ON BILINEAR FORMS OF HEIGHT 2 AND DEGREE 1 OR 2 IN CHARACTERISTIC 2

AHMED LAGHRIBI AND ULF REHMANN

Abstract. Our aim in this paper is to complete some results given in [15] on the classification of symmetric bilinear forms of height 2 and degree $d = 1$ or 2 over fields of characteristic 2, i.e., those whose anisotropic parts over their own function fields are similar to d-fold bilinear Pfister forms.

1. Introduction and Main results

Let F be a field of characteristic 2. Throughout this paper, the expression "bilinear form" means "finite dimensional regular symmetric bilinear form".

For a field extension L/F and a bilinear (or quadratic) form B over L, we say that B is *definable over* F if B is isometric to C_L for some bilinear (or quadratic) form C over F. If moreover, C is unique, then we say that B is *defined (by C) over F*.

To a bilinear form B with underlying vector space V, we associate a unique quadratic form $\widetilde{B}$ given on V by: $\widetilde{B}(v) = B(v, v)$ for $v \in V$. The function field of B, denoted by $F(B)$, is by definition the function field of $\widetilde{B}$. The standard splitting tower of a nonzero bilinear form B is the sequence of forms and fields defined as follows:

$$\begin{cases} F_0 = F \quad \text{and} \quad B_0 = B_{\mathrm{an}} \\ \text{For } n \geq 1: \quad F_n = F_{n-1}(B_{n-1}) \quad \text{and} \quad B_n = ((B_{n-1})_{F_n})_{\mathrm{an}}, \end{cases}$$

where C_{an} denotes the anisotropic part of a bilinear form C. The height $\mathrm{h}(B)$ of B is the smallest integer h such that $\dim B_h \leq 1$, where $\dim C$ denotes the dimension of a bilinear form C. As was done by the first author in [15], we associate to the form B another numerical invariant $\deg(B)$, called the degree of B, as follows: If $h = \mathrm{h}(B)$ and $(B_i, F_i)_{0 \leq i \leq h}$ is the standard splitting tower of B, then the form $B_{\mathrm{h}(B)-1}$ is of height 1. By the classification of height 1 bilinear forms [15, Th. 4.1], there

Key words and phrases. Bilinear and quadratic forms, function fields of quadrics, standard splitting theory, norm degree.

MSC 2000. 11E04; 11E81.

exists a unique bilinear Pfister form π over $F_{\mathrm{h}(B)-1}$ such that $B_{\mathrm{h}(B)-1}$ is similar to π or to the pure part of π according as $\dim B$ is even or odd. If $\dim B$ is even, then we put $\deg(B) = d$ where $\dim \pi = 2^d$. Otherwise, we put $\deg(B) = 0$.

We call π the leading form of B. The form B is called good if π is definable over F, and in this case, we know from [15, Prop. 5.3] that π is defined over F by a d-fold bilinear Pfister form. For example, if B is of even dimension and nontrivial determinant, then it is good of degree 1 and leading form $(\langle 1, \det B\rangle_b)_{F_{h-1}}$.

An important problem considered in [15] is the classification of bilinear forms by height and degree. Bilinear forms of height 1 are completely classified as we said before in the definition of the degree. For good bilinear forms of height 2, the first author gave in [15] a complete classification of those of degree 0, and a partial classification of those of degree ≥ 1. In this paper, we complete the classification of bilinear forms which are good of height 2 and degree 1 or 2, and with [15, Th. 5.10] we get the following theorem:

Theorem 1.1. *Let B be an anisotropic bilinear form over F which is good of height 2 and degree $d = 1$ or 2. Let λ be the unique d-fold bilinear Pfister form over F such that $\lambda_{F(B)}$ is the leading form of B. Then, we are in one of the following cases:*
(1) $\dim B = 2^n$ with $n \geq d+1$: In this case, there exists $\alpha \in F^* := F \setminus \{0\}$ and π similar to an n-fold bilinear Pfister form such that $B \perp \alpha\lambda \perp \pi$ is metabolic.
(2) $\dim B = 2^m - 2^d$ with $m \geq d+2$: In this case, $B \simeq \rho \otimes \lambda$ such that $\dim \rho$ is odd and $B \perp \langle \det \rho\rangle_b \otimes \lambda$ is similar to an m-fold bilinear Pfister form.
Conversely, any anisotropic bilinear form satisfying the conditions described in (1) or (2) is good of height 2 and degree d.

Moreover, the first author gave a formula on the possible dimensions of bilinear forms of height 2 (good or not) [15, Cor 5.20]. As a consequence of it, we get that the dimension of any anisotropic bilinear form of height and degree 2 (good or not) can be 2^n, $2^n - 2$, or $2^n - 4$ for some $n \geq 3$ [15, Comment after Remark 5.21]. Note that Theorem 1.1 shows that the integers 2^n for $n \geq 3$, and $2^n - 4$ for $n \geq 4$ occur as dimensions of good anisotropic bilinear forms of height and degree 2. We know by [15] that any Albert bilinear form, i.e., a 6-dimensional bilinear form of trivial determinant, is of height and degree 2 but not good. Before this work and except for the integer 6, we did not know other integers which really do occur as dimensions of anisotropic nongood bilinear forms of height and degree 2. Here, we clarify this point by proving the following theorem:

Theorem 1.2. *(1) There are* 8*-dimensional anisotropic nongood bilinear forms of height and degree* 2*.*
(2) An anisotropic bilinear form B *over* F *is nongood of height and degree* 2 *iff one of the following conditions holds:*
(i) B *is an Albert form.*
(ii) $\dim B = 8$ *and there exists an anisotropic Albert bilinear form* θ*, unique up to similarity, that becomes isotropic over* $F(B)$ *and satisfies* $B \perp \theta \in I^3F$*.*

The existence of 8-dimensional anisotropic bilinear forms of height and degree 2 which are not good is new in comparison with what is known in characteristic $\neq 2$. In fact, in this case, Kahn proved that a nongood anisotropic quadratic form of height and degree 2 is necessarily of dimension 6 and trivial discriminant, i.e., an Albert quadratic form [7]. Kahn's proof is based on the index reduction theorem of Merkurjev [18], [21]. But we do not have such a theorem for bilinear forms in characteristic 2. In our case, we will be inspired from a descent method due to Kahn [8], and we will use a result of Aravire and Baeza [1] to get the following theorem which is essential for the proofs of Theorems 1.1 and 1.2:

Theorem 1.3. *Let* B *be an anisotropic bilinear form over* F *of dimension* ≥ 3*, and* τ *an anisotropic bilinear form similar to a* d*-fold bilinear Pfister form over* $F(B)$*, with* $d = 1$ *or* 2*. Let* C *be a bilinear form over* F*.*
(1) Suppose that $d = 2$ *and* $\tau \perp C_{F(B)} \in I^3F(B)$*. Then, there exists an Albert bilinear form* θ *over* F *such that* $\tau \perp \theta_{F(B)} \in I^3F(B)$*. Furthermore, if* $\dim B > 8$*, then there exists a unique* 2*-fold bilinear Pfister form* λ *over* F *such that* τ *is similar to* $\lambda_{F(B)}$*.*
(2) Suppose that $\dim B > 2^{d+1}$ *and* $\tau \perp C_{F(B)} \in I^{d+2}F(B)$*. Let* λ *be as in (1) if* $d = 2$*, or* $\lambda = \langle 1, \det C\rangle_b$ *if* $d = 1$*. Moreover, suppose that* $B_{F(\lambda)}$ *is anisotropic or* $C_{F(\lambda)}$ *is metabolic. Then,* τ *is defined over* F *by a form similar to* λ*.*

Obviously, statement (1) of Theorem 1.3 implies that an anisotropic nongood bilinear form of height and degree 2 is of dimension 6 or 8. Moreover, as we see in Theorem 1.2, a complete classification of such bilinear forms consists in studying the isotropy of Albert bilinear forms over function fields of quadrics. This is an affair of norm field and norm degree (see subsection 2.3 for the definitions). More precisely, for θ an

Albert bilinear form, there exists scalars $x, u, v, r, s \in F^*$ such that $\theta \simeq x\langle r, s, rs, u, v, uv\rangle$ (because θ is of trivial determinant). The norm field of $\widetilde{\theta}$ is $F^2(r, s, u, v)$, and if θ is anisotropic then $[F^2(r, s, u, v) : F^2] = 8$ or 16, since this degree is a power of 2, and it is at least equal to $\dim \theta$. The condition that this degree equals 8 is equivalent to say that the 4-fold bilinear Pfister form $\langle\langle r, s, u, v\rangle\rangle$ is isotropic. In this case, the form $\widetilde{\theta}$ is a quasi-Pfister neighbor of a quasi-Pfister form π ([5, Def. 8.8], [5, Prop. 8.9(ii)]), and θ becomes isotropic over the function field of an anisotropic bilinear form B iff π is isotropic over $F(B)$ iff $\widetilde{B}$ is similar to a subform of π (we use [5, Prop. 8.9(iii)] and [12, Prop. 2.4]). If $[F^2(r, s, u, v) : F^2] = 16$ we ask the following question:

Question 1.4. *Let θ be an anisotropic Albert bilinear form over F such that the norm field of $\widetilde{\theta}$ is of degree 16 over F^2. Let B be an anisotropic bilinear form of dimension ≥ 2 such that $\theta_{F(B)}$ is isotropic and $\widetilde{B}$ is not similar to a 2-fold quasi-Pfister form. Is it true that $\widetilde{B}$ is similar to a subform of $\widetilde{\theta}$?*

We have a partial answer to this question:

Proposition 1.5. *Question 1.4 has a positive answer if $\dim B = 2$ or 3.*

For 8-dimensional nongood bilinear forms of height and degree 2, Theorem 1.2 can be refined for a special class of fields as follows:

Proposition 1.6. *Let F be a field of characteristic 2 satisfying one of the following conditions:*

(C1) Any 4-fold bilinear Pfister form over F is isotropic.
(C2) Question 1.4 has a positive answer.

Then, an 8-dimensional anisotropic bilinear form B over F is of height and degree 2 but not good iff there exists an anisotropic Albert bilinear form C, a 3-fold bilinear Pfister form π, and scalars $x, y, z \in F^*$ such that $xB \perp yC \perp z\pi$ is metabolic, and the forms $\widetilde{B}$ and $\widetilde{C}$ are similar to subforms of $\widetilde{\pi}$.

The rest of this paper is organized as follows. In the next section we recall some definitions, notions and results on bilinear forms and totally singular quadratic forms, like Witt decompositions, the notion of norm degree, and some facts on transfer for bilinear forms. After that, we give the proofs of the results announced in this section. We start with the proof of Theorem 1.3 since we will need it for the proofs of Theorems 1.1 and 1.2.

2. Backgrounds on bilinear forms

The details of the most results that we present in this section can be found in [2], [5] and [20].

2.1. **Some definitions.** A quadratic form φ is called totally singular if it is isometric to $\widetilde{B}$ for some bilinear form B.

A bilinear (or quadratic) form C is called a subform of B, denoted by $C \subset B$, if $B \simeq C \perp C'$ for some bilinear (or quadratic) form C'.

Two forms (bilinear or quadratic) B and C are called similar if $B \simeq \alpha C$ for some scalar $\alpha \in F^*$.

For $a_1, \cdots, a_n \in F^*$, the diagonal bilinear form B given by the polynomial $\sum_{i=1}^n a_i x_i y_i$ will be denoted by $\langle a_1, \cdots, a_n \rangle_b$, and the quadratic form $\widetilde{B}$ will be denoted by $\langle a_1, \cdots, a_n \rangle$.

For any integer $n \geq 1$ and $a_1, \cdots, a_n \in F^*$, the form $\langle 1, a_1 \rangle_b \otimes \cdots \otimes \langle 1, a_n \rangle_b$ is called an n-fold bilinear Pfister form, denoted by $\langle\langle a_1, \cdots, a_n \rangle\rangle_b$. ($\otimes$ means the product of bilinear forms.) The 0-fold bilinear Pfister form is just $\langle 1 \rangle_b$.

The pure part of a bilinear Pfister B is the unique form B' satisfying $B \simeq \langle 1 \rangle_b \perp B'$.

Let IF be the ideal of the Witt ring $W(F)$ of bilinear forms of even dimension, and $I^nF = (IF)^n$ for any $n \geq 0$ (with $I^0F = W(F)$). The ideal I^nF is additively generated by n-fold bilinear Pfister forms.

A basic result that we will use, called the Arason-Pfister Hauptsatz or simply the Hauptsatz, asserts that any anisotropic bilinear form B in I^nF is of dimension $\geq 2^n$, and if $\dim B = 2^n$ then B is similar to an n-fold bilinear Pfister form [15, Lem. 4.8].

For any integer $n \geq 0$, let $\overline{I^n}F$ denote the quotient $I^nF/I^{n+1}F$.

A quasi-Pfister form is a totally singular quadratic form φ such that $\varphi \simeq \widetilde{B}$ for some bilinear Pfister form B. A totally singular form ψ is called a quasi-Pfister neighbor if it is similar to a subform of a quasi-Pfister form φ and $2\dim\psi > \dim\varphi$.

2.2. **Witt decompositions.** A quadratic (or bilinear) form B with underlying vector space V is called isotropic if $B(v) = 0$ (or $\widetilde{B}(v) = 0$) for some nonzero vector $v \in V$, and it is called anisotropic otherwise.

For any scalar $a \in F$, the 2-dimensional bilinear form given by the matrix $\begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}$ is called a metabolic plane. Such a bilinear form is denoted by $\mathbb{M}(a)$. An orthogonal sum of metabolic planes is called a metabolic bilinear form.

Two bilinear forms B and C are called equivalent, denoted by $B \sim C$, if $B \perp M \simeq C \perp M'$ for some metabolic forms M and M'.

For totally singular quadratic forms, the Witt decomposition states that any such quadratic form φ decomposes as follows: $\varphi \simeq \psi \perp i \times \langle 0 \rangle$ for some integer $i \geq 0$, and an anisotropic form ψ which is unique up to isometry [5]. We call ψ (*resp.* i) the anisotropic part of φ, denoted by φ_{an} (*resp.* the defect index of φ, denoted by $i_d(\varphi)$).

The same decomposition is also known for bilinear forms and states that any such a form is isometric to an orthogonal sum of an anisotropic form and a metabolic form [9], [19]. In this paper we need the following refinement of this decomposition:

Proposition 2.1. ([15, Prop. 5.15]) *Let B be a bilinear form over F of dimension ≥ 1. Then, there exists a bilinear form C, a unique pair of integers (m, n), and scalars $a_1, \cdots, a_m \in F^*$ such that: $B \simeq C \perp \mathbb{M}(a_1) \perp \cdots \perp \mathbb{M}(a_m) \perp n \times \mathbb{M}(0)$ and $C \perp \langle a_1, \cdots, a_m \rangle_b$ is anisotropic. Consequently:*

1. $(\widetilde{B})_{an} \simeq \widetilde{C} \perp \langle a_1, \cdots, a_m \rangle$.
2. $m + \dim C = \dim(\widetilde{B})_{an}$.

With the same notations and hypotheses as in Proposition 2.1, the bilinear form C is unique, we call it the anisotropic part of B, denoted by B_{an}. The Witt index of B is the integer $m + n$, denoted by $i_W(B)$.

2.3. **Norm degree.** The norm field of a nonzero totally singular form φ, denoted by $N_F(\varphi)$, is the field $F^2(\alpha\beta \mid \alpha, \beta \in D_F(\varphi))$, where $D_F(\varphi)$ is the set of scalars in F^* represented by φ. We denote by $\mathrm{ndeg}_F(\varphi)$ the integer $[N_F(\varphi) : F^2]$, and we call it the norm degree of φ. It is clear that $N_F(\varphi) = N_F(\alpha\varphi)$ for any scalar $\alpha \in F^*$. If φ is anisotropic and $2^n < \dim \varphi \leq 2^{n+1}$, then $\mathrm{ndeg}_F(\varphi) \geq 2^{n+1}$, and $\mathrm{ndeg}_F(\varphi) = 2^{n+1}$ if and only if φ is a quasi-Pfister neighbor. If φ and φ' are anisotropic quadratic forms such that φ is totally singular and $\varphi_{F(\varphi')}$ is isotropic, then φ' is also totally singular and $N_F(\varphi') \subset N_F(\varphi)$. We refer to [5, Section 8] for more details on norm field and some of its applications.

2.4. **Transfer.** For a finite extension K/F, a bilinear form B over K with underlying vector space V, and a nonzero F-linear map $s : K \longrightarrow F$, we get an F-bilinear form $s_*(B) : V \times V \longrightarrow F$, given by: $(v, v') \mapsto s(B(v, v'))$. We call it the transfer of B by s. As was proved in [20, Lem. 5.5, Page 47], the bilinear form $s_*(B)$ is also regular. Moreover, the transfer operation is compatible with isometry and orthogonal sum. Since the transfer of a metabolic form is a also a metabolic form, it is clear that we get a group homomorphism $s_* : W(K) \longrightarrow W(F)$. By the same argument as in [20, Th. 5.6, page 48], we also have the Frobenius reciprocity which means that for a bilinear form B over F and a bilinear form B' over K, there is an isometry $s_*(B_K\otimes B')\simeq B\otimes s_*(B')$. Finally, let us denote by $i_*: W(F)\longrightarrow W(K)$ the ring homomorphism induced by the inclusion $F\subset K$.

3. Proof of Theorem 1.3

We start with preliminary results.

Lemma 3.1. *Let $d\in F^*\setminus F^{*2}$ and τ a bilinear form over $F(\sqrt{d})$ similar to a 2-fold bilinear Pfister form. If τ is definable over F, then there exists θ similar to a 2-fold bilinear Pfister form over F such that $\tau\simeq\theta_{F(\sqrt{d})}$.*

Proof. Let $p,q,r,s\in F^*$ be such that $\tau\simeq\langle p,q,r,s\rangle_{F(\sqrt{d})}$. By comparing determinants, we get $s=u^2pqr$ for some $u\in F(\sqrt{d})^*$. Then, $\tau\simeq\langle p,q,r,pqr\rangle_{F(\sqrt{d})}$. ∎

We need a computation due to Aravire and Baeza, and another one due to the first author:

Theorem 3.2. ([1, Cor. 3.3]) *Let $B=\langle\langle a_1,\cdots,a_n\rangle\rangle$ be an anisotropic n-fold bilinear Pfister form. Then, the kernel of the natural homomorphism $\overline{I^m}F\longrightarrow\overline{I^m}F(B)$ is trivial if $n>m$, and it is equal to $\overline{\{\psi\otimes\langle\langle x_1,\cdots,x_n\rangle\rangle\mid\psi\in I^{m-n}F, \text{and } x_1,\cdots,x_n\in F^2(a_1,\cdots,a_n)^*\}}$ if $n\leq m$.*

Proposition 3.3. ([15, Prop. 4.13]) *Let $n\geq 1$ be an integer, and B be an anisotropic bilinear form of dimension $>2^n$. Then, the kernel of the natural homomorphism $\overline{I^n}F\longrightarrow\overline{I^n}F(B)$ is trivial.*

We give two lemmas on transfer which are well-known for quadratic forms in characteristic $\neq 2$:

Lemma 3.4. *Let $L=F(\sqrt{k})$ with $k\in F^*\setminus F^{*2}$, and $s:L\longrightarrow F$ the F-linear map given by: $1\mapsto 0$ and $\sqrt{k}\mapsto 1$. Then, an anisotropic bilinear form $B\in W(L)$ satisfies $s_*(B)=0$ if and only if B belongs to the image of the homomorphism i_*.*

Proof. We use the same argument as for the proof of [20, Th. 5.10, page 50]. ∎

Lemma 3.5. *Let $L=F(\sqrt{k})$ with $k\in F^*\setminus F^{*2}$, and $s:L\longrightarrow F$ a nonzero F-linear map. Then, $s_*(I^nL)\subset I^nF$ for any integer $n\geq 0$.*

Proof. We use the same proof as for [20, Cor. 14.9] after generalizing without difficulty [20, Lem. 14.8, page 92] to the case of bilinear forms in characteristic 2. ∎

Proof of Theorem 1.3. Let B be an anisotropic bilinear form over F of dimension ≥ 3, τ be an anisotropic bilinear form similar to a d-fold bilinear Pfister form over $F(B)$, with $d = 1$ or 2. Let C be a bilinear form over F.

(1) Suppose that $d = 2$ and $\tau \perp C_{F(B)} \in I^3F(B)$. Note that $C \in I^2F$ since $C_{F(B)} \in I^2F(B)$ and $\dim B \geq 3$. We may suppose, modulo $I^3F(B)$, that τ is isometric to a 2-fold bilinear Pfister form.

We have $F(B) = L(\sqrt{k})$ for a purely transcendental extension L/F and $k \in L^* \setminus L^{*2}$. Let $s : F(B) \longrightarrow L$ be the L-linear map given by: $1 \mapsto 0$ and $\sqrt{k} \mapsto 1$. We have $\tau \perp C_{F(B)} \in I^3F(B)$. It follows from Lemma 3.5 that $s_*(\tau \perp C) = s_*(\tau) \in I^3L$. Since $\langle 1\rangle_b \subset \tau$, we deduce by the Hauptsatz that $s_*(\tau) = 0$. By Lemma 3.4, there exists θ_1 a bilinear form over L such that $\tau \sim (\theta_1)_{F(B)}$. By [13, Cor. 3.5], we may suppose that $\dim \theta_1 = 4$. By Lemma 3.1 and the multiplicativity of bilinear Pfister forms, we may suppose that θ_1 is isometric to a 2-fold bilinear Pfister. Since $(\theta_1)_{F(B)} \perp C_{F(B)} \in I^3F(B)$, we deduce that $\theta_1 \perp C_L$ belongs to the kernel of the natural homomorphism $\overline{I^2}L \longrightarrow \overline{I^2}L(\sqrt{k})$. By applying Theorem 3.2 in the case $m = 2$ and $n = 1$, we conclude that $\theta_1 \perp C_L \perp D \otimes \langle 1, x^2 + y^2k\rangle_b \in I^3L$, where $D \in IL$ and $x, y \in L$ such that $x^2 + y^2k \neq 0$. Hence, $\theta_1 \perp C_L \perp \theta_2 \in I^3L$, where $\theta_2 = \langle 1, \det D\rangle_b \otimes \langle 1, x^2 + y^2k\rangle$. Now, by specializing the variables defining the field L to suitable scalars in F, we deduce by [10] that

$$C \perp \gamma_1 \perp \gamma_2 \in I^3F \tag{1}$$

where γ_1 and γ_2 are 2-fold bilinear Pfister forms over F. If we extend (1) to the field $F(B)$, we get $\tau \perp (\gamma_1 \perp \gamma_2)_{F(B)} \in I^3F(B)$. Hence, the Albert bilinear form θ that we need is the orthogonal sum of the pure parts of γ_1 and γ_2.

Suppose that $\dim B > 8$ and $\tau \perp \theta_{F(B)} \in I^3F(B)$ for an Albert bilinear form θ over F. Let $\rho = \theta_{\mathrm{an}}$. Since $\tau \not\sim 0$, it follows from the Hauptsatz that $\dim \rho \in \{4, 6\}$. By [6, Th. 1.1] the bilinear form $\rho_{F(B)}$ is anisotropic. By the Hauptsatz, we have $\rho_{F(B)(\tau)} \sim 0$, and thus, by [13, Th. 1.2] $\dim \rho_{F(B)}$ is divisible by 4. Hence, $\dim \rho_{F(B)} = 4$, i.e., $\dim \rho = 4$. Hence, ρ is similar to a 2-fold bilinear Pfister form, denoted by λ. By the Hauptsatz $\tau \simeq \lambda_{F(B)}$, and thus τ is definable over F. For the uniqueness of λ, let δ be another 2-fold bilinear Pfister form satisfying $\tau \simeq \delta_{F(B)}$, then $(\lambda \perp \delta)_{F(B)} \sim 0$. Since $\dim(\lambda \perp \delta)_{\mathrm{an}} < 8$, it follows from [13, Prop. 1.1] that $\lambda \simeq \delta$.

(2) Suppose that $\dim B > 2^{d+1}$ and $\tau \perp C_{F(B)} \in I^{d+2}F(B)$. Let λ be as in (1) if $d = 2$, or $\lambda = \langle 1, \det C\rangle_b$ if $d = 1$. Suppose that one of the following conditions holds:

(C1) $B_{F(\lambda)}$ is anisotropic.
(C2) $C_{F(\lambda)}$ is metabolic.

The form $\lambda_{F(B)}$ is similar to τ. Hence, $(C \perp \lambda)_{F(B)} \in I^{d+1}F(B)$. By using $\dim B > 2^{d+1}$ and Proposition 3.3, we get $C \perp \lambda \in I^{d+1}F$. Moreover,

$$(C \perp \lambda)_{F(B)(\lambda)} \equiv (\tau \perp \lambda)_{F(B)(\lambda)} \pmod{I^{d+2}F(B)(\lambda)}.$$

Since $(\tau \perp \lambda)_{F(B)(\lambda)}$ is metabolic, it follows that $C \perp \lambda + I^{d+2}F(\lambda)$ belongs to the kernel of the homomorphism $\overline{I^{d+1}}F(\lambda) \longrightarrow \overline{I^{d+1}}F(\lambda)(B)$.

Now, if the condition (C1) is satisfied, then Proposition 3.3 with the hypothesis $\dim B > 2^{d+1}$ implies that $(C \perp \lambda)_{F(\lambda)} \in I^{d+2}F(\lambda)$. If the condition (C2) is satisfied, then it is clear that $(C \perp \lambda)_{F(\lambda)} \in I^{d+2}F(\lambda)$. Hence, $C \perp \lambda + I^{d+2}F$ belongs to the kernel of $\overline{I^{d+1}}F \longrightarrow \overline{I^{d+1}}F(\lambda)$. By Theorem 3.2, $C \perp \lambda \perp \nu \otimes \mu \in I^{d+2}F$ for suitable $\nu \in IF$ and μ a d-fold bilinear Pfister form over F. Hence, $C \perp \lambda \perp \langle 1, \beta\rangle_b \otimes \mu \in I^{d+2}F$, where $\beta = \det \nu$. The form $\lambda \perp \langle 1, \beta\rangle \otimes \mu$ is isotropic (because λ and $\langle 1, \beta\rangle \otimes \mu$ represent 1). Hence, by the Hauptsatz, and after extending scalars to $F(B)$, we get

$$\tau \sim (\lambda \perp \langle 1, \beta\rangle \otimes \mu)_{F(B)} \sim (\lambda' \perp \langle \beta\rangle_b \perp \langle 1, \beta\rangle \otimes \mu')_{F(B)},$$

where λ' and μ' denote the pure parts of λ and μ, respectively. Hence, $i_W((\lambda' \perp \langle \beta\rangle_b \perp \langle 1, \beta\rangle \otimes \mu')_{F(B)}) = 2^d - 1$, and thus, any subform of $\lambda' \perp \langle \beta\rangle_b \perp \langle 1, \beta\rangle \otimes \mu'$ of dimension 2^{d+1} becomes isotropic over $F(B)$ [6, Lem. 2.11]. Since $\dim B > 2^{d+1}$, it follows from [6, Th. 1.1] that any subform of $\lambda' \perp \langle \beta\rangle_b \perp \langle 1, \beta\rangle \otimes \mu'$ of dimension 2^{d+1} is isotropic. Hence, $\dim(\lambda' \perp \langle \beta\rangle_b \perp \langle 1, \beta\rangle \otimes \mu')_{\mathrm{an}} < 2^{d+1}$. Again, by [6, Th. 1.1], we conclude that τ is definable over F. Now, if ν and ν' are bilinear forms over F such that $\tau \simeq \nu_{F(B)} \simeq \nu'_{F(B)}$, then $(\nu \perp \nu')_{F(B)}$ is metabolic. Since $\dim B > 2^{d+1}$, it follows from [13, Prop. 1.1] that $\nu \perp \nu'$ is metabolic, i.e., $\nu \simeq \nu'$. Hence, τ is defined over F. Moreover, since $\dim B > 2^{d+1}$, it follows that the unique bilinear form ν over F satisfying $\tau \simeq \nu_{F(B)}$ is similar to λ. ∎

4. Proof of Theorem 1.1

We give a lemma:

Lemma 4.1. *Let B be an anisotropic bilinear form over F which is good of height 2 and degree > 0, and let $\lambda_{F(B)}$ be its leading form with λ a bilinear Pfister form over F. If $B_{F(\lambda)}$ is isotropic, then $B_{F(\lambda)}$ is metabolic.*

Proof. Let $\alpha \in F(B)^*$ be such that $B_{F(B)} \sim \alpha(\lambda_{F(B)})$. Let $[1, t^{-1}]$ be the quadratic form given by the polynomial $x^2 + xy + t^{-1}y^2$ over the rational function field $F(t)$ in one variable t. The relation $B_{F(B)} \sim \alpha(\lambda_{F(B)})$ implies that

$$B \otimes [1, t^{-1}]_{F(t)(B)} \sim \alpha(\varphi_{F(t)(B)}),$$

where $\otimes$ means the module action of $W(F(t))$ on the Witt group $W_q(F(t))$ of nonsingular quadratic forms over $F(t)$ [2], and $\varphi = \lambda \otimes [1, t^{-1}]$. Hence, $B \otimes [1, t^{-1}]$ is hyperbolic over $F(t)(B)(\varphi)$. Since $B_{F(\lambda)}$ is isotropic, it follows from [13, Prop. 3.9] that $B \otimes [1, t^{-1}]$ is also hyperbolic over $F(t)(\lambda)(\varphi)$. Moreover, the extension $F(t)(\lambda)(\varphi)/F(t)(\lambda)$ is purely transcendental, since φ is isotropic over $F(t)(\lambda)$. Hence, $B \otimes [1, t^{-1}]$ is hyperbolic over $F(t)(\lambda)$. By [15, Lem. 4.6], we conclude that $B_{F(\lambda)} \sim 0$. ∎

Proof of Theorem 1.1. Let B be an anisotropic bilinear form over F, good of height 2 and degree $d = 1$ or 2. Let λ be the unique d-fold bilinear Pfister form over F such that $\lambda_{F(B)}$ is the leading form of B.

We know from [15, Th. 5.10] that $\dim B = 2^n$ for some $n \geq d+1$, or $\dim B = 2^m - 2^d$ for some $m \geq d+2$. Moreover, the classification given in the theorem in the case $n = d+1$ or $m \geq d+2$ already exist in [15, Th. 5.10]. So, to complete our proof we have to consider the remaining case $n > d+1$.

Suppose that $\dim B = 2^n > 2^{d+1}$, and let $\alpha \in F(B)^*$ be such that $B_{F(B)} \sim \alpha(\lambda_{F(B)})$. If $B_{F(\lambda)}$ is isotropic, then it is metabolic by Lemma 4.1. Hence, by applying statement (2) of Theorem 1.3 for the forms λ and $C := B$, we conclude that $\alpha(\lambda_{F(B)})$ is definable over F. Without loss of generality, we may suppose that $\alpha \in F^*$. Hence, $(B \perp \alpha\lambda)_{F(B)} \sim 0$. Since $\dim(B \perp \alpha\lambda)_{\mathrm{an}} < 2^{n+1}$, it follows from [13, Th. 1.2] that $B \perp \alpha\lambda \sim \pi$, where π is similar to an n-fold bilinear Pfister form.

Conversely, if we proceed as in the proof of [15, Th. 5.10], we prove that any anisotropic bilinear form satisfying the conditions given in statement (1) or (2) of Theorem 1.1 is good of height 2 and degree d.∎

5. Proof of Theorem 1.2

(1) Let $F = \mathbb{F}_2(x, y, z)$ be the rational function field in the variables x, y, z over $\mathbb{F}_2$. Let us consider the following forms:

$$\pi = \langle\langle x, y, z\rangle\rangle\,,$$

$$\theta = \langle x, y, xy, 1+x, z, z(1+x)\rangle\,,$$

$$B = \langle\langle x, y\rangle\rangle \perp (x+y+z)\,\langle\langle 1+x, z\rangle\rangle\,.$$

We verify without difficult that the quadratic forms $\widetilde{B}$ and $\widetilde{\theta}$ are subforms of the quadratic form $\widetilde{\pi}$ (it suffices to use the well-known isometry $\langle a\rangle \perp \langle b\rangle \simeq \langle a\rangle \perp \langle a+b\rangle$ for any scalars a, b). Hence, $\widetilde{B}$ and $\widetilde{\theta}$ are quasi-Pfister neighbors of $\widetilde{\pi}$. By [5, Prop. 8.9] $\widetilde{\theta}_{F(B)}$ is isotropic, i.e., $\theta_{F(B)}$ is isotropic. We easily check that $B \perp \theta \in I^3F$, and by statement (2), B is of height and degree 2 but not good.

(2) Let B be an anisotropic bilinear form, and τ its leading form.

(a) Suppose that B is of height and degree 2 but not good: We have $B_{F(B)} \sim \alpha\tau$ for a suitable scalar $\alpha \in F(B)^*$. In particular, $B_{F(B)} \perp \tau \in I^3F(B)$. Since τ is not definable over F and $\dim B > 4$, it follows from statement (1) of Theorem 1.3 that $\dim B \in \{6, 8\}$. We discuss the two cases:

– Suppose $\dim B = 6$: Then B is an Albert bilinear form since $\det B = 1$.

– Suppose $\dim B = 8$: By Theorem 1.3, there exists an Albert bilinear form θ such that $\tau \perp \theta_{F(B)} \in I^3F(B)$. The form θ is anisotropic, otherwise θ_{an} would be similar to a 2-fold bilinear Pfister form, and by the Hauptsatz τ would be definable over F. Since $(B \perp \theta)_{F(B)} \in I^3F(B)$ and $\dim B > 4$, it follows from Proposition 3.3 that $B \perp \theta \in I^3F$. By the Hauptsatz, $\theta_{F(B)(\tau)} \sim 0$. By [13, Th. 1.2] the form $\theta_{F(B)}$ can not be metabolic, and $\dim(\theta_{F(B)})_{\text{an}}$ is divisible by 4. Hence, $\dim(\theta_{F(B)})_{\text{an}} = 4$, which means that $\theta_{F(B)}$ is isotropic. The uniqueness of θ, up to similarity, is a consequence of [16]. ■

(b) Conversely, if B is an Albert bilinear form, then we know by [15, Th. 5.10] that B is of height and degree 2 but not good. So, suppose that $\dim B = 8$ and $B \perp \theta \in I^3F$ for some anisotropic Albert bilinear form θ that becomes isotropic over $F(B)$.

Let λ be a bilinear form over $F(B)$ similar to a 2-fold bilinear Pfister form such that $\theta_{F(B)} \sim \lambda$. The bilinear form B is not similar to a 3-fold bilinear Pfister form, otherwise $\theta \in I^3F$, and by the Hauptsatz θ would be isotropic. Consequently, $B_{F(B)}$ is not metabolic [15, Cor. 5.5], and thus $0 < \dim(B_{F(B)})_{\text{an}} \leq 6$. Again by the Hauptsatz, the bilinear form $(B_{F(B)})_{\text{an}}$ is metabolic over $F(B)(\lambda)$, and thus $(B_{F(B)})_{\text{an}}$ is similar to a 2-fold bilinear Pfister form [13, Th. 1.2]. Hence, B is of height and degree 2. Moreover, if B is good and δ is the unique 2-fold bilinear Pfister form over F such that $\tau \simeq \delta_{F(B)}$, then we get $B \perp \delta \in I^3F$ [15, Prop. 5.3]. This implies that $\theta \perp \delta \in I^3F$, and, again by [15, Prop. 5.3], the bilinear Albert form θ is good, a contradiction. Hence, B is of height and degree 2, but not good. ■

6. Proof of Proposition 1.5

We start with some preliminary results.

Lemma 6.1. *Let $L = F(\sqrt{d})$ with $d \in F^*\backslash F^{*2}$. An anisotropic bilinear form B becomes isotropic over L if and only if B contains a subform similar to $\langle 1, x^2+d\rangle_b$ for suitable $x \in F$ such that $x^2 + d \neq 0$ (this is equivalent to saying that $\langle 1, d\rangle$ is similar to a subform of $\widetilde{B}$).*

Proof. It is clear that the condition given in the lemma is sufficient. Conversely, suppose that B_L is isotropic. By [13, Lem. 3.4], there exists $a \neq 0, b \in F$ such that the bilinear form B' given by the matrix $\begin{pmatrix} a & b \\ b & ad \end{pmatrix}$ is a subform of B. It is easy to show that B' is similar to $\langle 1, x^2+d\rangle_b$, where $x = 0$ or $x = b^2a^{-2}$ according as $b = 0$ or not. Since B is anisotropic we have $x^2 + d \neq 0$.

Since $\langle 1, x^2+d\rangle \simeq \langle 1, d\rangle$, it follows that the isotropy of B_L is equivalent to say that $\langle 1, d\rangle$ is similar to a subform of $\widetilde{B}$. ■

Proposition 6.2. *([14, Cor. 2.4]) If an anisotropic totally singular form φ over F represents a nonzero polynomial $p(x_1, \cdots, x_n)$ over $F(x_1, \cdots, x_n)$, and if $c = (c_1, \cdots, c_n) \in F^n$ satisfies $p(c) \neq 0$, then $p(c) \in D_F(\varphi)$.*

To prove Proposition 1.5 in dimension 3, we will adapt to our case some arguments used by Leep in his complete answer to the isotropy of Albert quadratic forms over function fields of quadrics in characteristic $\neq 2$ [17] (*cf.* PhD Thesis of Hoffmann [3]). Another important ingredient that we will use is the norm theorem for bilinear forms due to Knebusch [10].

Proof of Proposition 1.5. Let θ be an anisotropic Albert bilinear form such that $\mathrm{ndeg}_F(\widetilde{\theta}) = 16$. Let B be an anisotropic bilinear form such that $\dim B \in \{2, 3\}$. Obviously, if $\widetilde{B}$ is similar to a subform of $\widetilde{\theta}$, then $\theta_{F(B)}$ is isotropic. Conversely, suppose that $\theta_{F(B)}$ is isotropic.

(1) The case $\dim B = 2$: By Lemma 6.1 the quadratic form $\widetilde{B}$ is similar to a subform of $\widetilde{\theta}$.

(2) The case $\dim B = 3$: We may suppose that $B \simeq \langle 1, \alpha, \beta\rangle_b$. Let $F[t]$ be the polynomial ring in one variable t, and $F(t)$ its quotient field. It is well-known that the fields $F(t)(B)$ and $F(t)(\langle 1, \alpha + \beta t^2\rangle_b)$ are isomorphic. By Lemma 6.1, there exists polynomials $f, g, h, h_1, \cdots, h_4 \in F[t]$ such that:

$$\theta_{F(t)} \simeq f\left\langle 1, g^2 + h^2(\alpha + \beta t^2)\right\rangle_b \perp \langle h_1, \cdots, h_4\rangle_b . \tag{2}$$

This implies the following:

$$\widetilde{\theta}_{F(t)} \simeq f\left\langle 1, \alpha+\beta t^2\right\rangle \perp \langle h_1, \cdots, h_4\rangle . \tag{3}$$

It is clear that we may suppose g and h coprime, and the polynomials $f, h_1, \cdots, h_4$ are square free.

Our aim is to reduce in equation (3) to the case where the polynomial f is constant, and after that we conclude, by Proposition 6.2, that $\widetilde{\theta}$ represents the scalars f, αf and βf. Since $f\langle 1, \alpha, \beta\rangle$ is anisotropic, we deduce that $f\langle 1, \alpha, \beta\rangle = f\widetilde{B}$ is a subform of $\widetilde{\theta}$.

Suppose that $\deg f > 0$, and let p be a monic irreducible factor of f. Let F_p denote the residue field of the p-adic valuation of $F(t)$, and $\partial^1 : W(F(t)) \longrightarrow W(F_p)$ the first residue homomorphism.

• Suppose that $\alpha+\beta t^2$ is a square in $F_p \simeq F[t]/(p)$, then $\alpha+\beta t^2 = r^2 + p\cdot s$ for suitable $r, s \in F[t]$. We may suppose that $\deg r < \deg p$. If p is linear, then for $a \in F$ such that $p(a) = 0$, we get $\alpha + a^2\beta = r(a)^2$, which implies that $\langle 1, \alpha, \beta\rangle_b$ is isotropic, a contradiction. Hence, $\deg p \geq 2$. Consequently, $\deg s \leq \deg p - 2$ since $\deg(p\cdot s) = \deg(\alpha + \beta t^2 + r^2) \leq 2\deg p - 2$. Moreover, $p \cdot s$ is represented by $\langle 1, \alpha+\beta t^2\rangle_b$. By the multiplicativity of quasi-Pfister forms, we get $f\langle 1, \alpha+\beta t^2\rangle \simeq f_1\langle 1, \alpha+\beta t^2\rangle$, where $f_1 = \frac{s\cdot f}{p}$ is of degree smaller than $\deg f$.

• Suppose that $\alpha+\beta t^2$ is not a square in F_p. This implies that p does not divide $g^2 + h^2(\alpha+\beta t^2)$. In fact, if p divides $g^2+h^2(\alpha+\beta t^2)$, then p does not divide h since g and h are coprime, which implies that $\alpha+\beta t^2$ is a square in F_p, a contradiction.

Since $\det\theta_{F(t)} = (g^2+h^2(\alpha+\beta t^2))\, h_1h_2h_3h_4 \in F(t)^2$, we have three possibilities:

(a) $p \mid h_i$ for $1 \leq i \leq 4$.

(b) p only divides two polynomials among $h_1, \cdots, h_4$, say h_1 and h_2.

(c) $p \nmid h_i$ for $1 \leq i \leq 4$.

– Case (a) or case (b): By applying the homomorphism ∂^1 to equation (2), we get

$$\partial^1(\theta_{F(t)}) \sim \theta_{F_p} \sim 0$$

or

$$\partial^1(\theta_{F(t)}) \sim \theta_{F_p} \sim \left\langle \overline{h_3}, \overline{h_4}\right\rangle_b$$

according as we are in case (a) or (b) (here $\overline{u}$ denote the class of $u \in F[t]$ in F_p). In case (b) we get $\overline{h_3h_4} \in F_p^2$, and thus $\left\langle \overline{h_3}, \overline{h_4}\right\rangle_b \sim 0$. Hence, in both cases the form θ becomes metabolic over F_p. By the norm theorem, we deduce that p is a norm of $\theta_{F(t)}$. Hence,

$$\widetilde{\theta}_{F(t)} \simeq f_2\left\langle 1, \alpha+\beta t^2\right\rangle \perp \langle f\cdot h_1, \cdots, f\cdot h_4\rangle ,$$

where $f_2 = \frac{f}{p}$ is of degree smaller than $\deg f$.

– Case (c): We have $\partial^1(\theta_{F(t)}) \sim \theta_{F_p} \sim \left\langle \overline{h_1}, \cdots, \overline{h_4} \right\rangle_b$, which implies that $D := \left\langle \overline{h_1}, \cdots, \overline{h_4} \right\rangle_b$ is similar to a 2-fold bilinear Pfister form. If D is isotropic, then it is metabolic, and we may conclude as in the previous cases.

So we suppose that D is anisotropic. Then, $\mathrm{ndeg}_{F_p}(\widetilde{D}) = 4$.

Claim: $N_{F_p}(\widetilde{\theta}_{F_p}) \subset N_{F_p}(\widetilde{D})$, and thus $\mathrm{ndeg}_{F_p}(\widetilde{\theta}_{F_p}) \leq 4$.

To prove the claim its suffices to verify that $D_F(\theta) \subset D_{F_p}(D)$. In fact, for every $a \in D_F(\theta)$, we get the following by equation (3):

$$k^2 a = f\left(l^2 + (\alpha + \beta t^2)m^2\right) + \sum_{i=1}^{4} n_i^2 h_i \tag{4}$$

for suitable polynomials $k, l, m, n_1, \cdots, n_4 \in F[t]$, which we may suppose coprime. If p divides the polynomials $k, n_1, \cdots, n_4$, then p also divides $l^2 + (\alpha + \beta t^2)m^2$ since f is square free. Since $k, l, m, n_1, \cdots, n_4$ are coprime, the polynomial p does not divide m. In particular, $\alpha + \beta t^2$ becomes a square in F_p, which is excluded. Hence, at least one polynomial among $k, n_1, \cdots, n_4$ is not divided by p. Moreover, since

$$\overline{k}^2 a = \sum_{i=1}^{4} \overline{n_i}^2 \overline{h_i}$$

and D is anisotropic, we conclude that p does not divide k. Hence, $a \in D_{F_p}(D)$. This finishes the proof of the claim.

Moreover, since θ_{F_p} is isotropic, the polynomial p is inseparable, i.e., $\frac{\partial p}{\partial t} = 0$. Hence, we may write $F_p = S(\sqrt[2^n]{d})$ for an integer $n \geq 1$, a separable extension S/F and $d \in S$. Then, the norm degree of $\widetilde{\theta}$ decreases to 8 after extending scalars to F_p. But by, the claim above, we have $\mathrm{ndeg}_{F_p}(\widetilde{\theta}_{F_p}) \leq 4$, which is not possible. Hence, the case (c) does not happen.

Now in equation (3) we may change f by another polynomial f' such that $\deg f' < \deg f$. Let $u_1, \cdots, u_4 \in F[t]$ be such that

$$\widetilde{\theta}_{F(t)} \simeq f' \left\langle 1, \alpha + \beta t^2 \right\rangle \perp \left\langle u_1, \cdots, u_4 \right\rangle.$$

It is clear that this isometry implies the following:

$$\theta_{F(t)} \simeq f' \left\langle 1, (g')^2 + (\alpha + \beta t^2)(h')^2 \right\rangle_b \perp \left\langle v_1, \cdots, v_4 \right\rangle_b \tag{5}$$

for suitable polynomials $g', h', v_1, \cdots, v_4 \in F[X]$.

If $\deg f' = 0$ then we are done, if not, we apply to f' the same argument used for f to change f' with another polynomial of degree smaller than $\deg f'$. By continuing this process we get the desired conclusion. ■

7. Proof of Proposition 1.6

Let F be a field satisfying one of the following conditions:

(C1) Any 4-fold bilinear Pfister form over F is isotropic.

(C2) Question 1.4 has a positive answer.

Let B be an anisotropic bilinear form over F of dimension 8, and let τ be its leading form.

(1) $\Longrightarrow$ (2) Suppose that B is of height and degree 2 but not good. By Theorem 1.2 there exists an anisotropic Albert bilinear form θ which becomes isotropic over $F(B)$ and satisfies $B \perp \theta \in I^3F$. Then, by condition (C1) or (C2), the form $\widetilde{\theta}$ is a quasi-Pfister neighbor since $\operatorname{ndeg}_F(\widetilde{\theta}) = 8$.

Moreover, the isotropy of $\theta_{F(B)}$ implies that $N_F(\widetilde{B}) \subset N_F(\widetilde{\theta})$. Since $\operatorname{ndeg}_F(\widetilde{B}) \geq 8$, we conclude that $N_F(\widetilde{B}) = N_F(\widetilde{\theta})$. In particular, $\operatorname{ndeg}_F(\widetilde{B}) = 8$ and $\widetilde{B}$ is also a quasi-Pfister neighbor. By [5, Th. 8.11], we have $\dim(\widetilde{B}_{F(B)})_{\mathrm{an}} = \dim(\widetilde{\theta}_{F(B)})_{\mathrm{an}} = 4$. Moreover, $\theta_{F(B)}$ can not be metabolic since $\dim B > \dim \theta$. Then, $\dim(\theta_{F(B)})_{\mathrm{an}} = 4$. We also have $\dim(B_{F(B)})_{\mathrm{an}} = 4$ since $\mathrm{h}(B) = 2$. Now all this data with Proposition 2.1 imply the following:

$$\begin{aligned} B_{F(B)} &\simeq x\theta_1 \perp 2 \times \mathbb{M}(0) \\ \theta_{F(B)} &\simeq y\theta_2 \perp \mathbb{M}(0) \end{aligned} \tag{6}$$

for suitable scalars $x, y \in F(B)^*$ and 2-fold bilinear Pfister forms θ_1 and θ_2 over $F(B)$.

On the one hand, $\widetilde{x\theta_1}$ and $\widetilde{y\theta_2}$ are definable over F since, by equation (6), $\widetilde{x\theta_1} \simeq (\widetilde{B}_{F(B)})_{\mathrm{an}}$ and $\widetilde{y\theta_2} \simeq (\widetilde{\theta}_{F(B)})_{\mathrm{an}}$ (recall that the anisotropic part of a totally singular quadratic form over any field extension is definable over the ground field). In particular, $x\theta_1$ and $y\theta_2$ represent scalars in F^*, and hence we may suppose that $x, y \in F^*$.

On the other hand, the condition $B \perp \theta \in I^3F$ implies that $\theta_1 \perp \theta_2 \in I^3F(B)$, and by the Hauptsatz we have $\theta_1 \simeq \theta_2$.

Then we may conclude from equation (6) that $(xB \perp y\theta)_{F(B)} \sim 0$. Since $\operatorname{ndeg}_F(\widetilde{B}) = 8$, and $xB \perp y\theta$ is not metabolic (because B is anisotropic), it follows from [13, Th. 1.2] that $xB \perp y\theta \sim z\pi$ for some $z \in F^*$ and π a 3-fold bilinear Pfister form such that $\widetilde{\pi}$ is similar to $\widetilde{B}$. In particular, $N_F(\widetilde{B}) = N_F(\widetilde{\pi})$. Since $N_F(\widetilde{\theta}) = N_F(\widetilde{B}) = N_F(\widetilde{\pi})$, it follows that $\widetilde{\theta}$ is similar to a subform of $\widetilde{\pi}$.

(2) $\Longrightarrow$ (1) Suppose we have an anisotropic Albert bilinear form θ, a 3-fold bilinear Pfister form π, and scalars $x, y, z \in F^*$ such that $xB \perp y\theta \perp z\pi$ is metabolic and the forms $\widetilde{B}$ and $\widetilde{C}$ are similar to subforms of $\widetilde{\pi}$. Then, $xB \perp y\theta \in I^3F$ and the forms $\widetilde{B}$ and $\widetilde{\theta}$ are quasi-Pfister neighbors of $\widetilde{\pi}$. Consequently, $\widetilde{\theta}_{F(B)}$ is isotropic, i.e., $\theta_{F(B)}$ is isotropic and $B \perp \theta \in I^3F$ (because $B \perp xB \in I^3F$ and $\theta \perp y\theta \in I^3F$). By Theorem 1.2 we get that B is of height and degree 2 but not good.

Acknowledgment. This work was completed during a visit of the first author at the Universität Bielefeld for the period April-May 2008. The first author would like to thank the second author for organizing this visit, and he is grateful to the SFB 701 *"Spektrale Strukturen und topologische Methoden in der Mathematik"* for this support.

References

[1] R. Aravire, R. Baeza, *Linkage of fields in characteristic* 2, Comm. Algebra **31** (2003), 463–473.
[2] R. Baeza, *Quadratic forms over semilocal rings*, Lect. Notes Math. vol. **655**, Berlin, Heidelberg, New York: Springer 1978.
[3] D. W. Hoffmann, *Function fields of quadratic forms,* PhD Thesis, University of Berkely at California, 1992.
[4] D. W. Hoffmann, *Diagonal forms of degree p in characteristic p,* Contemporary Mathematics **344** (2004), 135-183.
[5] D. W. Hoffmann, A. Laghribi, *Quadratic forms and Pfister neighbors in characteristic* 2, Trans. Amer. Math. Soc. **356** (2004), 4019-4053.
[6] D. W. Hoffmann, A. Laghribi, *Isotropy of quadratic forms over the function field of a quadric in characteristic* 2, J. Algebra **295** (2006), 362-386.
[7] B. Kahn, *Formes quadratiques de hauteur et degré* 2, Indag. Math. N. S. **7** (1996), 47-66.
[8] B. Kahn, *A descent problem for quadratic forms,* Duke Math. J. **80** (1995), 139–159.
[9] M. Knebusch, *Grothendieck und Wittringe von nichtausgearteten symmetrischen Bilinearformen*, S.-B. Heidelberger Akad. Wiss. Math.-Natur. Kl. (1969-1970) 93–157.
[10] M. Knebusch, *Specialization of quadratic and symmetric bilinear forms, and a norm theorem*, Acta Arithm. **24** (1973), 279–299.
[11] A. Laghribi, *Quasi-hyperbolicity of totally singular quadratic forms,* Contemp. Math. **344** (2004), 237-248.
[12] A. Laghribi, *On splitting of totally singular quadratic forms,* Rend. Circ. Mat. Palermo. **53** (2004), 325-336.
[13] A. Laghribi, *Witt kernels of function field extensions in characteristic* 2, J. Pure Appl. Algebra **199** (2005), 167-182.
[14] A. Laghribi, *The norm theorem for totally singular quadratic forms,* Rocky Mountain J. Math. **36** (2006), 575-592.
[15] A. Laghribi, *Sur le déploiement des formes bilinéaires en caractéristique* 2, Pacific J. Math. **232** (2007), 207-232.

[16] A. Laghribi, *Jacobson's theorem for bilinear forms in characteristic* 2, http://www.mathematik.uni-bielefeld.de/LAG

[17] D. Leep, *Function field results,* Handwritten notes taken by T.Y. Lam, 1989.

[18] A. S. Merkurjev, *Algèbres simples et formes quadratiques*, (en russe). Izv. Akad. Nauk SSSR **55** (1991), 218–224. Traduction anglaise : Math. USSR Izv. **38** (1992), 215–221.

[19] J. Milnor, D. Husemoller, *Symmetric bilinear forms,* New York, Heidelberg: Springer 1973.

[20] W. Scharlau, *Quadratic and hermitian forms,* Springer, 1985.

[21] J.-P. Tignol, *Réduction de l'indice d'une algèbre simple centrale sur le corps des fonctions d'une quadrique*, Bull. Soc. Math. Belgique **42**, série A, (1990), 735–745.

Faculté Jean Perrin, Laboratoire de mathématiques de Lens EA 2462, rue Jean Souvraz SP-18, F-62307 Lens

E-mail address: `laghribi@euler.univ-artois.fr`

Universität Bielefeld, Fakultät für Mathematik, Postfach 100131, D-33501, Bielefeld

E-mail address: `rehmann@math.uni-bielefeld.de`