 = L$.*

Proof. We inductively construct fields $L_s \supset L$ and places $\mu_s\colon K_{t(s)} \to L_s \cup \infty$ as follows.

$s = 0$: We let $L_0 := L$. We choose for μ_0 any place from $K_{t(0)}$ to L which extends γ. This is possible by §12, Scholium 2. Obviously we have $\mu_0(K_r) \cdot L = L$ for every r with $0 \leq r \leq t(0)$.

$s \to s+1$: By the induction hypothesis, $(L_j \mid 0 \leq j \leq s)$ is the beginning of a generic splitting tower of $\overline{\varphi}$. In particular, L_s is a generic splitting field of $\overline{\varphi}$ of level s. Furthermore there is a place $\mu_s\colon K_{t(s)} \to L_s \cup \infty$. According to §12 the form $\varphi_{t(s)}$ has FR with respect to μ_s and $(\mu_s)_*(\varphi_{t(s)}) \cong \psi_s$.

[21] Now we are *not* using the notation established in §13, Definition 6.
[22] cf. §13, Definition 5.

Now we extend μ_s to a place $\tilde{\mu}_s$ from $K_{t(s)}(\varphi_{t(s)})$ to $L_s(\psi_s)$ (cf. §9, the proof of Theorem 3). Let $u := t(s)+1$ and let $\alpha: K_u \to K_{t(s)}(\varphi_{t(s)}) \cup \infty$ be a place over $K_{t(s)}$, which does indeed exist since K_u is a generic zero field of $\varphi_{t(s)}$. Then we have a place $\tilde{\mu}_s \circ \alpha$ from K_u to $L_s(\psi_s)$ and define L_{s+1} to be the composite of the fields L_s and $(\tilde{\mu}_s \circ \alpha)(K_u)$ in $L_s(\psi_s)$,

$$L_{s+1} := L \cdot (\tilde{\mu}_s \alpha)(K_u).$$

Let $\nu_s: K_u \to L_{s+1} \cup \infty$ be the place obtained from $\tilde{\mu}_s \circ \alpha$ by restricting the range. We have

$$\operatorname{ind}(\overline{\varphi} \otimes L_{s+1}) \geq \operatorname{ind}(\varphi \otimes K_u) = i_0 + \ldots + i_u > i_0 + \ldots + i_{t(s)},$$

and thus $\operatorname{ind}(\overline{\varphi} \otimes L_{s+1}) \geq i_0 + \ldots + i_{t(s+1)}$. This implies

$$\operatorname{ind}(\psi_s \otimes L_{s+1}) \geq i_u + \ldots + i_{t(s+1)} > 0.$$

Thus $\psi_s \otimes L_{s+1}$ is isotropic. Since $L_{s+1} \subset L_s(\psi_s)$, we conclude that L_{s+1} is a generic zero field of ψ_s. By §12, Scholium 2(3) we now have that ν_s can be extended to a place from $K_{t(s+1)}$ to L_{s+1}. We choose $\mu_{s+1}: K_{t(s+1)} \to L_{s+1} \cup \infty$ to be such a place. Since $L_{s+1} = \nu_s(K_u) \cdot L$, we have $L_{s+1} = \mu_{s+1}(K_r) \cdot L$ for every r with $u \leq r \leq t(s)$.

Eventually our construction yields a generic splitting tower

$$(L_s \mid 0 \leq s \leq e)$$

of $\overline{\varphi}$ and a place $\lambda := \mu_e$ from K_h to L_e which satisfy properties (i) and (ii) of the theorem.

Let us check that the tower $(L_s \mid 0 \leq s \leq e)$ is regular! For $0 \leq s \leq e-1$ our construction gives $L_s \subset L_{s+1} \subset L_s(\psi_s)$. If $\dim \psi_s > 2$, then $L_s(\psi_s)$ is regular over L_s, and thus L_{s+1} is regular over L_s. If $\dim \psi_s = 2$, then we must have $s = e-1$ and $t(s) = h-1$. Next, we modify the last step of the construction above in the sense that we replace $K_{h-1}(\varphi_{h-1})$ by the algebraic closure E of K_{h-1} in this field and also $K_{e-1}(\psi_{e-1})$ by the algebraic closure F of K_{e-1} in $K_{e-1}(\psi_{e-1})$.

Let $\mu := \mu_{h-1}$. We have $\varphi_{h-1} \cong \langle a \rangle \otimes \left[\begin{smallmatrix} 1 & 1 \\ 1 & b \end{smallmatrix}\right]$ with $a \in \mathfrak{o}_\mu^*$, $b \in \mathfrak{o}_\mu^*$, and $\psi_{e-1} \cong \langle \overline{a} \rangle \otimes \left[\begin{smallmatrix} 1 & 1 \\ 1 & \overline{b} \end{smallmatrix}\right]$ with $\overline{a} := \mu(a)$, $\overline{b} := \mu(b)$. We obtain $E = K_{h-1}(\xi)$ and $F = L_{e-1}(\eta)$ with generators ξ, η which satisfy the minimal equations $\xi^2 + \xi + b = 0$, $\eta^2 + \eta + \overline{b} = 0$. The extension K_h/E is inessential. This can be verified as in the proof of §13, Theorem 3 towards the end. (There for the case $\dim \varphi = 2$.)

The place $\mu = \mu_{e-1}$ from K_{h-1} to L_{e-1} has exactly one extension $\tilde{\mu}_{e-1}: E \to F \cup \infty$ with $\tilde{\mu}_{e-1}(\xi) = \eta$ (cf. §9, Lemma 2). If we prepend a place from K_h to E over E to $\tilde{\mu}_{e-1}$, we obtain a place $\mu_e: K_h \to F \cup \infty$ which extends μ_{e-1}. The field $L_e := \mu_e(K_h) \cdot L$ differs from L_{e-1} since it splits the form $\overline{\varphi}$ totally. Hence we must have $L_e = F$. Our generic splitting tower $(L_s \mid 0 \leq s \leq e)$ is regular. □

What we have obtained by now can perhaps best be understood in the important special case where the place $\gamma: k \to L \cup \infty$ is surjective (i.e. $\gamma(k) =$

L). For the given generic splitting tower $(K_r \mid 0 \le r \le h)$ of φ we have constructed a place $\lambda\colon K_h \to L_e \cup \infty$ which extends γ and for which the tower $(\lambda(K_r) \mid 0 \le r \le h)$ is an almost regular generic splitting tower of $\overline{\varphi}$ "with repetitions", i.e. a generic splitting tower in which the storeys are listed possibly more than once: $\lambda(K_r) = \lambda(K_{t(s)})$ for $t(s-1) < r \le t(s)$. The place λ furnishes a connection between the tower $(K_r | 0 \le r \le h)$ and the generic splitting tower $(L_s | 0 \le s \le e)$ of $\overline{\varphi}$, $L_s = \lambda(K_{t(s)})$ which reflects in an obvious way how the splitting behaviour of $\overline{\varphi}$ over L is a coarsening of the splitting behaviour of φ over K.

In many situations it is more natural or, for some given problem, more useful (see e.g. [KR]) to associate to the tower $(K_r \mid 0 \le r \le h)$ a different generic splitting tower of $\overline{\varphi}$. Such a situation (with γ trivial) was depicted in §13. Next we give a further construction which applies to an arbitrary place γ with respect to which φ has FR and with $\overline{\varphi} = \gamma_*(\varphi)$ nondegenerate, as above.

Specifically, let $(K_r \mid 0 \le r \le h)$ be the projective standard tower of φ (see §13, Definition 4). We define a field tower $(L_r \mid 0 \le r \le h)$ and a sequence of places $(\lambda_r\colon K_r \to L_r \cup \infty \mid 0 \le r \le h)$ with $L_r \supset L$ and $L_r = \lambda_r(K_r) \cdot L$ for all r, $0 \le r \le h$, as follows.[23]

We start with $L_0 := L$, $\lambda_0 := \gamma$. If $\lambda_r\colon K_r \to L_r \cup \infty$ is already defined and $r < h$, then φ_r has FR with respect to λ_r by §12. Let $\overline{\varphi}_r := (\lambda_r)_*(\varphi_r)$. We set $L_{r+1} := L_r\{\overline{\varphi}_r\}$. Let $\tilde{\lambda}_r\colon K_r(\varphi_r) \to L_r(\overline{\varphi}_r) \cup \infty$ be an extension of the place λ_r, obtained using the procedure in the proof of §9, Theorem 3.

Next, let $\overline{\varphi}_r \not\cong H$. The place $\tilde{\lambda}_r$ maps $K_{r+1} = K_r\{\varphi_r\}$ into $L_{r+1} \cup \infty$. From the construction in §9 it is clear that $L_{r+1} = \tilde{\lambda}_r(K_{r+1}) \cdot L$. We define $\lambda_{r+1}\colon K_{r+1} \to L_{r+1} \cup \infty$ by restricting the place $\tilde{\lambda}_r$. We have $L_{r+1} = \lambda_{r+1}(K_{r+1}) \cdot L$.

Finally, let $\overline{\varphi}_r \cong H$ (and so $r = h-1$). Now we have $L_{r+1} = L_r$. From the construction in §9, we have $\tilde{\lambda}_r(K_r(\varphi_r)) \cdot L = L_r$. Again we define λ_{r+1}, i.e. λ_h, to be the restriction of $\tilde{\lambda}_r$ to a place from K_{r+1} to $L_{r+1} = L_r$. Then, clearly, $\lambda_{r+1}(K_{r+1}) \cdot L = L_{r+1}$.

Definition. We call the tower $(L_r \mid 0 \le r \le h)$ the *transfer* of the projective standard tower by the place γ and we call $\lambda := \lambda_h\colon K_h \to L_h \cup \infty$ a *transferring place.*

Remark. By the remark in §9, following the proof of Theorem 3, the places $\lambda_r\colon K_r \to L_r \cup \infty$ for $0 \le r \le h-1$ in the construction above are uniquely determined by γ. If $\overline{\varphi}_{h-1} \not\cong H$ this is also the case for $\lambda_h = \lambda$. If $\overline{\varphi}_{h-1} \cong H$, then there are at most two possibilities for λ_h, since K_h is a quadratic extension of K_{h-1} in this situation. The fields $L_r = \lambda_r(K_r) \cdot L$ are all uniquely determined by φ and γ.

Theorem 3. *Let $(L_r \mid 0 \le r \le h)$ be the transfer of the projective standard tower $(K_r \mid 0 \le r \le h)$ of φ by the place $\gamma\colon k \to L \cup \infty$ and let $\lambda\colon K_h \to L_h \cup \infty$*

[23] The equation $L_r = \lambda_r(K_r) \cdot L$ only signifies that L_r is generated as a field by both subfields $\lambda_r(K_r)$ and L.

be a transferring place. Let $\overline{\varphi}_r := (\lambda_r)_*(\varphi_r)$ *be the specialization of* φ_r *with respect to the restriction* $\lambda_r\colon K_r \to L_r \cup \infty$ *of* λ $\quad (0 \le r \le h)$.

Claim: $(L_r \mid r \in J)$ *is a generic splitting tower of* $\overline{\varphi}$ *and for every* $r \in J$ *we have* $\overline{\varphi}_r = \ker(\overline{\varphi} \otimes L_r)$. *For* $r \notin J$, L_{r+1}/L_r *is purely transcendental. In particular, the extension* $L_{t(0)}/L$ *is purely transcendental.*

Proof. Let $r \in \{0, \dots, h\}$ be given. From the Witt decomposition

$$\varphi \otimes K_r \cong (i_0 + \ldots + i_r) \times H \ \perp\ \varphi_r$$

we obtain

$$\overline{\varphi} \otimes L_r \cong (i_0 + \ldots + i_r) \times H \ \perp\ \overline{\varphi}_r,$$

since $(\lambda_r)_*(\varphi \otimes K_r) \cong \overline{\varphi} \otimes L_r$.

If $r \notin J$, then $\operatorname{ind}(\overline{\varphi} \otimes L_r) > i_0 + \ldots + i_r$. Now $\overline{\varphi}_r$ is isotropic and hence $L_{r+1} = L_r\{\overline{\varphi}_r\}$ is purely transcendental over L_r.

Let a number s be given with $0 \le s \le e$. Then $L_{t(s)+1} = L_{t(s)}\{(\overline{\varphi}_{t(s)})\}$ and $L_{t(s+1)}/L_{t(s)+1}$ is purely transcendental. Hence $L_{t(s+1)}/L_{t(s)}$ is a generic zero field of $\overline{\varphi}_{t(s)}$. Furthermore, $L_{t(0)}/L$ is purely transcendental. Thus

$$\operatorname{ind}(\overline{\varphi} \otimes L_{t(0)}) = \operatorname{ind}(\overline{\varphi}) = i_0 + \ldots + i_{t(0)}.$$

This shows that $\overline{\varphi}_{t(0)}$ is the kernel form of $\overline{\varphi} \otimes L_{t(0)}$. Suppose now that for some $s \in \{1, \dots, e\}$ we already showed that $\overline{\varphi}_{t(k)}$ is the kernel form of $\overline{\varphi} \otimes L_{t(k)}$ for $0 \le k < s$. Then $(L_{t(k)} \mid 0 \le k \le s)$ is the beginning of a generic splitting tower of $\overline{\varphi}$. Hence $\overline{\varphi} \otimes L_{t(s)}$ has index $i_0 + \ldots + i_{t(s)}$, and it follows that $\overline{\varphi}_{t(s)}$ is the kernel form of $\overline{\varphi} \otimes L_{t(s)}$. Thus, this holds for all $s \in \{0, \dots, e\}$. Now it is clear that $(L_{t(s)} \mid 0 \le s \le e)$ is a generic splitting tower of $\overline{\varphi}$. $\square$

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