TITS INDICES OVER SEMILOCAL RINGS

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Abstract. We present a simplified version of Tits' proof of the classification of semisimple algebraic groups, which remains valid over semilocal rings. We also provide explicit conditions on anisotropic groups to appear as anisotropic kernels of semisimple groups of a given index.

1. Introduction

In his famous paper [Ti66] Tits has shown that any semisimple group G over a field is determined by its anisotropic kernel and a combinatorial structure called the *Tits index* of G. Some arguments were sketched or omitted there, and in full details appeared in [Ti90]. The goal of the present paper is to show that this result remains true over arbitrary connected semilocal rings. We do not use the case of fields, but rather provide a shortened and simplified version of Tits' arguments.

Our argument relies on the fact that, given a good enough representation of an algebraic group over a splitting covering, one can twist it to a representation into some Azumaya algebra, called a *Tits algebra* (cf. [Ti71]), over the base ring. We show that this in fact holds over arbitrary schemes (Theorem 1). Then in Theorem 2 we give a necessary and sufficient criterion for a semisimple group scheme H to be the derived subgroup of a Levi subgroup of a parabolic subgroup of a given type. After that, the list of all possible indices is easily obtained by induction, and the existence of a group with a given index shows to be equivalent to the existence of an anisotropic group subject to some explicitly stated restrictions (§ 5, Theorem 3).

2. Semisimple group schemes

In this section we reproduce some definitions and results of [SGA].

Let S be a scheme (not necessarily separated). A group scheme G over S is called *reductive* if it is affine and smooth over S, and its geometric fibers $G_{\overline{k(s)}}$ are connected reductive groups in the usual sense for all $s \in S$ (Exp. XIX Déf. 4.7 [1]). When S is reduced, the smoothness can be replaced by the condition that G is finitely presented over S and the dimension of a fiber is locally constant (see Exp. $\mathrm{VI_B}$, Cor. 4.4). The *type* of G at $s \in S$ is the root datum of $G_{\overline{k(s)}}$. The type is locally constant (Exp. XXII Prop. 2.8). To simplify the exposition, in the sequel we consider reductive group schemes of constant type only. Thus the type of a reductive group scheme G is a root datum $\mathcal{R} = (\Phi, \Lambda, \Phi^*, \Lambda^*)$, where Φ is a root system, called the *root system* of G, Λ is a $\mathbb{Z}$-lattice containing Φ, called the *lattice of weights* of G, and Φ^* and Λ^* are the dual objects (Exp. XXI Déf. 1.1.1). We usually include in the type the Dynkin diagram D of Φ, whose vertices can be identified with a system of simple roots of Φ.

Over any scheme S there exists a unique *split* group scheme G_0 of a given type $\mathcal{R}$, which actually comes from a group scheme over $\operatorname{Spec}\mathbb{Z}$ known as the Chevalley – Demazure group scheme (Exp. XXV Thm. 1.1). *Quasi-split* group schemes over S of the same type as G_0 are parametrized by $\mathrm{H}^1(S, \operatorname{Aut}(\mathcal{R}))$ (Exp. XXIV Thm. 3.11). All cohomology groups we consider are with respect to the fpqc topology (but note that $\mathrm{H}^1(S, H) = \mathrm{H}^1_{\text{ét}}(S, H)$ when H is smooth).

[1] All references starting with Exp. YZ refer to [SGA].

Every semisimple group scheme G is an inner twisted form of a uniquely determined quasi-split group G_{qs}, given by a cocycle $\xi \in \mathrm{Z}^1(S,\, G_{qs}^{ad})$, where G_{qs}^{ad} is the adjoint group acting on G_{qs} by inner automorphisms; cocycles in the same class in $\mathrm{H}^1(S,\, G_{qs}^{ad})$ produce isomorphic group schemes (Exp. XXIV 3.12.1).

A Dynkin diagram D is nothing but a finite set of vertices together with a subset $E \subseteq D \times D$ of edges and a length function $D \to \{1,\, 2,\, 3\}$ (in other words, a colored graph). The scheme-theoretic counterpart of this notion is called a *Dynkin scheme* (Exp. XXIV § 3). So a Dynkin scheme over S is a twisted finite scheme $\mathcal{D}$ over S together with a subscheme $\mathcal{E} \subseteq \mathcal{D} \times_S \mathcal{D}$ and a map $\mathcal{D} \to \{1,\, 2,\, 3\}_S$. Isomorphisms, base extensions and constant Dynkin schemes are defined in an obvious way. We denote by D_S the constant Dynkin scheme over S corresponding to a Dynkin diagram D. By $\mathrm{Aut}\,(\mathcal{D})$ we always mean the scheme of automorphisms of $\mathcal{D}$ over S as a Dynkin scheme; it is a twisted constant group scheme over S.

With each semisimple group scheme G one associates the Dynkin scheme $\mathrm{Dyn}(G)$, in such a way that $\mathrm{Dyn}(G)$ is isomorphic to $\mathrm{Dyn}(G_{qs})$; if G_{qs} is given by a cocycle $\xi \in \mathrm{Z}^1(S,\, \mathrm{Aut}\,(\mathcal{R}))$ then $\mathrm{Dyn}(G_{qs})$ is a twisted form of D_S corresponding to the image of ξ in $\mathrm{Z}^1(S,\, \mathrm{Aut}\,(D))$ induced by the canonical map $\mathrm{Aut}\,(\mathcal{R}) \to \mathrm{Aut}\,(D)$ (Exp. XXIV 3.7). When G_{qs} is simply connected or adjoint, the latter map is an isomorphism.

Let T/S be a Galois covering that splits $\mathrm{Dyn}(G)$, i.e. $\mathrm{Dyn}(G)_T \simeq D_T$. For example, one can take as T the torsor corresponding to the cocycle in $\mathrm{Z}^1(S,\, \mathrm{Aut}\,(D))$. Every element $\sigma \in \mathrm{Aut}\,(T/S)$ acts on $\mathrm{Dyn}(G)_T$ and therefore defines some $\varphi_\sigma \in \mathrm{Aut}\,(D)(T)$ such that the diagram

$$\begin{array}{ccc} D_T & \xrightarrow{\varphi_\sigma} & D_T \\ \downarrow & & \downarrow \\ T & \xrightarrow{\sigma} & T \end{array}$$

commutes. By Galois descent this action (which will be called the $*$-*action*) completely determines $\mathrm{Dyn}(G)$. If S is connected, the $*$-action can be considered as an action of $\mathrm{Aut}\,(T/S)$ on the Dynkin diagram D, and extends by linearity to the $*$-action on Λ.

A subgroup scheme P of G is called *parabolic* if it is smooth and $P_{\overline{k(s)}}$ is a parabolic subgroup of $G_{\overline{k(s)}}$ in the usual sense for every $s \in S$ (Exp. XXVI Déf. 1.1). To a parabolic subgroup P one can attach the *type* $\mathfrak{t}(P)$ of P which is a clopen subscheme of $\mathrm{Dyn}(G)$ (Exp. XXVI 3.2). Note that the clopen subschemes of $\mathrm{Dyn}(G)$ are in one-to-one correspondence with the $*$-invariant clopen subschemes of D_T, where T/S is as above.

3. Representations

By a *representation* of a group scheme G over S we mean a homomorphism of algebraic monoids $\rho\colon G \to A$, where A is an Azumaya algebra (more formally, a sheaf of Azumaya algebras) of constant rank over S.

We will say that a representation $\rho : G \to A$ is *absolutely irreducible* if it satisfies the following property: for any extension S'/S and any element $a \in A(S')$ there exists a faithfully flat extension S''/S' such that a considered as an element of $A(S'')$ belongs to the S''-span of $G(S'')$. It is easy to see that this property is stable under a base extension and is local in the fpqc topology.

Lemma 1. *Let $\rho\colon G \to A$ be a representation of a semisimple group scheme G over a scheme S. Then ρ is absolutely irreducible if and only if for any extension* $\mathrm{Spec}\, k \to S$*, where k is separably closed field, $\rho_k : G_k \to A_k$ is irreducible in the usual sense.*

Proof. If ρ_k is reducible for some separably closed field k, then all elements of $G(k)$ are presented by block-triangular matrices in $A_k \simeq \mathrm{M}_n(k)$ after an appropriate choice of basis. Clearly, the representation is not absolutely irreducible in this case.

Conversely, assume that ρ_k is irreducible for every separably closed field k. Since the property of being absolutely irreducible is local, we may pass to an étale neighbourhood, and therefore assume that the base scheme is a spectrum of a strictly Henselian ring R with the maximal ideal M. Note that A is isomorphic to the matrix ring $\mathrm{M}_n(R)$, and it suffices to show that the R-span of $G(R)$ coincides with $\mathrm{M}_n(R)$. But $k = R/M$ is separably closed, so the representation of G in $\mathrm{M}_n(k)$ is irreducible, hence the k-span of $G(k)$ coincides with $\mathrm{M}_n(k)$. Now the result follows from the Nakayama lemma and the fact that the map $G(R) \to G(k)$ is surjective. □

Let G_0 be a split semisimple group scheme over a scheme S, and let $G_0 \to \mathrm{End}\,(V)$ be a representation of G_0 on a projective module (more formally, a locally free sheaf of modules) V of constant rank over S. Fix a maximal split torus T_0 of G_0 and let Λ and Λ_{r} be its lattices of weights and roots respectively. Then V decomposes into a direct sum $\bigoplus_{\lambda\in\Lambda} V_\lambda$ such that for any $t \in T_0(S')$, where S' is a scheme over S, and any $v \in V_\lambda(S')$ one has $\rho(t)v = \lambda(t)v$ (Exp. I Prop. 4.7.3). A character λ with $V_\lambda \neq 0$ is called a *weight* of V.

Lemma 2. *For any $g \in G_0(S')$ one has $\rho(g)(V_\lambda) \leq \bigoplus_{\mu-\lambda\in\Lambda_{\mathrm{r}}} V_\mu$.*

Proof. Changing the base we may assume $S' = S$. Let g be an element of $G_0(S)$. We need to show that $\mathrm{pr}_\mu(\rho(g)(V_\lambda)) = 0$ when $\mu - \lambda \notin \Lambda_{\mathrm{r}}$, pr_μ stands for the projection onto V_μ. Since V_λ is finitely generated, we may assume that $S = \mathrm{Spec}\, k$ is the spectrum of an algebraically closed field.

Let v be an element of V_λ. Set $v_\nu = \mathrm{pr}_\nu(v)$; for every $c \in \mathrm{Cent}(G_0)(S)$ we have

$$\rho(g)\rho(c)v = \rho(g)(\lambda(c)v) = \sum_\nu \lambda(c)v_\nu$$

and

$$\rho(c)\rho(g)v = \sum_\nu \nu(c)v_\nu.$$

But $gc = cg$, therefore $(\mu(c) - \lambda(c))v_\mu = 0$ for any $c \in \mathrm{Cent}(G_0)(S)$. If $v_\mu \neq 0$ then the restrictions of μ and λ to $\mathrm{Cent}(G_0)$ coincide, i.e. $\mu - \lambda$ belongs to Λ_{r} (see Exp. XXII Rem. 4.1.8). □

Now let $\rho\colon G_{qs} \to \mathrm{End}\,(V)$ be a representation of a quasi-split group G_{qs} on a projective module V of constant rank over S. As above, fix a maximal torus T of G_{qs}, and let Λ and Λ_r be the lattices of weights and roots of G_{qs}. Let $\coprod S_\tau \to S$ be an fpqc covering by connected schemes, such that each $(G_{qs})_{S_\tau}$ is split. Over each S_τ the weights of $V \times_S S_\tau$ with respect to T_{S_τ} form a $*$-invariant subset of Λ. From now on, we will consider only representations of *constant type* in the sense that this subset doesn't depend on S_τ (the condition automatically holds when S is connected); its elements are called the *weights* of V. We say that a weight $\lambda \in \Lambda$ is $*$*-invariant*, if it is $*$-invariant over each S_τ. This is the case if and only if λ descends to a character of T over the base scheme. We denote by Λ^* and ${\Lambda_{\mathrm{r}}}^*$ the sublattices of $*$-invariant elements in Λ and Λ_{r} respectively. By [Ti71, 3.1] we have $(\Lambda\,/\,\Lambda_{\mathrm{r}})^* = \Lambda^*\,/\,{\Lambda_{\mathrm{r}}}^*$.

A representation $\rho\colon G_{qs} \to \mathrm{End}\,(V)$ will be called *center preserving* if $\rho(\mathrm{Cent}(G_{qs})) \subseteq \mathrm{Cent}(\mathrm{End}\,(V))$. In this case ρ induces a homomorphism $\rho^{ad}\colon G_{qs}^{ad} \to \mathrm{PGL}(V)$.

Lemma 3. (1) *V is center preserving if and only if every two weights of V differ by an element of Λ_{r}.*
(2) *The dual V^* to a center preserving representation V is center preserving.*
(3) *Tensor product $V_1 \otimes V_2$ of center preserving representations V_1 and V_2 is center preserving.*
(4) *The class of a weight λ of a center preserving representation V in $\Lambda\,/\,\Lambda_{\mathrm{r}}$ is $*$-invariant.*

(5) *If λ be a weight of a representation V of a quasi-split group such that the class of λ in $\Lambda\,/\,\Lambda_{\mathrm{r}}$ is $*$-invariant, then V decomposes as a direct sum of two represetations $V = W \oplus W'$, such that λ is a weight of W and W is center preserving.*

Proof. **1.** Since the condition $\rho(\mathrm{Cent}(G_{qs})) \subseteq \mathrm{Cent}(\mathrm{End}\,(V))$ is local with respect to fpqc topology, we can assume that $G_{qs} = G_0$ is split. Then V is center preserving if and only if the restrictions of every two weights λ and μ to $\mathrm{Cent}(G_0)$ coincide. This exactly means that $\lambda - \mu$ belongs to Λ_{r} (Exp. XXII Rem. 4.1.8).

2,3,4. Follows from **1.**

5. Consider a covering $\coprod S_\tau \to S$ by connected schemes that splits G_{qs}. Lemma 2 implies that the submodules $W = \bigoplus_{\mu\in\lambda+\Lambda_{\mathrm{r}}} V_\mu$ and $W' = \bigoplus_{\mu\notin\lambda+\Lambda_{\mathrm{r}}} V_\mu$ are $(G_{qs})_{S_\tau}$-invariant. Since $\lambda+\Lambda_{\mathrm{r}}$ is $*$-invariant, these modules are also $*$-invariant. By the faithfully flat descent we see that W and W' are defined over the base scheme S and G_{qs}-invariant, and V is the direct sum of W and W'. The summand W is center preserving by **1.** □

Lemma 4. *Let G_{qs} be a quasi-split group over a scheme S. Then any weight $\lambda \in \Lambda^*$ is congruent modulo ${\Lambda_{\mathrm{r}}}^*$ to a weight of some absolutely irreducible representation $G_{qs} \to \mathrm{End}\,(V)$.*

Proof. It is known (see [B, Ch. VI, Exerc. 5 du §2]) that any weight is equivalent modulo Λ_{r} to a minuscule weight; on the other hand by [Ti71, 3.1] we have $(\Lambda\,/\,\Lambda_{\mathrm{r}})^* = \Lambda^*\,/\,{\Lambda_{\mathrm{r}}}^*$. So we may assume that λ is a $*$-invariant minuscule weight.

Consider first the split group G_0 over $\mathbb{Z}$. Recall briefly the construction of the Weyl module $V(\lambda)$ (see [Jan] for details). We start from the finite dimensional irreducible $(G_0)_{\mathbb{C}}$-module with the highest weight λ; we fix a vector v_+ of the weight λ (which is unique up to a scalar). Denote by $\mathfrak{U}$ the universal enveloping algebra of the Lie algebra of $(G_0)_{\mathbb{C}}$, by $\mathfrak{U}^+$ and $\mathfrak{U}^-$ its subalgebras generated by the positive (respectively, negative) root subspaces, and by $\mathfrak{U}_{\mathbb{Z}}$, $\mathfrak{U}_{\mathbb{Z}}^+$, $\mathfrak{U}_{\mathbb{Z}}^-$ their $\mathbb{Z}$-forms used in the Chevalley construction. Then $V(\lambda)$ is defined as $\mathfrak{U}_{\mathbb{Z}}^-\,v_+$. If λ is minuscule, $V(\lambda)_k$ is irreducible for any field k, so $V(\lambda)$ is an absolutely irreducible representation of G_0.

Let Γ be a group of outer automorphisms of G_0 preserving some fixed *épinglage* of G_0 and λ. Then any element $\gamma \in \Gamma$ induces an automorphism of $\mathfrak{U}_{\mathbb{Z}}$ which preserves $\mathfrak{U}_{\mathbb{Z}}^+$ and $\mathfrak{U}_{\mathbb{Z}}^-$. Since γ preserves λ, the representations $d\rho\colon\,\mathfrak{U} \to \mathrm{End}\,(V(\lambda)_{\mathbb{C}})$ and $d\rho\circ\gamma\colon\,\mathfrak{U} \to \mathrm{End}\,(V(\lambda)_{\mathbb{C}})$ are equivalent, therefore there exists $\varphi \in \mathrm{GL}(V(\lambda)_{\mathbb{C}})$ such that $\gamma(g)\varphi(v) = \varphi(gv)$ for every $v \in V(\lambda)_{\mathbb{C}}$ and $g \in \mathfrak{U}$; moreover, φ is unique up to a scalar. It is easy to see that φ preserves the line generated by v_+, and we can normalize φ in such a way that $\varphi(v_+) = v_+$. Now

$$\varphi(\mathfrak{U}_{\mathbb{Z}}^-\,v_+) \le \gamma(\mathfrak{U}_{\mathbb{Z}}^-)\varphi(v_+) = \mathfrak{U}_{\mathbb{Z}}^-\,v_+,$$

so φ induces an automorphism of $V(\lambda)$ compatible with γ and preserving v_+. Since $\mathfrak{U}_{\mathbb{Z}}$ is the universal enveloping algebra of the Lie algebra of G_0, φ is an equivalence of the representations $\rho\colon G_0 \to \mathrm{End}\,(V(\lambda))$ and $\rho\circ\gamma\colon G_0 \to \mathrm{End}\,(V(\lambda))$. Moreover, since φ is uniquely determined by γ, we obtain a homomorphism $\psi\colon \Gamma \to \mathrm{GL}(V(\lambda))$.

Now G_{qs} is constructed by a cocycle $\xi \in \mathrm{Z}^1(S, \Gamma)$, and the cocycle $\psi_*(\xi)$ allows us to construct a projective module V together with a representation $G_{qs} \to \mathrm{End}\,(V)$, as claimed. □

Lemma 5. *If $G_{qs} \to A$ is an absolutely irreducible representation of a quasi-split group G_{qs} over a scheme S, then $A \simeq \mathrm{End}\,(V)$ for some projective module V over S, and the representation is center preserving.*

Proof. Consider an affine covering $\coprod S_\tau \to S$ which splits G. Passing to a finer covering, we can also assume that for each τ A_{S_τ} is isomorphic to $\mathrm{M}_n(S_\tau)$, a matrix algebra over S_τ. The decomposition of S_τ^n into a direct sum of the 1-dimensional highest weight subspace and its complement determines an idempotent e_τ in $\mathrm{M}_n(S_\tau)$. These idempotents can be glued together to an idempotent $e \in A$ defined over the base scheme S. Now the natural map $A \to \mathrm{End}\,(Ae)$ is a homomorphism between Azumaya algebras of the same rank, so it is an isomorphism by [K, Ch. III, Cor. 5.1.18]. Finally, Lemma 1, together with the Schur lemma, implies that an absolutely irreducible representation of a quasi-split group into $\mathrm{End}\,(V)$ is center preserving. □

4. Tits algebras

Theorem 1. *Let G be a semisimple group scheme of constant type over S, and let G_{qs} be the corresponding quasi-split group with the lattice of weights Λ.*

1. *Suppose that G is given by a cocycle $\xi \in \mathrm{Z}^1(S,\, G_{qs}^{ad})$. For every center preserving representation $\rho\colon G_{qs} \to \mathrm{End}\,(V)$ there exists a representation $\rho_\xi\colon G \to A_\rho$ into an Azumaya algebra A_ρ over S which becomes equal to ρ over a covering of S that quasi-splits G.*
2. *For every absolutely irreducible representation $\rho'\colon G \to A$ and every cocycle $\xi \in \mathrm{Z}^1(S,\, G_{qs}^{ad})$ that gives G, there exists an absolutely irreducible representation $\rho\colon G_{qs} \to \mathrm{End}\,(V)$ such that $\rho' = \rho_\xi$.*
3. *In the setting of 1, the class $[A_\rho]$ in the Brauer group $\mathrm{Br}(S)$ depends only on G and on the class of any weight λ of ρ in $\Lambda\,/\,\Lambda_{\mathrm{r}}$, and not on the particular choices of ρ and ξ. Its image in $\mathrm{H}^2(S,\, \mathbf{G}_{\mathrm{m}})$ coincides with $\lambda_*(\delta([\xi]))$, where $\lambda_*\colon\ \mathrm{H}^2(S,\, \mathrm{Cent}(G_{qs})) \to \mathrm{H}^2(S,\, \mathbf{G}_{\mathrm{m}})$ corresponds to the restriction of λ to $\mathrm{Cent}(G_{qs})$, and δ is the connecting homomorphism in the long exact sequence associated to the sequence*

$$1 \longrightarrow \mathrm{Cent}(G_{qs}) \longrightarrow G_{qs} \longrightarrow G_{qs}^{ad} \longrightarrow 1.$$

Proof. **1.** The cocycle ξ is presented by elements $g_{\sigma\tau} \in G_{qs}^{ad}(S_\sigma \times_S S_\tau)$ for some covering $\coprod S_\tau \to S$ that quasi-splits G. Now the diagram (we write $S_{\sigma\tau}$ instead of $S_\sigma \times_S S_\tau$ for brevity)

$$\begin{array}{ccc}
(G_{qs})_{S_{\sigma\tau}} & \xrightarrow{\mathrm{id}_{S_\sigma} \times \rho_{S_\tau}} & \mathrm{End}\,(V)_{S_{\sigma\tau}} \\
{\scriptstyle g_{\sigma\tau}}\downarrow & & \downarrow{\scriptstyle \rho^{ad}_{S_{\sigma\tau}}(g_{\sigma\tau})} \\
(G_{qs})_{S_{\sigma\tau}} & \xrightarrow{\rho_{S_\sigma} \times \mathrm{id}_{S_\tau}} & \mathrm{End}\,(V)_{S_{\sigma\tau}}.
\end{array}$$

commutes, and the faithfully flat descent gives the homomorphism $\rho_\xi\colon G \to A_\rho$, where A_ρ is a twisted form of $\mathrm{End}\,(V)$, that is an Azumaya algebra.

2. Let η be some cocycle in $\mathrm{Z}^1(S,\, G^{ad})$ corresponding to G_{qs}; η is presented by elements $h_{\sigma\tau} \in G^{ad}(S_\sigma \times_S S_\tau)$. Using the same construction as in the first part (with interchanged G and G_{qs}) we obtain an absolutely irreducible representation $\rho\colon G_{qs} \to B$, where $B \simeq \mathrm{End}\,(V)$ for some V by Lemma 5. We will show that, for a suitable η, the representation ρ_ξ of G coincides with ρ'.

Denote by φ_σ the fixed isomorphisms from $(G_{qs})_{S_\sigma}$ to G_{S_σ}, and by ψ_σ the isomorphisms between A_{S_σ} and $\mathrm{End}\,(V)_{S_\sigma}$ constructed by the descent. We have the following commutative diagram:

$$\begin{array}{ccccccc}
(G_{qs})_{S_{\sigma\tau}} & \xrightarrow{\mathrm{id}_{S_\sigma} \times \varphi_\tau} & G_{S_{\sigma\tau}} & \xrightarrow{\mathrm{id}_{S_\sigma} \times \rho'_{S_\tau}} & A_{S_{\sigma\tau}} & \xrightarrow{\mathrm{id}_{S_\sigma} \times \psi_\tau} & \mathrm{End}\,(V)_{S_{\sigma\tau}} \\
\| & & {\scriptstyle h_{\sigma\tau}}\downarrow & & \downarrow{\scriptstyle {\rho'}^{ad}_{S_{\sigma\tau}}(h_{\sigma\tau})} & & \| \\
(G_{qs})_{S_{\sigma\tau}} & \xrightarrow{\varphi_\sigma \times \mathrm{id}_{S_\tau}} & G_{S_{\sigma\tau}} & \xrightarrow{\rho'_{S_{\sigma\tau}}} & A_{S_{\sigma\tau}} & \xrightarrow{\psi_\sigma \times \mathrm{id}_{S_\tau}} & \mathrm{End}\,(V)_{S_{\sigma\tau}},
\end{array}$$

and we want to show that the rightmost square in the diagram

$$\begin{array}{ccccccc}
G_{S_{\sigma\tau}} & \xrightarrow{\mathrm{id}_{S_\sigma} \times \varphi_\tau^{-1}} & (G_{qs})_{S_{\sigma\tau}} & \xrightarrow{\mathrm{id}_{S_\sigma} \times \rho_{S_\tau}} & \mathrm{End}\,(V)_{S_{\sigma\tau}} & \xrightarrow{\mathrm{id}_{S_\sigma} \times \psi_\tau^{-1}} & A_{S_{\sigma\tau}} \\
\| & & {\scriptstyle g_{\sigma\tau}}\downarrow & & \downarrow{\scriptstyle \rho^{ad}_{S_{\sigma\tau}}(g_{\sigma\tau})} & & \| \\
G_{S_{\sigma\tau}} & \xrightarrow{\varphi_\sigma^{-1} \times \mathrm{id}_{S_\tau}} & (G_{qs})_{S_{\sigma\tau}} & \xrightarrow{\rho_{S_{\sigma\tau}}} & \mathrm{End}\,(V)_{S_{\sigma\tau}} & \xrightarrow{\psi_\sigma^{-1} \times \mathrm{id}_{S_\tau}} & A_{S_{\sigma\tau}}
\end{array}$$

is commutative. For a suitable choice of η, the first two squares are commutative, and the big rectangle is commutative as well, since the top and the bottom maps coincide with ρ'. Now the claim follows from the fact that ρ' is absolutely irreducible.

3. The cohomological class in $\mathrm{H}^1(S,\,\mathrm{PGL}(V))$ corresponding to A_ρ is nothing but $\rho^{ad}_*([\xi])$. Now the last assertion of the Theorem follows from the commutativity of the diagram

$$\begin{array}{ccc} \mathrm{H}^1(S,\,G^{ad}_{qs}) & \xrightarrow{\delta} & \mathrm{H}^2(S,\,\mathrm{Cent}(G_{qs})) \\ \downarrow{\scriptstyle \rho^{ad}_*} & & \downarrow{\scriptstyle \lambda_*} \\ \mathrm{H}^1(S,\,\mathrm{PGL}(V)) & \longrightarrow & \mathrm{H}^2(S,\,\mathbf{G}_\mathrm{m}), \end{array}$$

which comes from the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{Cent}(G_{qs}) & \longrightarrow & G_{qs} & \longrightarrow & G^{ad}_{qs} & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho^{ad}} & & \\ 1 & \longrightarrow & \mathbf{G}_\mathrm{m} & \longrightarrow & \mathrm{GL}(V) & \longrightarrow & \mathrm{PGL}(V) & \longrightarrow & 1. \end{array}$$

So once ξ is fixed, the class of A_ρ depends only on the class of λ in $\Lambda/\,\Lambda_\mathrm{r}$. By Lemma 3 the class of λ is $*$-invariant, and hence λ is congruent modulo Λ_r to a $*$-invariant weight $\mu \in \Lambda$. Now Lemma 4 and part 2 imply that $\lambda_*(\delta([\xi])) = \mu_*(\delta([\xi]))$ doesn't depend on the particular choice of ξ, since any representation $\rho : G_{qs} \to \mathrm{End}\,(V)$ one choses has the same set of weights. □

The Azumaya algebra A_ρ will be called the *Tits algebra* of a center preserving representation ρ of G_{qs}. We denote by β_G the corresponding homomorphism from $\Lambda^*\,/\,{\Lambda_\mathrm{r}}^* \simeq (\Lambda\,/\,\Lambda_\mathrm{r})^*$ to $\mathrm{Br}(S)$,

$$\beta_G\colon\ \Lambda^*\,/\,{\Lambda_\mathrm{r}}^* \to \mathrm{Br}(S)$$
$$\lambda \mapsto [A_\rho].$$

To see that β_G is indeed a homomorphism one can use either the tensor product of representations or the fact that $\mathrm{Br}(S)$ is a subgroup in $\mathrm{H}^2(S,\,\mathbf{G}_\mathrm{m})$.

$\mathrm{Dyn}(G)$ is the disjoint union of its minimal clopen subschemes which will be called the *orbits* for brevity; they indeed correspond to the orbits of the $*$-action on the set of simple roots.

Assume that G is simply connected. Let T be a fixed torus of G_{qs}. Over a splitting covering we have two canonical homomorphisms

$$\alpha,\,\omega\colon\ \mathrm{Dyn}(G) \to \mathrm{Hom}\,(T, \mathbf{G}_\mathrm{m})$$

which associate to each vertex i of the Dynkin diagram the simple root α_i (resp. the fundamental weight ω_i); the faitfully flat descent shows that these homomorphisms are defined over the base scheme S as well.

Let O be an orbit in $\mathrm{Dyn}(G)$; taking composition with the inclusion map $O \to \mathrm{Dyn}(G)$ we obtain a root $\alpha_O\colon T_O \to \mathbf{G}_\mathrm{m}$ (resp., a weight $\omega_O\colon T_O \to \mathbf{G}_\mathrm{m}$) of G_O which will be called the *canonical root* (resp., the *canonical weight*) corresponding to O (cf. Exp. XXIV 3.8). It is easy to see that α_O and ω_O are $*$-invariant weights of G_O.

Note that we have homomorphisms[2]

$$\bar\alpha_O,\,\bar\omega_O\colon T \to R_{O/S}(\mathbf{G}_\mathrm{m})$$

which are the compositions of $R_{O/S}(\alpha_O)$ and $R_{O/S}(\omega_O)$ with the canonical homomorphism $T \to R_{O/S}(T_O)$.

[2]For the Weil restriction we use a more common notation $R_{S'/S}$ instead of $\prod_{S'/S}$ of [SGA].

Proposition 1. *Let G be simply connected, L be a Levi subgroup of a parabolic subgroup P in G and L' be the corresponding Levi subgroup of the corresponding parabolic subgroup P' in G^{ad}. Then we have the isomorphisms*

$$\prod_{O\colon O\subset \mathfrak{t}(P)} \bar{\omega}_O\colon \mathrm{Cent}(L) \simeq \prod_{O\colon O\subset \mathfrak{t}(P)} R_{O/S}(\mathbf{G}_\mathrm{m});$$

$$\prod_{O\colon O\subset \mathfrak{t}(P)} \bar{\alpha}_O\colon \mathrm{Cent}(L') \simeq \prod_{O\colon O\subset \mathfrak{t}(P)} R_{O/S}(\mathbf{G}_\mathrm{m})$$

(cf. Exp. XXIV Prop. 3.13).

Proof. Indeed, composing these maps with the isomorphism

$$\prod_{O\colon O\subset \mathfrak{t}(P)} R_{O/S}(\mathbf{G}_\mathrm{m}) \to R_{\mathfrak{t}(P)/S}(\mathbf{G}_\mathrm{m})$$

we obtain the maps $\mathrm{Cent}(L) \to R_{\mathfrak{t}(P)/S}(\mathbf{G}_\mathrm{m})$ and $\mathrm{Cent}(L') \to R_{\mathfrak{t}(P)/S}(\mathbf{G}_\mathrm{m})$. It is easy to see that they are isomorphisms over a splitting covering of G. □

Proposition 2. *In the settings of Theorem* 1 *assume that G is simply connected and* $\mathrm{Pic}(\mathrm{Dyn}(G)) = 0$. *Then $[\xi]$ comes from an element in $\mathrm{H}^1(S, G_{qs})$ if and only if $\beta_{G_O}(\omega_O) = 0$ for each orbit O.*

Proof. If $[\xi]$ belongs to the image of $\mathrm{H}^1(S, G_{qs}) \to \mathrm{H}^1(S, G^{ad}_{qs})$ then $\delta([\xi]_O) = 0$ and therefore $\beta_{G_O} = 0$ for each O. Conversely, assume that $\beta_{G_O}(\omega_O) = 0$ for each O. Proposition 1 applied to the Borel subgroup implies that $T_{qs} \simeq \prod_O R_{O/S}(\mathbf{G}_\mathrm{m})$ and $T_{ad} \simeq \prod_O R_{O/S}(\mathbf{G}_\mathrm{m})$. Now the Shapiro lemma (cf. Exp. XXIV Prop. 8.2) implies that the image of $\delta([\xi])$ in $\mathrm{H}^2(S, T_{qs})$ is trivial, while $\mathrm{H}^1(S, T^{ad}_{qs}) = \mathrm{Pic}(\mathrm{Dyn}(G)) = 0$. Now the claim follows from the exact sequence

$$\mathrm{H}^1(S, T^{ad}_{qs}) \longrightarrow \mathrm{H}^2(S, \mathrm{Cent}(G_{qs})) \longrightarrow \mathrm{H}^2(S, T_{qs}),$$

which comes from the sequence

$$1 \longrightarrow \mathrm{Cent}(G_{qs}) \longrightarrow T_{qs} \longrightarrow T^{ad}_{qs} \longrightarrow 1.$$

□

Theorem 2. (1) *Let G be a semisimple group scheme of constant type over S, P be its parabolic subgroup admitting a Levi subgroup L, H be the derived subgroup of L. Denote by G_{qs} and H_{qs} the corresponding quasi-split groups and by Λ the lattice of weights of G_{qs}. For every $\lambda \in \Lambda^*$ denote by λ' the restriction of λ to the maximal torus of H_{qs}. Then $\beta_G(\lambda) = \beta_H(\lambda')$. In particular, for any $\alpha \in {\Lambda_\mathrm{r}}^*$ one has $\beta_H(\alpha') = 0$.*

(2) *Let G_{qs} be a quasi-split simply connected group, P_{qs} be a standard parabolic subgroup of G_{qs}, L_{qs} be its standard Levi part, H_{qs} be the derived subgroup of L_{qs}. Assume that H is an inner form of H_{qs}, satisfying the conditions $\beta_{H_O}(\alpha'_O) = 0$ for all $O \subset \mathfrak{t}(P_{qs})$. Then there exist an inner form G of G_{qs} and its parabolic subgroup P admitting a Levi subgroup L, such that over a quasi-splitting covering the pair $L \le G$ becomes isomorphic to $L_{qs} \le G_{qs}$, and the derived subgroup of L is isomorphic to H.*

(3) *In the setting of 2, assume that* $\mathrm{Pic}(\mathrm{Dyn}(S)) = 0$. *Then such a G is unique up to an isomorphism.*

Proof. **1.** Let ξ be a cocycle in $\mathrm{Z}^1(S, G^{ad}_{qs})$ corresponding to G, given by elements $g_{\sigma\tau} \in G^{ad}_{qs}(S_\sigma \times_S S_\tau)$ for some covering $\coprod S_\tau \to S$ that quasi-splits G. Over each S_τ one can (possibly, passing to a finer covering) conjugate P_{S_τ} and L_{S_τ} by some element of G^{ad}_{qs} to P_{qs} and L_{qs}, where P_{qs} is a standard parabolic subgroup of G_{qs} and L_{qs} is its standard Levi subgroup. Adjusting ξ by the coboundary given by these elements, we can assume that all

$g_{\sigma\tau}$'s belong to L'_{qs}, where L'_{qs} is the image of L_{qs} in G^{ad}_{qs}, by Exp. XXVI Prop. 1.15 and Cor. 1.8 (cf. Exp. XXVI 3.21)

Let $\rho\colon G_{qs} \to \mathrm{GL}(V)$ be a center preserving representation with a weight λ. Consider its restriction to H_{qs} and denote by U the center preserving direct summand corresponding to the weight λ' and by U' its complement invariant under H_{qs} (see Lemma 3, part 5). Denote by T_{qs} the standard maximal torus of L_{qs} and by T'_{qs} its intersection with H_{qs}. Note that U and U' being sums of weight subspaces of T'_{qs} are stable under T_{qs} and, therefore, are invariant under the action of L_{qs}. Therefore, the map $\mathrm{H}^1(S,\, L'_{qs}) \to \mathrm{H}^1(S,\, \mathrm{PGL}(V)$ factors through $\mathrm{H}^1(S,\, \mathrm{GL}(U)\times\mathrm{GL}(U')/\,\mathbf{G}_{\mathrm{m}})$, where $\mathbf{G}_{\mathrm{m}}$ is embedded into $\mathrm{GL}(U)\times\mathrm{GL}(U')$ diagonally.

Now the claim is obtained by comparing the diagrams

$$\begin{array}{ccc}
\mathrm{H}^1(S,\, (\mathrm{GL}(U)\times\mathrm{GL}(U'))/\,\mathbf{G}_{\mathrm{m}}) & \longrightarrow & \mathrm{H}^2(S,\, \mathbf{G}_{\mathrm{m}}) \\
\downarrow & & \| \\
\mathrm{H}^1(S,\, \mathrm{PGL}(V)) & \longrightarrow & \mathrm{H}^2(S,\, \mathbf{G}_{\mathrm{m}})
\end{array}$$

and

$$\begin{array}{ccc}
\mathrm{H}^1(S,\, (\mathrm{GL}(U)\times\mathrm{GL}(U'))/\,\mathbf{G}_{\mathrm{m}}) & \longrightarrow & \mathrm{H}^2(S,\, \mathbf{G}_{\mathrm{m}}) \\
\downarrow & & \| \\
\mathrm{H}^1(S,\, \mathrm{PGL}(U)) & \longrightarrow & \mathrm{H}^2(S,\, \mathbf{G}_{\mathrm{m}}),
\end{array}$$

which come from the sequences

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbf{G}_{\mathrm{m}} & \longrightarrow & \mathrm{GL}(U)\times\mathrm{GL}(U') & \longrightarrow & (\mathrm{GL}(U)\times\mathrm{GL}(U'))/\,\mathbf{G}_{\mathrm{m}} & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbf{G}_{\mathrm{m}} & \longrightarrow & \mathrm{GL}(V) & \longrightarrow & \mathrm{PGL}(V) & \longrightarrow & 1.
\end{array}$$

and

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbf{G}_{\mathrm{m}} & \longrightarrow & \mathrm{GL}(U)\times\mathrm{GL}(U') & \longrightarrow & (\mathrm{GL}(U)\times\mathrm{GL}(U'))/\,\mathbf{G}_{\mathrm{m}} & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbf{G}_{\mathrm{m}} & \longrightarrow & \mathrm{GL}(U) & \longrightarrow & \mathrm{PGL}(U) & \longrightarrow & 1.
\end{array}$$

2. Let ζ be a cocycle in $\mathrm{Z}^1(S,\, H^{ad}_{qs}) = \mathrm{Z}^1(S,\, L^{ad}_{qs})$ corresponding to H. Denote by L'_{qs} and H'_{qs} the images of L_{qs} and H_{qs} in G^{ad}_{qs}. Let us compute the image $\delta([\zeta]) \in \mathrm{H}^2(S,\, \mathrm{Cent}(L'_{qs}))$. Using the assumption, Theorem 1, and the commutative diagram

$$\begin{array}{ccc}
\mathrm{H}^1(S,\, H^{ad}_{qs}) & \xrightarrow{\ \delta\ } & \mathrm{H}^2(S,\, \mathrm{Cent}(H'_{qs})) \\
\| & & \downarrow \\
\mathrm{H}^1(S,\, L^{ad}_{qs}) & \xrightarrow{\ \delta\ } & \mathrm{H}^2(S,\, \mathrm{Cent}(L'_{qs})),
\end{array}$$

we see that $\alpha_O(\delta([\zeta_O])) = 0$ for any $O \subset \mathfrak{t}(P_{qs})$. Now Proposition 1 and the Shapiro lemma show that $\delta([\zeta]) = 0$. It means that $[\zeta]$ comes from some $[\xi] \in \mathrm{H}^1(S,\, L'_{qs})$, and the image of $[\xi]$ in $\mathrm{H}^1(S,\, G^{ad}_{qs})$ defines a desired group G.

3. Let G be such a group; denote by ξ a corresponding cocycle in $\mathrm{Z}^1(S,\, G^{ad}_{qs})$. As we have seen earlier, $[\xi]$ comes from an element of $\mathrm{H}^1(S,\, L'_{qs})$, say $[\zeta]$. We have to show that $[\zeta]$ (and a fortioti $[\xi]$) is completely determined by its image in $\mathrm{H}^1(S,\, L^{ad}_{qs})$, or, in other words, that the canonical map $\pi_*\colon\ \mathrm{H}^1(S,\, L'_{qs}) \to \mathrm{H}^1(S,\, L^{ad}_{qs})$ is injective. Since $\mathrm{Cent}(L'_{qs})$ is central in L'_{qs}, $\mathrm{H}^1(S,\, \mathrm{Cent}(L'_{qs}))$ acts on $\mathrm{H}^1(S,\, L'_{qs})$, and the orbits of the action coincide with the fibers of π_*. But $\mathrm{H}^1(S,\, \mathrm{Cent}(L'_{qs}))$ by Proposition 1 and the Shapiro lemma injects to $\mathrm{Pic}(\mathrm{Dyn}(G))$, which is trivial by the assumption. □

5. Tits indices

From now on we assume that $S = \operatorname{Spec} R$, where R is a connected semilocal ring. Recall that in this case all minimal parabolic subgroups $P_{\min}$ of G are conjugate under $G(S)$ and hence have the same type $\mathfrak{t}_{\min} = \mathfrak{t}(P_{\min})$, which is a clopen subscheme of $\mathrm{Dyn}(G)$. If T/S is a Galois covering that splits $\mathrm{Dyn}(G)$, $(\mathfrak{t}_{\min})_T$ is a clopen $*$-invariant subscheme of D_T. By Exp. XXVI Lemme 3.8 $P \mapsto \mathfrak{t}(P)$ is a bijection between parabolic subgroups P of G containing $P_{\min}$ and clopen subschemes $\mathfrak{t}$ of $\mathrm{Dyn}(G)$ containing $\mathfrak{t}_{\min}$.

Since S is affine, for any parabolic subgroup P of G there exists a Levi subgroup L (Exp. XXVI Cor. 2.3) of P, and a unique parabolic subgroup P^- which is opposite to P with respect to L, i.e. satisfies $P^- \cap P = L$ (Exp. XXVI Th. 4.3.2). The type $\mathfrak{t}(P^-)$ is the image $s_G(\mathfrak{t}(P))$ of $\mathfrak{t}(P)$ under some automorphism s_G of $\mathrm{Dyn}(G)$, called the *opposition involution* (Exp. XXIV Prop. 3.16.6 and Exp. XXVI 4.3.1; cf. [Ti66] 1.5.1). The corresponding automorphism $s_G \in \mathrm{Aut}\,(D)$ is induced by the automorphism $\alpha \mapsto -w_0(\alpha)$ of the root system Φ of G_0, where w_0 is the unique element of the maximal length in the Weyl group of Φ. In fact s_G acts nontrivially only on components of types A_n, $n \geq 2$, D_{2n+1}, $n \geq 1$, or E_6, in which cases it acts as the unique nontrivial automorphism of this component.

Clearly, we have $\mathfrak{t}_{\min} = s_G(\mathfrak{t}_{\min})$, since if $P = P_{\min}$ is a minimal parabolic subgroup, then P^- is also minimal, and we must have both $\mathfrak{t}_{\min} \subseteq s_G(\mathfrak{t}_{\min})$ and $s_G(\mathfrak{t}_{\min}) \subseteq \mathfrak{t}_{\min}$. It follows that $J = s_G(J)$.

We now start the classification of semisimple algebraic groups over $S = \operatorname{Spec} R$. The problem allows two immediate reductions. First, every semisimple group G is completely determined by its root datum and the corresponding simply connected group G^{sc}, so we can assume that G is simply connected.

Second, if the Dynkin diagram D of G is not connected (as a graph), we can present D as the disjoint union of its *isotypic* components D_t (it means that we collect isomorphic components together), and then we have a canonical decomposition $G \simeq \prod G_t$, where G_t is a group over S with the Dynkin diagram D_t (Exp. XXIV Prop. 5.5). Further, if D_t is the disjoint union of n_t copies of a connected graph $D_{0,\,t}$, there exists a canonical étale extension S_t/S of degree n_t and a group $G_{0,\,t}$ over S_t such that $G_t \simeq R_{S_t/S}(G_{0,\,t})$ (Exp. XXIV Prop. 5.9). So we can assume that D is connected, that is, G is a *simple* algebraic group.

The assumption that $S = \operatorname{Spec} R$ is connected allows us to identify D_T with D, and a clopen $*$-invariant subscheme of D_T with a $*$-invariant subset of D. Let $J \subseteq D$ be the complement of the subset corresponding to $\mathfrak{t}_{\min}$. Then the *Tits index* of G is the pair $(D,\, J)$, together with a $*$-action on D, represented by a subgroup Γ of $\mathrm{Aut}\,(D)$. Usually we indicate Γ by writing its order as the upper left index attached to D (for example, ${}^2E_6$, ${}^6D_4$ and so on). The group is of *inner type*, if $\Gamma = \{1\}$. The group G is quasi-split, if $J = D$, and split, if it is quasi-split and the $*$-action is trivial. When $J = \emptyset$ we say that G is *anisotropic*.

Our reasoning will be based on Theorem 2, which implies that a semisimple algebraic group is determined, up to an isomorphism, by the type $\mathfrak{t}_{min} = J$ of a minimal parabolic subgroup, the anisotropic semisimple group (the *anisotropic kernel* of G) subject to certain conditions on Tits algebras, which is the derived subgroup of a Levi subgroup of a minimal parabolic, and by the quasi-split group G_{qs}. In its turn, G_{qs} is determined by the Dynkin diagram D and the $*$-action on it. Thus the classification consists in listing all possible Tits indices of simple algebraic groups, and, for any given index, the conditions on the corresponding anisotropic kernels. Whenever it does not require any extra technique, we describe the isomorphism classes of anisotropic kernels in terms of more intuitive algebraic structures, like Azumaya algebras over R or étale extensions R'/R of a given degree.

Our numbering of the vertices of Dynkin diagrams follows [B]. We represent Tits indices graphically by Dynkin diagrams D with the vertices in J being circled. We also use the

Tits notation ${}^mX^k_{n,r}$ for the groups of specific indices (see [Ti66]). Unless explicitly stated otherwise, E denotes an Azumaya algebra over R.

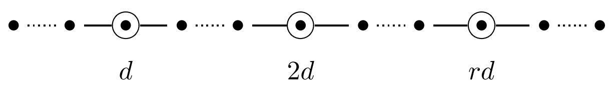
We begin with simple groups of type A_n. The split simple simply connected group scheme of type A_n over R is $\mathrm{SL}_{n+1}(R)$; the corresponding adjoint group is $\mathrm{PGL}_{n+1}(R) = \mathrm{Aut}\,(\mathrm{M}_{n+1}(R))$. So the simple simply connected groups of inner type A_n are of the form $\mathrm{SL}_1(A)$, where A is an Azumaya algebra over R of degree $n+1$, uniquely determined up to an isomorphism. Obviously A is the Tits algebra of $\mathrm{SL}_1(A)$ corresponding to the natural representation of $\mathrm{SL}_{n+1}(R)$ in R^{n+1}; so $[A] = \beta_{\mathrm{SL}_1(A)}(\omega_1)$.

Lemma 6. *Assume that* $\mathrm{SL}_1(E)$ *and* $\mathrm{SL}_1(E')$ *are anisotropic, and* $[E] = [E']$ *in* $\mathrm{Br}(R)$. *Then* $E \simeq E'$.

Proof. Since projective modules over R are free, $[E] = [E']$ means that $\mathrm{M}_n(E) \simeq \mathrm{M}_m(E')$ for some n and m. Consider the simple group $G = \mathrm{SL}_n(E) \simeq \mathrm{SL}_m(E')$. Then $\mathrm{SL}_1(E)^n$ and $\mathrm{SL}_1(E')^m$ are anisotropic groups which are the derived subgroups of Levi subgroups of some parabolic subgroups of G. Hence these parabolic subgroups are minimal, and $\mathrm{SL}_1(E)^n$ and $\mathrm{SL}_1(E')^m$ are semisimple groups of the same type; in particular, $m = n$ and the degrees of E and E' are equal. This implies $E \simeq E'$ by [K, Ch. III Prop. 5.2.3 2)]. □

Theorem 3 ($\mathbf{{}^1A_n}$). *Every simple simply connected group G of inner type A_n over R has the form $\mathrm{SL}_{r+1}(E)$ for a uniquely determined $r \geq 0$ and an Azumaya algebra E over R such that $\mathrm{SL}_1(E)$ is anisotropic. The Tits index of G is $({}^1A_n,\, J)$, where $J = \{d,\, 2d,\, \ldots,\, rd\}$, d is the degree of E and $n+1 = (r+1)d$:*

$({}^1A^{(d)}_{n,r})$



Proof. The proof goes by induction on n. Let $({}^1A_n,\, J)$ be the Tits index of G and m be the least element of J (in the case $J = \emptyset$ G is anisotropic, and we can set $d = n$, $r = 0$). Denote the derived subgroup of the Levi part of the parabolic subgroup of type $D \setminus \{m\}$ by G'. By Theorem 2, we have $\beta_{G'}(\alpha'_m) = 0$, and G is uniquely determined by G' satisfying this condition.

Assume first that $1 < m < n$. Then $G' \simeq \mathrm{SL}_1(E) \times H$, where E has degree m, $\mathrm{SL}_1(E)$ is anisotropic, and H is a simple simply connected group of inner type A_{n-d}. By the induction hypothesis, $H \simeq \mathrm{SL}_r(E')$ for some r, where $\mathrm{SL}_1(E')$ is anisotropic, and the Tits index of H is $({}^1A_{n-m},\, J')$ with $J' = \{d,\, 2d,\, \ldots,\, (r-1)d\}$, where $d = \deg E'$, $n - m + 1 = rd$. The Cartan matrix of A_n shows that $\alpha_m = 2\omega_m - \omega_{m-1} - \omega_{m+1}$, so

$$0 = \beta_{G'}(\alpha'_m) = \beta_{\mathrm{SL}_1(E)}(\omega_1) - \beta_H(\omega_1) = [E] - [E'].$$

By Lemma 6, $E \simeq E'$ and hence $m = d$. In this case $G \simeq \mathrm{SL}_{r+1}(E)$.

In the case $m = 1$ G' is a simply simply connected group of inner type A_{n-1}. By the induction hypothesis, $G' \simeq \mathrm{SL}_r(E')$ for some r and E'. The Cartan matrix shows that $\alpha_m = 2\omega_1 - \omega_2$, and

$$0 = \beta_{G'}(\alpha'_m) = -\beta_{G'}(\omega_1) = -[E'],$$

therefore G' splits and so does G. In this case $G \simeq \mathrm{SL}_{n+1}(R)$.

In the case $m = n$ $G' = \mathrm{SL}_1(E)$ is anisotropic. But the Cartan matrix shows that $\alpha_m = 2\omega_n - \omega_{n-1}$, and

$$0 = \beta_{\mathrm{SL}_1(E)}(\alpha'_m) = \beta_{\mathrm{SL}_1(E)}(\omega_1) = [E],$$

a contradiction. □

The above result implies that for any Azumaya algebra A over R, the group $G = \mathrm{SL}_1(A)$ is isomorphic to $\mathrm{SL}_{r+1}(E)$, with E an Azumaya algebra such that $\mathrm{SL}_1(E)$ is anisotropic. In this case the degree of E is called the *index* of A and is denoted by $\mathrm{ind}\, A$; obviously $\mathrm{ind}\, A$

divides $\deg A$. The *exponent* $\exp A$ of A is the order of $[A]$ in $\mathrm{Br}(R)$. We will need the following result:

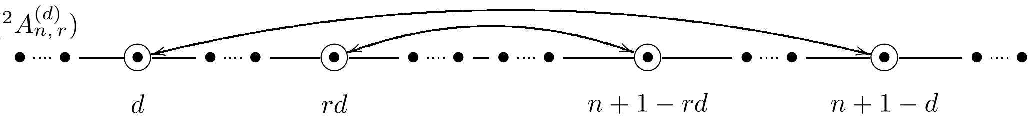
Proposition 3. *For any A $\exp A$ divides $\operatorname{ind} A$, and they have the same prime factors.*

Proof. The first part follows from the fact that $[A] = [E] = \beta_{\mathrm{SL}_1(E)}(\omega_1)$, and $(\deg E)\omega_1$ belongs to Λ_r. The second part follows from [Gab, Ch. II, Thm. 1]. □

Let R'/R be an étale extension of degree n. We can interpret the corestriction homomorphism $\mathrm{cores}_{R'/R}\colon \mathrm{Br}(R') \to \mathrm{Br}(R)$ as follows. If A is an Azumaya algebra over R' of degree d, $R_{R'/R}(\mathrm{SL}_1(A))$ is a group of type nA_{d-1} over R, with the $*$-action permuting the copies of A_{d-1}. Now $\mathrm{cores}_{R'/R}([A]) = \beta_{R_{R'/R}(\mathrm{SL}_1(A))}(\omega)$, where ω is the sum of the fundamental weights ω_1 corresponding to each copy of A_{d-1} (cf. [Ti71, § 5.3]).

Theorem 4 (^{2}A$_n$). *Every simple simply connected group G of type ^2A_n over R has the Tits index $(^2A_n,\ J)$, where $J = \{d,\ 2d,\ \ldots,\ rd,\ n+1-rd,\ \ldots,\ n+1-2d,\ n+1-d\}$ for some $r \geq 0$, $d > 0$ such that $d \mid n+1$, $2rd \leq n+1$:*

$(^2A^{(d)}_{n,r})$



Isomorphism classes of groups of a given index bijectively correspond to isomorphism classes of:

- *simple simply connected anisotropic groups H of type $^2A_{n-2rd}$ over R with $\beta_{H_O}(\omega_1) = [E]$, $\operatorname{ind} E = d$, O is the orbit corresponding to $\{1,\ n-2rd\}$, when $n-2rd \geq 2$;*
- *pairs consisting of an Azumaya algebras A over R and a connected quadratic étale extension R'/R such that $\operatorname{ind} A = \deg A = 2$ and $\operatorname{ind} A_{R'} = d$, when $n - 2rd = 1$;*
- *Azumaya algebras E over a connected quadratic étale extension R'/R with $\operatorname{ind} E = \deg E = d$, $\mathrm{cores}_{R'/R}([E]) = 0$, when $n-2rd \leq 0$.*

Proof. Let $(^2A_n,\ J)$ be the Tits index of G. Note that if j belongs to J then so does $n+1-j$. Let m be the greatest element of J which is not greater than $\frac{n+1}{2}$ (in the case $J = \emptyset$ G is anisotropic, and we can set $r = 0$). Denote the orbit corresponding to m by O, and the derived subgroup of the Levi part of the parabolic subgroup of type $D \setminus O$ by G'. By Theorem 2 we have $\beta_{G'_O}(\alpha'_O) = 0$, and G is uniquely determined by G' satisfying this condition.

Assume first that $1 < m < \frac{n-1}{2}$. Then O as a scheme is a connected quadratic étale extension of R, $G' \simeq H_1 \times H$, where H is a simple simply connected anisotropic group of type $^2A_{n-2m}$, H_1 is a group of outer type $A_{m-1} + A_{m-1}$ with the $*$-action permuting the copies of A_{m-1}. Conseqeuently, $H_1 \simeq R_{O/R}(\mathrm{SL}_r(E))$ for some Azumaya algebra E over O, $\operatorname{ind} E = \deg E = d$, $dr = m$. The Cartan matrix shows that $\alpha_m = 2\omega_m - \omega_{m-1} - \omega_{m+1}$, so

$$0 = \beta_{G'_O}(\alpha'_O) = \beta_{\mathrm{SL}_r(E)}(\omega_1) - \beta_{H_O}(\omega_1) = [E] - \beta_{H_O}(\omega_1).$$

Note that H_O is of inner type A_{n-2m}, and therefore d must divide $n-2m+1 = n-2rd+1$. We have $J = \{d, \ldots,\ rd,\ n+1-rd,\ \ldots,\ n+1-d\}$.

In the case $m = \frac{n-1}{2}$, O as a scheme is a connected quadratic étale extension R'/R, $G' \simeq R_{R'/R}(\mathrm{SL}_r(E)) \times \mathrm{SL}_1(A)$, E is an Azumaya algebra over R', $\operatorname{ind} E = \deg E = d$, $dr = m$, A is an Azumaya algebra over R, $\operatorname{ind} A = \deg A = 2$. The Cartan matrix shows that $\alpha_m = 2\omega_m - \omega_{m-1} - \omega_{m+1}$, so

$$0 = \beta_{G'_O}(\alpha'_O) = \beta_{\mathrm{SL}_r(E)}(\omega_1) - \beta_{\mathrm{SL}_1(A)_O}(\omega_1) = [E] - [A_{R'}].$$

Since $\operatorname{ind} A = 2$, $d = 1$ or 2 in this case, so $d \mid n+1$. We have $J = \{d, \ldots,\ rd,\ n+1-rd,\ \ldots,\ n+1-d\}$ again.

In the case $m = \frac{n}{2}$, O as a scheme is a connected quadratic étale extension R'/R, $G' \simeq R_{R'/R}(\mathrm{SL}_r(E))$, E is an Azumaya algebra over R', $\operatorname{ind} E = \deg E = d$, $dr = m$. The Cartan matrix shows that $\alpha_m = 2\omega_m - \omega_{m-1} - \omega_{m+1}$, so

$$0 = \beta_{G'_O}(\alpha'_O) = \beta_{\mathrm{SL}_r(E)}(\omega_1) = [E].$$

Hence $E = R'$ and $d = 1$. G is quasi-split in this case.

In the case $m = \frac{n+1}{2}$, O as a scheme is isomorphic to R, $G' \simeq R_{R'/R}(\mathrm{SL}_r(E))$ for some Azumaya algebra E over a connected quadratic étale extension R'/R with $\operatorname{ind} E = \deg E = d$, $rd = m$. The Cartan matrix shows that $\alpha_m = 2\omega_m - \omega_{m-1} - \omega_{m+1}$, so

$$0 = \beta_{G'_O}(\alpha'_O) = \mathrm{cores}_{R'/R}(\beta_{\mathrm{SL}_r(E)}(\omega_1)) = \mathrm{cores}_{R'/R}([E]).$$

We have $J = \{d, \ldots, (r-1)d, rd = (n+1) - rd, \ldots, n+1-d\}$.

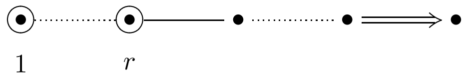
In the case $m = 1$, O as a scheme is a connected quadratic étale extension of R, $G' = H$ is a simple simply connected anisotropic group of type ${}^2A_{n-2}$. The Cartan matrix show that $\alpha_m = 2\omega_1 - \omega_2$, so

$$0 = \beta_{G'_O}(\alpha'_O) = \beta_{H_O}(\omega_1).$$

We have $J = \{1, n\}$ in this case. □

Theorem 3 ($\mathbf{B_n}$). *Every simple simply connected group of type B_n over R, $n \geq 2$, has the Tits index (B_n, J), where $J = \{1, 2, \ldots, r\}$ for some $r \geq 0$:*

$(B_{n,r})$



Isomorphism classes of groups of a given index bijectively correspond to isomorphism classes of:

- *simple simply connected anisotropic groups of type B_{n-r} over R, when $n - r \geq 2$;*
- *Azumaya algebras A over R with $\operatorname{ind} A = \deg A = 2$, when $n - r = 1$.*

If $n = r$ then G is split.

Proof. Let (B_n, J) be the Tits index of G and m be the greatest element of J (in the case $J = \emptyset$ G is anisotropic, and we can set $r = 0$). Denote the derived subgroup of the Levi part of the parabolic subgroup of type $D \setminus \{m\}$ by G'. By Theorem 2 we have $\beta_{G'}(\alpha'_m) = 0$, and G is uniquely determined by G' satisfying this condition.

Assume first that $1 < m < n-1$. Then $G' \simeq \mathrm{SL}_r(E) \times H$, where $\operatorname{ind} E = \deg E = d$, $dr = m$, H is a simple simply connected anisotropic group of type B_{n-m}. The Cartan matrix shows that $\alpha_m = 2\omega_m - \omega_{m-1} - \omega_{m+1}$, so

$$0 = \beta_{G'}(\alpha'_m) = \beta_{\mathrm{SL}_r(E)}(\omega_1) = [E],$$

that is $E = R$, $d = 1$, $J = \{1, \ldots, r\}$.

In the case $m = n-1$ $G' \simeq \mathrm{SL}_r(E) \times \mathrm{SL}_1(A)$, where $\operatorname{ind} E = \deg E = d$, $dr = n-1$, $\operatorname{ind} A = \deg A = 2$. The Cartan matrix shows that $\alpha_m = 2\omega_{n-1} - \omega_{n-2} - 2\omega_n$, so

$$0 = \beta_{G'}(\alpha'_m) = \beta_{\mathrm{SL}_r(E)}(\omega_1) = [E],$$

that is $E = R$, $d = 1$, $J = \{1, \ldots, n-1\}$.

In the case $m = n$ $G' \simeq \mathrm{SL}_r(E)$, where $\operatorname{ind} E = \deg E = d$, $dr = n$. The Cartan matrix shows that $\alpha_m = 2\omega_n - \omega_{n-1}$, so

$$0 = \beta_{G'}(\alpha'_m) = \beta_{\mathrm{SL}_r(E)}(\omega_1) = [E],$$

that is $E = R$, $d = 1$, $J = \{1, \ldots, n\}$. G is split in this case.

In the case $m = 1$ $G' = H$ is a simple simply connected anisotropic group of type B_{n-1}. The Cartan matrix shows that $\alpha_m = 2\omega_1 - \omega_2$, and the condition $\beta_{G'}(\alpha'_m) = 0$ is vacuous. We have $J = \{1\}$ in this case. □

The split simple simply connected group scheme of type C_n over R is $\mathrm{Sp}_{2n}(R)$.

Proposition 4. *Assume that G is a simple simply connected group of type C_n over R, $\beta_G(\omega_1) = [E]$, $\operatorname{ind} E = d$. Then $d = 2^k$ for some $k \geq 0$ and $d \mid 2n$. If $d = 1$ then G is split.*

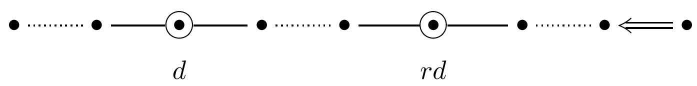
Proof. We have $2[E] = 0$, since $2\omega_1$ belongs to Λ_r. Now Proposition 3 implies that $d = 2^k$.

The vector representation $\rho\colon \mathrm{Sp}_{2n}(R) \to \mathrm{End}\,(R^{2n})$ is center preserving and has a weight ω_1; so $[A_\rho] = [E]$. But A_ρ has degree $2n$, so $d \mid 2n$.

If $d = 1$ then by Proposition 2 G corresponds to an element of $\mathrm{H}^1(R, \mathrm{Sp}_{2n})$, and the latter is trivial by [K, Ch. I, Cor. 4.1.2]. □

Theorem 3 ($\mathbf{C_n}$). *Every simple simply connected group G of type C_n over R, $n \geq 2$, has the Tits index $(C_n,\, J)$, where $J = \{d,\, 2d,\, \ldots,\, rd\}$ for some $r \geq 0$, $d > 0$ such that $d = 2^k \mid 2n$, $rd \leq n$, and $r = n$ when $d = 1$:*

$(C_{n,\,r}^{(d)})$



Isomorphism classes of groups of a given index bijectively correspond to isomorphism classes of:

- *simple simply connected anisotropic groups H of type C_{n-rd} over R with $\beta_H(\omega_1) = [E]$, $\operatorname{ind} E = 2$, when $n - rd \geq 2$;*
- *Azumaya algebras E over R with $\operatorname{ind} E = \deg E = d$ and $\exp E \leq 2$, when $n-rd \leq 1$.*

Proof. Let $(C_n,\, J)$ be the Tits index of G and m be the greatest element of J (in the case $J = \emptyset$ G is anisotropic, and we can set $r = 0$). Denote the derived subgroup of the Levi part of the parabolic subgroup of type $D \setminus \{m\}$ by G'. By Theorem 2 we have $\beta_{G'}(\alpha'_m) = 0$, and G is uniquely determined by G' satisfying this condition.

Assume first that $1 < m < n - 1$. Then $G' \simeq \mathrm{SL}_r(E) \times H$, where $\operatorname{ind} E = \deg E = d$, $dr = m$, H is a simple imply connected anisotropic group of type C_{n-m}. Note that $[E] = \beta_G(\omega_1)$, hence $d = 2^k \mid 2n$ and $d > 1$ by Proposition 4. The Cartan matrix shows that $\alpha_m = 2\omega_m - \omega_{m-1} - \omega_{m+1}$, so

$$0 = \beta_{G'}(\alpha'_m) = \beta_{\mathrm{SL}_r(E)}(\omega_1) - \beta_H(\omega_1) = [E] - \beta_H(\omega_1).$$

We have $J = \{d,\, \ldots,\, rd\}$.

In the case $m = n - 1$ $G' \simeq \mathrm{SL}_r(E) \times \mathrm{SL}_1(A)$, where $\operatorname{ind} E = \deg E = d$, $dr = n - 1$, $d = 2^k \mid 2n$, $d > 1$, $\operatorname{ind} A = \deg A = 2$. The Cartan matrix shows that $\alpha_m = 2\omega_{n-1} - \omega_{n-2} - \omega_n$, so

$$0 = \beta_{G'}(\alpha'_m) = \beta_{\mathrm{SL}_r(E)}(\omega_1) - \beta_{\mathrm{SL}_1(A)}(\omega_1) = [E] - [A].$$

Hence $[E] = [A]$ and $d = 2$, so $d \mid 2n$. We have $J = \{d, \ldots,\, rd\}$ again.

In the case $m = n$ $G' \simeq \mathrm{SL}_r(E)$, where $\operatorname{ind} E = \deg E = d$, $dr = n$, $d = 2^k$. The Cartan matrix shows that $\alpha_m = 2\omega_n - 2\omega_{n-1}$, so

$$0 = \beta_{G'}(\alpha'_m) = 2\beta_{\mathrm{SL}_r(E)}(\omega_1) = 2[E],$$

that is $\exp E \leq 2$. We have $J = \{d, \ldots,\, rd = n\}$.

In the case $m = 1$ $G' = H$ is a simple simply connected anisotropic group of type C_{n-1}. The Cartan matrix shows that $\alpha_m = 2\omega_1 - \omega_2$, so

$$0 = \beta_{G'}(\alpha'_m) = -\beta_H(\omega_1).$$

Now Proposition 4 implies that H is split, a contradiction. □

The split simple simply connected group scheme of type D_n over R is $\mathrm{Spin}_{2n}(R)$.

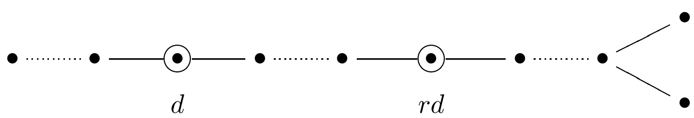
Proposition 5. *Assume that G is a simple simply connected group of type ${}^1D_n$ or ${}^2D_n$ over R, $n \geq 4$, $\beta_G(\omega_1) = [E]$, $\operatorname{ind} E = d$. Then $d = 2^k$ for some $k \geq 0$ and $d \mid 2n$.*

Proof. We have $2[E] = 0$, since $2\omega_1$ belongs to Λ_r. Now Proposition 3 implies that $d = 2^k$.

The vector representation $\rho\colon \mathrm{Spin}_{2n}(R) \to \mathrm{End}\,(R^{2n})$ is center preserving and has a weight ω; so $[A_\rho] = [E]$. But A_ρ has degree $2n$, so $d \mid 2n$. □

Theorem 3 ($^1\mathbf{D_n}$). *Every simple simply connected group G of inner type D_n over R, $n \geq 4$, has the Tits index $({}^1D_n,\, J)$, where $J = \{d,\, 2d,\, \dots,\, rd\}$ (possibly, after interchanging $n-1$ and n) for some $r \geq 0$, $d > 0$ such that $d = 2^k \mid 2n$, $rd \leq n$, $n \neq rd+1$:*

$({}^1D^{(d)}_{n,\,r})$



Isomorphism classes of groups of a given index bijectively correspond to isomorphism classes of:

- *simple simply connected anisotropic groups H of inner type D_{n-rd} over R with $\beta_H(\omega_1) = [E]$,* $\operatorname{ind} E = 2$, *when $n - rd \geq 4$;*
- *Azumaya algebras A over R with* $\operatorname{ind} A = \deg A = 4$ *and* $2[A] = [E]$, $\operatorname{ind} E = d$, *when $n - rd = 3$;*
- *pairs of Azumaya algebras A_1 and A_2 over R with* $\operatorname{ind} A_1 = \deg A_1 = \operatorname{ind} A_2 = \deg A_2 = 2$ *and* $[A_1] + [A_2] = [E]$, $\operatorname{ind} E = d$, *when $n - rd = 2$;*
- *Azumaya algebras E over R with* $\operatorname{ind} E = \deg E = d$ *and* $\exp E \leq 2$, *when $n = rd$.*

Proof. Let $({}^1D_n,\, J)$ be the Tits index of G and m be the greatest element of J (in the case $J = \emptyset$ G is anisotropic, and we can set $r = 0$). Denote the derived subgroup of the Levi part of the parabolic subgroup of type $D \setminus \{m\}$ by G'. By Theorem 2 we have $\beta_{G'}(\alpha'_m) = 0$, and G is uniquely determined by G' satisfying this condition.

Assume first that $1 < m < n-3$. Then $G' \simeq \mathrm{SL}_r(E) \times H$, where $\operatorname{ind} E = \deg E = d$, $dr = m$, H is a simple simply connected anisotropic group of type D_{n-m}. Note that $[E] = \beta_G(\omega_1)$, hence $d = 2^k \mid 2n$ by Proposition 5. The Cartan matrix shows that $\alpha_m = 2\omega_m - \omega_{m-1} - \omega_{m+1}$, so

$$0 = \beta_{G'}(\alpha'_m) = \beta_{\mathrm{SL}_r(E)}(\omega_1) - \beta_H(\omega_1) = [E] - \beta_H(\omega_1).$$

We have $J = \{d,\, \dots,\, rd\}$.

In the case $m = n-3$ $G' \simeq \mathrm{SL}_r(E) \times \mathrm{SL}_1(A)$, where $\operatorname{ind} E = \deg E = d$, $dr = n-3$, $d = 2^k \mid 2n$ (so $d = 1$ or 2), $\operatorname{ind} A = \deg A = 4$. The Cartan matrix shows that $\alpha_m = 2\omega_{n-3} - \omega_{n-4} - \omega_{n-2}$, so

$$0 = \beta_{G'}(\alpha'_m) = \beta_{\mathrm{SL}_r(E)}(\omega_1) - \beta_{\mathrm{SL}_1(A)}(\omega_2) = [E] - 2[A].$$

We have $J = \{d, \dots,\, rd = n-3\}$.

In the case $m = n-2$ $G' \simeq \mathrm{SL}_r(E) \times \mathrm{SL}_1(A_1) \times \mathrm{SL}_1(A_2)$, where $\operatorname{ind} E = \deg E = d$, $dr = n-2$, $d = 2^k \mid 2n$ (so $d = 1,\, 2$, or 4), $\operatorname{ind} A_1 = \deg A_1 = \operatorname{ind} A_2 = \deg A_2 = 2$. The Cartan matrix shows that $\alpha_m = 2\omega_{n-2} - \omega_{n-3} - \omega_{n-1} - \omega_n$, so

$$0 = \beta_{G'}(\alpha'_m) = \beta_{\mathrm{SL}_r(E)}(\omega_1) - \beta_{\mathrm{SL}_1(A_1)}(\omega_1) - \beta_{\mathrm{SL}_1(A_2)}(\omega_1) = [E] - [A_1] - [A_2].$$

We have $J = \{d, \dots,\, rd = n-2\}$.

In the case $m = n-1$ we can interchange $n-1$ and n and so assume that $m = n$.

In the case $m = n$ $G' \simeq \mathrm{SL}_r(E)$, where $\operatorname{ind} E = \deg E = d$, $dr = n$, $d = 2^k$. The Cartan matrix shows that $\alpha_m = 2\omega_n - \omega_{n-2}$, so

$$0 = \beta_{G'}(\alpha'_m) = \beta_{\mathrm{SL}_r(E)}(\omega_2) = 2[E],$$

hence $\exp E \leq 2$. We have $J = \{d, \dots,\, rd = n\}$.

In the case $m = 1$ $G' = H$ is a simple simply connected anisotropic group of type D_{n-1}. The Cartan matrix shows that $\alpha_m = 2\omega_1 - \omega_2$, so

$$0 = \beta_{G'}(\alpha'_m) = -\beta_H(\omega_1).$$

We have $J = \{1\}$ in this case. ☐

Proposition 6. *Assume that G is a simple simply connected group of inner type D_5 over R with $\beta_G(\omega_4) = 0$. Then G is of index ${}^1D^{(1)}_{5,\,1}$ or ${}^1D^{(1)}_{5,\,5}$.*

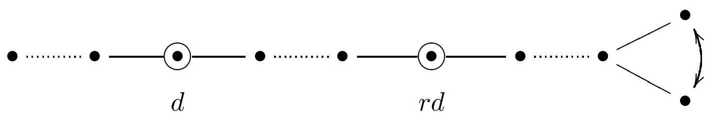
Proof. Indeed, by Theorem 2, part 2, we can construct a group G' of type E_6 over R such that G is the derived subgroup of the Levi subgroup of the parabolic subgroup corresponding to the subset $\{2, 3, 4, 5, 6\}$. Let ${}^1E_6$ be the Tits index of G'; then 1 belongs to J and therefore $6 = s_{G'}(1)$ belongs to J as well. Therefore, G has the index ${}^1D^{(1)}_{5,\,r}$ for some $r \geq 1$. If $r \geq 2$, consider the parabolic subgroup of type $\{2, 3, 4\}$ in G; the derived subgroup of its Levi part is $\mathrm{SL}_1(A)$ for some Asumaya algebra A over R with $\deg A = 4$. By Theorem 2, part 1, we have

$$0 = \beta_G(\omega_4) = \beta_{\mathrm{SL}_1(A)}(\omega_1) = [A],$$

so G is split and has the index ${}^1D^{(1)}_{5,\,5}$ in this case. □

Theorem 3 (^{2}D$_\mathbf{n}$). *Every simple simply connected group G of type ${}^2D_n$, $n \geq 4$, has the Tits index $({}^2D_n,\, J)$, where $J = \{d,\, 2d,\, \ldots,\, rd\}$, except for $J = \{d,\, 2d,\, \ldots,\, (r-1)d,\, n-1,\, n\}$ in the case $n - rd = 1$, for some $r \geq 0$, $d > 0$ such that $d = 2^k \mid 2n$, $rd \leq n-1$:*

$({}^2D^{(d)}_{n,\,r})$



Isomorphism classes of groups of a given index bijectively correspond to isomorphism classes of:

- *simple simply connected anisotropic groups H of type ${}^2D_{n-rd}$ over R with $\beta_H(\omega_1) = [E]$, $\operatorname{ind} E = d$, when $n - rd \geq 4$;*
- *simple simply connected anisotropic groups H of type ${}^2A_3$ over R with $\beta_H(\omega_2) = [E]$, $\operatorname{ind} E = d$, when $n - rd = 3$;*
- *Azumaya algebras A over a connected quadratic étale extension R'/R with $\operatorname{ind} A = \deg A = 2$ and $\operatorname{cores}_{R'/R}([A]) = [E]$, $\operatorname{ind} E = d$, when $n - rd = 2$;*
- *pairs consisting of an Azumaya algebra E over R with $\operatorname{ind} E = \deg E = d$ and a connected quadratic étale extension R'/R such that $[E_{R'}] = 0$, when $n - rd = 1$.*

Proof. Let $({}^2D_n,\, J)$ be the Tits index of G and m be the greatest element of J (in the case $J = \emptyset$ G is anisotropic, and we can set $r = 0$). Denote the orbit corresponding to m by O, and the derived subgroup of the Levi part of the parabolic subgroup of type $D \setminus O$ by G'. By Theorem 2 we have $\beta_{G'_O}(\alpha'_O) = 0$, and G is uniquely determined by G' satisfying this condition.

Assume first that $1 < m < n-3$. Then O as a scheme is isomorphic to R, $G' \simeq \mathrm{SL}_r(E) \times H$, where $\operatorname{ind} E = \deg E = d$, $dr = m$, H is a simple simply connected anisotropic group of type ${}^2D_{n-m}$. Note that $[E] = \beta_G(\omega_1)$, hence $d = 2^k \mid 2n$ by Proposition 5. The Cartan matrix shows that $\alpha_m = 2\omega_m - \omega_{m-1} - \omega_{m+1}$, so

$$0 = \beta_{G'_O}(\alpha'_O) = \beta_{\mathrm{SL}_r(E)}(\omega_1) - \beta_H(\omega_1) = [E] - \beta_H(\omega_1).$$

We have $J = \{d,\, \ldots,\, rd\}$.

In the case $m = n-3$ $G' \simeq \mathrm{SL}_r(E) \times H$, where $\operatorname{ind} E = \deg E = d$, $dr = n-3$, $d = 2^k \mid 2n$ (so $d = 1$ or $d = 2$), H is a simple simply connected anisotropic group of type ${}^2A_3$. The Cartan matrix shows that $\alpha_m = 2\omega_{n-3} - \omega_{n-4} - \omega_{n-2}$, so

$$0 = \beta_{G'_O}(\alpha'_O) = \beta_{\mathrm{SL}_r(E)}(\omega_1) - \beta_H(\omega_2) = [E] - \beta_H(\omega_2).$$

We have $J = \{d, \ldots,\, rd = n-3\}$.

In the case $m = n-2$ O as a scheme is isomorphic to R, $G' \simeq \mathrm{SL}_r(E) \times R_{R'/R}(\mathrm{SL}_1(A))$, where $\operatorname{ind} E = \deg E = d$, $dr = n-2$, $d = 2^k \mid 2n$ (so $d = 1$, 2, or 4), A is an Azumaya algebra over a connected quadratic étale extension R'/R, $\operatorname{ind} A = \deg A = 2$. The Cartan matrix shows that $\alpha_m = 2\omega_{n-2} - \omega_{n-3} - \omega_{n-1} - \omega_n$, so

$$0 = \beta_{G'_O}(\alpha'_O) = \beta_{\mathrm{SL}_r(E)}(\omega_1) - \operatorname{cores}_{R'/R}(\beta_{\mathrm{SL}_1(A)}(\omega_1)) = [E] - \operatorname{cores}_{R'/R}([A]).$$

We have $J = \{d, \dots, rd = n-2\}$.

In the case $m = n-1$ n also belongs to J, a contradiction.

In the case $m = n$ O as a scheme is a connected quadratic étale extension R'/R, $G' \simeq \mathrm{SL}_r(E)$, where $\operatorname{ind} E = \deg E = d$, $dr = n-1$, $d = 2^k \mid 2n$ (so $d = 1$ or 2). The Cartan matrix shows that $\alpha_m = 2\omega_n - \omega_{n-2}$, so

$$0 = \beta_{G'_O}(\alpha'_O) = \beta_{\mathrm{SL}_r(E)_O}(\omega_1) = [E_{R'}].$$

We have $J = \{d, \dots, (r-1)d, n-1, n\}$ in this case.

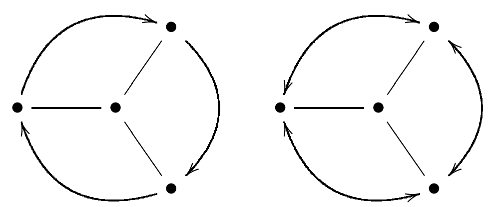
In the case $m = 1$ O as a scheme is isomorphic to R, $G' = H$ is a simple simply connected anisotropic group of type ${}^2D_{n-1}$. The Cartan matrix shows that $\alpha_m = 2\omega_1 - \omega_2$, so

$$0 = \beta_{G'_O}(\alpha'_O) = -\beta_H(\omega_1).$$

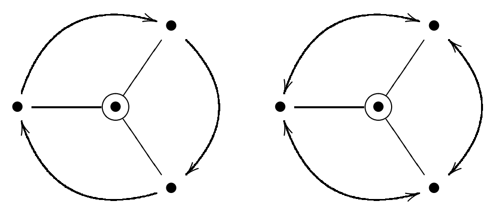
We have $J = \{1\}$ in this case. □

Theorem 3 (${}^3\mathbf{D_4}$ and ${}^6\mathbf{D_4}$). *Every simple simply connected group G of type ${}^3D_4$ or ${}^6D_4$ over R has one of the following Tits indices:*

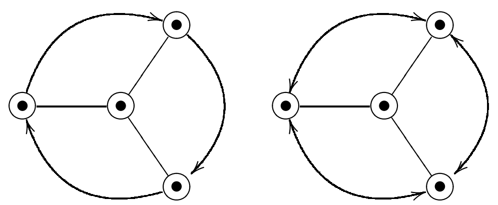
(${}^3D^{28}_{4,0}$, ${}^6D^{28}_{4,0}$)



(${}^3D^{9}_{4,1}$, ${}^6D^{9}_{4,1}$)



(${}^3D^{2}_{4,2}$, ${}^6D^{2}_{4,2}$)



Isomorphism classes of groups of a given index bijectively correspond to isomorphism classes of:

- *Azumaya algebras A over a connected cubic cyclic (resp., noncyclic) étale extension R'/R with* $\operatorname{ind} A = \deg A = 2$ *and* $\operatorname{cores}_{R'/R}([A]) = 0$*, in the case of ${}^3D^9_{4,1}$ (resp., ${}^6D^9_{4,1}$);*
- *connected cubic cyclic (resp., noncyclic) étale extensions R'/R, in the case of ${}^3D^2_{4,2}$ (resp., ${}^6D^2_{4,2}$).*

Proof. Let $({}^3D_4, J)$ (or $({}^6D_4, J)$) be the Tits index of G and m be the least element of J (in the case $J = \emptyset$ G is anisotropic, that is of index ${}^3D^{28}_{4,0}$ or ${}^6D^{28}_{4,0}$). Denote the orbit corresponding to m by O, and the derived subgroup of the Levi part of the parabolic subgroup of type $D \setminus O$ by G'. By Theorem 2 we have $\beta_{G'_O}(\alpha'_O) = 0$, and G is uniquely determined by G' satisfying this condition.

In the case $m = 1$ O as a scheme is a connected cubic étale extension R'/R, $G' \simeq \mathrm{SL}_1(E)$, where $\operatorname{ind} E = \deg E = 2$. The Cartan matrix shows that $\alpha_m = 2\omega_1 - \omega_2$, so

$$0 = \beta_{G'_O}(\alpha'_O) = -\beta_{\mathrm{SL}_1(E)_O}(\omega_1) = [E_{R'}].$$

But then $3[E] = \mathrm{cores}_{R'/R}([E_{R'}]) = 0$, and $2[E] = 0$ by Proposition 3, hence $E = R$. In this case G is quasi-split, that is of index ${}^3D^2_{4,\,4}$ or ${}^6D^2_{4,\,4}$.

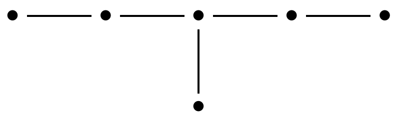
In the case $m = 2$ O as a scheme is isomorphic to R, $G' \simeq R_{R'/R}(\mathrm{SL}_1(A))$, where A is an Azumaya algebra over a connected cubic étale extension R'/R, $\mathrm{ind}\, A = \deg A = 2$. The Cartan matrix shows that $\alpha_m = 2\omega_2 - \omega_1 - \omega_3 - \omega_4$, so

$$0 = \beta_{G'_O}(\alpha'_O) = -\,\mathrm{cores}_{R'/R}(\beta_{\mathrm{SL}_1(A)}(\omega_1)) = -\,\mathrm{cores}_{R'/R}([A]).$$

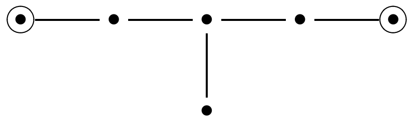
In this case G is of index ${}^3D^9_{4,\,1}$ or ${}^6D^9_{4,\,1}$. □

Theorem 3 (${}^\mathbf{1}\mathbf{E_6}$)**.** *Every simple simply connected group G of inner type E_6 over R has one of the following Tits indices:*

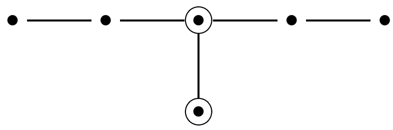
(${}^1E^{78}_{6,\,0}$)



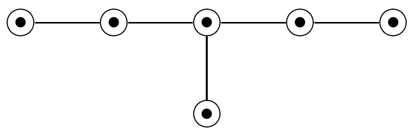
(${}^1E^{28}_{6,\,2}$)



(${}^1E^{16}_{6,\,2}$)



(${}^1E^{0}_{6,\,6}$)



Isomorphism classes of groups of a given index bijectively correspond to isomorphism classes of:

- *simple simply connected anisotropic groups H of type D_4 over R with $\beta_H = 0$, in the case of ${}^1E^{28}_{6,\,2}$;*
- *Azumaya algebras A over R with* $\mathrm{ind}\, A = \deg A = 3$, *in the case of ${}^1E^{16}_{6,\,2}$.*

The only group of index ${}^1E^{0}_{6,\,6}$ is split.

Proof. Let $({}^1E_6,\, J)$ be the Tits index of G and m be the least element of J (in the case $J = \emptyset$ G is anisotropic, that is of index ${}^1E^{78}_{6,\,0}$). Denote the derived subgroup of the Levi part of the parabolic subgroup of type $D \setminus \{m\}$ by G'. By Theorem 2 we have $\beta_{G'}(\alpha'_m) = 0$, and G is uniquely determined by G' satisfying this condition. Since J is invariant under s_G, $m \neq 5, 6$.

In the case $m = 1$ G' is a simple simply connected group of inner type D_5. The Cartan matrix shows that $\alpha_m = 2\omega_1 - \omega_3$, so

$$0 = \beta_{G'}(\alpha'_m) = \beta_{G'}(\omega_4).$$

By Proposition 6 G' is of index ${}^1D^{(1)}_{5,\,1}$ or ${}^1D^{(1)}_{5,\,5}$. In the first case G has the index ${}^1E^{28}_{6,\,2}$ and is uniquely determined by a simple simply connected group H of inner type D_4 with $\beta_H(\omega_1) = 0$ and $\beta_H(\omega_3) = \beta_G(\omega_4) = 0$, that is $\beta_H = 0$. In the second case G is split, that is of index ${}^1E^{0}_{6,\,6}$.

In the case $m = 2$ $G' \simeq \mathrm{SL}_r(A)$, where A is an Azumaya algebra over R with $\operatorname{ind} A = \deg A = d$, $d \neq 1$ $dr = 6$. The Cartan matrix shows that $\alpha_m = 2\omega_2 - \omega_4$, so

$$0 = \beta_{G'}(\alpha'_m) = \beta_{\mathrm{SL}_r(A)}(\omega_3) = 3[A].$$

Therefore, $\exp A = 3$, and by Proposition 3 $d = 3$. The index of G is ${}^1E^{16}_{6,\,2}$ in this case.

In the case $m = 3$ $G' \simeq \mathrm{SL}_1(A) \times \mathrm{SL}_1(B)$, where A and B are Azumaya algebras over R with $\deg A = \operatorname{ind} A = 2$ and $\deg B = 5$. The Cartan matrix shows that $\alpha_m = 2\omega_3 - \omega_1 - \omega_4$, so

$$0 = \beta_{G'}(\alpha'_m) = -\beta_{\mathrm{SL}_1(A)}(\omega_1) - \beta_{\mathrm{SL}_1(B)}(\omega_2) = -[A] - 2[B].$$

But by Proposition 3 $5[B] = 0$ and $2[A] = 0$; hence $[A] = 0$, a contradiction.

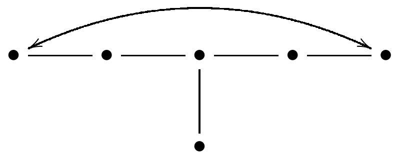
In the case $m = 4$ $G' \simeq \mathrm{SL}_1(A) \times \mathrm{SL}_1(B) \times \mathrm{SL}_1(C)$, where A, B, C are Azumaya algebras over R with $\deg A = \deg B = 3$, $\deg C = \operatorname{ind} C = 2$. The Cartan matrix shows that $\alpha_m = 2\omega_4 - \omega_2 - \omega_3 - \omega_5$, so

$$0 = \beta_{G'}(\alpha'_m) = \beta_{\mathrm{SL}_1(A)}(\omega_1) - \beta_{\mathrm{SL}_1(B)}(\omega_1) - \beta_{\mathrm{SL}_1(C)}(\omega_1) = [A] - [B] - [C].$$

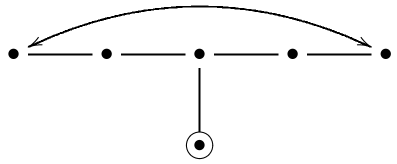
But by Proposition 3 $3[A] = 3[B] = 0$ and $2[C] = 0$; hence $[C] = 0$, a contradiction. □

Theorem 3 (^{2}E$_6$). *Every simple simply connected group G of type ${}^2E_6$ over R has one of the following Tits indices:*

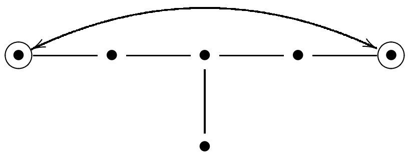
(${}^2E^{78}_{6,\,0}$)



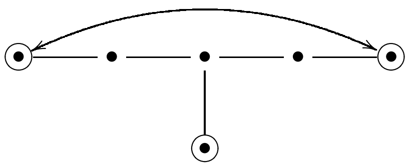
(${}^2E^{35}_{6,\,1}$)



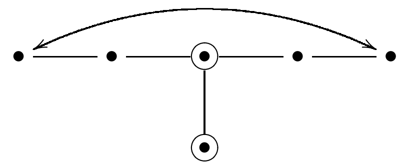
(${}^2E^{29}_{6,\,1}$)



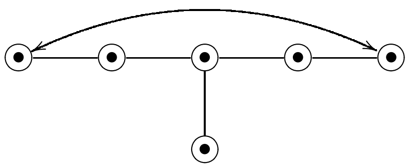
(${}^2E^{16'}_{6,\,2}$)



(${}^2E^{16''}_{6,\,2}$)



(${}^2E^{2}_{6,\,4}$)



Isomorphism classes of groups of a given index bijectively correspond to isomorphism classes of:

- *simple simply connected anisotropic groups H of type ${}^2A_5$ over R with $\beta_H(\omega_3)=0$, in the case of ${}^2E^{35}_{6,1}$;*
- *simple simply connected anisotropic groups H of type ${}^2D_4$ over R with $\beta_{H_O}(\omega_3)=0$, O is the orbit corresponding to $\{3,4\}$, in the case of ${}^2E^{29}_{6,1}$;*
- *simple simply connected anisotropic groups H of type ${}^2A_3$ over R with $\beta_H(\omega_2)=0$ and $\beta_{H_O}(\omega_1)=0$, O is the orbit corresponding to $\{1,3\}$, in the case of ${}^2E^{16'}_{6,2}$;*
- *Azumaya algebras A over a connected quadratic étale extension R'/R with* $\operatorname{ind} A = \deg A = 3$ *and* $\operatorname{cores}_{R'/R}([A])=0$*, in the case ${}^2E^{16''}_{6,2}$;*
- *connected quadratic étale extensions R'/R, in the case of ${}^2E^{2}_{6,4}$.*

Proof. Let $({}^2E_6,\, J)$ be the Tits index of G and m be the least element of J (in the case $J=\emptyset$ G is anisotropic, that is of index ${}^2E^{78}_{6,0}$). Denote the orbit corresponding to m by O, and the derived subgroup of the Levi part of the parabolic subgroup of type $D\setminus O$ by G'. By Theorem 2 we have $\beta_{G'_O}(\alpha'_O)=0$, and G is uniquely determined by G' satisfying this condition.

In the case $m=1$ O as a scheme is a connected quadratic étale extension R'/R, G' is a simple simply connected group of type ${}^2D_4$. The Cartan matrix shows that $\alpha_m=2\omega_1-\omega_3$, so

$$0=\beta_{G'_O}(\alpha'_O)=-\beta_{G'_O}(\omega_3).$$

By Theorem 3, case ${}^2D_n$, G' has one of the following indices: ${}^2D_{4,0}$, ${}^2D^{(1)}_{4,1}$, ${}^2D^{(1)}_{4,2}$, ${}^2D^{(2)}_{4,1}$, ${}^2D^{(1)}_{4,4}$.

In the case of ${}^2D_{4,0}$ G has the index ${}^2E^{29}_{6,1}$ and is uniquely determined by $H=G'$.

In the case of ${}^2D^{(1)}_{4,1}$ G has the index ${}^2E^{16'}_{6,2}$ and is uniquely determined by the a simple simply connected group H of type ${}^2A_3$ with $\beta_H(\omega_2)=0$ and $\beta_{H_O}(\omega_1)=\beta_{G'_O}(\omega_3)=0$.

In the cases of ${}^2D^{(1)}_{4,2}$ and ${}^2D^{(2)}_{4,1}$ G' is determined by an Azumaya algebra A over R' with $\deg A=\operatorname{ind} A=2$. But $[A]=\beta_{G'_O}(\omega_3)=0$, a contradiction.

In the case of ${}^2D^{(1)}_{4,4}$ G is quasi-split, that is of index ${}^2E^{2}_{6,4}$.

In the case $m=2$ O as a scheme is isomorphic to R, G' is a simple simply connected group of type ${}^2A_5$. The Cartan matrix shows that $\alpha_m=2\omega_2-\omega_4$, so

$$0=\beta_{G'_O}(\alpha'_O)=-\beta_{G'}(\omega_3).$$

By Theorem 3, case ${}^2D_n$, G' has one of the following indices: ${}^2A_{5,0}$, ${}^2A^{(1)}_{5,2}$, ${}^2A^{(1)}_{5,3}$.

In the case of ${}^2A_{5,0}$ G has the index ${}^2E^{35}_{6,1}$ and is uniquely determined by $H=G'$.

In the case of ${}^2A^{(1)}_{5,2}$ G' is determined by an Azumaya algebra A over R with $\deg A=\operatorname{ind} A=2$. But $[A]=\beta_{G'_O}(\omega_3)=0$, a contradiction.

In the case of ${}^2A^{(1)}_{5,3}$ G has the index ${}^2E^{16''}_{6,2}$ and is uniquely determined by an Azumaya algebra A over a connected quadratic étale extension R'/R with $\operatorname{ind} A=\deg A=3$ and $\operatorname{cores}_{R'/R}([A])=0$.

In the case $m=3$ O as a scheme is a connected quadratic étale extension R'/R, $G'\simeq R_{R'/R}(\mathrm{SL}_1(A))\times\mathrm{SL}_1(E)$, where A is an Azumaya algebra over R' with $\operatorname{ind} A=\deg A=2$, E is an Azumaya algebra over R with $\operatorname{ind} E=\deg E=3$. The Cartan matrix shows that $\alpha_m=2\omega_3-\omega_1-\omega_4$, so

$$0=\beta_{G'_O}(\alpha'_O)=-\beta_{\mathrm{SL}_1(A)}(\omega_1)-\beta_{\mathrm{SL}_1(E)_O}(\omega_1)=-[A]-[D_{R'}].$$

But by Proposition 3 $3[E]=0$ and $2[A]=0$, so $[A]=0$, a contradiction.

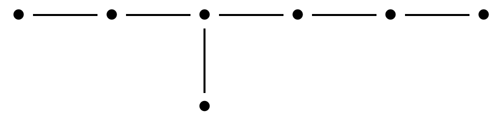
In the case $m=4$ O as a scheme is isomorphic to R, $G'\simeq R_{R'/R}(\mathrm{SL}_1(A))\times\mathrm{SL}_1(E)$, where A is an Azumaya algebra over R' with $\operatorname{ind} A=\deg A=3$, E is an Azumaya algebra over R with $\operatorname{ind} E=\deg E=2$. The Cartan matrix shows that $\alpha_m=2\omega_4-\omega_2-\omega_3-\omega_5$, so

$$0=\beta_{G'_O}(\alpha'_O)=\operatorname{cores}_{R'/R}(\beta_{\mathrm{SL}_1(A)}(\omega_1))-\beta_{\mathrm{SL}_1(E)}(\omega_1)=\operatorname{cores}_{R'/R}([A])-[E].$$

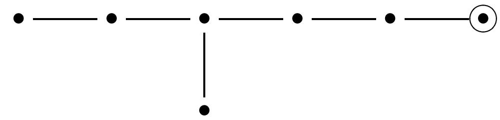
But by Proposition 3 $3[A]=0$ and $2[E]=0$, so $[E]=0$, a contradiction. ☐

Theorem 3 ($\mathbf{E_7}$). *Every simple simply connected group G of type E_7 over R has one of the following Tits indices:*

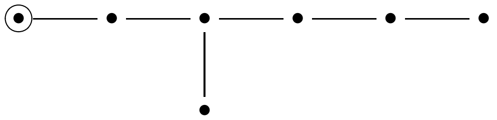
($E^{133}_{7,0}$)



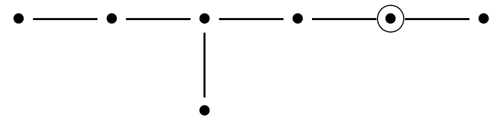
($E^{78}_{7,1}$)



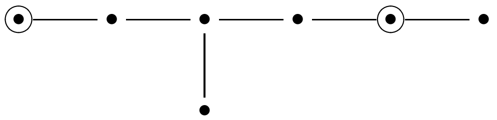
($E^{66}_{7,1}$)



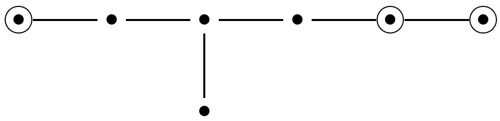
($E^{48}_{7,1}$)



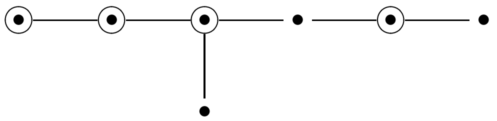
($E^{31}_{7,2}$)



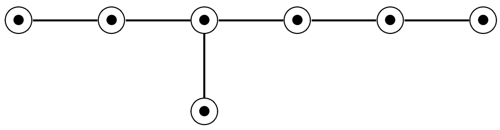
($E^{28}_{7,3}$)



($E^{9}_{7,4}$)



($E^{0}_{7,7}$)



Isomorphism classes of groups of a given index bijectively correspond to isomorphism classes of:

- *simple simply connected anisotropic groups H of type ${}^1E_6$ over R with $\beta_H = 0$, in the case of $E^{78}_{7,1}$;*
- *simple simply connected anisotropic groups H of type ${}^1D_6$ over R with $\beta_H(\omega_5) = 0$, in the case of $E^{66}_{7,1}$;*
- *simple simply connected anisotropic groups H of type ${}^1D_5$ over R with $\beta_H(\omega_4) = [E]$, $\operatorname{ind} E = 2$, in the case of $E^{48}_{7,1}$;*
- *simple simply connected anisotropic groups H of type ${}^1D_4$ over R with $\beta_H(\omega_1) = 0$ and $\beta_H(\omega_3) = [E]$, $\operatorname{ind} E = 2$, in the case of $E^{31}_{7,2}$;*

- *simple simply connected anisotropic groups H of type ${}^1D_4$ over R with $\beta_H = 0$, in the case of $E^{28}_{7,3}$;*
- *Azumaya algebras A over R with* $\operatorname{ind} A = \deg A = 2$*, in the case of $E^{9}_{7,4}$.*

The only group of index $E^{0}_{7,7}$ is split.

Proof. Let (E_7, J) be the Tits index of G and m be the least element of J (in the case $J = \emptyset$ G is anisotropic, that is of index $E^{133}_{7,0}$). Denote the derived subgroup of the Levi part of the parabolic subgroup of type $D \setminus \{m\}$ by G'. By Theorem 2 we have $\beta_{G'}(\alpha'_m) = 0$, and G is uniquely determined by G' satisfying this condition.

In the case $m = 1$ G' is a simple simply connected group of inner type D_6. The Cartan matrix shows that $\alpha_m = 2\omega_1 - \omega_3$, so

$$0 = \beta_{G'}(\alpha'_m) = \beta_{G'}(\omega_5).$$

By Theorem 3, case ${}^1D_n$, together with Proposition 6, G' has one of the following indices: ${}^1D_{6,0}$, ${}^1D^{(1)}_{6,2}$, ${}^1D^{(2)}_{6,1}$, ${}^1D^{(2)}_{6,2}$, ${}^1D^{(2)}_{6,3}$, ${}^1D^{(4)}_{6,1}$, ${}^1D^{(1)}_{6,6}$.

In the case of ${}^1D_{6,0}$ G has the index $E^{78}_{7,1}$ and is uniquely determined by $H = G'$.

In the case of ${}^1D^{(1)}_{6,2}$ G has the index $E^{28}_{7,3}$ and is uniquely determined by a simple simply connected group H of inner type D_4 with $\beta_H(\omega_1) = 0$ and $\beta_H(\omega_3) = \beta_{G'}(\omega_5) = 0$, that is $\beta_H = 0$.

In the case of ${}^1D^{(2)}_{6,1}$ G has the index $E^{48}_{7,1}$ and is uniquely determined by a simple simply connected group H of inner type D_4 with $\beta_H(\omega_1) = [E]$, $\operatorname{ind} E = 2$, and $\beta_H(\omega_3) = \beta_{G'}(\omega_5) = 0$. We may change enumeration of weights in such a way that $\beta_H(\omega_3) = [E]$ and $\beta_H(\omega_1) = 0$.

In the cases of ${}^1D^{(2)}_{6,2}$ and ${}^1D^{(4)}_{6,1}$ G' is determined by Azumaya algebras A_1 and A_2 over R with $\operatorname{ind} A_1 = \deg A_1 = \operatorname{ind} A_2 = \deg A_2 = 2$. But $[A_1] = \beta_{G'}(\omega_5) = 0$, a contradiction.

In the case of ${}^1D^{(2)}_{6,3}$ G' is determined by an Azumaya algebra A over R with $\operatorname{ind} A = \deg A = 2$. There are two possibilities: $J = \{1, 3, 4, 6\}$ and $J = \{1, 2, 4, 6\}$. In the first case G has the index $E^{9}_{7,4}$. In the second case $[A] = \beta_{G'}(\omega_5) = 0$, a contradiction.

In the case of ${}^1D^{(1)}_{6,6}$ G is split, that is of index $E^{0}_{7,7}$.

In the case $m = 2$ $G' \simeq \mathrm{SL}_1(A)$, where A is an Azumaya algebra over R with $\deg A = 7$. The Cartan matrix shows that $\alpha_m = 2\omega_2 - \omega_4$, so

$$0 = \beta_{G'}(\alpha'_m) = -\beta_{\mathrm{SL}_1(A)}(\omega_3) = -3[A].$$

But by Proposition 3 $7[A] = 0$, hence $[A] = 0$, a contradiction.

In the case $m = 3$ $G' \simeq \mathrm{SL}_1(A) \times \mathrm{SL}_1(B)$, where A and B are Azumaya algebras over R with $\deg A = \operatorname{ind} A = 2$ and $\deg B = 6$. The Cartan matrix shows that $\alpha_m = 2\omega_3 - \omega_1 - \omega_4$, so

$$0 = \beta_{G'}(\alpha'_m) = -\beta_{\mathrm{SL}_1(A)}(\omega_1) - \beta_{\mathrm{SL}_1(B)}(\omega_2) = -[A] - 2[B].$$

But by Proposition 3 $6[B] = 0$ and $2[A] = 0$; hence $[A] = 0$, a contradiction.

In the case $m = 4$ $G' \simeq \mathrm{SL}_1(A) \times \mathrm{SL}_1(B) \times \mathrm{SL}_1(C)$, where A, B, C are Azumaya algebras over R with $\deg A = \operatorname{ind} A = 3$, $\deg B = 4$, $\deg C = \operatorname{ind} C = 2$. The Cartan matrix shows that $\alpha_m = 2\omega_4 - \omega_2 - \omega_3 - \omega_5$, so

$$0 = \beta_{G'}(\alpha'_m) = \beta_{\mathrm{SL}_1(A)}(\omega_1) - \beta_{\mathrm{SL}_1(B)}(\omega_1) - \beta_{\mathrm{SL}_1(C)}(\omega_1) = [A] - [B] - [C].$$

But by Proposition 3 $3[A] = 0$, $4[B] = 0$ and $2[C] = 0$; hence $[A] = 0$, a contradiction.

In the case $m = 5$ $G' \simeq \mathrm{SL}_1(A) \times \mathrm{SL}_1(B)$, where A and B are Azumaya algebras over R with $\deg A = \operatorname{ind} A = 5$ and $\deg B = 3$. The Cartan matrix shows that $\alpha_m = 2\omega_3 - \omega_1 - \omega_4$, so

$$0 = \beta_{G'}(\alpha'_m) = -\beta_{\mathrm{SL}_1(A)}(\omega_3) - \beta_{\mathrm{SL}_1(B)}(\omega_1) = -3[A] - [B].$$

But by Proposition 3 $3[B] = 0$ and $5[A] = 0$; hence $[A] = 0$, a contradiction.

In the case $m = 6$ $G' \simeq H \times \mathrm{SL}_1(E)$, where H is a simple simply connected anisotropic group of inner type D_5, E is an Azumaya algebra over R with $\deg E = 2$. The Cartan matrix shows that $\alpha_m = 2\omega_6 - \omega_5 - \omega_7$, so

$$0 = \beta_{G'}(\alpha'_m) = -\beta_H(\omega_4) - \beta_{\mathrm{SL}_1(E)}(\omega_1) = -\beta_H(\omega_4) - [E].$$

If $[E] = 0$ then H is isotropic by Proposition 6, a contradiction. Therefore, $\operatorname{ind} E = 2$. The index of G is $E^{48}_{7,1}$ in this case.

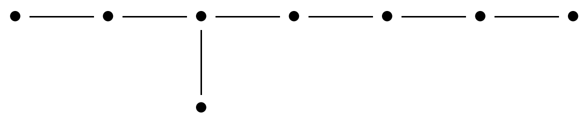
In the case $m = 7$ $G' = H$ is a simple simply connected anisotropic group of inner type E_6. The Cartan matrix shows that $\alpha_m = 2\omega_7 - \omega_6$, so

$$0 = \beta_{G'}(\alpha'_m) = -\beta_H(\omega_6),$$

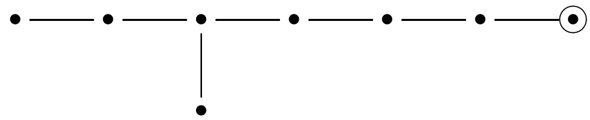
that is $\beta_H = 0$. The index of G is $E^{78}_{7,1}$ in this case. □

Theorem 3 ($\mathbf{E_8}$). *Every simple simply connected group G of type E_8 over R has one of the following Tits indices:*

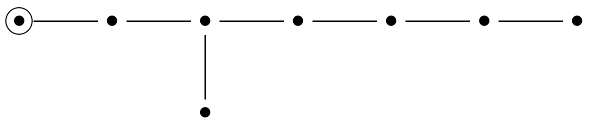
($E^{248}_{8,0}$)



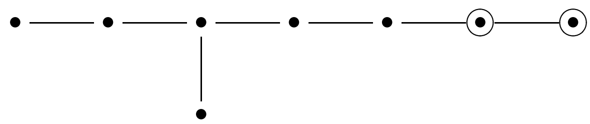
($E^{133}_{8,1}$)



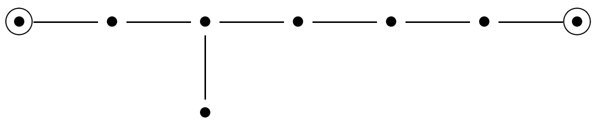
($E^{91}_{8,1}$)



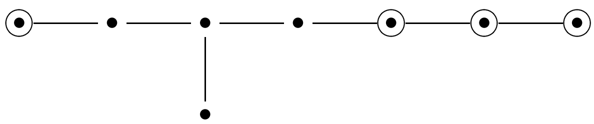
($E^{78}_{8,2}$)



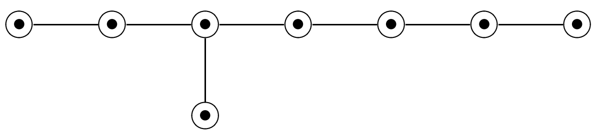
($E^{66}_{8,2}$)



($E^{28}_{8,4}$)



($E^{0}_{8,8}$)



Isomorphism classes of groups of a given index bijectively correspond to isomorphism classes of:

- *simple simply connected anisotropic groups H of type E_7 over R with $\beta_H = 0$, in the case of $E^{133}_{8,1}$;*
- *simple simply connected anisotropic groups H of type ${}^1D_7$ over R with $\beta_H = 0$, in the case of $E^{91}_{8,1}$;*

- *simple simply connected anisotropic groups H of type ${}^1E_6$ over R with $\beta_H = 0$, in the case of $E^{78}_{8,2}$;*
- *simple simply connected anisotropic groups H of type ${}^1D_6$ over R with $\beta_H = 0$, in the case of $E^{66}_{8,2}$;*
- *simple simply connected anisotropic groups H of type ${}^1D_4$ over R with $\beta_H = 0$, in the case of $E^{28}_{8,4}$.*

The only group of index $E^0_{8,8}$ is split.

Proof. Let $(E_8,\, J)$ be the Tits index of G and m be the least element of J (in the case $J = \emptyset$ G is anisotropic, that is of index $E^{248}_{8,0}$). Denote the derived subgroup of the Levi part of the parabolic subgroup of type $D \setminus \{m\}$ by G'. By Theorem 2 we have $\beta_{G'}(\alpha'_m) = 0$, and G is uniquely determined by G' satisfying this condition.

In the case $m = 1$ G' is a simple simply connected group of inner type D_7. The Cartan matrix shows that $\alpha_m = 2\omega_1 - \omega_3$, so

$$0 = \beta_{G'}(\alpha'_m) = \beta_{G'}(\omega_6),$$

that is $\beta_{G'} = 0$. By Theorem 3, case ${}^1D_n$, together with Proposition 6, G' has one of the following indices: ${}^1D_{7,0}$, ${}^1D^{(1)}_{7,1}$, ${}^1D^{(1)}_{7,3}$, ${}^1D^{(1)}_{7,7}$.

In the case of ${}^1D_{7,0}$ G has the index $E^{91}_{8,1}$ and is uniquely determined by $H = G'$.

In the case of ${}^1D^{(1)}_{7,1}$ G has the index $E^{66}_{8,2}$ and is uniquely determined by a simple simply connected group H of inner type D_6 with $\beta_H(\omega_1) = 0$ and $\beta_H(\omega_5) = \beta_{G'}(\omega_6) = 0$, that is $\beta_H = 0$.

In the case of ${}^1D^{(1)}_{7,3}$ G has the index $E^{28}_{8,4}$ and is uniquely determined by a simple simply connected group H of inner type D_4 with $\beta_H(\omega_1) = 0$ and $\beta_H(\omega_3) = \beta_{G'}(\omega_6) = 0$, that is $\beta_H = 0$.

In the case of ${}^1D^{(1)}_{7,7}$ G is split, that is of index $E^0_{8,8}$.

In the case $m = 2$ $G' \simeq \mathrm{SL}_1(A)$, where A is an Azumaya algebra over R with $\deg A = 8$. The Cartan matrix shows that $\alpha_m = 2\omega_2 - \omega_4$, so

$$0 = \beta_{G'}(\alpha'_m) = -\beta_{\mathrm{SL}_1(A)}(\omega_3) = -3[A].$$

But by Proposition 3 $8[A] = 0$, hence $[A] = 0$, a contradiction.

In the case $m = 3$ $G' \simeq \mathrm{SL}_1(A) \times \mathrm{SL}_1(B)$, where A and B are Azumaya algebras over R with $\deg A = \operatorname{ind} A = 2$ and $\deg B = 7$. The Cartan matrix shows that $\alpha_m = 2\omega_3 - \omega_1 - \omega_4$, so

$$0 = \beta_{G'}(\alpha'_m) = -\beta_{\mathrm{SL}_1(A)}(\omega_1) - \beta_{\mathrm{SL}_1(B)}(\omega_2) = -[A] - 2[B].$$

But by Proposition 3 $7[B] = 0$ and $2[A] = 0$; hence $[A] = 0$, a contradiction.

In the case $m = 4$ $G' \simeq \mathrm{SL}_1(A) \times \mathrm{SL}_1(B) \times \mathrm{SL}_1(C)$, where A, B, C are Azumaya algebras over R with $\deg A = \operatorname{ind} A = 3$, $\deg B = 5$, $\deg C = \operatorname{ind} C = 2$. The Cartan matrix shows that $\alpha_m = 2\omega_4 - \omega_2 - \omega_3 - \omega_5$, so

$$0 = \beta_{G'}(\alpha'_m) = \beta_{\mathrm{SL}_1(A)}(\omega_1) - \beta_{\mathrm{SL}_1(B)}(\omega_1) - \beta_{\mathrm{SL}_1(C)}(\omega_1) = [A] - [B] - [C].$$

But by Proposition 3 $3[A] = 0$, $5[B] = 0$ and $2[C] = 0$; hence $3[B] + [C] = 0$ and $[C] = 0$, a contradiction.

In the case $m = 5$ $G' \simeq \mathrm{SL}_1(A) \times \mathrm{SL}_1(B)$, where A and B are Azumaya algebras over R with $\deg A = \operatorname{ind} A = 5$ and $\deg B = 4$. The Cartan matrix shows that $\alpha_m = 2\omega_3 - \omega_1 - \omega_4$, so

$$0 = \beta_{G'}(\alpha'_m) = -\beta_{\mathrm{SL}_1(A)}(\omega_3) - \beta_{\mathrm{SL}_1(B)}(\omega_1) = -3[A] - [B].$$

But by Proposition 3 $4[B] = 0$ and $5[A] = 0$; hence $[A] = 0$, a contradiction.

In the case $m = 6$ $G' \simeq H \times \mathrm{SL}_1(E)$, where H is a simple simply connected anisotropic group of inner type D_5, E is an Azumaya algebra over R with $\deg E = 3$. The Cartan matrix shows that $\alpha_m = 2\omega_6 - \omega_5 - \omega_7$, so

$$0 = \beta_{G'}(\alpha'_m) = -\beta_H(\omega_4) - \beta_{\mathrm{SL}_1(E)}(\omega_1) = -\beta_H(\omega_4) - [E].$$

But by Proposition 3 $3[E]=0$, hence $\beta_H(\omega_4)=0$. Therefore, H is isotropic by Proposition 6, a contradiction.

In the case $m=7$ $G'\simeq H\times \mathrm{SL}_1(E)$, where H is a simple simply connected anisotropic group of inner type E_6, E is an Azumaya algebra over R with $\deg A=2$. The Cartan matrix shows that $\alpha_m=2\omega_7-\omega_6-\omega_8$, so

$$0=\beta_{G'}(\alpha'_m)=-\beta_H(\omega_6)-\beta_{\mathrm{SL}_1(E)}(\omega_1)=-\beta_H(\omega_6)-[E].$$

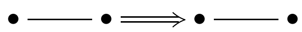
But by Proposition 3 $2[E]=0$, hence $\beta_H=0$ and $[E]=0$. The index of G is $E^{78}_{8,\,2}$ in this case.

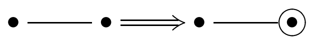
In the case $m=8$ $G'=H$ is a simple simply connected anisotropic group of type E_7. The Cartan matrix shows that $\alpha_m=2\omega_8-\omega_7$, so

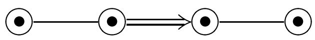
$$0=\beta_{G'}(\alpha'_m)=-\beta_H(\omega_7),$$

that is $\beta_H=0$. The index of G is $E^{133}_{8,\,1}$ in this case. □

Theorem 3 ($\mathbf{F_4}$). *Every simple simply connected group G of type F_4 over R has one of the following Tits indices:*

($F^{52}_{4,\,0}$) 

($F^{21}_{4,\,1}$) 

($F^{0}_{4,\,4}$) 

Isomorphism classes of groups of index $F^{21}_{4,\,1}$ bijectively correspond to isomorphism classes of simple simply connected anisotropic groups H of type B_3 over R with $\beta_H=0$. The only group of index $F^{0}_{4,\,4}$ is split.

Proof. Let $(F_4,\,J)$ be the Tits index of G and m be the least element of J (in the case $J=\emptyset$ G is anisotropic, that is of index $F^{52}_{4,\,0}$). Denote the derived subgroup of the Levi part of the parabolic subgroup of type $D\setminus\{m\}$ by G'. By Theorem 2 we have $\beta_{G'}(\alpha'_m)=0$, and G is uniquely determined by G' satisfying this condition.

In the case $m=1$ G' is a simple simply connected group of type C_3. The Cartan matrix shows that $\alpha_1=2\omega_1-\omega_2$, so

$$0=\beta_{G'}(\alpha'_m)=-\beta_{G'}(\omega_1).$$

By Proposition 4 G' is split and so is G. The index of G is $F^{0}_{4,\,4}$ in this case.

In the case $m=2$ $G'\simeq \mathrm{SL}_1(A)\times\mathrm{SL}_1(B)$, where A and B are Azumaya algebras over R with $\deg A=\operatorname{ind}A=2$ and $\deg B=3$. The Cartan matrix shows that $\alpha_m=2\omega_2-\omega_1-2\omega_3$, so

$$0=\beta_{G'}(\alpha'_m)=-\beta_{\mathrm{SL}_1(A)}(\omega_1)-2\beta_{\mathrm{SL}_1(B)}(\omega_1)=-[A]-2[B].$$

But by Proposition 3 $3[B]=0$ and $2[A]=0$; hence $[A]=0$, a contradiction.

In the case $m=2$ $G'\simeq \mathrm{SL}_1(A)\times\mathrm{SL}_1(B)$, where A and B are Azumaya algebras over R with $\deg A=\operatorname{ind}A=3$ and $\deg B=2$. The Cartan matrix shows that $\alpha_m=2\omega_3-\omega_2-\omega_4$, so

$$0=\beta_{G'}(\alpha'_m)=\beta_{\mathrm{SL}_1(A)}(\omega_1)-\beta_{\mathrm{SL}_1(B)}(\omega_1)=[A]-[B].$$

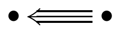
But by Proposition 3 $2[A]=0$ and $3[B]=0$; hence $[A]=0$, a contradiction.

In the case $m=4$ $G'=H$ is a simple simply connected group of type B_3. The Cartan matrix shows that $\alpha_1=2\omega_4-\omega_3$, so

$$0=\beta_{G'}(\alpha'_m)=-\beta_H(\omega_3),$$

that is $\beta_H=0$. The index of G is $F^{21}_{4,\,1}$ in this case. □

Theorem 3 ($\mathbf{G_2}$)**.** *Every simple simply connected group G of type G_2 over R has one of the following Tits indices:*

$(G^{14}_{2,0})$ 

$(G^{0}_{2,2})$ 

The only group of index $G^{0}_{2,2}$ is split.

Proof. Let $(G_2,\, J)$ be the Tits index of G and m be the least element of J (in the case $J = \emptyset$ G is anisotropic, that is of index $G^{14}_{2,0}$). Denote the derived subgroup of the Levi part of the parabolic subgroup of type $D \setminus \{m\}$ by G'. By Theorem 2 we have $\beta_{G'}(\alpha'_m) = 0$.

In the case $m = 1$ $G' \simeq \mathrm{SL}_1(A)$, where A is an Azumaya algebra over R with $\deg A = 2$. The Cartan matrix shows that $\alpha_m = 2\omega_1 - \omega_2$, so

$$0 = \beta_{G'}(\alpha'_m) = -\beta_{\mathrm{SL}_1(A)}(\omega_1) = -[A].$$

Therefore, G' is split and so is G. The index of G is $G^{0}_{2,2}$ in this case.

In the case $m = 1$ $G' \simeq \mathrm{SL}_1(A)$, where A is an Azumaya algebra over R with $\operatorname{ind} A = \deg A = 2$. The Cartan matrix shows that $\alpha_m = 2\omega_2 - 3\omega_1$, so

$$0 = \beta_{G'}(\alpha'_m) = -3\beta_{\mathrm{SL}_1(A)}(\omega_1) = -3[A].$$

But by Proposition 3 $2[A] = 0$, hence $[A] = 0$, a contradiction. □

References

[B] N. Bourbaki, *Groupes et algèbres de Lie. Chapitres 4–6*, Hermann, Paris, 1968.

[SGA] M. Demazure, A. Grothendieck, *Schémas en groupes*, Lecture Notes in Mathematics, Vol. 151–153, Springer-Verlag, Berlin-Heidelberg-New York, 1970.

[Gab] O. Gabber, *Some theorems on Azumaya algebras*, Lecture Notes in Mathematics, Vol. 844, Springer, Berlin-New York, 1981, 129–209.

[Jan] J.C. Jantzen, *Darstellungen halbeinfacher Gruppen und kontravariante Formen*, J. Reine Angew. Math. **290** (1977), 117–141.

[K] M.-A. Knus, *Quadratic and Hermitian forms over rings*, Springer-Verlag, Berlin, 1991.

[Ti66] J. Tits, *Classification of algebraic semisimple groups*, Algebraic groups and discontinuous subgroups, Proc. Sympos. Pure Math. **9**, Amer. Math. Soc., Providence RI, 1966, 33–62.

[Ti71] J. Tits. *Représentations linéares irréductibles d'un groupe réductif sur un corps quelconque*, J. Reine Angew. Math. **247** (1971), 196–220.

[Ti90] J. Tits. *Strongly inner anisotropic forms of simple algebraic groups*, J. of Algebra **131** (1990), 648–677.

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