WITT GROUPS OF GRASSMANN VARIETIES

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Abstract. We compute the total Witt groups of (split) Grassmann varieties, over any regular base X. The answer is a free module over the total Witt ring of X. We provide an explicit basis for this free module, which is indexed by a special class of Young diagrams, that we call *even* Young diagrams.

Contents

Introduction

At first glance, it might be surprising for the non-specialist that more than thirty years after the definition of the Witt group of a scheme, by Knebusch [13], the Witt group of such a classical variety as a Grassmannian has not been computed yet. This is especially striking since analogous results for ordinary cohomologies, for K-theory and for Chow groups have been settled a long time ago. The goal of this article is to explain how Witt groups differ from these sister theories and to prove the following:

Main Theorem (See Thm. 6.1)**.** *Let X be a regular noetherian and separated scheme over $\mathbb{Z}[\frac{1}{2}]$, of finite Krull dimension. Let $0 < d < n$ be integers and let $\mathrm{Gr}_X(d,n)$ be the Grassmannian of d-dimensional subbundles of the trivial n-dimensional vector bundle $\mathcal{V} = \mathcal{O}_X^n$ over X. (More generally, we treat any vector bundle $\mathcal{V}$ admitting a complete flag of subbundles.)*

Then the total Witt group of $\mathrm{Gr}_X(d,n)$ is a free graded module over the total Witt group of X with an explicit basis indexed by so-called "even" Young diagrams. The basis element corresponding to an even Young diagram is essentially the push-forward of the unit along the inclusion of the corresponding Schubert variety. The cardinal of this basis equals $2 \cdot \dfrac{(d'+e')!}{d'! \cdot e'!}$ where $d' = \big[\frac{d}{2}\big]$ and $e' = \big[\frac{n-d}{2}\big]$.

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Let us explain this statement. The *total* Witt group refers to the sum of the Witt groups $\mathrm{W}^i(X,L)$

$$\mathrm{W}^{\mathrm{tot}}(X) = \bigoplus_{\substack{i\in\mathbb{Z}/4\\ L\in \mathrm{Pic}(X)/2}} \mathrm{W}^i(X,L)$$

for all possible shifts $i\in\mathbb{Z}/4$ and all possible twists $L\in\mathrm{Pic}(X)/2$ in the duality, see Appendix A. For $X=\mathrm{Spec}(F)$, the spectrum of a field, the total Witt group boils down to the classical Witt group $\mathrm{W}(F)$ but even in that case the above Theorem is new and the total Witt group of $\mathrm{Gr}_F(d,n)$ involves non-trivial shifted and twisted Witt groups. The result has a very round form when stated for total Witt groups but the classical unshifted Witt groups $\mathrm{W}^0(X,L)$ can be isolated, as well as the unshifted and untwisted Witt group $\mathrm{W}(X)=\mathrm{W}^0(X,\mathcal{O}_X)$. Indeed, the announced basis consists of homogeneous elements and we describe below how to read their explicit shifts and twists on the Young diagrams. For instance, it is worth noting that there are no new interesting antisymmetric forms in the Witt groups of $\mathrm{Gr}_X(d,n)$, that is, except for those extended from X, see Corollary 6.6.

To describe our basis explicitly, we need to introduce *even* Young diagrams. We first consider ordinary Young diagrams sitting in the upper left corner of a rectangle with d rows and e columns, which we call the *frame* of the diagram. See Figure 1.

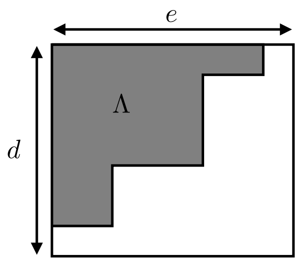


FIGURE 1. Young diagram Λ in $(d\times e)$-frame

We say that such a framed Young diagram Λ is *even* if all the segments of the boundary of Λ which are strictly inside the frame have even length. That is, we allow Λ to have odd-length segments on its boundary only where it touches the outside frame. See Figure 2 for examples. (In Figures 11, 12 and 13 we further give all even diagram in $(d\times e)$-frame for $(d,e)=(4,4)$, $(4,5)$ and $(5,5)$, respectively.)

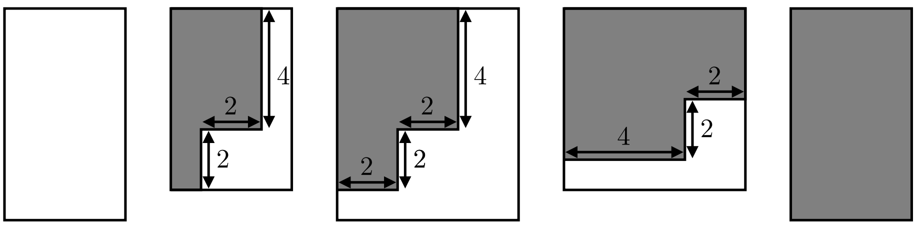


FIGURE 2. Five examples of even Young diagrams

We shall see that basis elements are in bijective correspondence with even Young diagrams in $(d\times e)$-frame, for $e:=n-d$. Moreover, as explained in Section 3, the Witt class $\phi_{d,e}(\Lambda)$ corresponding to such a diagram Λ lives in the Witt group $\mathrm{W}^i\big(\mathrm{Gr}_X(d,n),L\big)$ for the shift $i=|\Lambda|\in\mathbb{Z}/4$ equal to the *area* of the diagram and for the twist $L \in \mathrm{Pic}(\mathrm{Gr}_X(d,n))/2$ equal to the class of $t(\Lambda)\cdot\Delta$ where $\Delta \in \mathrm{Pic}(\mathrm{Gr}_X(d,n))$ is the determinant of the tautological bundle and where the integer $t(\Lambda)$ is half the *perimeter* of the diagram Λ (see Figure 8 in Section 3). More generally, when $\mathcal{V}$ is not free but admits a complete flag of subbundles, the twist of $\phi_{d,e}(\Lambda)$ also involves a multiple of the determinant of $\mathcal{V}$, in the direct summand of $\mathrm{Pic}(\mathrm{Gr}_X(d,n))/2$ coming from $\mathrm{Pic}(X)/2$.

Let us put our result in perspective. In its modern form, see for instance Laksov [14], the computation of the cohomology (or K-theory, or Chow groups) of a cellular variety usually relies on four important ingredients, (a)-(d) below. Let $Z \subset V$ be a closed subscheme with open complement U (think of U as an open cell of V which is an $\mathbb{A}^*$-bundle over some smaller scheme). These results are:

(a) A long exact sequence of localization relating the cohomology of V, the cohomology of V with support in Z and the cohomology of U.
(b) A dévissage theorem, which identifies the cohomology of Z with the cohomology of V with support in Z.
(c) Homotopy invariance, that is the fact that the cohomology of the total space of an $\mathbb{A}^*$-bundle is the same as the cohomology of the base, which helps identifying the cohomology of U with something simpler.
(d) A way of splitting the localization long exact sequence for particular cellular decompositions, usually proving the connecting homomorphism to vanish.

Looking back, it is now less surprising that Witt groups of Grassmann varieties could not be computed in the 70's, because most of these tools have only been constructed recently for Witt groups of schemes. With the notable exception of projective space over a field, see Arason [2], most of the recent computations for projective varieties have awaited those tools to be available.

The long exact sequence could only be established by defining first "higher" or "shifted" Witt groups, as it was done in [3] and [4] in 2000 by the first author using the framework of triangulated categories. This enabled Walter [19] to compute the Witt groups of projective bundles by decomposing their derived categories. The homotopy invariance was proved by Gille [10]. The dévissage theorem is usually a consequence of the definition of push-forwards and was obtained in the form we need here by Hornbostel and the second author in [7], although several forms of dévissage were previously considered by various authors. Finally, point (d) was investigated by the authors in [6] where it is shown that, under some geometric assumptions, the connecting homomorphism can be computed by geometric means, using only push-forwards and pull-backs. This is the part where Witt groups differ in the most significant way from the sister theories mentioned above. Unlike the case of oriented cohomology theories in the sense of Levine-Morel [15] or Panin [17], the connecting homomorphism of point (d) is in general *not* zero for Witt groups.

We use all the above techniques here, in the case of Grassmann varieties and their usual cellular decompositions.

Let us now comment on the organization of the paper. Sections 1 and 2 contain preparatory material on Grassmann varieties, desingularizations of Schubert varieties and even Young diagrams.

Our generators of $\mathrm{W}^{\mathrm{tot}}\big(\mathrm{Gr}_X(d,n)\big)$ are defined in Section 3 as push-forwards of the unit forms of certain desingularized Schubert varieties. The reader should keep in mind that pushing the unit form is not always possible, due to the presence of line bundles in the definition of the push-forward. Indeed, for a proper morphism $f : X \to Y$ of equidimension $\dim(f)$, between regular noetherian schemes (think $\dim f = \dim X - \dim Y$), the push-forward along f is defined between the following

Witt groups:

$$\mathrm{W}^{i+\dim f}(X,\omega_f\otimes f^*L)\longrightarrow \mathrm{W}^i(Y,L)\,,\tag{1}$$

where the line bundle ω_f is the so-called *relative line bundle*. So, when $\mathcal{O}_X$ is not (up to squares) of the form $\omega_f\otimes f^*L$ for some line bundle L over Y, one *cannot* push-forward the unit form of X, which lives in $\mathrm{W}^0(X,\mathcal{O}_X)$. This is why we start Section 3 by discussing the "parity" of the relevant canonical bundles ω_f. Although somewhat heavy, these computations are elementary and are all based on a repeated application of the computation of the relative canonical bundle of a Grassmann bundle (Prop. 1.5). The condition for a Young diagram to be even implies the existence of such a push-forward for the unit of the desingularized Schubert cell into the Grassmannian. In fact, we could push-forward the unit forms for more Schubert cells but these additional generators would be redundant. The even Young diagrams are chosen so that the corresponding forms are also linearly independent.

Then, in Section 4, we recall the classical relative cellular structure of the Grassmann varieties and in Section 5 the long exact sequence associated to it. In the final Section 6, we compute how our candidate-generators behave under the morphisms in the long exact sequence, especially under the connecting homomorphism, which is most of the time not zero (Cor. 6.7). The proof of the main theorem (Thm. 6.1) then follows by induction on the rank of the vector bundle $\mathcal{V}$.

This article is written in the language of "functors of points", which means that we describe schemes in terms of their points (which are here flags) and morphisms of schemes as how they act on those points. This method is completely rigorous in this case. The original source is [8, § I.1] and we also refer the reader to [12, Part 2] for general considerations on this subject. This language is customary when dealing with flag varieties, see for instance [14] in which it is used for the computation of Chow groups of Grassmann varieties.

Convention. Throughout the paper, X stands for a regular noetherian separated scheme over $\mathbb{Z}[\frac{1}{2}]$, of finite Krull dimension.

1. Combinatorics of Grassmann and flag varieties

We recall elementary facts about Grassmann varieties and desingularizations of Schubert cells. We also provide the necessary material about canonical bundles to treat the push-forward homomorphisms for Witt groups in Section 3.

1.1. **Definition.** A *subbundle* $\mathcal{P}\lhd\mathcal{V}$ of a vector bundle $\mathcal{V}$ over a scheme X is an $\mathcal{O}_X$-submodule which is locally a direct summand, *i.e.* $\mathcal{P}$ and $\mathcal{V}/\mathcal{P}$ are vector bundles.

1.2. **Definition.** Let $\mathcal{V}$ be a vector bundle of rank $n>0$ over a scheme X and let d be an integer $0\leqslant d\leqslant n$. We denote by $\mathrm{Gr}_X(d,\mathcal{V})$ the Grassmann bundle over X parameterizing the subbundles of rank d of $\mathcal{V}$. In the language of functors of points, it means that for any morphism $f:\mathrm{Spec}(R)\to X$, the set $\mathrm{Gr}_X(d,\mathcal{V})(R)$ consists of the R-submodules $P\lhd\mathcal{V}(R)=f^*(\mathcal{V})$ which are direct summands of rank d.

The scheme $\mathrm{Gr}_X(d,\mathcal{V})$ comes equipped with a smooth structural morphism $\pi:\mathrm{Gr}_X(d,\mathcal{V})\to X$ and a tautological bundle $\mathcal{T}_d=\mathcal{T}_d^{\mathrm{Gr}_X(d,\mathcal{V})}$ of rank d, whose determinant we denote by Δ_d.

1.3. **Proposition.** *The scheme $\mathrm{Gr}_X(d,\mathcal{V})$ is smooth over X of relative dimension $d(n-d)$. For $0<d<n$, the Picard group of $\mathrm{Gr}_X(d,\mathcal{V})$ is given by*

$$\begin{array}{ccc}\mathrm{Pic}(X)\oplus\mathbb{Z} & \cong & \mathrm{Pic}(\mathrm{Gr}_X(d,\mathcal{V}))\\ (\ell,m) & \mapsto & \pi^*(\ell)+m\,\Delta_d.\end{array}$$

In case $d = 0$ or $d = n$, the morphism $\pi : \mathrm{Gr}_X(d, \mathcal{V}) \to X$ is the identity.

Proof. The Picard group of a regular scheme coincides with its Chow group CH^1, which is computed in [14] for Grassmannians: see Theorem 16 for the case where X is a field, and §13 to work over a regular base X. Using the Plücker embedding, one checks that the generator in *loc. cit.* is indeed Δ_d. □

1.4. **Corollary.** *We have* $\mathrm{Pic}\left(\mathrm{Gr}_X(d, \mathcal{V})\right)/2 \ = \ \mathrm{Pic}(X)/2 \ \oplus \ \mathbb{Z}/2 \cdot \Delta_d$. □

1.5. **Proposition.** *The relative canonical bundle $\omega_{\mathrm{Gr}_X(d,\mathcal{V})/X}$ of the projection $\pi : \mathrm{Gr}_X(d, \mathcal{V}) \to X$ is $\omega_{\mathrm{Gr}_X(d,\mathcal{V})/X} = -d \det \mathcal{V} + n\Delta_d$ in $\mathrm{Pic}(\mathrm{Gr}_X(d, \mathcal{V}))$. In particular, if $\mathcal{V} = \mathcal{O}_X^n$ is trivial, $\omega_{\mathrm{Gr}_X(d,\mathcal{V})/X} = n\,\Delta_d$.*

Proof. The morphism π is smooth, so $\omega_{\mathrm{Gr}_X(d,\mathcal{V})/X}$ is the determinant (highest exterior power) of the relative cotangent bundle of π. This cotangent bundle is the tautological bundle tensored by the dual of the tautological quotient bundle (see [9, Appendix B.5.8]). Taking the determinant, we get the result. □

* * *

We now extend the previous results from Grassmannians to some flag varieties.

1.6. **Definition.** Let $k \geqslant 1$ and $(\underline{d}, \underline{e})$ be a pair of k-tuples of non-negative integers $\underline{d} = (d_1, \dots, d_k)$ and $\underline{e} = (e_1, \dots, e_k)$ satisfying

$$0 < d_1 < \cdots < d_k \qquad \text{and} \qquad e_1 + d_1 \leqslant \cdots \leqslant e_k + d_k\,. \tag{2}$$

(The second condition holds in particular if we have $e_1 \leqslant \cdots \leqslant e_k$.) Consider a flag

$$\mathcal{V}_{d_1+e_1} \lhd \cdots \lhd \mathcal{V}_{d_i+e_i} \lhd \cdots \lhd \mathcal{V}_{d_k+e_k} \tag{3}$$

of vector bundles over X, where $\lhd$ indicates subbundles in the strong sense of Definition 1.1 and where the rank is given by the index: $\mathrm{rk}_X(\mathcal{V}_r) = r$.

We associate to this data the scheme $\mathcal{F}l_X(\underline{d}, \underline{e}, \mathcal{V}_\bullet)$ over X, which parameterizes the flags of vector bundles $\mathcal{P}_{d_1} \lhd \mathcal{P}_{d_2} \lhd \cdots \lhd \mathcal{P}_{d_k}$ such that $\mathrm{rk}\,\mathcal{P}_{d_j} = d_j$ and $\mathcal{P}_{d_j} \lhd \mathcal{V}_{d_j+e_j}$. As a functor of points, this gives for any morphism $f : Y \to X$

$$\mathcal{F}l_X(\underline{d}, \underline{e}, \mathcal{V}_\bullet)(Y) := \left\{ \begin{array}{ccccccccc} 0 & \lhd & \mathcal{P}_{d_1} & \lhd & \mathcal{P}_{d_2} & \lhd & \cdots & \lhd & \mathcal{P}_{d_k} \\ & & \triangle\, e_1 & & \triangle\, e_2 & & & & \triangle\, e_k \\ 0 & \lhd & f^*\mathcal{V}_{d_1+e_1} & \lhd & f^*\mathcal{V}_{d_2+e_2} & \lhd & \cdots & \lhd & f^*\mathcal{V}_{d_k+e_k} \end{array} \right\}, \tag{4}$$

where all $\mathcal{P}_{d_i}$ are vector bundles over Y of rank d_i such that all inclusions are subbundles in the sense of Definition 1.1. The integers along inclusions indicate codimensions. Following general practice, we shall drop the mention of f^* in the sequel. Moreover, to avoid cumbersome notations, unless the original flag (3) varies, we drop the mention of $\mathcal{V}_\bullet$ from the notation: $\mathcal{F}l_X(\underline{d}, \underline{e}) = \mathcal{F}l_X(\underline{d}, \underline{e}, \mathcal{V}_\bullet)$.

1.7. **Example.** For $k = 1$, the scheme $\mathcal{F}l_X(\underline{d}, \underline{e})$ is simply $\mathrm{Gr}_X(d_1, \mathcal{V}_{d_1+e_1})$.

1.8. **Remark.** For any choice J of k' indices among $\{1, \dots, k\}$, one can consider the pair of k'-tuples $(\underline{d}', \underline{e}')$ obtained from $(\underline{d}, \underline{e})$ by keeping d_i and e_i only for indices $i \in J$. There is a natural morphism $\mathcal{F}l_X(\underline{d}, \underline{e}) \to \mathcal{F}l_X(\underline{d}', \underline{e}')$ over X, obtained by dropping the $\mathcal{P}_{d_j}$ for the non-chosen indices j.

Furthermore, for any vector bundle $\mathcal{V}$ such that $\mathcal{V}_{d_k+e_k} \lhd \mathcal{V}$, there is a natural morphism $f_{\underline{d},\underline{e},\mathcal{V}}$ of schemes over X as follows:

$$f_{\underline{d},\underline{e},\mathcal{V}} : \quad \begin{array}{ccccc} \mathcal{F}l_X(\underline{d}, \underline{e}) & \longrightarrow & \mathrm{Gr}_X(d, \mathcal{V}_{d_k+e_k}) & \hookrightarrow & \mathrm{Gr}_X(d, \mathcal{V}) \\ (\mathcal{P}_{d_1}, \dots, \mathcal{P}_{d_k}) & \longmapsto & \mathcal{P}_{d_k} & \longmapsto & \mathcal{P}_{d_k}\,, \end{array} \tag{5}$$

where the first morphism is as above and the second is a closed immersion.

1.9. **Definition.** The scheme $\mathcal{F}l_X(\underline{d},\underline{e})$ is equipped with *tautological bundles* $\mathcal{T}_{d_i}$, $1 \leqslant i \leqslant k$, of rank d_i, whose determinants we denote by $\Delta_{d_i} := \det(\mathcal{T}_{d_i})$. The stalk of $\mathcal{T}_{d_i}$ at a point $(\mathcal{P}_{d_1}, \ldots, \mathcal{P}_{d_k})$ is $\mathcal{P}_{d_i}$. In ambiguous cases, the full notation for $\mathcal{T}_{d_i}$ would be $\mathcal{T}_{d_i}^{\mathcal{F}l_X(\underline{d},\underline{e},\mathcal{V}_\bullet)}$.

1.10. **Remark.** If $e_i = 0$ then the vector bundles $\mathcal{T}_{d_i} = \mathcal{V}_{d_i}$ and $\Delta_{d_i} = \det \mathcal{V}_{d_i}$ are both extended from X.

1.11. **Lemma.** *Let $k \geqslant 2$ and let $(\underline{d},\underline{e})$ be a pair of k-tuples satisfying (2). Let $\mathcal{V}_\bullet$ be a flag as in (3). Define the $(k-1)$-tuples $\underline{d}_{|k-1}$ and $\underline{e}_{|k-1}$ as the restrictions of $\underline{d}$ and $\underline{e}$ to the first $k-1$ entries. Consider the scheme*

$$Y := \mathcal{F}l_X(\,\underline{d}_{|k-1}\,,\,\underline{e}_{|k-1}\,,\,\mathcal{V}_\bullet\,)\,,$$

which only "uses" the first $k-1$ bundles $\mathcal{V}_{d_1+e_1} \lhd \cdots \lhd \mathcal{V}_{d_{k-1}+e_{k-1}}$. Consider the pull-back to Y of the remaining bundle, still denoted $\mathcal{V}_{d_k+e_k}$. Observe that $\mathcal{T}^Y_{d_{k-1}} \lhd \mathcal{V}_{d_k+e_k}$ and consider the quotient bundle

$$\tilde{\mathcal{V}} := \mathcal{V}_{d_k+e_k}/\mathcal{T}^Y_{d_{k-1}}$$

over Y. It has rank $d_k - d_{k-1} + e_k$. We then have a canonical isomorphism of schemes over Y (hence over X):

$$\mathcal{F}l_X(\underline{d},\underline{e},\mathcal{V}_\bullet) \quad\cong\quad \mathrm{Gr}_Y\left(d_k - d_{k-1}, \tilde{\mathcal{V}}\right). \tag{6}$$

Under this identification, we have $\mathcal{T}_{d_i}^{\mathcal{F}l(\underline{d},\underline{e},\mathcal{V}_\bullet)} = \mathcal{T}^Y_{d_i}$ for all $1 \leqslant i \leqslant k-1$ and

$$\mathcal{T}_{d_k}^{\mathcal{F}l_Y(\underline{d},\underline{e},\mathcal{V}_\bullet)}/\mathcal{T}_{d_{k-1}}^{\mathcal{F}l_Y(\underline{d},\underline{e},\mathcal{V}_\bullet)} = \mathcal{T}_{d_k-d_{k-1}}^{\mathrm{Gr}_Y(d_k-d_{k-1},\tilde{\mathcal{V}})}\,. \tag{7}$$

Proof. This simply amounts to the bijective correspondence between a flag $\mathcal{P}_{d_1} \lhd \cdots \lhd \mathcal{P}_{d_{k-1}} \lhd \mathcal{P}_{d_k}$ satisfying $\mathcal{P}_{d_i} \lhd \mathcal{V}_{d_i+e_i}$ for all $1 \leqslant i \leqslant k$ and the following data:

(a) the beginning of this flag $\mathcal{P}_{d_1} \lhd \cdots \lhd \mathcal{P}_{d_{k-1}}$ satisfying $\mathcal{P}_{d_i} \lhd \mathcal{V}_{d_i+e_i}$ for all $1 \leqslant i \leqslant k-1$,

(b) the bundle $\mathcal{P}_{d_k}$ such that $\mathcal{P}_{d_{k-1}} \lhd \mathcal{P}_{d_k} \lhd \mathcal{V}_{d_k+e_k}$

and to observe that (b) is equivalent to a subbundle $\tilde{\mathcal{P}} \lhd \mathcal{V}_{d_k+e_k}/\mathcal{P}_{d_{k-1}}$ of rank $d_k - d_{k-1}$, where $\tilde{\mathcal{P}} := \mathcal{P}_{d_k}/\mathcal{P}_{d_{k-1}}$. Details are left to the reader. □

1.12. **Convention.** When using k-tuples $\underline{d} = (d_1, \ldots, d_k)$, it will unify several formulas to simply define $d_0 = 0$.

1.13. **Proposition.** *Let $\underline{d}$ and $\underline{e}$ be two k-tuples as in (2) and $\mathcal{V}_\bullet$ be a flag as in (3). Then $\mathcal{F}l_X(\underline{d},\underline{e})$ is smooth over X of relative dimension $\sum_{i=1}^k (d_i - d_{i-1})\,e_i$. The Picard group of $\mathcal{F}l_X(\underline{d},\underline{e})$ is generated by $\mathrm{Pic}(X)$ and the "new" bundles Δ_{d_i}:*

$$\mathrm{Pic}(\mathcal{F}l_X(\underline{d},\underline{e})) \cong \mathrm{Pic}(X) \;\oplus \bigoplus_{\substack{1 \leqslant i \leqslant k \\ s.t.\ e_i \neq 0}} \mathbb{Z}\,\Delta_{d_i}\,. \tag{8}$$

The relative canonical bundle $\omega_{\mathcal{F}l_X(\underline{d},\underline{e})/X}$ is given by the formula

$$\begin{aligned} \omega_{\mathcal{F}l_X(\underline{d},\underline{e})/X} = \; & \sum_{i=1}^{k} (-d_i + d_{i-1})\; \det \mathcal{V}_{d_i+e_i} + \\ & + \sum_{i=1}^{k-1} (d_i - d_{i-1} + e_i - e_{i+1})\,\Delta_{d_i} + (d_k - d_{k-1} + e_k)\,\Delta_{d_k}\,, \end{aligned} \tag{9}$$

where $\Delta_{d_i} = \det \mathcal{V}_{d_i}$ if $e_i = 0$ by Remark 1.10 and where we use Convention 1.12. In particular, for $k = 1$, this reads $\omega_{\mathcal{F}l_X(\underline{d},\underline{e})/X} = -d_1\,\det \mathcal{V}_{d_1+e_1} + (d_1 + e_1)\,\Delta_{d_1}$.

Proof. By induction on k. The case $k = 1$ is that of a Grassmannian over X (Example 1.7) so the result follows from Propositions 1.3 and 1.5.

Let now $k \geqslant 2$. Consider $Y = \mathcal{F}l_X(\underline{d}|_{k-1}, \underline{e}|_{k-1}, \mathcal{V}_\bullet)$ and the bundle $\tilde{\mathcal{V}} = \mathcal{V}_{d_k+e_k}/\mathcal{T}_{d_{k-1}}$ over Y, as in Lemma 1.11. Recall that $\mathrm{rk}_Y(\tilde{\mathcal{V}}) = d_k - d_{k-1} + e_k$, which is always strictly positive ($d_k > d_{k-1}$) and which is bigger than or equal to $d_k - d_{k-1}$ with equality if and only if $e_k = 0$. Equation (6) and Propositions 1.3 and 1.5 immediately give smoothness, the formula for the relative dimension and that for the Picard group (8). Finally, to prove (9), observe that

$$\begin{aligned} \omega_{\mathcal{F}l_X(\underline{d},\underline{e})/Y} &= \omega_{\mathrm{Gr}_Y(d_k-d_{k-1},\tilde{\mathcal{V}})\,/\,Y} \\ &= (-d_k + d_{k-1})\det(\tilde{\mathcal{V}}) + \mathrm{rk}(\tilde{\mathcal{V}})\,\Delta^{\mathrm{Gr}_Y(d_k-d_{k-1},\tilde{\mathcal{V}})}_{d_k-d_{k-1}} \\ &= (-d_k + d_{k-1})\det(\tilde{\mathcal{V}}) + \mathrm{rk}(\tilde{\mathcal{V}})\,\big(\Delta^{\mathcal{F}l_X(\underline{d},\underline{e})}_{d_k} - \Delta^{\mathcal{F}l_X(\underline{d},\underline{e})}_{d_{k-1}}\big) \\ &= (-d_k + d_{k-1})\det\mathcal{V}_{d_k+e_k} - e_k\,\Delta_{d_{k-1}} + (d_k - d_{k-1} + e_k)\,\Delta_{d_k}\,. \end{aligned}$$

The first equality uses (6), the second comes from Proposition 1.5 and the third from (7). The last equality is a direct computation (in which we drop the mention of $\mathcal{F}l_X(\underline{d},\underline{e})$ for readability). By induction hypothesis, we get $\omega_{Y/X}$ from Equation (9) applied for $k-1$, that is, for the flag variety Y. Since $\omega_{\mathcal{F}l_X(\underline{d},\underline{e})/X} = \omega_{\mathcal{F}l_X(\underline{d},\underline{e})/Y} + \omega_{Y/X}$ over $\mathcal{F}l_X(\underline{d},\underline{e})$, we get (9) for k by adding the above. □

1.14. **Corollary.** *Let $\underline{d}$ and $\underline{e}$ be two k-tuples as in (2) and $\mathcal{V}_\bullet$ be a flag as in (3). Let $\mathcal{V}$ be a vector bundle of rank $d+e$ such that $\mathcal{V}_{d_k+e_k} \lhd \mathcal{V}$. The relative canonical bundle for the morphism $f_{\underline{d},\underline{e},\mathcal{V}} : \mathcal{F}l_X(\underline{d},\underline{e}) \to \mathrm{Gr}_X(d,\mathcal{V})$ of (5) is given by*

$$(10)\quad \begin{aligned} \omega_{\mathcal{F}l_X(\underline{d},\underline{e})/\,\mathrm{Gr}_X(d,\mathcal{V})} \;=\; & \sum_{i=1}^{k}(-d_i + d_{i-1})\,\det\mathcal{V}_{d_i+e_i} + d_k\,\det\mathcal{V} \,+ \\ & + \sum_{i=1}^{k-1}\;(d_i - d_{i-1} + e_i - e_{i+1})\,\Delta_{d_i} + (-d_{k-1} + e_k - e)\,\Delta_{d_k}\,, \end{aligned}$$

where $\Delta_{d_i} = \det\mathcal{V}_{d_i}$ if $e_i = 0$ by Remark 1.10 and where we use Convention 1.12. For $k = 1$, this reads $\omega_{\mathcal{F}l_X(\underline{d},\underline{e})/\,\mathrm{Gr}_X(d,\mathcal{V})} = d_1\,\det(\mathcal{V}/\mathcal{V}_{d_1+e_1}) + (e_1 - e)\,\Delta_{d_1}$.

Proof. Subtract $(f_{\underline{d},\underline{e},\mathcal{V}})^*\,\omega_{\mathrm{Gr}_X(d,\mathcal{V})/X} = -d_k\,\det\mathcal{V} + (d_k + e)\,\Delta_{d_k}$ (Prop. 1.5) from the bundle $\omega_{\mathcal{F}l_X(\underline{d},\underline{e})/X}$ given in (9). □

1.15. **Remark.** When $\mathcal{V}_\bullet = \mathcal{O}_X^\bullet$, all the formulas are simpler, since all the $\det\mathcal{V}_i$ are zero. This applies in particular when $X = \mathrm{Spec}(R)$ for a local ring R .

2. Even Young diagrams

We introduce *even* Young diagrams that will parameterize the basis of the total Witt group of the Grassmann variety, to be constructed in Section 3.

2.1. **Definition.** Let $d, e \geqslant 1$. We shall call *Young diagram in* $(d\times e)$*-frame*, or simply (d,e)*-diagram*, any d-tuple $\Lambda = (\Lambda_1, \Lambda_2, \ldots, \Lambda_d)$ of integers such that:

$$e \geqslant \Lambda_1 \geqslant \Lambda_2 \geqslant \ldots \geqslant \Lambda_d \geqslant 0.$$

See Figure 1 in the Introduction. The *area* of Λ is $|\Lambda| = \Lambda_1 + \Lambda_2 + \ldots + \Lambda_d$. These (d,e)-diagrams are just ordinary Young diagrams displayed in the upper left corner of a rectangle with d rows and e columns, possibly leaving empty rows below and empty columns to the right of the Young diagram. So, an ordinary Young diagram with ρ rows and γ columns defines a (d,e)-diagram for any $d \geqslant \rho$ and $e \geqslant \gamma$.

2.2. **Example.** We denote by $\varnothing$ the empty diagram $\varnothing = (0,\ldots,0)$. We denote by $[d\times e]$ the full $d\times e$-rectangle $[d\times e] = (e,\ldots,e) \in \mathbb{N}^d$.

2.3. **Definition.** Let $d, e \geqslant 1$ and let Λ be a Young diagram in $(d \times e)$-frame. The decreasing sequence $\Lambda_1 \geqslant \Lambda_2 \geqslant \ldots \geqslant \Lambda_d$ can be written in a unique way as a series of equalities and strict inequalities:

$$(11) \quad \underbrace{\Lambda_1 = \cdots = \Lambda_{d_1}}_{d_1 \text{ terms}} > \underbrace{\Lambda_{d_1+1} = \ldots = \Lambda_{d_2}}_{d_2 - d_1 \text{ terms}} > \cdots > \underbrace{\Lambda_{d_{k-1}+1} = \cdots = \Lambda_{d_k}}_{d_k - d_{k-1} \text{ terms}} = \Lambda_d \,.$$

Note that $d_k = d$. The integers $k \geqslant 1$ and $0 < d_1 < \ldots < d_k$ depend on Λ. If we need to stress this we shall write $k = k(\Lambda)$ and $d_i = d_i(\Lambda)$ for $1 \leqslant i \leqslant k(\Lambda)$.

For fixed d and e, there is a bijection (pictured on Figure 3) between the Young diagrams Λ in $(d \times e)$-frame and pairs of k-tuples of integers

$$(12) \qquad \begin{array}{lll} \underline{d} = (d_1, \ldots, d_k) & \text{such that} & 0 < d_1 < \cdots < d_k = d \\ \underline{e} = (e_1, \ldots, e_k) & \text{such that} & 0 \leqslant e_1 < \cdots < e_k \leqslant e \end{array}$$

with $1 \leqslant k \leqslant d$. The integers $k = k(\Lambda)$ and $d_i = d_i(\Lambda)$ are the above ones and we set $e_i = e_i(\Lambda) := e - \Lambda_{d_i}$ for all $i = 1, \ldots, k$. The converse construction is obvious.

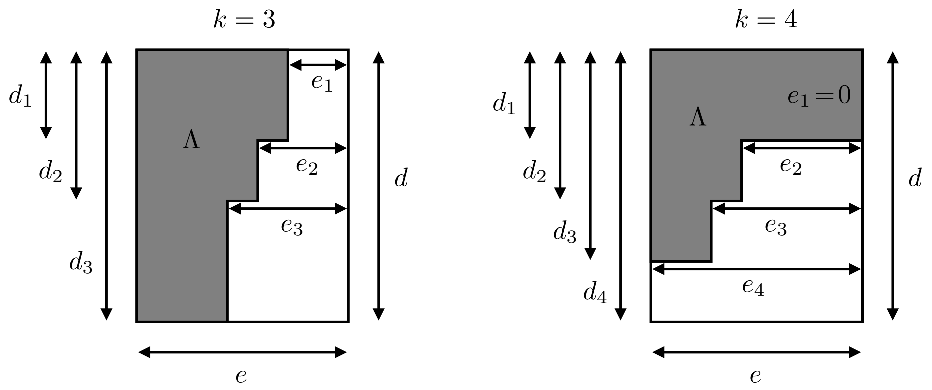


FIGURE 3. Two examples of the two k-tuples $(d_1, \ldots, d_k)$ and $(e_1, \ldots, e_k)$ corresponding to a Young diagram Λ in $(d \times e)$-frame.

2.4. **Definition.** Let $d, e \geqslant 1$. Fix a complete flag of vector bundles over X

$$(13) \qquad 0 = \mathcal{V}_0 \lhd \mathcal{V}_1 \lhd \cdots \lhd \mathcal{V}_i \lhd \cdots \lhd \mathcal{V}_{d+e} =: \mathcal{V} \,.$$

Note that we baptize $\mathcal{V}$ the bundle of dimension $d + e$, to lighten notation.

Let Λ be a Young diagram in $(d \times e)$-frame. By Definition 2.3, this amounts to a pair $(\underline{d}, \underline{e})$ of k-tuples of integers satisfying (12), and hence satisfying (2). We can now apply Definition 1.6 to $\underline{d}$ and $\underline{e}$ and the flag (3) taken from (13) above:

(14)

$$\mathcal{F}l_X(d, e, \mathcal{V}_\bullet; \Lambda) := \mathcal{F}l_X(\underline{d}, \underline{e}, \mathcal{V}_\bullet) = \left\{ \begin{array}{ccccccccc} 0 & \lhd & \mathcal{P}_{d_1} & \lhd & \mathcal{P}_{d_2} & \lhd & \cdots & \lhd & \mathcal{P}_{d_k} \\ & & \triangle\, e_1 & & \triangle\, e_2 & & & & \triangle\, e_k \\ 0 & \lhd & \mathcal{V}_{d_1+e_1} & \lhd & \mathcal{V}_{d_2+e_2} & \lhd & \cdots & \lhd & \mathcal{V}_{d_k+e_k} \end{array} \right\} .$$

As usual, instead of $\mathcal{F}l_X(d, e, \mathcal{V}_\bullet; \Lambda)$, we might simply write $\mathcal{F}l(\Lambda)$ or anything "in between" depending on what is obvious from the context.

As in (5), there is a natural morphism f_Λ from $\mathcal{F}l_X(d, e; \Lambda)$ to $\mathrm{Gr}(d, \mathcal{V})$

$$(15) \qquad f_\Lambda = f_{d,e;\Lambda} := f_{\underline{d},\underline{e},\mathcal{V}} : \quad \begin{array}{ccc} \mathcal{F}l_X(d, e; \Lambda) & \longrightarrow & \mathrm{Gr}(d, \mathcal{V}) \\ (\mathcal{P}_{d_1}, \ldots, \mathcal{P}_{d_k}) & \longmapsto & \mathcal{P}_{d_k} \,. \end{array}$$

When $X = \mathrm{Spec}(F)$ is a field, one can understand the image of f_Λ as the subset of those subspaces $\mathcal{P}_d \lhd \mathcal{V}$ whose intersection with each $\mathcal{V}_{d_i+e_i}$ is of dimension at least d_i. This is the classical Schubert cell associated to the diagram Λ. It is pretty clear that f_Λ is a birational morphism. The advantage of $\mathcal{F}l_X(d,e;\Lambda)$ over the Schubert cell is that $\mathcal{F}l_X(d,e;\Lambda)$ is not singular by Proposition 1.13.

2.5. **Example.** Following up on Example 1.7, when $\Lambda=\varnothing$ is the empty diagram, that is for $k=1$ and $e_1=e$, we have $\mathcal{F}l_X(\varnothing)=\mathrm{Gr}_X(d,\mathcal{V})$ and $f_\varnothing$ is the identity. At the other end, for $\Lambda=[d\times e]$ the whole $(d\times e)$-rectangle, that is for $k=1$ and $e_1=0$, we have $\mathcal{F}l_X(d,e;\Lambda)=\mathrm{Gr}_X(d,\mathcal{V}_d)=X$ and f_Λ is a closed immersion.

2.6. **Definition.** Let Λ be a Young diagram in $(d\times e)$-frame. We define $\rho(\Lambda)\in\{0,\ldots,d\}$ to be the number of non-zero rows of Λ. Complementarily, we define $\zeta(\Lambda)=d-\rho(\Lambda)$ to be the number of zero rows at the end of Λ, that is

$$\begin{array}{llll}\rho(\Lambda)=d & \text{and} & \zeta(\Lambda)=0 & \text{if } \Lambda_d>0\,,\\ \rho(\Lambda)=d_{k-1} & \text{and} & \zeta(\Lambda)=d-d_{k-1} & \text{if } \Lambda_d=0\,.\end{array}$$

For the empty diagram, we have $\rho(\varnothing)=0$ and $\zeta(\varnothing)=d$.

We are going to use a certain class of (d,e)-diagrams, that we call the *even* (d,e)-diagrams. Defining them by a picture is very easy. The condition to be even is that any segment of the (d,e)-diagram which does not belong to the outer $(d\times e)$-frame must have even length. See Figure 2. The formal definition is the following.

2.7. **Definition.** Let Λ be a Young diagram in $(d\times e)$-frame and let $\underline{d}$ and $\underline{e}$ be the associated k-tuples as in Definition 2.3. We say that Λ is *even* if all the following conditions are satisfied:

- (i) $d_{i+1}-d_i$ is even for all $i=1,\ldots,k-2$ (for $k\geqslant 3$ otherwise no condition),
- (ii) $e_{i+1}-e_i$ is even, for all $i=1,\ldots,k-1$ (for $k\geqslant 2$),
- (iii) when $0<e_1<e$ we also require d_1 to be even,
- (iv) when $0<e_k<e$ we also require d_k-d_{k-1} to be even.

2.8. **Example.** For any $d,e\geqslant 1$, both the empty diagram $\varnothing$ and the full-rectangle $[d\times e]$ are even (d,e)-diagrams (see Ex. 2.2). Indeed, in both cases, $k=1$ and $\underline{d}=(d)$, whereas $\underline{e}(\varnothing)=(e)$ and $\underline{e}([d\times e])=(0)$; so there is no condition to check.

When $d=1$ or $e=1$, these are the only even Young diagram in $(d\times e)$-frame.

For more examples, the reader can find all even Young diagrams in the cases $(d,e)=(4,4)$, $(4,5)$ and $(5,5)$ in Figures 11, 12 and 13, at the end of the paper.

2.9. **Remark.** Definition 2.7 depends on d and e as well as on the Young diagram Λ. For an even (d,e)-diagram Λ to remain even in a bigger frame, we might have one or two more conditions to check, namely (iii) or (iv) in Definition 2.7, in the case where Λ was touching the right border or the bottom border of its $(d\times e)$-frame.

2.10. **Remark.** For each even (d,e)-diagram we will construct a basis element in the total Witt group $\mathrm{W}^{\mathrm{tot}}\big(\mathrm{Gr}_X(d,\mathcal{V})\big)$ of the Grassmannian. The proof that these Witt classes actually form a basis will proceed by induction on $d+e=\mathrm{rk}(\mathcal{V})$, using the long exact sequence of localization associated to a natural "cellular" decomposition of the Grassmannians. In that proof, we shall need the description in terms of Young diagrams of the various Witt group homomorphisms appearing in that long exact sequence. As we shall see, these are the ones of the next propositions. This explains why the following constructions are relevant here.

2.11. **Proposition.** *Let $d,e\geqslant 2$.*

(a) *There is a bijection*

$$\left\{\begin{array}{c}\textit{even Young } (d,e-1)\textit{-diagrams}\\ \Lambda' \textit{ such that } \zeta(\Lambda') \textit{ is even}\end{array}\right\}\overset{\simeq}{\longleftrightarrow}\left\{\begin{array}{c}\textit{even Young } (d,e)\textit{-diagrams}\\ \Lambda \textit{ such that } \Lambda_d>0\end{array}\right\}$$

$$\begin{array}{ccc}\Lambda' & \longmapsto & (\Lambda'_1+1,\ldots,\Lambda'_d+1)\\ (\Lambda_1-1,\ldots,\Lambda_d-1) & \longleftarrow\!\shortmid & \Lambda\end{array}$$

(b) *There is a bijection*

$$\left\{\begin{array}{c}\textit{even Young } (d,e)\textit{-diagrams}\\ \Lambda \textit{ such that } \Lambda_d=0\end{array}\right\} \overset{\simeq}{\longleftrightarrow} \left\{\begin{array}{c}\textit{even Young } (d-1,e)\textit{-diagrams}\\ \Lambda'' \textit{ such that } \Lambda''_{d-1} \textit{ is even}\end{array}\right\}$$

$$\Lambda \longmapsto \Lambda_{|_{d-1,e}}$$

$$(\Lambda''_1,\ldots,\Lambda''_{d-1},0) \longleftarrow\!\shortmid\ \Lambda''$$

(c) *There is a bijection*

$$\left\{\begin{array}{c}\textit{even Young } (d-1,e)\textit{-diagrams}\\ \Lambda'' \textit{ such that } \Lambda''_{d-1} \textit{ is odd}\end{array}\right\} \overset{\simeq}{\longleftrightarrow} \left\{\begin{array}{c}\textit{even Young } (d,e-1)\textit{-diagrams}\\ \Lambda' \textit{ such that } \zeta(\Lambda') \textit{ is odd}\end{array}\right\}$$

$$\Lambda'' \longmapsto (\Lambda''_1-1,\ldots,\Lambda''_{d-1}-1,0)$$

$$(1+\Lambda'_1,\ldots,1+\Lambda'_{d-1}) \longleftarrow\!\shortmid\ \Lambda'$$

Proof. The proof essentially consists in checking that the announced constructions are well-defined and that they preserve even diagrams. Checking that they are mutually inverse constructions is straightforward. The notation $\Lambda_{|_{d-1,e}}$ is the obvious one: we view a diagram with empty last row in a smaller frame. All this is most easily performed and followed on pictures. For instance, the maps from left to right are pictured in the upper parts of Figures 4, 5 and 6 below. □

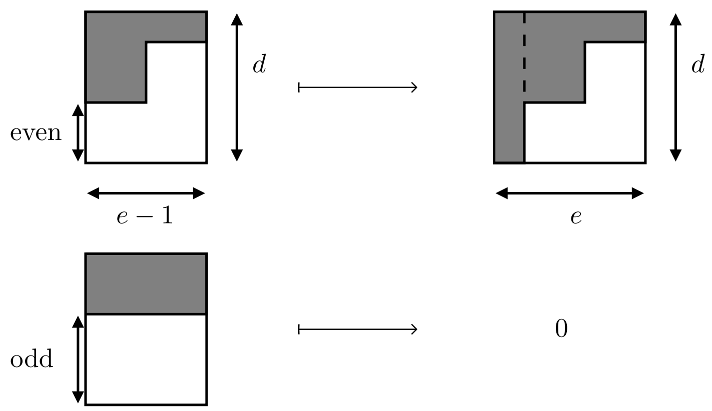


FIGURE 4. Morphism $\bar{\iota}$ (and later ι_*) on various $(d,e-1)$-diagrams Λ'.

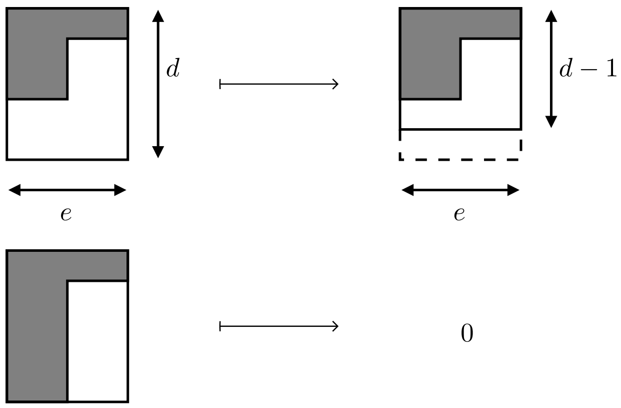


FIGURE 5. Morphism $\bar{\kappa}$ (and later κ) on various (d,e)-diagrams Λ.

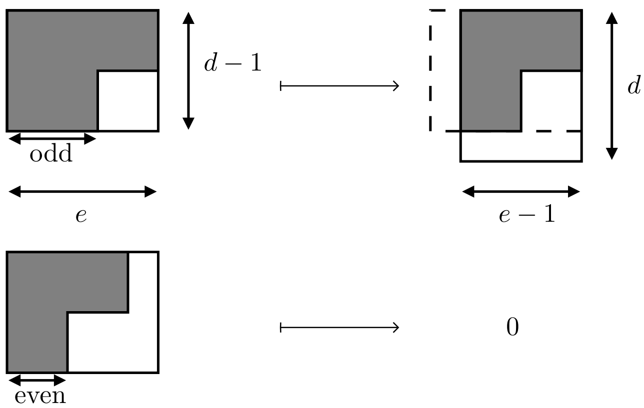


FIGURE 6. Morphism $\bar{\partial}$ (and later ∂) on various $(d-1,e)$-diagrams Λ''.

2.12. **Corollary.** *Let R be a ring, for instance $R=\mathbb{Z}$ or later $R=\mathrm{W}^{\mathrm{tot}}(X)$. For each $d,e\in\mathbb{N}$ let $\mathbb{F}(d,e)$ be the free R-module on the set of even (d,e)-diagrams. Then for any $d,e\geqslant 1$, the following cyclic sequence is exact everywhere*

(16)
$$\mathbb{F}(d,e-1)\xrightarrow{\ \bar{\iota}\ }\mathbb{F}(d,e)\xrightarrow{\ \bar{\kappa}\ }\mathbb{F}(d-1,e)\xrightarrow{\ \bar{\partial}\ }\mathbb{F}(d,e-1)$$

where the homomorphisms $\bar{\iota}$, $\bar{\kappa}$ and $\bar{\partial}$ are defined on the basis elements by the constructions of Proposition 2.11, whenever they make sense, and by zero otherwise. See Figures 4, 5 and 6. More precisely, we have:

The homomorphism $\bar{\iota}:\mathbb{F}(d,e-1)\to\mathbb{F}(d,e)$ is defined on the basis by

(17)
$$\bar{\iota}(\Lambda')=\begin{cases}(\Lambda'_1+1,\ldots,\Lambda'_d+1) & \text{if } \zeta(\Lambda') \text{ is even}\\ 0 & \text{if } \zeta(\Lambda') \text{ is odd.}\end{cases}$$

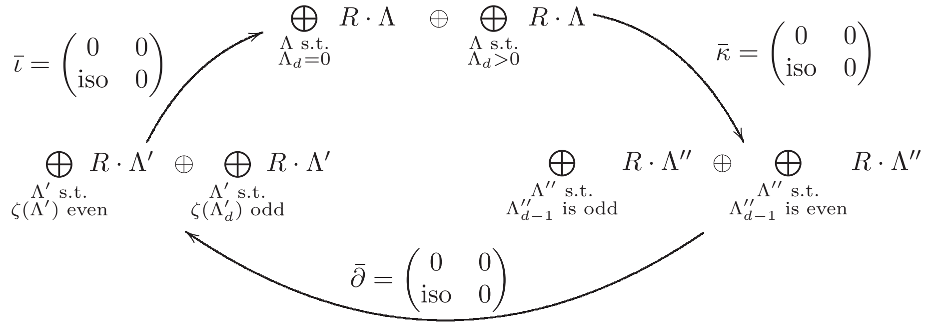
The homomorphism $\bar{\kappa}:\mathbb{F}(d,e)\to\mathbb{F}(d-1,e)$ is defined on the basis by

(18)
$$\bar{\kappa}(\Lambda)=\begin{cases}\Lambda_{|_{d-1,e}} & \text{if } \Lambda_d=0\\ 0 & \text{if } \Lambda_d>0\,.\end{cases}$$

The homomorphism $\bar\partial : \mathbb{F}(d-1,e) \to \mathbb{F}(d,e-1)$ *is defined on the basis by*

$$\bar\partial(\Lambda'') = \begin{cases} (\Lambda''_1 - 1, \ldots, \Lambda''_{d-1} - 1, 0) & \text{if } \Lambda''_{d-1} \text{ is odd} \\ 0 & \text{if } \Lambda''_{d-1} \text{ is even.} \end{cases} \tag{19}$$

Proof. Indeed, the sequence (16) is even "split" exact as follows:



where "iso" indicates an isomorphism induced by a bijection of the corresponding basis elements, according to Proposition 2.11. ☐

3. Generators of the total Witt group

For this section, let Λ be a Young diagram in $(d\times e)$-frame and recall the k-tuples $\underline{d}$ and $\underline{e}$ associated to Λ in Definition 2.3.

3.1. **Remark.** Our goal is to construct classes in the total Witt group of $\mathrm{Gr}_X(d,\mathcal{V})$ by pushing-forward the unit form $1 \in \mathrm{W}^0(\mathcal{F}l_X(\Lambda))$ along the morphism $f_\Lambda : \mathcal{F}l_X(\Lambda) \to \mathrm{Gr}_X(d,\mathcal{V})$ of (15). As recalled in (1), this push-forward only exists conditionally, namely only when the class of $\omega_{\mathcal{F}l_X(\Lambda)/\mathrm{Gr}_X(d,\mathcal{V})}$ in $\mathrm{Pic}(\mathcal{F}l_X(\Lambda))/2$ belongs to the image of

$$(f_\Lambda)^* \ : \ \mathrm{Pic}(\mathrm{Gr}_X(d,\mathcal{V}))/2 \longrightarrow \mathrm{Pic}(\mathcal{F}l_X(\Lambda))/2\,.$$

This is true if and only if the following conditions are satisfied:

(a) $d_i - d_{i-1} + e_{i+1} - e_i$ is even for every $i = 2, \ldots, k-1$ (for $k \geqslant 3$)
(b) when $0 < e_1 < e$, require moreover $d_1 + e_2 - e_1$ even.

We shall be more precise in Proposition 3.8 below but the reader can verify our claim using (10) in Corollary 1.14. In that expression, every term coming from X will also come from $\mathrm{Gr}_X(d,\mathcal{V})$ since f_Λ is a morphism over X. This also applies to $\Delta_{d_1} = \det \mathcal{V}_{d_1}$ in case $e_1 = 0$. Finally, the term in Δ_d also comes from $\mathrm{Gr}_X(d,\mathcal{V})$ since $(f_\Lambda)^*\big(\Delta_d^{\mathrm{Gr}_X(d,\mathcal{V})}\big) = \Delta_d^{\mathcal{F}l_X(\Lambda)}$, as can be checked on the tautological bundles.

Conditions (a) and (b) hold in particular when Λ is *even* in the sense of Definition 2.7. Indeed, for such Λ not only the sum $d_i - d_{i-1} + e_{i+1} - e_i$ is even but actually each term $d_i - d_{i-1}$ and $e_{i+1} - e_i$ is. Compare Figure 7.

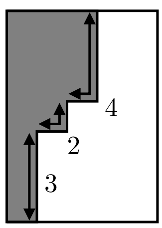


Figure 7. Framed Young diagram satisfying Conditions (a) and (b) of Remark 3.1 but which is not even (at all).

When (a) and (b) hold (*e.g.* for Λ even), there exists a line bundle L on $\mathrm{Gr}_X(d,\mathcal{V})$ such that $\omega_{f_\Lambda} + f_\Lambda^*(L) = 0$ in $\mathrm{Pic}(\mathcal{F}l_X(\Lambda))/2$. Therefore, the push-forward (1) applied to $f = f_\Lambda$ and $i = -\dim(f_\Lambda)$ yields a homomorphism:

$$\mathrm{W}^0\left(\mathcal{F}l(\Lambda), 0\right) = \mathrm{W}^0\left(\mathcal{F}l(\Lambda), \omega_{f_\Lambda} + f_\Lambda^*(L)\right)\right) \xrightarrow{(f_\Lambda)_*} \mathrm{W}^{|\Lambda|}\left(\mathrm{Gr}_X(d,\mathcal{V}), L\right).$$

(Use that $\dim(f_\Lambda) = \dim \mathcal{F}l_X(\Lambda) - \dim \mathrm{Gr}_X(d,\mathcal{V}) = -|\Lambda|$ by Propositions 1.3 and 1.13.) Consequently, we can produce a Witt class $(f_\Lambda)_*(1)$ over $\mathrm{Gr}_X(d,\mathcal{V})$. This is what we are going to do below for Λ even, making the class of L in $\mathrm{Pic}(\mathrm{Gr}_X(d,\mathcal{V}))/2$ more explicit in terms of the diagram Λ.

3.2. **Remark.** The perimeter of a Young diagram Λ is an even integer. Indeed, from the lower-left corner of Λ to its upper-right corner, there are two paths which follow the boundary (the upper path and the lower path) and they have the same length, namely the lattice distance between these two corners.

3.3. **Definition.** Let Λ be a Young diagram. We define $t(\Lambda) \in \mathbb{Z}/2$ to be the class of half the perimeter of Λ. From the above remark, $t(\Lambda)$ is also the class of the (lattice) distance from the lower-left corner of Λ to its upper-right corner. That is:

$$t(\Lambda) = [\Lambda_1 + \rho(\Lambda)] \in \mathbb{Z}/2$$

where $\rho(\Lambda)$ is the number of non-zero rows of Λ (Def. 2.6). Note that this Definition does not depend on an ambient frame.

3.4. **Remark.** On an even Young diagram Λ in $(d\times e)$-frame, there is another way to read $t(\Lambda) \in \mathbb{Z}/2$ on the diagram. Add the (parity of) the length of the segments where Λ touches the right and the bottom of the frame. See Figure 8. This is justified and generalized in Proposition 3.5.

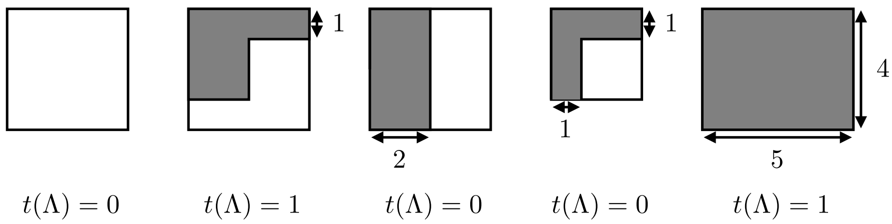


FIGURE 8. Class $t(\Lambda) \in \mathbb{Z}/2$ for different Λ.

3.5. **Proposition.** *Let Λ be an even Young diagram in $(d\times e)$-frame and let $\underline{d}, \underline{e}$ be the associated k-tuples (Def. 2.3). Then $t(\Lambda) = [d_i + (e - e_j)] \in \mathbb{Z}/2$ for any $i, j \in \{1, \ldots, k\}$ such that $e_i < e$ (only $i = k$ should be avoided when $e_k = e$).*

Proof. Measure the half-perimeter of Λ as the length of the lower boundary of Λ, from the lower-left corner of Λ to its upper-right corner (see Rem. 3.2). Since Λ is even, all segments on this lower half-perimeter which are not on the outside frame have even length. So, the only two segments to contribute to the lower half-perimeter with possible odd length, are on the outside $(d\times e)$-frame, *i.e.* :

- the vertical segment most to the right, which has length d_1, and
- the lowest horizontal segment, which has length $e - e_k$ when Λ touches the lower part of the $(d\times e)$-frame (otherwise $e_k = e$ and this length is even).

In any case, this shows that $t(\Lambda) = [d_1 + (e - e_k)] \in \mathbb{Z}/2$, that is, the announced formula for $i = 1$ and $j = k$. The other formulas follow from this one since by Definition 2.7 the successive differences $d_i - d_{i-1}$ and $e_{j+1} - e_j$ are even for all $i = 2, \ldots, k-1$, for all $j = 1, \ldots, k-1$, and also for $i = k$ when $e_k < e$. $\square$

3.6. **Definition.** Let Λ be an even Young diagram in $(d\times e)$-frame. We define *the twist $T(\Lambda)$ of Λ* as the following class in $\mathrm{Pic}(\mathrm{Gr}_X(d,\mathcal{V}))/2 = \mathrm{Pic}(X)/2 \oplus \mathbb{Z}/2\Delta_d$ (see Cor. 1.4):

$$T(\Lambda) = T(\Lambda, d, e) = \rho(\Lambda)\cdot \det \mathcal{V} + t(\Lambda)\cdot \Delta_d\,,$$

where we recall that $\rho(\Lambda)$ is the number of non-zero rows of Λ (Def. 2.6).

3.7. **Remark.** The important part of $T(\Lambda)$ is of course $t(\Lambda)\,\Delta_d$, which is not coming from the base X. For instance, when $\mathcal{V}$ is trivial, the other term disappears; this holds in particular when $X = \mathrm{Spec}(R)$ for a local ring R

3.8. **Proposition.** *Let Λ be an even Young diagram in $(d\times e)$-frame. Then*

$$\omega_{f_\Lambda} + f_\Lambda^*(T(\Lambda)) = 0 \quad \textit{in} \quad \mathrm{Pic}\big(\mathcal{F}l_X(\Lambda)\big)/2\,.$$

Proof. Suppose $k(\Lambda) \geqslant 2$. Remove from (10) all even coefficients coming from the fact that Λ is even and use Proposition 3.5 for $i = k-1$ and $j = k$. This gives in $\mathrm{Pic}\big(\mathcal{F}l_X(\Lambda)\big)/2$:

$$\omega_{f_\Lambda} = d_1 \det \mathcal{V}_{d_1+e_1} + (d+d_{k-1}) \det \mathcal{V}_{d+e_k} + d \det \mathcal{V} + d_1\,\Delta_{d_1} + t(\Lambda)\,\Delta_d\,.$$

Now observe that $d_1 \det \mathcal{V}_{d_1+e_1} + d_1\,\Delta_{d_1} = 0$ in $\mathrm{Pic}\big(\mathcal{F}l_X(\Lambda)\big)/2$. Indeed, either $e_1 > 0$ hence d_1 is even, or $e_1 = 0$ hence $\Delta_{d_1} = \det \mathcal{V}_{d_1}$ by Remark 1.10. So, we can simplify the above equation in $\mathrm{Pic}\big(\mathcal{F}l_X(\Lambda)\big)/2$:

$$\omega_{f_\Lambda} = (d+d_{k-1}) \det \mathcal{V}_{d+e_k} + d \det \mathcal{V} + t(\Lambda)\,\Delta_d\,.$$

Now, if $e_k < e$ then $d_k - d_{k-1}$ is even and $\rho(\Lambda) = d$; on the other hand, if $e_k = e$, then $\rho(\Lambda) = d_{k-1}$. In any case, the above expression becomes $\rho(\Lambda) \det \mathcal{V} + t(\Lambda)\,\Delta_d$, which is $T(\Lambda)$ by Definition 3.6.

Similarly, the case $k(\Lambda) = 1$ is an easy consequence of Corollary 1.14. □

3.9. **Proposition.** *Let Λ be an even Young diagram in $(d\times e)$-frame. Then the push-forward for Witt groups induced by $f_\Lambda : \mathcal{F}l_X(\Lambda) \longrightarrow \mathrm{Gr}_X(d,\mathcal{V})$ yields a homomorphism:*

$$\mathrm{W}^0\left(\mathcal{F}l_X(\Lambda), 0\right) = \mathrm{W}^0\left(\mathcal{F}l_X(\Lambda)\,,\, \omega_{f_\Lambda} + f_\Lambda^*(T(\Lambda))\right) \xrightarrow{(f_\Lambda)_*} \mathrm{W}^{|\Lambda|}\left(\mathrm{Gr}_X(d,\mathcal{V}), T(\Lambda)\right)$$

where the first identification comes from Proposition 3.8 (see Appendix A).

Proof. Apply push-forward (1), for $i = -\dim(f_\Lambda) = |\Lambda|$ and $L = T(\Lambda)$, as explained in Remark 3.1. □

3.10. **Definition.** Let Λ be an even Young diagram in $(d\times e)$-frame. By Proposition 3.9, we can push-forward along $f_\Lambda : \mathcal{F}l_X(\Lambda) \to \mathrm{Gr}_X(d,\mathcal{V})$ the unit form $1 \in \mathrm{W}^0(\mathcal{F}l_X(\Lambda), 0)$ to our announced Witt class

$$\phi_{d,e}(\Lambda) := (f_\Lambda)_*(1) \in \mathrm{W}^{|\Lambda|}\left(\mathrm{Gr}_X(d,\mathcal{V})\,,\, T(\Lambda)\right). \tag{20}$$

3.11. **Example.** Let $\Lambda = \varnothing$ be the empty Young diagram in $(d\times e)$-frame, which is even, as we know. Then $\phi_{d,e}(\varnothing) = 1 \in \mathrm{W}^0(\mathrm{Gr}_X(d,\mathcal{V}), 0)$ is the unit form on the Grassmannian. Indeed, $f_\varnothing : \mathcal{F}l_X(d,e;\varnothing) \to \mathrm{Gr}_X(d,\mathcal{V})$ is the identity.

3.12. **Definition.** Let $d, e \geqslant 1$. Consider the free $\mathrm{W}^{\mathrm{tot}}(X)$-module

$$\mathbb{F}_X(d,e) := \bigoplus_{\Lambda \text{ even}} \mathrm{W}^{\mathrm{tot}}(X)\cdot \Lambda$$

with (formal) basis given by the even Young diagrams Λ in $(d\times e)$-frame. We have an obvious $\mathbb{Z}/2$-grading on $\mathbb{F}_X(d,e)$ given by $t(\Lambda) \in \mathbb{Z}/2 = \mathbb{Z}/2\,\Delta_d$ (Def. 3.3):

$$\mathbb{F}_X(d,e) = \bigoplus_{t\in\mathbb{Z}/2}\ \bigoplus_{\substack{\Lambda \text{ even}\\ t(\Lambda)=t\,\Delta_d}} \mathrm{W}^{\mathrm{tot}}(X)\cdot \Lambda\,.$$

Now since $\mathrm{W}^{\mathrm{tot}}(X)$ is a graded ring over $\mathbb{Z}/4 \oplus \mathrm{Pic}(X)/2$ (see Appendix), we can extend this grading to $\mathbb{F}_X(d,e)$ by letting Λ have degree $|\Lambda| + \rho(\Lambda)\det\mathcal{V}$ in $\mathbb{Z}/4 \oplus \mathrm{Pic}(X)/2$. Combining both observations, $\mathbb{F}_X(d,e)$ becomes a graded abelian group over $\mathbb{Z}/4 \oplus \mathrm{Pic}(X)/2 \oplus \mathbb{Z}/2$. The latter is isomorphic to $\mathbb{Z}/4 \oplus \mathrm{Pic}(\mathrm{Gr}_X(d,\mathcal{V}))/2$ by Corollary 1.4. Explicitly, Λ has degree

$$\begin{array}{ccccccccc} |\Lambda| & + & \rho(\Lambda)\det\mathcal{V} & + & t(\Lambda)\,\Delta_d & = & |\Lambda| & + & T(\Lambda) \quad \text{in} \\ \mathbb{Z}/4 & \oplus & \mathrm{Pic}(X)/2 & \oplus & \mathbb{Z}/2\,\Delta_d & \cong & \mathbb{Z}/4 & \oplus & \mathrm{Pic}(\mathrm{Gr}_X(d,\mathcal{V}))/2\,. \end{array} \tag{21}$$

We $\mathrm{W}^{\mathrm{tot}}(X)$-linearly extend the $\phi_{d,e}(\Lambda)$ of Definition 3.10 into a homomorphism

$$\begin{array}{rccc} \phi_{d,e}: & \mathbb{F}_X(d,e) & \longrightarrow & \mathrm{W}^{\mathrm{tot}}\big(\mathrm{Gr}_X(d,\mathcal{V})\big) \\ & \Lambda & \longmapsto & \phi_{d,e}(\Lambda)\,, \end{array} \tag{22}$$

which is homogeneous (of degree zero) with respect to the above grading, by (20).

3.13. **Proposition.** *Let $d,e \geq 2$ and let $\mathcal{V}^1 := \mathcal{V}_{d+e-1}$ in the complete flag (13). With $\mathbb{F}_X(d,e)$ graded as above over $\mathbb{Z}/4 \oplus \mathrm{Pic}(X)/2 \oplus \mathbb{Z}/2\,\Delta_d$, the homomorphisms $\bar\iota$, $\bar\kappa$ and $\bar\partial$ of Corollary 2.12 are not always homogeneous (when $\mathcal{V}/\mathcal{V}^1$ is not trivial in $\mathrm{Pic}(X)/2$). More precisely, they behave as follows:*

- (a) *$\bar\iota$ is indeed homogeneous of degree $\big(d\,,\,d\,(\mathcal{V}/\mathcal{V}^1)\,,\,1\big)$.*
- (b) *$\bar\kappa$ sends degree (s,ℓ,t) into degree $\big(s\,,\,\ell + t\cdot(\mathcal{V}/\mathcal{V}^1)\,,\,t\big)$.*
- (c) *$\bar\partial$ sends degree (s,ℓ,t) into degree $\big(s-d+1\,,\,\ell+(t-d)\,(\mathcal{V}/\mathcal{V}^1)\,,\,t-1\big)$.*

Proof. This is a direct inspection of the behavior of $\deg(\phi_{d,e}(\Lambda))$ given in (21) under the combinatorial constructions of Corollary 2.12. We leave this to the reader. □

3.14. **Corollary.** *When $\mathcal{V}/\mathcal{V}^1$ is trivial (e.g. when $\mathcal{V}_\bullet = \mathcal{O}_X^\bullet$, e.g. when $X = \mathrm{Spec}(R)$ with R a field or a local ring) then $\bar\iota$, $\bar\kappa$ and $\bar\partial$ are homogeneous.* □

3.15. **Remark.** Our main result is that $\phi_{d,e}$ is an isomorphism for all $d,e \geqslant 1$. We shall prove it by induction on $d+e$. The case $d=1$ or $e=1$ is that of the projective bundle $\mathbb{P}(\mathcal{V})$. In that case, Walter [19] has proved

$$\mathrm{W}^{\mathrm{tot}}(\mathbb{P}(\mathcal{V})) = \mathrm{W}^{\mathrm{tot}}(X)\cdot 1_{\mathbb{P}^m} \oplus \mathrm{W}^{\mathrm{tot}}(X)\cdot \xi \tag{23}$$

where $\xi \in \mathrm{W}^{d+e-1}(\mathbb{P}(\mathcal{V}),\omega_\pi)$ is such that $\pi_*(\xi) = 1 \in \mathrm{W}^0(X)$ and $\pi : \mathbb{P}(\mathcal{V}) \to X$ is the structure morphism. We use the assumption that $\mathcal{V}$ has a complete flag to get the above closed formula, see [19, Proposition 8.1]. One can check that ξ is equal to $\phi(\Lambda)$, up to a unit, where $\Lambda = [d\times e]$ is the only non-empty even diagram in $(d\times e)$-frame in our case. Formula (23) was also proved in the case $\mathcal{V} = \mathcal{O}_X^{d+e}$ in [5], using methods closer to the present geometric philosophy. We could therefore assume the starting point of the induction (*i.e.* the case $d=1$ or $e=1$). Nevertheless, our proof of the induction step can be adapted to give this initial step as well. So, to be self-contained, we indicate how to do this in Proposition 6.4.

4. Cellular decomposition

We describe the usual relative cellular decomposition of Grassmannians. Fix $d,e \geqslant 2$ for the whole section.

4.1. **Notation.** Fix a complete flag $\mathcal{V}_\bullet$ of vector bundles on X

$$0 = \mathcal{V}_0 \lhd \mathcal{V}_1 \lhd \cdots \lhd \mathcal{V}_{d+e} = \mathcal{V}\,,$$

as in (13). We set $\mathcal{V}^1 = \mathcal{V}_{d+e-1}$ to be the codimension one subbundle of $\mathcal{V}$. We have an obvious closed immersion $\mathrm{Gr}_X(d,\mathcal{V}^1) \hookrightarrow \mathrm{Gr}_X(d,\mathcal{V})$, of codimension d, whose open complement we denote by $U_X(d,\mathcal{V}_\bullet)$.

4.2. **Notation.** Let $\mathcal{P}_d \lhd \mathcal{V}$ be a subbundle of rank d. We write $\mathcal{P}_d \dot{\not\lhd} \mathcal{V}^1$ to express that $\mathcal{P}_d$ is not a subbundle of $\mathcal{V}^1$ but moreover satisfies the equivalent conditions:

(a) The natural map from $\mathcal{P}_d/(\mathcal{P}_d \cap \mathcal{V}^1) = (\mathcal{P}_d + \mathcal{V}^1)/\mathcal{V}^1$ into $\mathcal{V}/\mathcal{V}^1$ is an isomorphism.
(b) $\mathcal{P}_d \cap \mathcal{V}^1$ is a subbundle of $\mathcal{P}_d$ (in the strong sense of Definition 1.1).

Over a field, this amounts to $\mathcal{P}_d \not\subset \mathcal{V}^1$ but this is not sufficient in general.

4.3. **Definition.** Using notations of Section 1, we have a commutative diagram

$$\begin{array}{ccccc} \mathrm{Gr}_X(d,\mathcal{V}^1) & \overset{\iota}{\hookrightarrow} & \mathrm{Gr}_X(d,\mathcal{V}) & \overset{\upsilon}{\hookleftarrow} & U_X(d,\mathcal{V}_\bullet) \\ \uparrow{\scriptstyle\tilde\pi} & & \uparrow{\scriptstyle\pi} & \swarrow{\scriptstyle\tilde\upsilon} & \downarrow{\scriptstyle\alpha} \\ \mathcal{F}l_X((d-1,d),(e,e-1)) & \overset{\tilde\iota}{\hookrightarrow} & \mathcal{F}l_X((d-1,d),(e,e)) & \xrightarrow{\tilde\alpha} & \mathcal{F}l_X(d-1,\mathcal{V}^1)\,. \end{array} \tag{24}$$

which looks as follows on points:

$$\begin{array}{ccccc} \{\mathcal{P}_d \lhd \mathcal{V}^1\} & \overset{\iota}{\hookrightarrow} & \{\mathcal{P}_d \lhd \mathcal{V}\} & \overset{\upsilon}{\hookleftarrow} & \{\mathcal{P}_d \dot{\not\lhd} \mathcal{V}^1\} \\ \uparrow{\scriptstyle\tilde\pi} & & \uparrow{\scriptstyle\pi} & \swarrow{\scriptstyle\tilde\upsilon} & \downarrow{\scriptstyle\alpha} \\ \{\mathcal{P}_{d-1} \lhd \mathcal{P}_d \lhd \mathcal{V}^1\} & \overset{\tilde\iota}{\hookrightarrow} & \{\mathcal{P}_{d-1} \lhd \mathcal{P}_d \lhd \mathcal{V} \mid \mathcal{P}_{d-1} \lhd \mathcal{V}^1\} & \xrightarrow{\tilde\alpha} & \{\mathcal{P}_{d-1} \lhd \mathcal{V}^1\}\,. \end{array} \tag{25}$$

Here ι, $\tilde\iota$, π, $\tilde\pi$ and $\tilde\alpha$ are the obvious morphisms; the morphism $\tilde\upsilon$ maps $\mathcal{P}_d$ to the flag $\mathcal{P}_{d-1} \lhd \mathcal{P}_d$ with $\mathcal{P}_{d-1} := \mathcal{P}_d \cap \mathcal{V}^1$ (see Not. 4.2); finally α is defined as $\tilde\alpha \circ \tilde\upsilon$.

4.4. **Proposition.** *In Diagram (24), the scheme $\mathcal{F}l_X((d-1,d),(e,e))$ is the blow-up of $\mathrm{Gr}_X(d,\mathcal{V})$ along $\mathrm{Gr}_X(d,\mathcal{V}^1)$ with exceptional fiber $\mathcal{F}l_X((d-1,d),(e,e-1))$.*

Proof. We can reduce to the case where X is affine and $\mathcal{V}$ is free and then to $X = \mathrm{Spec}(\mathbb{Z})$ by compatibility of blow-ups with pull-back diagrams. We omit X in the notation in the rest of the proof. Now, we first show that there is a morphism from the announced blow-up B to $\mathcal{F}l((d-1,d),(e,e))$ and then that this morphism is an isomorphism. The inverse image of $\mathrm{Gr}(d,\mathcal{V})$ in $\mathcal{F}l((d-1,d),(e,e))$ is $\mathcal{F}l((d-1,d),(e,e-1)) = \mathbb{P}_Y(\mathcal{V}^1/\mathcal{P}_{d-1})$ where $Y = \mathrm{Gr}(d-1,\mathcal{V}^1)$, by construction. It includes in $\mathcal{F}l((d-1,d),(e,e)) = \mathbb{P}_Y(\mathcal{V}/\mathcal{P}_{d-1})$, so it is locally $\mathbb{P}^{e-1} \subset \mathbb{P}^e$ hence an effective Cartier divisor (*i.e.* a closed subscheme locally given by a principal ideal). By the universal property of the blow-up construction (see [16, § 8.1.2, Corollary 1.16]) we thus obtain a map $f : B \to \mathcal{F}l((d-1,d),(e,e))$, such that $\pi \circ f = \pi'$, where $\pi' : B \to \mathrm{Gr}(d,\mathcal{V})$ is the structural morphism of the blow-up. In particular, f is a finite (check over $\mathrm{Gr}(d,\mathcal{V}^1)$ and $U(d,\mathcal{V})$) birational morphism to a normal variety, so it is an isomorphism: it is locally a finite morphism $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ where R is integrally closed. □

4.5. **Definition.** Let $B_X(d,\mathcal{V}_\bullet) = \mathcal{F}l_X((d-1,d),(e,e))$ be the blow-up of $\mathrm{Gr}_X(d,\mathcal{V})$ along $\mathrm{Gr}_X(d,\mathcal{V}^1)$ and let $E_X(d,\mathcal{V}_\bullet) = \mathcal{F}l_X((d-1,d),(e,e-1))$ be the exceptional fiber. By (24), $\mathrm{Gr}_X(d,\mathcal{V})$ admits a decomposition like in [6, Hypothesis 1.2], namely

$$\begin{array}{ccccc} \mathrm{Gr}_X(d,\mathcal{V}^1) & \overset{\iota}{\hookrightarrow} & \mathrm{Gr}_X(d,\mathcal{V}) & \overset{\upsilon}{\hookleftarrow} & U_X(d,\mathcal{V}_\bullet) \\ \uparrow{\scriptstyle\tilde\pi} & & \uparrow{\scriptstyle\pi} & \swarrow{\scriptstyle\tilde\upsilon} & \downarrow{\scriptstyle\alpha} \\ E_X(d,\mathcal{V}_\bullet) & \overset{\tilde\iota}{\hookrightarrow} & B_X(d,\mathcal{V}_\bullet) & \xrightarrow{\tilde\alpha} & \mathrm{Gr}_X(d-1,\mathcal{V}^1)\,. \end{array} \tag{26}$$

It is easy to see that α is an $\mathbb{A}^e$-bundle because $U_X(d,\mathcal{V}_\bullet)$ is canonically isomorphic to $\mathbb{P}_Y(\mathcal{V}/\mathcal{T}^Y_{d-1}) \setminus \mathbb{P}_Y(\mathcal{V}^1/\mathcal{T}^Y_{d-1})$ where $Y = \mathrm{Gr}_X(d-1,\mathcal{V}^1)$, with α corresponding to the structural morphism to Y.

4.6. **Remark.** We can compute the relevant Picard groups and canonical bundles, via the methods of Section 1. Let us start with Picard groups, using (8). Since

$\mathrm{Pic}(X)$ is a direct summand of the Picard group of all schemes in (26), we focus on the relative Picard groups $\mathrm{Pic}(-)/\mathrm{Pic}(X)$. In short "$\mathrm{Pic}(-)/\mathrm{Pic}(X)$ of (26)" is

$$(27)\qquad \begin{array}{ccccc}
\mathbb{Z}\Delta_d & \xleftarrow{\iota^*=1} & \mathbb{Z}\Delta_d & \xrightarrow{\upsilon^*=1} & \mathbb{Z}\upsilon^*(\Delta_d)=\mathbb{Z}\alpha^*(\Delta_{d-1}) \\
{\scriptstyle \tilde{\pi}^*=\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)}\downarrow & & {\scriptstyle \pi^*=\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)}\downarrow & {\scriptstyle \tilde{\upsilon}^*=(1\ 1)}\nearrow & \uparrow{\scriptstyle \alpha^*=1} \\
\mathbb{Z}\Delta_{d-1}\oplus\mathbb{Z}\Delta_d & \xleftarrow[\tilde{\iota}^*=\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)]{} & \mathbb{Z}\Delta_{d-1}\oplus\mathbb{Z}\Delta_d & \xleftarrow[\tilde{\alpha}^*=\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)]{} & \mathbb{Z}\Delta_{d-1}
\end{array}$$

(In the case $X=\mathrm{Spec}(R)$ for a local ring R, the Picard groups are exactly as above.) Here we used that the closed subscheme $\mathrm{Gr}_X(d,\mathcal{V}^1)$ is of codimension $d\geqslant 2$ in $\mathrm{Gr}_X(d,\mathcal{V})$ to see that $\upsilon^*:\mathrm{Pic}(\mathrm{Gr}_X(d,\mathcal{V}))\cong\mathrm{Pic}(U_X(d,\mathcal{V}_\bullet))$. We also used that $e\geqslant 2$, otherwise $e-1=0$ and $\Delta_{d-1}\in\mathrm{Pic}(X)$ by Remark 1.10. (When $d=1$, resp. $e=1$, we loose all components $\mathbb{Z}\Delta_{d-1}$, resp. all components $\mathbb{Z}\Delta_d$, in the previous diagram.) Alternatively, (27) follows from the computation of the Picard groups provided in [6, Proposition A.6]. Finally, the maps into the upper corner of Diagram (27) are obtained from

$$(28)\qquad \upsilon^*(\Delta_d)-\alpha^*(\Delta_{d-1})=\mathcal{V}/\mathcal{V}^1\,,$$

which follows from Condition (a) in Notation 4.2.

We shall use push-forward along some morphisms of (26). The relevant relative canonical bundles are given by Corollary (1.14):

$$\begin{array}{llcl}
(29) & \omega_\iota & = & -\Delta_d+d\,(\mathcal{V}/\mathcal{V}^1)\\
(30) & \omega_\pi & = & (d-1)\Delta_{d-1}+(1-d)\Delta_d+(d-1)\,\mathcal{V}/\mathcal{V}^1\\
(31) & \omega_{\tilde{\iota}} & = & \mathcal{V}/\mathcal{V}^1+\Delta_{d-1}-\Delta_d\\
(32) & \omega_{\tilde{\pi}} & = & d\,\Delta_{d-1}+(1-d)\Delta_d\,.
\end{array}$$

Again, when the fixed complete flag $\mathcal{V}_\bullet=\mathcal{O}_X^\bullet$ is trivial, the "noise" $\mathcal{V}/\mathcal{V}^1$ vanishes.

We end this Section with two geometric lemmas which will be useful in the proof of the main theorem.

4.7. **Lemma.** *Let $d,e\geqslant 2$ and let Λ be an even Young (d,e)-diagram with empty last row (i.e. $\Lambda_k=0$, i.e. $\zeta(\Lambda)>0$). Hence $\Lambda_{|d-1,e}$ is an even $(d-1,e)$-diagram. Then the base-change to $U_X(d,\mathcal{V}_\bullet)$ of the morphisms $f_{d,e;\Lambda}$ and $f_{d-1,e;\Lambda_{|d-1,e}}$ coincide, that is, we have two cartesian squares:*

$$(33)\qquad \begin{array}{ccccc}
\mathrm{Gr}_X(d,\mathcal{V}) & \xleftarrow{\upsilon} & U_X(d,\mathcal{V}_\bullet) & \xrightarrow{\alpha} & \mathrm{Gr}_X(d-1,\mathcal{V}^1)\\
{\scriptstyle f_{d,e;\Lambda}}\uparrow & \square & \uparrow & \square & \uparrow{\scriptstyle f_{d-1,e;\Lambda_{|d-1,e}}}\\
\mathcal{F}l_X(d,e;\Lambda) & \longleftarrow & U' & \xrightarrow{\alpha'} & \mathcal{F}l_X(d-1,e;\Lambda_{|d-1,e})\,.
\end{array}$$

Proof. Let us check this on points. Let $\underline{d}$ and $\underline{e}$ be the k-tuples associated to Λ as usual (Def. 2.3). We need to distinguish two cases, namely $d_k-d_{k-1}>1$ and $d_k-d_{k-1}=1$.

When $d_k>d_{k-1}+1$, that is, when there is more than one zero line at the end of Λ (*i.e.* $\zeta(\Lambda)>1$), we then have $k(\Lambda_{|d-1,e})=k(\Lambda)=k$ and the k-tuples $\underline{d}(\Lambda_{|d-1,e})$ and $\underline{e}(\Lambda_{|d-1,e})$ are almost the same as $\underline{d}$ and $\underline{e}$ except for the last entry of $\underline{d}(\Lambda_{|d-1,e})$ which becomes $d-1$. Diagram (33) then looks as follows on points (as usual the

$\mathcal{P}_i$ and $\mathcal{P}'_j$ are "variables" whereas the $\mathcal{V}_i$ belong to the fixed complete flag):

$$
\begin{array}{ccccc}
\{\mathcal{P}_d \lhd \mathcal{V}\} & \hookleftarrow & \{\mathcal{P}_d \dot{\ntriangleleft} \mathcal{V}^1\} & \xrightarrow{\alpha} & \{\mathcal{P}'_{d-1} \lhd \mathcal{V}^1\} \\
\uparrow{\scriptstyle f_{d,e;\Lambda}} & \square & \uparrow & \square & \uparrow{\scriptstyle f_{d-1,e;\Lambda_{|d-1,e}}} \\
\left\{\begin{matrix} \cdots \lhd \mathcal{P}_{d_{k-1}} \lhd \mathcal{P}_d \\ {\scriptstyle e_{k-1}}\triangle \quad {\scriptstyle e}\,\triangle \\ \cdots \lhd \mathcal{V}_{d_{k-1}+e_{k-1}} \lhd \mathcal{V} \end{matrix}\right\} & \hookleftarrow & \left\{\begin{matrix} \cdots \lhd \mathcal{P}_{d_{k-1}} \lhd \mathcal{P}_d \;\overset{\mathcal{V}^1}{\dot{\triangledown}} \\ {\scriptstyle e_{k-1}}\triangle \quad {\scriptstyle e}\,\triangle \\ \cdots \lhd \mathcal{V}_{d_{k-1}+e_{k-1}} \lhd \mathcal{V} \end{matrix}\right\} & \xrightarrow{\alpha'} & \left\{\begin{matrix} \cdots \lhd \mathcal{P}_{d_{k-1}} \lhd \mathcal{P}'_{d-1} \\ {\scriptstyle e_{k-1}}\triangle \quad {\scriptstyle e}\,\triangle \\ \cdots \lhd \mathcal{V}_{d_{k-1}+e_{k-1}} \lhd \mathcal{V}^1 \end{matrix}\right\}
\end{array}
$$

where the morphisms α sends $\mathcal{P}_d$ to $\mathcal{P}'_{d-1} := \mathcal{P}_d \cap \mathcal{V}^1$ and similarly for α' .

On the other hand, when $d_k = d_{k-1} + 1$, that is, when Λ has only one zero line (*i.e.* $\zeta(\Lambda) = 1$), then we have $k(\Lambda_{|d-1,e}) = k(\Lambda) - 1 = k - 1$ and the $(k-1)$-tuples $\underline{d}(\Lambda_{|d-1,e})$ and $\underline{e}(\Lambda_{|d-1,e})$ are respectively $\underline{d}$ and $\underline{e}$ truncated from their last entry. Diagram (33) then looks as follows on points:

$$
\begin{array}{ccccc}
\{\mathcal{P}_d \lhd \mathcal{V}\} & \hookleftarrow & \{\mathcal{P}_d \dot{\ntriangleleft} \mathcal{V}^1\} & \xrightarrow{\alpha} & \{\mathcal{P}'_{d-1} \lhd \mathcal{V}^1\} \\
\uparrow{\scriptstyle f_{d,e;\Lambda}} & \square & \uparrow & \square & \uparrow{\scriptstyle f_{d-1,e;\Lambda_{|d-1,e}}} \\
\left\{\begin{matrix} \cdots \lhd \mathcal{P}_{d_{k-1}} \lhd \mathcal{P}_d \\ {\scriptstyle e_{k-1}}\triangle \quad {\scriptstyle e}\,\triangle \\ \cdots \lhd \mathcal{V}_{d_{k-1}+e_{k-1}} \lhd \mathcal{V} \end{matrix}\right\} & \hookleftarrow & \left\{\begin{matrix} \cdots \lhd \mathcal{P}_{d_{k-1}} \lhd \mathcal{P}_d \;\overset{\mathcal{V}^1}{\dot{\triangledown}} \\ {\scriptstyle e_{k-1}}\triangle \quad {\scriptstyle e}\,\triangle \\ \cdots \lhd \mathcal{V}_{d_{k-1}+e_{k-1}} \lhd \mathcal{V} \end{matrix}\right\} & \xrightarrow{\alpha'} & \left\{\begin{matrix} \cdots \lhd \mathcal{P}_{d_{k-1}} \\ \triangle\, {\scriptstyle e_{k-1}} \\ \cdots \lhd \mathcal{V}_{d_{k-1}+e_{k-1}} \end{matrix}\right\}
\end{array}
$$

where α still sends $\mathcal{P}_d$ to $\mathcal{P}'_{d-1} := \mathcal{P}_d \cap \mathcal{V}^1$ and where $f_{d-1,e;\Lambda_{|d-1,e}}$ sends a flag to $\mathcal{P}_{d_{k-1}}$. Note that in this case, α' drops the last subspace $\mathcal{P}_d$ in the flag.

In both cases, it is easy to check that the two squares are cartesian. □

4.8. **Lemma.** *Let $d, e \geqslant 2$ and let Λ'' be an even $(d-1, e)$-diagram such that Λ''_{d-1} is odd. Hence we can consider the even $(d, e-1)$-diagram $\Lambda' = (\Lambda''_1 - 1, \ldots, \Lambda''_{d-1} - 1, 0)$. Then, there exists a commutative diagram:*

$$
\begin{array}{ccccc}
\mathrm{Gr}_X(d, \mathcal{V}^1) & \xleftarrow{\tilde{\pi}} & E_X(d, \mathcal{V}_\bullet) & \xrightarrow{\tilde{\alpha}\,\tilde{\iota}} & \mathrm{Gr}_X(d-1, \mathcal{V}^1) \\
\uparrow{\scriptstyle f_{d,e-1;\Lambda'}} & & \uparrow{\scriptstyle f'} & \square & \uparrow{\scriptstyle f_{d-1,e;\Lambda''}} \\
\mathcal{F}l_X(d, e-1; \Lambda') & \xleftarrow[\pi']{} & F' & \longrightarrow & \mathcal{F}l_X(d-1, e; \Lambda'')
\end{array}
\tag{34}
$$

where $E_X(d, \mathcal{V}_\bullet)$ is the exceptional fiber of Diagram (26) and where the right-hand square is cartesian. Moreover, either π' is an isomorphism or the scheme F' (with the morphism π') identifies with the blow-up of $\mathcal{F}l_X(d, e-1; \Lambda')$ along a closed regular subscheme of odd codimension.

Proof. Let $k = k(\Lambda'')$, $\underline{d} = \underline{d}(\Lambda'')$ and $\underline{e} = \underline{e}(\Lambda'')$ as usual (Def. 2.3). We need to distinguish two cases, namely $\Lambda''_{d-1} > 1$ and $\Lambda''_{d-1} = 1$.

Suppose first that $\Lambda''_{d-1} > 1$. Then $k(\Lambda') = k + 1$ and $\underline{d}(\Lambda')$ and $\underline{e}(\Lambda')$ are just $\underline{d}(\Lambda'')$ and $\underline{e}(\Lambda'')$ with one more entry at the end, namely d and $e - 1$ respectively.

We can describe the pull-back in Diagram (34) as follows (on points):

$$\begin{array}{ccc}
\{\mathcal{P}_{d-1} \lhd \mathcal{P}_d \lhd \mathcal{V}^1\} & \longrightarrow & \{\mathcal{P}_{d-1} \lhd \mathcal{V}^1\} \\
\uparrow{\scriptstyle f'} & \square & \uparrow \\
\left\{\begin{array}{ccccc} \mathcal{P}_{d_1} & \lhd \cdots \lhd & \mathcal{P}_{d-1} & \lhd & \mathcal{P}_d \\ \triangle & & \triangle & & \triangle \\ \mathcal{V}_{d_1+e_1} & \lhd \cdots \lhd & \mathcal{V}_{d-1+e_k} & \lhd & \mathcal{V}^1 \end{array}\right\} & \longrightarrow & \left\{\begin{array}{ccc} \mathcal{P}_{d_1} & \lhd \cdots \lhd & \mathcal{P}_{d-1} \\ \triangle & & \triangle \\ \mathcal{V}_{d_1+e_1} & \lhd \cdots \lhd & \mathcal{V}_{d-1+e_k} \end{array}\right\}
\end{array}$$

This pull-back F' is $\mathcal{F}l_X(d, e-1; \Lambda')$. So, take $\pi' = \mathrm{id}$. The morphism f' composed with $\tilde{\pi}$ sends a flag $\mathcal{P}_{d_1} \lhd \cdots \lhd \mathcal{P}_{d-1} \lhd \mathcal{P}_d$ to $\mathcal{P}_d$ and so does $f_{d,e-1;\Lambda'}$.

Suppose now that $\Lambda''_{d-1} = 1$. Then $k(\Lambda') = k$ and $\underline{e}(\Lambda') = \underline{e}(\Lambda'')$, whereas $\underline{d}(\Lambda')$ is obtained from $\underline{d}(\Lambda'')$ by replacing its last entry by d. We can describe the pull-back in Diagram (34) as follows:

$$\begin{array}{ccc}
\{\mathcal{P}_{d-1} \lhd \mathcal{P}_d \lhd \mathcal{V}^1\} & \longrightarrow & \{\mathcal{P}_{d-1} \lhd \mathcal{V}^1\} \\
\uparrow{\scriptstyle f'} & \square & \uparrow \\
\left\{\begin{array}{ccccccc} \mathcal{P}_{d_1} & \lhd \cdots \lhd & \mathcal{P}_{d_{k-1}} & \lhd & \mathcal{P}_{d-1} & \lhd & \mathcal{P}_d \\ \triangle & & & & \triangle & & \triangle \\ \mathcal{V}_{d_1+e_1} & \lhd \cdots \lhd & \mathcal{V}_{d_{k-1}+e_{k-1}} & \lhd & \mathcal{V}_{d+e-2} & \lhd & \mathcal{V}^1 \end{array}\right\} & \longrightarrow & \left\{\begin{array}{ccc} \mathcal{P}_{d_1} & \lhd \cdots \lhd & \mathcal{P}_{d-1} \\ \triangle & & \triangle \\ \mathcal{V}_{d_1+e_1} & \lhd \cdots \lhd & \mathcal{V}_{d+e-2} \end{array}\right\}
\end{array}$$

where f' is the obvious morphism. The left-hand square of (34) is defined by:

$$\begin{array}{ccc}
\{\mathcal{P}_d \lhd \mathcal{V}^1\} & \xleftarrow{\ \tilde{\pi}\ } & \{\mathcal{P}_{d-1} \lhd \mathcal{P}_d \lhd \mathcal{V}^1\} \\
\uparrow & & \uparrow{\scriptstyle f'} \\
\left\{\begin{array}{ccccc} \mathcal{P}_{d_1} & \lhd \cdots \lhd & \mathcal{P}_{d_{k-1}} & \lhd & \mathcal{P}_d \\ \triangle & & \triangle & & \triangle \\ \mathcal{V}_{d_1+e_1} & \lhd \cdots \lhd & \mathcal{V}_{d_{k-1}+e_{k-1}} & \lhd & \mathcal{V}^1 \end{array}\right\} & \xleftarrow{\ \pi'\ } & \left\{\begin{array}{ccccccc} \mathcal{P}_{d_1} & \lhd \cdots \lhd & \mathcal{P}_{d_{k-1}} & \lhd & \mathcal{P}_{d-1} & \lhd & \mathcal{P}_d \\ \triangle & & & & \triangle & & \triangle \\ \mathcal{V}_{d_1+e_1} & \lhd \cdots \lhd & \mathcal{V}_{d_{k-1}+e_{k-1}} & \lhd & \mathcal{V}_{d+e-2} & \lhd & \mathcal{V}^1 \end{array}\right\}
\end{array}$$

The morphism $\pi' : F' \to \mathcal{F}l_X(d, e-1; \Lambda')$ simply drops $\mathcal{P}_{d-1}$ in this case.

Let $\mathcal{V}^2 := \mathcal{V}_{d+e-2}$ and $Y := \mathcal{F}l_X((d_1, \ldots, d_{k-1}), (e_1, \ldots, e_{k-1}), \mathcal{V}_\bullet)$. We have $\mathcal{F}l_X(d, e-1; \Lambda') = \mathrm{Gr}_Y(d-d_{k-1}, \mathcal{V}^1/\mathcal{T}_{d_{k-1}})$ by Lemma 1.11. As in Definition 4.5, we consider the blow-up $B_Y(d-d_{k-1}\,,\, \mathcal{V}^2/\mathcal{T}_{d_k-1} \lhd \mathcal{V}^1/\mathcal{T}_{d_k-1})$ of $\mathrm{Gr}_Y(d-d_{k-1}, \mathcal{V}^1/\mathcal{T}_{d_{k-1}})$ along the closed regular immersion of $\mathrm{Gr}_Y(d-d_{k-1}, \mathcal{V}^2/\mathcal{T}_{d_{k-1}})$. By Proposition 4.4, this blow-up coincides with F' and the morphism π of (25) here becomes the above morphism π'. In other words, the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{F}l_X(d, e-1; \Lambda') & \xleftarrow{\qquad \pi' \qquad} & F' \\
\| & & \| \\
\mathrm{Gr}_Y(d-d_{k-1}, \mathcal{V}^1/\mathcal{T}_{d_{k-1}}) & \xleftarrow{\text{“}\pi\text{”}} & B_Y(d-d_{k-1},\, \mathcal{V}^2/\mathcal{T}_{d_k-1} \lhd \mathcal{V}^1/\mathcal{T}_{d_{k-1}})\,.
\end{array}$$

The closed immersion $\mathrm{Gr}_Y(d-d_{k-1}, \mathcal{V}^2/\mathcal{T}_{d_{k-1}}) \hookrightarrow \mathrm{Gr}_Y(d-d_{k-1}, \mathcal{V}^1/\mathcal{T}_{d_{k-1}})$ is of odd codimension equal to $d - d_{k-1}$. □

5. Push-forward, pull-back and connecting homomorphism

We describe the long exact sequence on Witt groups associated to the geometric decomposition $\mathrm{Gr}_X(d, \mathcal{V}) = \mathrm{Gr}_X(d, \mathcal{V}^1) \cup U_X(d, \mathcal{V}_\bullet)$, considered in Section 4. Recall the important Diagram (26):

$$(26)\qquad \begin{array}{ccccc} \mathrm{Gr}_X(d,\mathcal{V}^1) & \overset{\iota}{\hookrightarrow} & \mathrm{Gr}_X(d,\mathcal{V}) & \overset{\upsilon}{\hookleftarrow} & U_X(d,\mathcal{V}_\bullet) \\ \uparrow{\scriptstyle\tilde\pi} & & \uparrow{\scriptstyle\pi} & \swarrow{\scriptstyle\tilde\upsilon} & \downarrow{\scriptstyle\alpha} \\ E_X(d,\mathcal{V}_\bullet) & \overset{\tilde\iota}{\hookrightarrow} & B_X(d,\mathcal{V}_\bullet) & \xrightarrow{\tilde\alpha} & \mathrm{Gr}_X(d-1,\mathcal{V}^1)\,. \end{array}$$

5.1. **Notation.** On Witt groups, $\alpha^* : \mathrm{W}^{\mathrm{tot}}(\mathrm{Gr}_X(d-1,\mathcal{V}^1)) \xrightarrow{\sim} \mathrm{W}^{\mathrm{tot}}(U_X(d,\mathcal{V}_\bullet))$ is an isomorphism by homotopy invariance and we define

$$\kappa := (\alpha^*)^{-1}\upsilon^* : \mathrm{W}^{\mathrm{tot}}(\mathrm{Gr}_X(d,\mathcal{V})) \longrightarrow \mathrm{W}^{\mathrm{tot}}(\mathrm{Gr}_X(d-1,\mathcal{V}^1))\,.$$

5.2. **Proposition.** *Let $L \in \mathrm{Pic}(\mathrm{Gr}_X(d,\mathcal{V}))/2$. Then define $L' := \omega_\iota + \iota^*(L)$ in $\mathrm{Pic}(\mathrm{Gr}_X(d,\mathcal{V}^1))/2$ and $L'' := (\alpha^*)^{-1}(\upsilon^*(L))$ in $\mathrm{Pic}(\mathrm{Gr}_X(d-1,\mathcal{V}^1))/2$. There is a long exact sequence*

$$\cdots \mathrm{W}^{i-d}(\mathrm{Gr}_X(d,\mathcal{V}^1),L') \xrightarrow{\iota_*} \mathrm{W}^{i}(\mathrm{Gr}_X(d,\mathcal{V}),L) \xrightarrow{\kappa} \mathrm{W}^{i}(\mathrm{Gr}_X(d-1,\mathcal{V}^1),L'') \xrightarrow{\partial}$$
$$\mathrm{W}^{i-d+1}(\mathrm{Gr}_X(d,\mathcal{V}^1),L') \xrightarrow{\iota_*} \mathrm{W}^{i+1}(\mathrm{Gr}_X(d,\mathcal{V}),L) \xrightarrow{\kappa} \mathrm{W}^{i+1}(\mathrm{Gr}_X(d-1,\mathcal{V}^1),L'') \cdots$$

Proof. This is the long exact sequence of localization [3] associated to the codimension d closed immersion $\mathrm{Gr}_X(d,\mathcal{V}^1) \subset \mathrm{Gr}_X(d,\mathcal{V})$ in which we used dévissage to replace $\mathrm{W}^i_{\mathrm{Gr}(d,\mathcal{V}^1)}(\mathrm{Gr}_X(d,\mathcal{V}),L)$ by $\mathrm{W}^{i-d}(\mathrm{Gr}_X(d,\mathcal{V}^1),L')$ and homotopy invariance to replace $\mathrm{W}^i(U_X(d,\mathcal{V}_\bullet),\upsilon^*L)$ by $\mathrm{W}^i(\mathrm{Gr}_X(d-1,\mathcal{V}^1),L'')$. See [6, Sequence (11)]. □

5.3. **Corollary.** *For $d,e \geqslant 2$, there is a 3-term exact sequence of total Witt groups*

$$(35)\qquad \begin{array}{c} \mathrm{W}^{\mathrm{tot}}(\mathrm{Gr}_X(d,\mathcal{V}^1)) \xrightarrow{\iota_*} \mathrm{W}^{\mathrm{tot}}(\mathrm{Gr}_X(d,\mathcal{V})) \xrightarrow{\kappa} \mathrm{W}^{\mathrm{tot}}(\mathrm{Gr}_X(d-1,\mathcal{V}^1)) \\ \uparrow\!\!\underline{\qquad\qquad\qquad\quad \partial \quad\qquad\qquad\qquad}\!\!\lrcorner \end{array}$$

Proof. Add the long exact sequences of Proposition 5.2 over $L \in \mathrm{Pic}(\mathrm{Gr}_X(d,\mathcal{V}))/2 = \mathrm{Pic}(X)/2 \oplus \mathbb{Z}/2\,\Delta_d$ and $i \in \mathbb{Z}/4$. All three schemes have the same Picard groups ($d,e \geqslant 2$) by (27). So when L runs among all possible values of $\mathrm{Pic}(\mathrm{Gr}_X(d,\mathcal{V}))/2$, so does L' in $\mathrm{Pic}(\mathrm{Gr}_X(d,\mathcal{V}^1))/2$ and L'' in $\mathrm{Pic}(\mathrm{Gr}_X(d-1,\mathcal{V}^1))/2$. □

5.4. **Remark.** When $d = 1$, one can define the sum of the above proof but for each $L'' \in \mathrm{Pic}(\mathrm{Gr}_X(d-1,\mathcal{V}^1)/2$, and each $i \in \mathbb{Z}/4$, the Witt group $\mathrm{W}^i(\mathrm{Gr}_X(d-1,\mathcal{V}^1),L'')$ will appear twice. This is explained in Remark 4.6: the kernel of the homomorphism $\mathrm{Pic}(\mathrm{Gr}_X(d,\mathcal{V}))/2 \longrightarrow \mathrm{Pic}(\mathrm{Gr}_X(d-1,\mathcal{V}^1))/2$ is $\mathbb{Z}/2\cdot\Delta_d$. In this case, the third "total" Witt group of (35) must be replaced by two copies of it.

Similarly, when $e = 1$, the total Witt group of $\mathrm{Gr}_X(d,\mathcal{V}^1)$ will appear twice in the first entry of (35). See more in Remark 6.3.

5.5. **Remark.** All three groups in (35) are graded over

$$\mathbb{Z}/4 \oplus \mathrm{Pic}(X)/2 \oplus \mathbb{Z}/2\,\Delta_d\,.$$

The homomorphisms ι_*, κ and ∂ are not always homogeneous when the line bundle $\mathcal{V}/\mathcal{V}^1$ is not trivial (in $\mathrm{Pic}(X)/2$). Indeed, the same happens as in Proposition 3.13:

(a) ι_* is indeed homogeneous of degree $\big(d\,,\,-d\cdot\mathcal{V}/\mathcal{V}^1\,,\,1\big)$.
(b) κ sends degree (s,ℓ,t) into degree $\big(s\,,\,\ell + t\cdot(\mathcal{V}/\mathcal{V}^1)\,,\,t\big)$.
(c) ∂ sends degree (s,ℓ,t) into degree $\big(s-d+1\,,\,\ell+(t-d)\,(\mathcal{V}/\mathcal{V}^1)\,,\,t-1\big)$.

Note that if we restrict attention to the significant part of the twist (the part $\mathbb{Z}/2\,\Delta_d$ not coming from X), κ leaves it unchanged, and ι_* and ∂ change it.

Finally, we shall also use the following general fact about push-forwards along blow-up morphisms.

5.6. **Proposition.** *Let X be a noetherian scheme (quasi-compact and quasi-separated is enough) and let $Z \hookrightarrow X$ be a regular immersion of pure codimension d. Let $\pi : B \to X$ be the blow-up of X along Z. Then:*

(a) *There is a natural isomorphism $R\pi_*(\mathcal{O}_B) \cong \mathcal{O}_X$ in the derived category of X.*

(b) *Assume further that X is regular and that $\omega_{B/X}$ is a square (which happens exactly when d is odd by* [6, Proposition A.11 (iii)]*). Then the push-forward $\pi_* : \mathrm{W}^0(B) \to \mathrm{W}^0(X)$ maps the unit class 1_B to the unit class 1_X. (This is even true on the level of forms, without taking Witt classes.)*

Proof. (a) can be found in SGA 6, see [1, Lemme VII.3.5, p. 441] or the more recent account in [18, Lemme 2.3 (a)].

(b) follows from (a). Indeed, when $d = 1$, we have $B = X$ and there is nothing to prove. When $d \geqslant 3$ then line bundles over X, and homomorphisms between them, are determined by their restriction to the open complement $U = X \smallsetminus Z$ of Z since Z is of codimension at least 2. By the base-change formula for push-forward, and since $\pi_{|_{\pi^{-1}(U)}} : \pi^{-1}(U) \to U$ is an isomorphism, the result follows. □

5.7. **Remark.** The main application of the previous result will appear at a slightly technical moment below. Yet, it also has a reassuring application, namely that our candidate-generators $\phi_{d,e}(\Lambda)$ do not really depend on the chosen desingularization $\mathcal{F}l(\Lambda)$ of the Schubert subvariety corresponding to the Young diagram Λ. Indeed the unit will remain the unit when pushed-forward between two such desingularizations, if one is obtained from the other by blow-up. (The condition on the relative bundle being even is automatically satisfied if they both have even relative bundle with respect to the Grassmannian.)

6. Main result

We are now ready to state and prove the main result of the paper.

6.1. **Theorem.** *Let $d, e \geqslant 1$. Let X be a regular scheme over $\mathbb{Z}[\frac{1}{2}]$. Let $\mathcal{V}$ be a vector bundle of rank $d + e$ which admits a complete flag* (13) *of subbundles. (For instance, when $\mathcal{V}$ is free.) Then, the elements $\phi_{d,e}(\Lambda)$ of Definition 3.10, for Λ even, form a basis of the graded $\mathrm{W}^{\mathrm{tot}}(X)$-module $\mathrm{W}^{\mathrm{tot}}(\mathrm{Gr}_X(d, \mathcal{V}))$.*

Proof. The proof occupies most of this Section. We fix a complete flag

$$0 = \mathcal{V}_0 \lhd \mathcal{V}_1 \lhd \cdots \lhd \mathcal{V}_{d+e} = \mathcal{V}\,.$$

We set as before $\mathcal{V}^1 := \mathcal{V}_{d+e-1}$.

The idea of the proof is very simple; it is an induction on $\mathrm{rk}(\mathcal{V}) = d + e$. The case $d = 1$ or $e = 1$ is slightly particular and treated on its own (see Remark 3.15 or Remark 6.3), so we assume $d, e \geqslant 2$ and the result proved for vector bundles of rank smaller than $d + e$.

The induction step is immediate by the 5-Lemma once we have established the commutativity of the following (horizontally-periodic) diagram:

(36)

$$\begin{array}{ccccccc}
\mathbb{F}_X(d,e-1) & \xrightarrow{\ \bar\iota\ } & \mathbb{F}_X(d,e) & \xrightarrow{\ \bar\kappa\ } & \mathbb{F}_X(d-1,e) & \xrightarrow{\ \bar\partial\ } & \mathbb{F}_X(d,e-1) \\
{\scriptstyle \phi_{d,e-1}}\downarrow\simeq & & {\scriptstyle \phi_{d,e}}\downarrow & & {\scriptstyle \phi_{d-1,e}}\downarrow\simeq & & {\scriptstyle \phi_{d,e-1}}\downarrow\simeq \\
\mathrm{W}^{\mathrm{tot}}(\mathrm{Gr}_X(d,\mathcal{V}^1)) & \xrightarrow{\iota_*} & \mathrm{W}^{\mathrm{tot}}(\mathrm{Gr}_X(d,\mathcal{V})) & \xrightarrow{\kappa} & \mathrm{W}^{\mathrm{tot}}(\mathrm{Gr}_X(d-1,\mathcal{V}^1)) & \xrightarrow{\partial} & \mathrm{W}^{\mathrm{tot}}(\mathrm{Gr}_X(d,\mathcal{V}^1))
\end{array}$$

whose first row is the exact sequence of Corollary 2.12 and whose second row is the exact sequence (35).

To check that Diagram (36) commutes amounts to control how the morphisms ι_*, κ and ∂ behave on the elements $\phi_{d,e}(\Lambda)$ constructed in Def. 3.10. Recall that the $\phi_{d,e}(\Lambda)$ are only defined when Λ is even. Note that Proposition 3.13 and Remark 5.5 imply that commutativity of (36) makes sense as far as degrees as concerned. So, rephrasing the commutativity of (36), the core of the proof is the following:

6.2. **Proposition.** *Let $d, e \geqslant 2$. Keep notations as above.*

(a) *Let Λ' be an even Young diagram in $(d\times(e-1))$-frame. The push-forward ι_* satisfies the following (see Figure 4):*
$$\iota_*(\phi_{d,e-1}(\Lambda')) = \phi_{d,e}(\bar{\iota}(\Lambda'))\,.$$

(b) *Let Λ be an even Young diagram in $(d\times e)$-frame. The "restriction" morphism κ satisfies the following (see Figure 5):*
$$\kappa(\phi_{d,e}(\Lambda)) = \phi_{d-1,e}(\bar{\kappa}(\Lambda))\,.$$

(c) *Let Λ'' be an even Young diagram $((d-1)\times e)$-frame. The connecting homomorphism ∂ satisfies the following (see Figure 6):*
$$\partial(\phi_{d-1,e}(\Lambda'')) = \phi_{d,e-1}(\bar{\partial}(\Lambda''))\,.$$

Proof. Recall from Corollary 2.12 that the definitions of $\bar{\iota}$, $\bar{\kappa}$ and $\bar{\partial}$ each involve two cases. The general organization of the proof is to start by proving the equations of (a), (b) and (c) in the first cases (the non-zero ones). The remaining cases will be easy to deduce from these.

Interesting case of (a). Let Λ' be an even $(d, e-1)$-diagram such that $\zeta(\Lambda')$ is even. Let $\underline{d}$ and $\underline{e}$ be the corresponding k-tuples (Def. 2.3). By assumption we can consider the even (d, e)-diagram $\bar{\iota}(\Lambda') := (\Lambda'_1 + 1, \ldots, \Lambda'_d + 1)$. Observe that it has the same associated k-tuples (with respect to the $(d\times e)$-frame). From (14), it is then easy to see that $\mathcal{F}l_X(d, e-1; \Lambda') = \mathcal{F}l_X(d, e; \bar{\iota}(\Lambda'))$ and that the diagram

$$\begin{array}{ccc}
\mathrm{Gr}_X(d,\mathcal{V}^1) & \xrightarrow{\ \iota\ } & \mathrm{Gr}_X(d,\mathcal{V}) \\
{\scriptstyle f_{d,e-1;\Lambda'}}\uparrow & & \uparrow{\scriptstyle f_{d,e,\bar{\iota}(\Lambda')}} \\
\mathcal{F}l_X(d,e-1;\Lambda') & = & \mathcal{F}l_X(d,e;\bar{\iota}(\Lambda'))
\end{array}$$

commutes. In that case, (a) follows by composition of push-forwards.

Interesting case of (b). Let Λ be an even (d, e)-diagram such that $\Lambda_d = 0$. Let $\underline{d}$ and $\underline{e}$ be the corresponding k-tuples (Def. 2.3). By assumption we can consider the even $(d-1, e)$-diagram $\bar{\kappa}(\Lambda) := \Lambda_{|_{d-1,e}}$. We then have two cartesian squares:

$$\begin{array}{ccccc}
\mathrm{Gr}_X(d,\mathcal{V}) & \xleftarrow{\ \upsilon\ } & U_X(d,\mathcal{V}_\bullet) & \xrightarrow{\ \alpha\ } & \mathrm{Gr}_X(d-1,\mathcal{V}^1) \\
{\scriptstyle f_{d,e;\Lambda}}\uparrow & \square & \uparrow & \square & \uparrow{\scriptstyle f_{d-1,e;\bar{\kappa}(\Lambda)}} \\
\mathcal{F}l_X(d,e;\Lambda) & \longleftarrow & U' & \longrightarrow & \mathcal{F}l_X(d-1,e;\bar{\kappa}(\Lambda))
\end{array}$$

by Lemma 4.7. Replacing κ by its definition (see 5.1) and using that α^* is an isomorphism, the equality in (b) claims that

$$\upsilon^*(\phi_{d,e}(\Lambda)) = \alpha^*(\phi_{d-1,e}(\Lambda_{|_{d-1,e}})) \qquad \text{in} \qquad \mathrm{W}^{\mathrm{tot}}(U_X(d,\mathcal{V}_\bullet))\,.$$

This follows by base change on the above two cartesian squares (see [7], the horizontal morphisms are smooth, so flat, and the vertical maps are proper, including $U' \to U_X(d,\mathcal{V}_\bullet)$). Both sides of the above equation are equal in $\mathrm{W}^{\mathrm{tot}}(U_X(d,\mathcal{V}_\bullet))$ to the push-forward of the same $1_{U'} \in \mathrm{W}^0(U')$ along $U' \to U_X(d,\mathcal{V}_\bullet)$.

Interesting case of (c). Let Λ'' be an even $(d-1,e)$-diagram such that Λ''_{d-1} is odd. Let $\underline{d}$ and $\underline{e}$ be the corresponding k-tuples (Def. 2.3). By assumption, we can consider the even $(d,e-1)$-diagram

$$\bar{\partial}(\Lambda'') := (\Lambda''_1 - 1, \ldots, \Lambda''_{d-1} - 1, 0)\,.$$

The twist $t(\Lambda'')$ is given by $t(\Lambda'') = [d+1] \in \mathbb{Z}/2$. Using Diagram (27) and Equation (30), one can easily check that this is the twist for which the connecting homomorphism ∂ can be computed by first pulling back to $E_X(d,\mathcal{V}_\bullet)$ and then pushing-forward to $\mathrm{Gr}_X(d,\mathcal{V}^1)$, see [6, Theorem 2.6]. By Lemma 4.8, we have the right-hand cartesian square in the following commutative diagram:

$$\begin{array}{ccccc}
\mathrm{Gr}_X(d,\mathcal{V}^1) & \xleftarrow{\ \tilde{\pi}\ } & E_X(d,\mathcal{V}_\bullet) & \xrightarrow{\ \tilde{\alpha}\tilde{\iota}\ } & \mathrm{Gr}_X(d-1,\mathcal{V}^1) \\
{\scriptstyle f_{d,e-1;\bar{\partial}(\Lambda'')}}\uparrow & & \uparrow & \square & \uparrow{\scriptstyle f_{d-1,e;\Lambda''}} \\
\mathcal{F}l_X(d,e-1;\bar{\partial}(\Lambda'')) & \xleftarrow{\ \pi'\ } & F' & \longrightarrow & \mathcal{F}l_X(d-1,e;\Lambda'')\,.
\end{array}$$

We can now compute $\partial(\phi_{d-1,e}(\Lambda''))$ by base-change on the right-hand square and composition of push-forwards, starting with the unit form on $\mathcal{F}l_X(d-1,e;\Lambda'')$. The key point is to check that π'_* preserves this unit form. By Lemma 4.8, we know that π' is either an isomorphism or a blow-up along a closed regular immersion of odd codimension. In both cases $\pi'_*(1) = 1$ by Proposition 5.6 and we get the result.

Remaining cases. At this stage, we have proved the equations of (a), (b) and (c) in the first cases (the non-zero ones). In the remaining cases, both sides of these equations are zero, as easily follows from the above cases and diagram chase in (36).

For instance, let us prove that both sides of (b) vanish for an even (d,e)-diagram Λ such that $\Lambda_d > 0$. Since $\bar{\kappa}(\Lambda) = 0$ it suffices to show $\kappa(\phi_{d,e}(\Lambda)) = 0$. By Corollary 2.12, there exists an even $(d,e-1)$-diagram Λ' such that $\zeta(\Lambda')$ is even and $\bar{\iota}(\Lambda') = \Lambda$. We have established equation (a) for such Λ'. Applying κ to this equation gives $\kappa\iota_*(\phi_{d,e-1}(\Lambda')) = \kappa(\phi_{d,e}(\Lambda))$ hence the result since $\kappa\,\iota_* = 0$. The other cases (Λ' with $\zeta(\Lambda')$ odd and Λ'' with Λ''_{d-1} even) can be proved similarly. $\square$

6.3. **Remark.** Following up on Remark 3.15, we now unfold the case $d = 1$ (resp. $e = 1$) of our constructions from Sections 4 and 5. The difference comes from the apparition of $\mathrm{Gr}_X(0,\mathcal{V}^1)$ (resp. $\mathrm{Gr}_X(d,\mathcal{V}_d)$), which is simply X, in which case the Picard groups do not look like in (27). This involves an artificial total Witt group of $\mathrm{Gr}_X(0,\mathcal{V}^1)$ (resp. $\mathrm{Gr}_X(d,\mathcal{V}_d)$) in the localization long exact sequence (35), not summing on its "own" twists but on the "ambient" twists of $\mathrm{Gr}_X(d,\mathcal{V})$, as explained in Remark 5.4. Proposition 6.2 then becomes the following (only the cases involving X are covered, the other ones having already been done).

6.4. **Proposition.** *Let $d = 1$ (resp. $e = 1$) and let 1_i, $i \in \mathbb{Z}/2$ be two copies of the unit form on X, one for each twist in $\mathbb{Z}/2\,\Delta_d$ of the ambient $\mathrm{Gr}_X(d,\mathcal{V})$. As before, $[d \times e]$ is the full diagram and $\varnothing$ is the empty one. When $e = 1$, the push-forward satisfies the following:*

$$(37) \qquad \iota_*(1_{d+1}) = 0 \ \textit{and}\ \iota_*(1_d) = \phi_{d,1}([d \times 1]).$$

When $d = 1$, the morphism κ satisfies the following:

$$(38) \qquad \kappa(\phi_{1,e}(\varnothing)) = 1_0$$

When $d = 1$ (resp. $e = 1$), the connecting homomorphism ∂ satisfies the following:

$$(39) \qquad \partial(1_1) = \phi_{1,e}(\varnothing) \ \textit{and}\ \partial(1_0) = 0 \quad (\textit{resp.}\ \partial([(d-1) \times 1]) = 1_{d+1})\,.$$

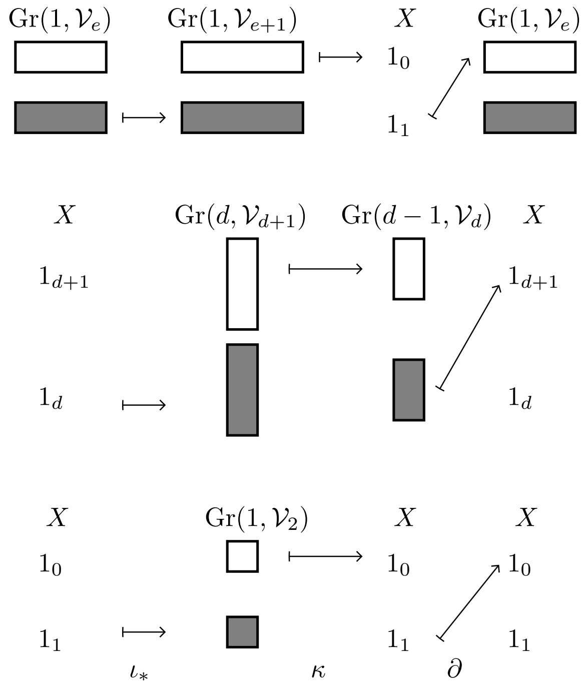


FIGURE 9. Image of the generating elements by the morphisms ι_*, κ and ∂ in three special cases when $d=1$, $e=1$ and $d=e=1$. No arrow means mapped to zero.

Since the proof works as the one of the previous proposition, we leave the details to the reader. See Figure 9.

The induction step in the proof of Theorem 6.1 then also works in the case $d=1$ or $e=1$, using $\mathbb{F}_X(0,e)=\mathbb{F}_X(d,0)=\mathrm{W}^{\mathrm{tot}}(X).1_0\oplus\mathrm{W}^{\mathrm{tot}}(X).1_1$. Its proof is therefore completed. □

6.5. **Remark.** Of course every $\phi_{d,e}(\Lambda)$ has a particular shift and twist. When X is local, and more generally when $\mathrm{W}^{\mathrm{tot}}(X)=\mathrm{W}^0(X)$, one obtains a basis of $\mathrm{W}^i(\mathrm{Gr}_X(d,\mathcal{V}),L)$, for $i\in\mathbb{Z}/4$ and $L\in\mathrm{Pic}/2$ fixed, by selecting the generators $\phi_{d,e}(\Lambda)$ with shift i and twist L.

6.6. **Corollary.** *Let d' (resp. e') be the integral part of $d/2$ (resp. $e/2$) and consider the binomial coefficient $\binom{a+b}{a}=\frac{(a+b)!}{a!b!}$. As modules over $\mathrm{W}^{\mathrm{tot}}(X)$, the total Witt group of $\mathrm{Gr}_X(d,\mathcal{V})$ is free of rank $2\binom{d'}{e'}$. If we assume moreover that $\mathrm{W}^{\mathrm{tot}}(X)=\mathrm{W}^0(X)$, for instance for X local, then*

- *the classical Witt group of symmetric forms $\mathrm{W}^0(\mathrm{Gr}_X(d,\mathcal{V}),0)$ has rank $\binom{d'+e'}{e'}$,*
- *the classical Witt group of antisymmetric forms $\mathrm{W}^2(\mathrm{Gr}_X(d,\mathcal{V}),0)$ is zero.*
- *the Witt groups $\mathrm{W}^1(\mathrm{Gr}_X(d,\mathcal{V}),1)$ and $\mathrm{W}^3(\mathrm{Gr}_X(d,\mathcal{V}),1)$ are zero.*

Proof. Let "4-block" mean "2×2 square". Every even Young diagram Λ is either

(a) a union of 4-blocks and $\phi(\Lambda)$ is in $\mathrm{W}^0(\mathrm{Gr}_X(d,\mathcal{V}),0)$,
(b) a single row plus 4-blocks (e even) and $\phi(\Lambda)$ is in $\mathrm{W}^e(\mathrm{Gr}_X(d,\mathcal{V}),1)$,
(c) a single column plus 4-blocks (d even) and $\phi(\Lambda)$ is in $\mathrm{W}^d(\mathrm{Gr}_X(d,\mathcal{V}),1)$,

(d) a single row and a single column plus 4-blocks (d and e odd) and $\phi(\Lambda)$ is in $\mathrm{W}^{d+e-1}(\mathrm{Gr}_X(d,\mathcal{V}),0)$.

All possibilities (a)-(d) are exclusive and can be enumerated easily by counting the diagrams of 4-blocks, which amounts to counting the usual Young diagrams in $(d'\times e')$-frame. We get $\binom{d'+e'}{e'}$ elements in case (a), and the other results depend on the parities of d and e but are also very easy to figure out in each case. □

6.7. **Corollary.** *The connecting homomorphism ∂ is zero (and thus the long exact sequence (35) for $\mathrm{Gr}_X(d,\mathcal{V})$ decomposes as split short exact sequences as for Chow groups) if and only if both d and e are even.*

Proof. Looking back at the proof of the main theorem, and at (19) (or Figure 6), we see that ∂ is zero if and only if there is no even $(d-1,e)$-diagram Λ'' such that Λ''_{d-1} is odd. This implies that e is even (otherwise $\Lambda''=[(d-1)\times e]$ would be an even diagram) and that $d-1$ is odd (otherwise $\Lambda''=(1,\ldots,1)$ would be an even diagram). Conversely, assume e even and the existence of an even $(d-1,e)$-diagram Λ'' such that Λ''_{d-1} is odd. Then e_k is odd (since $e=\Lambda''_{d-1}+e_k$ is even), hence all e_i are odd since Λ'' is an even diagram. In particular, e_1 is odd, hence $e_1>0$ and therefore d_1 is even. This implies that $d-1=d_k=(d_k-d_{k-1})+\ldots+(d_2-d_1)+d_1$ is even, *i.e.* d is odd, as was to be shown. □

6.8. **Notation.** For $d,e\geqslant 1$, we denote by $\mathbb{G}_X(d,e)$ the split Grassmannian

$$\mathbb{G}_X(d,e)=\mathrm{Gr}_X(d,\mathcal{O}_X^{d+e})\,.$$

6.9. **Example.** Figure 10 shows how the different generators map to each other by ι_*, κ and ∂ in the long exact sequence (35) for $\mathbb{G}(3,3)$.

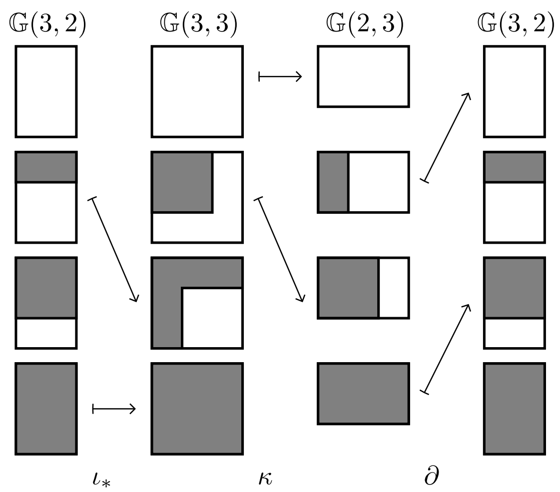


FIGURE 10. Total long exact sequence on generators (no arrow means mapped to zero)

6.10. **Example.** Figures 11, 12 and 13 give the even Young diagrams in $(d{\times}e)$-frame and the corresponding shifts in $\mathbb{Z}/4$ and twists in $\mathbb{Z}/2\,\Delta_d$ for the Grassmannians $\mathbb{G}(4,4)$, $\mathbb{G}(4,5)$ and $\mathbb{G}(5,5)$.

* * *

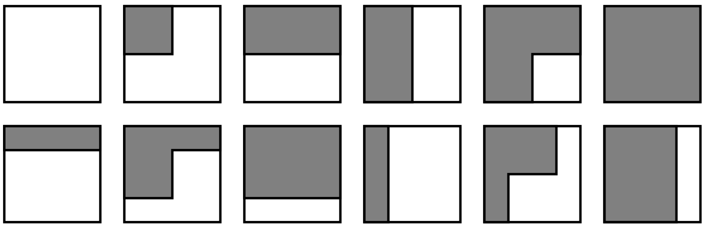


FIGURE 11. Diagrams of generators for $\mathbb{G}(4,4)$: first row in shift 0 and twist 0, second row in shift 0 and twist 1.

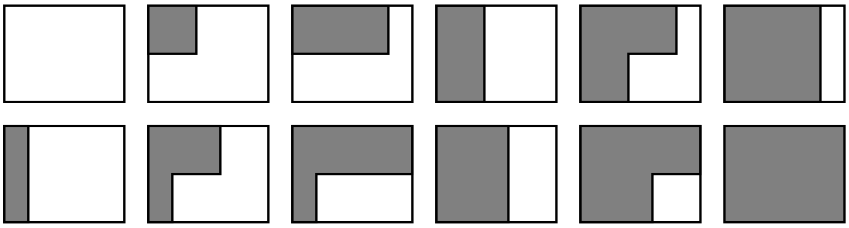


FIGURE 12. Diagrams of generators for $\mathbb{G}(4,5)$: first row in shift 0 and twist 0, second row in shift 0 and twist 1.

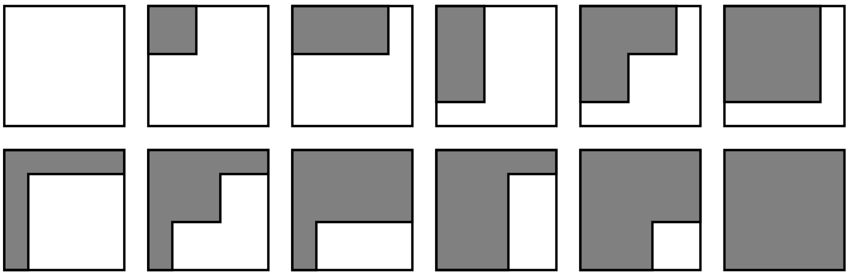


FIGURE 13. Diagrams of generators for $\mathbb{G}(5,5)$: first row in shift 0 and twist 0, second row in shift 1 and twist 1.

We conclude with a few comments.

6.11. **Remark.** We could have considered a larger set of elements $\phi_{d,e}(\Lambda)$ using Remark 3.1 instead of assuming Λ even. This larger set is also stable by applying ι_*, κ and ∂. Some of these extra elements are then easily seen to be zero from the exact sequence, but not all of them.

6.12. **Remark.** Part of the ring structure on the total Witt group can be computed at each induction step using the projection formula. Unfortunately, this is not enough for the whole computation. Despite the results for the Grothendieck and the Chow rings using basis of Schubert cells, it is unclear to the authors what kind of Littlewood-Richardson type rule one should expect.

Note however that all generators $\phi_{d,e}(\Lambda)$ are nilpotent except for $\Lambda = \varnothing$. This can be checked using homotopy invariance and a discussion on the supports or simply the fact that these Witt classes are generically trivial.

6.13. **Remark.** Although we don't need it here, it is possible to show that, for $\mathcal{V} = \mathcal{O}_X^{d+e}$, the isomorphism $\mathbb{G}(d,e) = \mathrm{Gr}(d,\mathcal{V}) \simeq \mathrm{Gr}(e,\mathcal{V}^\vee) = \mathbb{G}(e,d)$ sends $\phi_{d,e}(\Lambda)$ to $\phi_{e,d}(\Lambda^\vee)$ where $\Lambda^\vee$ is the dual partition.

Appendix A. Total Witt group

For $i \in \mathbb{Z}$ and for a line bundle L over X, the Witt group $\mathrm{W}^i(X, L)$ is the triangular Witt group of the bounded derived category of vector bundles over X with respect to the duality derived from $\mathcal{H}om_{\mathcal{O}_X}(-, L)$ and shifted i times. See [3, 4]. These groups are periodic in two elementary ways. First they are 4-periodic in shift, *i.e.* there is an isomorphism

$$\mathrm{W}^i(X, L) \simeq \mathrm{W}^{i+4}(X, L)$$

by [3, Proposition 2.14]. This isomorphism is completely canonical and easily goes through all other constructions, so we do not discuss it any further.

We then have a product

$$\mathrm{W}^i(X, L) \times \mathrm{W}^j(X, M) \to \mathrm{W}^{i+j}(X, L \otimes M)$$

by [11] that yields, for any line bundle M, a periodicity isomorphism

$$\mathrm{W}^i(X, L) \simeq \mathrm{W}^i(X, L \otimes M^{\otimes 2}) \tag{40}$$

given by multiplication with the Witt class $[M \xrightarrow{\sim} M^\vee \otimes M^{\otimes 2}] \in \mathrm{W}^0(X, M^{\otimes 2})$, where $M^\vee$ denotes the dual of M.

From these two periodicity isomorphisms, it is clear that all the information about the Witt groups of a scheme can be concentrated in a *total Witt group*, summing Witt groups indexed by elements in $\mathrm{Pic}/2$ and integers in $\mathbb{Z}/4$. However, this total group is not canonical since it involves the choice of a line bundle L for every class in $\mathrm{Pic}/2$. Furthermore, if we want to turn this total Witt group into a ring, using the above product, we need to choose isomorphisms between $L \otimes M$ and the line bundle representing $L + M$ in $\mathrm{Pic}/2$, including the choice of "square roots" (for the periodicity modulo 2), and so on. All this data should further satisfy some compatibilities, of the highest sex appeal. Unfolding these technicalities would simultaneously increase the size of the article and reduce its readability, beyond the taste of any potential reader. Therefore we make the following choice:

A.1. **Convention.** We treat all periodicity isomorphisms as identities:

$$\mathrm{W}^i(X, L) = \mathrm{W}^{i+4}(X, L) \quad \text{and} \quad \mathrm{W}^i(X, L) = \mathrm{W}^i(X, L \otimes M^{\otimes 2}).$$

We see two ways of providing a formal ground for this convention, at least in our discussion of the total Witt group of the Grassmann varieties over X.

The first possibility is to assume the above choices for the base scheme X and then use the explicit isomorphism $\mathrm{Pic}(\mathrm{Gr}_X(d, \mathcal{V})) \cong \mathrm{Pic}(X) \oplus \mathbb{Z} \cdot \Delta_d$ to extend the choices made for X to $\mathrm{Gr}(d, \mathcal{V})$, by using the tensor powers of Δ_d in a natural way.

The second possibility can be applied beyond Grassmann varieties. It starts with the observation that two different choices of the above conglomerate of line bundles and compatibility isomorphisms would only differ "by multiplication by a global unit". The fact that some collection of Witt classes forms a basis of the graded Witt ring is stable by multiplication of these basis elements by units. As long as the proof of such a fact never uses addition in $\mathrm{W}^{\mathrm{tot}}$ but only the 5-Lemma, then it is insensible to the ambiguity of the choices. Indeed, it is easily checked that the 5-Lemma holds if its 4 squares only commute "up to units".

Deciding which method is more pleasant is left to the insomniac reader.

References

1. *Théorie des intersections et théorème de Riemann-Roch*, Springer-Verlag, Berlin, 1971, Séminaire de Géométrie Algébrique du Bois-Marie 1966–1967 (SGA 6), Dirigé par P. Berthelot, A. Grothendieck et L. Illusie. Springer Lecture Notes in Mathematics, Vol. 225.
2. J. K. Arason, *Der Wittring projektiver Räume*, Math. Ann. **253** (1980), no. 3, 205–212.

3. P. Balmer, *Triangular Witt groups Part I: The 12-term localization exact sequence*, K-Theory **4** (2000), no. 19, 311–363.
4. ———, *Triangular Witt groups Part II: From usual to derived*, Math. Z. **236** (2001), no. 2, 351–382.
5. ———, *Products of degenerate quadratic forms*, Compositio Mathematica **6** (2005), no. 141, 1374–1404.
6. P. Balmer and B. Calmès, *Geometric description of the connecting homomorphism for Witt groups*, preprint, 2008.
7. B. Calmes and J. Hornbostel, *Witt motives, transfers and dévissage*, preprint, 2006.
8. M. Demazure and P. Gabriel, *Groupes algébriques. Tome I: Géométrie algébrique, généralités, groupes commutatifs*, Masson & Cie, Éditeur, Paris, 1970.
9. W. Fulton, *Intersection theory*, second ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, vol. 2, Springer-Verlag, Berlin, 1998.
10. S. Gille, *Homotopy invariance of coherent Witt groups*, Math. Z. **244** (2003), no. 2, 211–233.
11. S. Gille and A. Nenashev, *Pairings in triangular Witt theory*, J. Algebra **261** (2003), no. 2, 292–309.
12. N. A. Karpenko, *Cohomology of relative cellular spaces and of isotropic flag varieties*, Algebra i Analiz **12** (2000), no. 1, 3–69.
13. M. Knebusch, *Symmetric bilinear forms over algebraic varieties*, Queen's Papers in Pure and Appl. Math., No. 46, Queen's Univ., Kingston, Ont., 1977, pp. 103–283.
14. D. Laksov, *Algebraic Cycles on Grassmann Varieties*, Adv. in Math. **9** (1972), no. 3, 267–295.
15. M. Levine and F. Morel, *Algebraic cobordism*, Springer Monographs in Mathematics, Springer, Berlin, 2007.
16. Qing Liu, *Algebraic geometry and arithmetic curves*, Oxford Graduate Texts in Mathematics, vol. 6, Oxford University Press, Oxford, 2002.
17. I. Panin, *Riemann-Roch theorems for oriented cohomology*, Axiomatic, enriched and motivic homotopy theory, NATO Sci. Ser. II, vol. 131, Kluwer, Dordrecht, 2004, pp. 261–333.
18. R. W. Thomason, *Les K-groupes d'un schéma éclaté et une formule d'intersection excédentaire*, Invent. Math. **112** (1993), no. 1, 195–215.
19. C. Walter, *Grothendieck-Witt groups of projective bundles*, preprint, 2003.

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