SUMS OF SQUARES IN FUNCTION FIELDS OF QUADRICS AND CONICS

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Abstract. For a quadric Q over a real field k, we investigate whether finiteness of the Pythagoras number of the function field $k(Q)$ implies the existence of a uniform bound on the Pythagoras numbers of all finite extensions of k. We give a positive answer if the quadratic form that defines Q is weakly isotropic. In the case where Q is a conic, we show that the Pythagoras number of $k(Q)$ is 2 only if k is hereditarily pythagorean.

1. Introduction

For a field K we denote by $K^\times$ the multiplicative group, further by $K^{\times 2}$ the subgroup of nonzero squares and by $\sum K^2$ the subgroup of nonzero sums of squares. For $x \in K$ we set

$$\ell_K(x) = \inf\left\{n \in \mathbb{N} \mid \exists x_1, \dots, x_n \in K \text{ s.t. } x = x_1^2 + \cdots + x_n^2\right\}$$

and call it the *length of x in K*. Two interesting field invariants for the study of sums of squares in K are the *level*

$$s(K) = \ell_K(-1)$$

and the *Pythagoras number*

$$p(K) = \sup\left\{\ell_K(x) \mid x \in \sum K^2\right\}.$$

By the Artin-Schreier Criterion ([Lam, Chap. VIII, (1.11)]), K admits a field ordering if and only if $s(K) = \infty$. In this case we say that K is *real*, otherwise *nonreal*. For nonreal fields, Pfister [Pf, Chap. 3, Thm 1.3, Thm 1.4] showed that the possible values for the level are exactly the powers of 2. For the Pythagoras number, Hoffmann [H] showed that for any positive integer n, there exists a real (even uniquely ordered) field K with $p(K) = n$. Given a field extension L/K of a certain type (e.g. quadratic, finite, or finitely generated) little is known about the relation between $p(L)$ and $p(K)$. If L/K is a finite field extension, then it is

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known that $p(L) \leq [L:K]p(K)$ ([Pf, Chap. 7, Proposition 1.13]), but at present one doesn't know of any finite extension L/K where $p(L) > p(K) + 1$.
A field K is called *pythagorean* if $p(K) = 1$, i.e. if $\sum K^2 = K^{\times 2}$. A real field K is called *hereditarily pythagorean* if every finite real field extension of K is pythagorean.

Theorem 1.1 (Diller-Dress)**.** *Let L/K be a finite field extension. If L is pythagorean, then so is K.*

For a proof, see [Lam, Chap.VIII, Theorem 5.7]. Note that this 'Going Down' does not generalise to Pythagoras numbers other than 1, since for any field K with $p(K)$ arbitrarily large, we can find a quadratic extension L/K with $p(L) = 2$, just take $L = K(\sqrt{-1})$. Moreover and less trivially, in [Pr] examples of *real* quadratic extensions L/K are constructed, where $p(L) = 2$ and $p(K)$ is larger than an aribtrary given integer, and K is uniquely ordered.

We now fix a ground field k and consider function fields F/k. If k is real closed and the transcendence degree of F/k is n, then we have the Pfister bound $p(F) \leq 2^n$ ([Pf, Chap.6, Corollary 3.4]). This leads naturally to the search for weaker conditions on the field k that imply bounds on $p(F)$ in terms of $p(k)$ and the transcendence degree of F/k. It is not even known whether finiteness of the Pythagoras number of k goes up to $k(X)$. Conversely, $p(k(X)) < \infty$ implies $p(k) < \infty$.
In fact, $p(k(X)) < \infty$ is equivalent to the existence of a common upper bound on $p(L)$ for all finite extensions L/k by the following statement obtained in [BvG, Theorem 3.5].

Theorem 1.2. *Let k be a real field and $n \in \mathbb{N}$. Then the following are equivalent:*
  (i) $p(k(X)) \leq 2^n$
 (ii) $p(L) < 2^n$ *for every finite real extension* L/k.
(iii) $s(M) \leq 2^{n-1}$ *for every finite nonreal extension* M/k.

This leads us to the qualitative question, for what kind of function fields F/k does finiteness of $p(F)$ imply the existence of an $n \in \mathbb{N}$ such that $p(L) \leq n$ for all finite L/k ?
In this work, we focus on function fields F of (projective) quadrics. We show, that if the quadratic form defining the quadric is weakly isotropic in k, then $p(F) < \infty$ implies the existence of a common upper bound on the Pythagoras numbers of all finite extensions of k. In the case of certain projective conics, we also give explicit upper bounds on the Pythagoras numbers of the finite extensions of k.

Theorem 1.2 generalises a characterisation of hereditarily pythagorean fields given in [Bk].

Corollary 1.3 (Becker)**.** *For a real field k, we have $p(k(X)) = 2$ if and only if k is hereditarily pythagorean.*

We generalise the 'only if' implication of this result, replacing the rational function field in one variable by the function field of an arbitrary (not necessarily rational) conic.

Theorem 1.4. *Let k be a real field. If there exists a conic C over k such that $p(k(C)) = 2$, then k is hereditarily pythagorean.*

In [TY, Theorem 3], the converse implication in Becker's result was already generalised to real function fields of conics. Both results together yield the following generalisation of Corollary 1.3.

Corollary 1.5. *Let C be a conic over k with real function field $k(C)$. Then $p(k(C)) = 2$ if and only if k is hereditarily pythagorean.*

2. Valuations on Function Fields of Hyperelliptic Curves

In this section, we explain how k-valuations on the rational function field $k(X)$ extend to a quadratic extension (i.e. the function field of a hyperelliptic curve over k). More precisely, we determine the extension of the residue field. But first, we make a general observation for valuations with nonreal residue fields and value group tthat is not 2-divisible, namely that the Pythagoras number of the valued field is a strict upper bound on the level of the residue field.

Let w be a valuation on K with valuation ring $\mathcal{O}_w$, maximal ideal $\mathfrak{m}_w$, residue field κ_w and value group Γ_w. We call w a *non 2-divisible valuation*, if there is an isomorphism $\Gamma_w \neq 2\Gamma_w$.

Proposition 2.1. *Let w be a non 2-divisible valuation on K with nonreal residue field κ_w of characteristic different from 2. Then $p(K) > s(\kappa_w)$.*

Proof. Let $s = s(\kappa_w)$. Then there exist $x_0, \ldots, x_s \in \mathcal{O}_w^\times$ with $\bar{x}_0^2 + \ldots + \bar{x}_s^2 = 0$. We may assume that $w(x_0^2 + \ldots + x_s^2)$ is odd; in fact, if $w(x_0^2 + \ldots + x_s^2) \in 2\Gamma_w$, we simply replace x_s by $(x_s + t)$ for some $t \in K$ with $0 < w(t) < w(x_0^2 + \ldots + x_s^2)$ and $w(t) \notin 2\Gamma_w$. We show, that $x_0^2 + \ldots + x_s^2$ cannot be written as a sum of less than $s+1$ squares in K. Assume there are $y_1, \ldots, y_s \in K$ with $y_1^2 + \ldots + y_s^2 = x_0^2 + \ldots + x_s^2$, and where $w(y_1) \leq w(y_i)$ for $1 \leq i \leq s$, thus $z_i = \frac{y_i}{y_1} \in \mathcal{O}_w$ for $2 \leq i \leq s$. Since

$w(y_1^2 + \ldots + y_s^2)$ is odd, it follows that $w(1 + z_1^2 + \ldots z_s^2) > 0$. We obtain the equality $-1 = \overline{z}_2^2 + \ldots + \overline{z}_s^2$ in κ_w, contradicting $s = s(\kappa_w)$. $\square$

Let L/K be a field extension. Let v be a valuation on K. We call a valuation w on L an *extension of v to L* if $\Gamma_v \subseteq \Gamma_w$ and $w|_K = v$. By Chevalley's Theorem, such a extension always exists (a general reference for extensions of valuations is [EP, Chap. 3]). Consider now the situation where L/K is finite. If Γ_v is not 2-divisible, then so is Γ_w, since $[\Gamma_w : \Gamma_v] < \infty$. Also the inertia degree $[\kappa_w : \kappa_v]$ is finite. For quadratic separable extensions L/K, the following can be said about extensions of valuations.

Remark 2.2. Let v be a valuation on K, let L/K be a quadratic separable extension, and $\overline{} : L \to L$ the nontrivial K-automorphism of L. Let w be an extension of v to L. Then w and $w \circ \overline{}$ are all extensions of v to L, and if $w \circ \overline{} \neq w$ then the value groups of both extensions are just Γ_v (i.e. v is nonramified in L) and the residue fields of both extensions are just κ_v. In general the fundamental inequality $2 \geq [\Gamma_w : \Gamma_v][\kappa_w : \kappa_v]$ holds. (See [EP, Theorem 3.2.15, Theorem 3.3.5])

Consider the rational function field in one variable $k(X)$ over k and an irreducible polynomial $p \in k[X]$. Then there is a unique k-valuation $v_p : k(X) \to \mathbb{Z} \cup \{\infty\}$ with $v_p(p) = 1$. The residue field of v_p is $k[X]/(p)$, the root field of p. Now consider the hyperelliptic curve $Y^2 = g(X)$ for some square free polynomial $g \in k[X]$. (We call $g \in k[X]$ square free, if $h^2 \not| g$ for all $h \in k[X] \setminus k$). Let F denote its function field, i.e. the quadratic extension $k(X)(\sqrt{g})$ of the rational function field $k(X)$.

Lemma 2.3. *Let k be a field with* $\mathrm{char(k)} \neq 2$ *and $F = k(X)(\sqrt{g})$ for some square free polynomial $g \in k[X]$. Let $p \in k[X]$ be an irreducible polynomial and $v_p : k(X) \to \mathbb{Z} \cup \{\infty\}$ the corresponding k-valuation. Let α be a root of p. Let w be an extension of v_p to F. Then the residue field of w is $k(\alpha)(\sqrt{g(\alpha)})$.*

Proof. Note that $w(\sqrt{g}) = \frac{1}{2}v_p(g) \geq 0$ and in κ_w we have $(\overline{\sqrt{g}})^2 = \bar{g} = g(\alpha)$. This shows, that $k(\alpha)(\sqrt{g(\alpha)}) \subseteq \kappa_w$. From Remark 2.2 we know that $[\kappa_w : k(\alpha)] \leq 2$. Hence, if $\sqrt{g(\alpha)} \notin k(\alpha)$, then $k(\alpha)(\sqrt{g(\alpha)}) = \kappa_w$ follows immediately.
Suppose now that $g(\alpha)$ is a square in $k(\alpha)$. Let $\overline{}$ be the nontrivial $k(X)$-automorphism of F. If $w \neq w \circ \overline{}$, then $\kappa_w = \kappa_{v_p} = k(\alpha) = k(\alpha)(\sqrt{g(\alpha)})$ by Remark 2.2. Hence we can suppose that $w = w \circ \overline{}$. Since $g(\alpha) = \bar{g}$ is a square in the residue field of v_p, we can choose some $h \in k[X]$ with $v_p(h^2 - g) > 0$. Replacing h by $h + p$ if neccessary, we can even arrange that $v_p(h^2 - g) = 1$. (Note that $v_p(0^2 - g) = 1$ if $p \mid g$ in $k[X]$, since g is assumed to be square free).

We write $z = h + \sqrt{g} \in F$ and obtain $h^2 - g = z\overline{z}$. Since $w = w \circ \overline{}$, using Remark 2.2 we get

$$1 = v_p(h^2 - g) = w(z) + w(\overline{z}) = 2w(z).$$

We see that $\frac{1}{2} \in \Gamma_w$ and thus $[\Gamma_w : \Gamma_v] = 2$. With Remark 2.2 it follows that $[\kappa_w : \kappa_v] = 1$, i.e. $\kappa_w = \kappa_{v_p} = k(\alpha) = k(\alpha)(\sqrt{g(\alpha)})$. □

3. Certain primitive elements for nonpythagorean extensions

In this section, we will show that for a finite real nonpythagorean extensions L/k, one can always find primitive elements for L/k that are sums of squares in L with some additional properties. This result, together with our considerations on residue field extensions in the previous section, will enable us to construct a k-valuation on a function field of a give conic over k, that has a nonreal residue field where -1 is not a square. Applying Proposition 2.1 to this situation afterwards, this will show us that the Pythagoras number of the function field is larger than 3.

Proposition 3.1. *Let L/k be a finite field extension such that L is real and nonpythagorean.*

(i) Let $c \in L^\times$, $d \in L$ and $\nu \in L$ such that $L = k(\nu)$. Then
  (a) $L = k\left(c(\nu + x)^2 + d\right)$ for almost every[1] $x \in k$.
  (b) for almost every $\alpha \in k^\times$, there are infinitely many $x \in k$ such that $L = k\left(c\frac{\alpha^2}{(\nu+x)^2} + d\right)$.

(ii) There exists $\xi \in L$ such that $L = k(\xi^2)$ and $\xi^2 + 1 \notin L^{\times 2}$.

(iii) There exists $\sigma \in \sum L^2 \setminus L^{\times 2}$ such that $L = k(\sigma)$ and $\sigma + 1 \notin L^{\times 2}$.

Proof. (i): Suppose

$$\mathcal{G} = \left\{x \in k \mid k\left(c(\nu + x)^2 + d\right) \subsetneq L\right\}$$

is infinite. Since L/k has only finitely many intermediate extensions, there exists a proper subfield K of L that contains k, for which

$$\left\{x \in \mathcal{G} \mid c(\nu + x)^2 + d \in K\right\}$$

is infinite. Let $x_1, x_2, x_3 \in k$ be three distinct elements from this set. Then

$$\begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{pmatrix} \begin{pmatrix} c\nu^2 + d \\ 2c\nu \\ c \end{pmatrix} = \begin{pmatrix} c(\nu + x_1)^2 + d \\ c(\nu + x_2)^2 + d \\ c(\nu + x_3)^2 + d \end{pmatrix} \in K^3.$$

Since the Vandermonde matrix has determinant $(x_3 - x_2)(x_2 - x_1)(x_1 - x_3) \neq 0$, we can multiplying with its inverse in $M_3(K)$ from the left and obtain that

[1] 'almost every' in our context means 'all but finitely many'

$(c\nu^2 + d, 2c\nu, c) \in K^3$. So $c \in K^\times$, $2c\nu \in K$ contradicting the assumption that $\nu \notin K$.

For the second part, we assume there are infinitely many $\alpha \in k^\times$ such that $k(c\frac{\alpha^2}{(\nu+x)^2} + d) \neq L$ for all but finitely many $x \in k$. Let n denote the number of proper subfields of L containing k. Choose $\alpha_0, \ldots, \alpha_n$ such that the $\mathcal{N}_{\alpha_i} := \{x \in k \mid k(c\frac{\alpha_i^2}{(\nu+x)^2} + d) \neq L\}$ is cofinite in k for $0 \leq i \leq n$. Then $\mathcal{N}_{\alpha_0} \cap \cdots \cap \mathcal{N}_{\alpha_n}$ is cofinite in k. For every $x \in \mathcal{N}_{\alpha_0} \cap \cdots \cap \mathcal{N}_{\alpha_n}$ there are $0 \leq i_x < j_x \leq n$ such that $c\frac{\alpha_{i_x}^2}{(\nu+x)^2} + d$ and $c\frac{\alpha_{j_x}^2}{(\nu+x)^2} + d$ generates a common proper subfield of L over k. Renumbering $\alpha_0, ..., \alpha_n$, we can assume that $i_x = 0$ and $j_x = 1$ for infinitely many $x \in \mathcal{N}_{\alpha_0} \cap \cdots \cap \mathcal{N}_{\alpha_n}$. Furthermore, we can assume that, for those x, the elements $c\frac{\alpha_0^2}{(\nu+x)^2} + d$ and $c\frac{\alpha_1^2}{(\nu+x)^2} + d$ generate the same proper subfield $K \subsetneq L$. Hence

$$c\frac{\alpha_0^2}{(\nu+x)^2} + d - \left(c\frac{\alpha_1^2}{(\nu+x)^2} + d\right) = \frac{c(\alpha_0^2 - \alpha_1^2)}{\nu + x^2} \in K$$

for infinitely many $x \in k$, contradicting that $k\left(\frac{(\nu+x)^2}{c(\alpha_0^2-\alpha_1^2)}\right) = L$ for all but finitely many $x \in k$ by what we have shown in the first part of the proof.

(ii): If $L = k$, then the assertion is trivial. Let $k \subsetneq L$. As L/k is finite and separable, we can choose $\nu \in L$ with $k(\nu) = L$. Since L is nonpythagorean, we can choose $z \in L$ such that $z^2 + 1$ is not a square in L. For any $\alpha \in k^\times$ consider

$$\mathcal{H}_\alpha = \left\{x \in k \;\middle|\; \frac{(\nu+x)^2}{\alpha^2} + z^2 + 1 \notin L^{\times 2}\right\}$$
$$\text{and } \mathcal{H}'_\alpha = \left\{x \in k \;\middle|\; \frac{\alpha^2(z^2+1)^2}{(\nu+x)^2} + z^2 + 1 \notin L^{\times 2}\right\}.$$

Since $\frac{\alpha^2(z^2+1)}{(\nu+x)^2}$ is not a square in L, and with

$$\frac{\alpha^2(z^2+1)}{(\nu+x)^2}\left(\frac{(\nu+x)^2}{\alpha^2} + z^2 + 1\right) = \frac{\alpha^2(z^2+1)^2}{(\nu+x)^2} + z^2 + 1,$$

we conclude that $k = \mathcal{H}_\alpha \cup \mathcal{H}'_\alpha$.

Case 1: $\mathcal{H}_\alpha$ is infinite for some $\alpha \in k^\times$.

Let $\mathcal{M} = \left\{x \in \mathcal{H}_\alpha \mid \frac{(\nu+x)^2}{\alpha^2} + z^2 \in L^{\times 2}\right\}$.

Case 1.1 : $\mathcal{M}$ is infinite.

For any $x \in \mathcal{M}$, let $\xi_x \in L$ such that $\xi_x^2 = \frac{(\nu+x)^2}{\alpha^2} + z^2$. Then $k(\xi_x^2) = L$ for all but finitely many $x \in \mathcal{M}$ by (i). We choose one of the infinitely many $x \in \mathcal{M}$ with $k(\xi_x^2) = L$, and $\xi = \xi_x$ will satisfy the assertion of (ii).

Case 1.2: $\mathcal{M}$ is finite.

For all $x \in \mathcal{H}_\alpha \setminus \mathcal{M}$ we have $\frac{(\nu+x)^2}{\alpha^2} + z^2 \notin L^{\times 2}$, thus also $\frac{(\nu+x)^2}{\alpha^2 z^2} + 1 \notin L^{\times 2}$. With $\xi_x = \frac{(\nu+x)}{\alpha z}$, we have that $k(\xi_x^2) = L$ for all but finitely many $x \in \mathcal{H}_\alpha \setminus \mathcal{M}$, applying

(i) for $c = \frac{1}{\alpha^2 z^2}$, $d = 0$. Since $\mathcal{H}_\alpha \setminus \mathcal{M}$ is infinite, we can choose $x \in \mathcal{H}_\alpha \setminus \mathcal{M}$, such that $k(\xi_x^2) = L$. Then $\xi = \xi_x$ satisfies the assertion of (ii).

Case 2: $\mathcal{H}_\alpha$ is finite for each $\alpha \in k^\times$.

In particular $k \setminus \mathcal{H}'_\alpha$ is finite for all $\alpha \in k^\times$. For $\alpha \in k^\times$ and $x \in \mathcal{H}'_\alpha$ we denote $\zeta_{\alpha,x} = \frac{\alpha(z^2+1)}{(\nu+x)}$ and we set $\mathcal{M}'_\alpha = \{x \in \mathcal{H}'_\alpha \mid \zeta_{\alpha,x}^2 + z^2 \in L^{\times 2}\}$. For $x \in \mathcal{M}'_\alpha$ we set $\xi_{\alpha,x} = \sqrt{\zeta_{\alpha,x}^2 + z^2}$.

Case 2.1: $\mathcal{H}'_\alpha \setminus \mathcal{M}'_\alpha$ is finite for each $\alpha \in k^\times$.

Putting $c = z^2 + 1$, we get with (i), that for all but finitely many $\alpha \in k^\times$ there are infinitely many $x \in k$ such that $k(\frac{\alpha^2(z^2+1)}{(\nu+x)^2}) = L$. Since $\mathcal{M}'_\alpha$ is cofinite in k for each α, there is an appropriate α and $x \in \mathcal{M}'_\alpha$, such that $k(\xi_{\alpha,x}^2) = L$. Then the assertion of (ii) holds for $\xi_{\alpha,x}$, since

$$\xi_{\alpha,x}^2 + 1 = \frac{\alpha^2(z^2+1)}{(\nu+x)^2}\left(\frac{(\nu+x)^2}{\alpha^2} + z^2 + 1\right) \notin L^{\times 2}.$$

Case 2.2: $\mathcal{H}'_\alpha \setminus \mathcal{M}'_\alpha$ is infinite for some $\alpha \in k^\times$.

Let $\alpha \in k^\times$ be such that $\mathcal{H}'_\alpha \setminus \mathcal{M}'_\alpha$ is infinite. We have that $\frac{\zeta_{\alpha,x}^2}{z^2} + 1 \notin L^{\times 2}$ for all $x \in \mathcal{H}'_\alpha \setminus \mathcal{M}'_\alpha$. We show, that for some $x \in \mathcal{H}'_\alpha \setminus \mathcal{M}'_\alpha$ the element $\xi = \frac{\zeta_{\alpha,x}}{z}$ satisfies (ii), i.e. that $k(\xi^2) = L$. Assume on the contrary that $k\left(\frac{\zeta_{\alpha,x}^2}{z^2}\right) \subsetneq L$ for all $x \in \mathcal{H}'_\alpha \setminus \mathcal{M}'_\alpha$. Then there is a field extension K/k with $K \subsetneq L$ such that $\frac{z^2}{\zeta_{\alpha,x}^2} = \frac{z^2}{\alpha^2(z^2+1)^2}(\nu+x)^2 \in K$ for infinitely many $x \in \mathcal{H}'_\alpha \setminus \mathcal{M}'_\alpha$. This contradicts the assertion of (i), which says that for $c = \frac{z^2}{\alpha^2(z^2+1)^2}$, we have $k\left(c(\nu+x)^2\right) \subsetneq L$ only for finitely many $x \in k$.

(iii): Let $\xi \in L$ as in (ii), then $\eta = \xi^2 + 1 \in \sum L^2 \setminus L^{\times 2}$. Thus, either $\eta + 1 \notin L^{\times 2}$, or $\frac{1}{\eta} + 1 = \frac{1}{\eta}(\eta+1) \notin L^{\times 2}$. Since $k(\eta) = k(\frac{1}{\eta}) = L$, $\sigma = \eta$ or $\sigma = \frac{1}{\eta}$ satisfies the claim. □

Remark 3.2. The statement of Proposition 3.1 holds more generally for separable finite extensions L/k of infinite fields where L is nonpythagorean, but the proof for this becomes even more technical.

Corollary 3.3. *Let L/k be a finite real extension where L is nonpythagorean. Let $a, b \in k$ such that either $a, b \in L^{\times 2}$ or $a, b \in -L^{\times 2}$. Then there exists $\eta \in \sum L^2 \setminus L^{\times 2}$ such that $L = k(\eta)$ and $\sqrt{-1} \notin L(\sqrt{-\eta})(\sqrt{-a\eta + b})$.*

Proof. If $a, b \in L^{\times 2}$, we set $\eta = \frac{b}{a}(\sigma + 1)$, where $\sigma \in \sum L^2 \setminus L^{\times 2}$ is chosen such that $k(\sigma) = L$ and $\sigma + 1 \in \sum L^2 \setminus L^{\times 2}$; such σ exists by Proposition 3.1(iii).

If $a, b \in -L^{\times 2}$, we set $\eta = \frac{b}{a}(\xi^2 + 1)$, where $\xi \in L$ is such that $\xi^2 + 1 \in \sum L^2 \setminus L^{\times 2}$ and $L = k(\xi^2)$, which exists by Proposition 3.1(ii).

By the choice of η, we see that neither η, nor $(a\eta - b)$, nor $-\eta(a\eta - b)$ is a square in L. Therefore $\sqrt{-1} \notin L(\sqrt{-\eta})(\sqrt{-r\eta + t})$. $\square$

4. Results for function fields of quadrics

Let k be of characteristic $\neq 2$. A regular quadratic form $\varphi \in k[X_0, ..., X_n]$ of dimensions $n+1$ defines a $(n-1)$-dimensional projective quadric Q_φ by the homogeneous equation $\varphi(x_0, \ldots, x_n) = 0$. If φ is isotropic, i.e. if Q_φ has a k-rational point, then the function field $k(Q_\varphi)$ is the rational function field in $(n-1)$ variables ([Lam, Chap. X, Theorem 4.1]).

If $\dim(\varphi) = 3$, then $C = Q_\varphi$ is a projective conic over k, and since $q \sim \langle 1, -a, -b\rangle$ for some $a, b \in k^\times$, we get the representation

$$k(C) \cong k(X)(\sqrt{aX^2 + b})$$

for its function field.

We first prove some general 'Going Downs' for the finiteness Pythagoras number for function fields of quadrics defined by weakly isotropic forms. Later we skip the assumption that the defining form is weakly isotropic, but consider conics (i.e. one dimensional quadrics) only. The proof of Theorem 1.4 does not need any of the results for quadrics of arbitrary dimension, so readers that are interested in that part only should read on from Theorem 4.7.

Lemma 4.1. *K be a field. Let $a, b \in \sum K^2$ and $L = K(\sqrt{a})$. Let $r \in \mathbb{N}$ be such that either (i) $\ell_L(b) \leq 2^r$ or (ii) $\ell_K(a) \leq 2^r$.*

Let $m_1 \in \{0, \ldots, 2^r - 1\}$ such that $m_1 \equiv \ell_K(a) (\mod 2^r)$ and $m_2 \in \{0, \ldots, 2^r - 1\}$ such that $m_2 \equiv \ell_L(b) (\mod 2^r)$.

In case (i), we have $\ell_K(b) \leq \ell_L(b) + \ell_K(a) - m_1 + 2^r$ and in case (ii), we have $\ell_K(b) \leq 2\ell_L(b) - m_2 + 2^r$.

Remark 4.2. For each $n \in \mathbb{N}$, the set $D_K(2^n) = \{x \in \sum K^2 \mid \ell_K(x) \leq 2^n\}$ is a multiplicative subgroup of $\sum K^2$.

Proof of the lemma. Let $n = \ell_L(b)$, then there are $y_1, \ldots, y_n \in K$ and $x_1, \ldots, x_n \in K$ such that

$$\begin{aligned} b &= \sum_{1 \leq i \leq n} (x_i + y_i\sqrt{a})^2 \\ {}^2 &= \sum_{1 \leq i \leq n} x_1^2 + a \sum_{1 \leq i \leq n} y_i^2. \end{aligned}$$

We find some $z_1, \dots, z_s \in K$ with $a = z_1^2 + \dots z_s^2$ for $s = \ell_K(a)$, hence

$$b = \sum_{1 \le i \le n} x_1^2 + (\sum_{1 \le i \le s} z_i^2)(\sum_{1 \le i \le n} y_i^2).$$

(i): Suppose $\ell_L(b) \le 2^r$. We show $\ell_K(b) \le \ell_L(b) + \ell_K(a) - m_1 + 2^r$. We rearrange $a = z_1^2 + \dots z_s^2 = \sigma_1 + \dots + \sigma_t + \eta$, with $t \in \mathbb{N}_0$ and $\sigma_j, \eta \in \sum K^2$, such that $\ell_K(\sigma_j) = 2^r$ for $1 \le j \le t$ and $\ell_K(\eta) = m_1$ and $s = t2^r + m_2$. Then we get

$$b = \sum_{1 \le i \le n} x_i^2 + \sum_{1 \le j \le t} \left(\sigma_j \sum_{1 \le i \le n} y_i^2 \right) + \eta \sum_{1 \le i \le n} y_i^2.$$

By Remark 4.2, we get $\ell_K\left(\sigma_j \sum_{1 \le i \le n} y_i^2\right) \le 2^r = \ell_K(\sigma_j)$ for $1 \le j \le t$, as well as $\ell_K\left(\eta \sum_{1 \le i \le n} y_i^2\right) \le 2^r$. Hence, $\ell_K(b) \le n + t2^r + 2^r = \ell_L(b) + \ell_K(a) - m_1 + 2^r$.
(ii): Suppose $\ell_K(a) \le 2^r$. We rearrange $y_1^2 + \dots + y_n^2 = \sigma_1 + \dots \sigma_t + \eta$ for some $t \in \mathbb{N}_0$ and $\sigma_j, \eta \in \sum K^2$ with $\ell_K(\sigma_j) = 2^r$ and $\ell_(\eta) = m_2$ and $n = t2^r + m_2$. Then we get

$$b = \sum_{1 \le i \le n} x_i^2 + \sum_{1 \le j \le t} \sigma_j a + \eta a.$$

Thus, since $\ell_K(a) \le 2^r$ and $\ell_K(\sigma_j) = 2^r$, as well as $\ell_K(\eta) \le 2^r$, by Remark 4.2, we get $\ell_K(b) \le n + t2^r + 2^r = \ell_L(b) + \ell_L(b) - m_2 + 2^r$. ☐

Corollary 4.3. *Let K be a real field, $\sigma \in \sum K^2$, and $L = K(\sqrt{\sigma})$. Let $r \in \mathbb{N}$ such that $\ell_K(\sigma) \le 2^r$ and $p(L) \le 2^r$ Then $p(K) \le p(L) + 2^r \le 2^{r+1}$.*

Proposition 4.4. *Let k be a real field, Q a projective quadric defined by a weakly isotropic regular quadratic form over k. If $p(k(Q)) < \infty$, then there exists an $n \in \mathbb{N}$, such that $p(L) \le n$ for all finite extensions L/k.*

Proof. Let φ be a weakly isotropic regular form of dimension $n \ge 3$ such that $Q = Q_\varphi$. Since φ is weakly isotropic, there are certain sums of squares $\sigma_1, \dots, \sigma_n$, such that φ becomes isotropic over $K = k(\sqrt{\sigma_1}, \dots, \sqrt{\sigma_n})$. Hence $K(Q)$ is a rational function field in $n-2$ variables. We have $p(K(Q)) \le [K(Q) : k(Q)] p(k(Q)) < \infty$, and thus $p(K(X)) < \infty$ for the rational function field $K(X)$ in one variable.
Since $K(X) = k(X)(\sqrt{\sigma_1}) \dots (\sqrt{\sigma_n})$ is a multiquadratic totally positive extension of $k(X)$, corollary 4.3 yields that $p(k(X)) < \infty$. Now the rest follows with theorem 1.2. ☐

Presumably the strong condition 'weakly isotropic' on the quadratic form φ is not necessary for the statement. However, the weaker condition that φ is 'not totally

[2] since $b \in K$, the coefficient to $\sqrt{a}$ has to vanish.

definite in k' cannot be omitted, since for totally definite forms φ, the function field $k(Q_\varphi)$ is nonreal, and hence $p(k(Q_\varphi)) < \infty$, even if $p(k) = \infty$.
In the following, we look at quadrics of dimension 1 (i.e. conics), that are defined by a special kind of weakly isotropic 3-dimensional forms. We consider forms $\varphi = \langle 1, 1, -\sigma\rangle$ with $\sigma \in \sum k^2$. For the function fields of the corresponding conics, we get, in certain sense, two complementary quantitative bounds on the pythagoras number of all finite extensions L/k, in terms of the Pythagoras number of the function field.

Proposition 4.5. *Let k be a real field and $\sigma \in \sum k^2$ with $\ell_k(\sigma) \leq 2^n + 1$. Let C be the projective conic defined by $\langle 1, 1, -\sigma\rangle$ and $F = k(C)$. If $p(F) < 2^n$ then $p(L) < 2^{n+1}$ for all finite extensions L/k.*

Proof. We can assume, that $\sigma = 1 + u$ for some $u \in \sum k^2$ with $\ell_k(u) \leq 2^n$. The field $F' = F(\sqrt{u})$ is a rational function field over $k' = k(\sqrt{u})$, since $\langle 1, 1, -(1 + u)\rangle$ becomes isotropic over k', and $p(F') \leq 2^n$, following [BL, Prop 3.5, Thm 3.10].
Assume there is a finite extensions L/k with $p(L) \geq 2^{n+1}$. We can assume that L is real, by the equivalence $((ii) \Leftrightarrow (iii))$ of Theorem 1.2. Then $L' = L(\sqrt{u})$ is a finite real extension of k'. Thus we have $p(L') < 2^n$ by equivalence $((i) \Leftrightarrow (ii))$ of theorem 1.2, since $p(F') \leq 2^n$.
This implies $p(L) < 2^{n+1}$ by Corollary 4.3, since $\ell_L(u) \leq 2^n$. □

Proposition 4.6. *Let k be a real field and $\sigma \in \sum k^2$ with $\ell_k(\sigma) < 2^n$. Let C be the projective conic defined by $\langle 1, 1, -\sigma\rangle$ and $F = k(C)$. If $p(F) \leq 2^n$ then $p(L) \leq 2^{n+1}$ for all finite extensions L/k.*

Proof. We can assume that $\sigma - 1 \in \sum k^2$ with $\ell_k(\sigma - 1) = \ell_k(\sigma) - 1$. Let L be a finite real extension of k with $p(L) > 2^{n+1}$ (we can assume, as in the proof of the previous theorem, that L is real).
Let $\eta \in \sum L^2$ with $\ell_L(\eta) > 2^{n+1}$. Then also $\ell_L(\frac{4\eta}{(\eta-1)^2}) > 2^n$, since $4, (\eta - 1)^2 \in D_L(2^n)$. By

$$\eta = \frac{(\eta + 1)^2}{4} - \frac{(\eta - 1)^2}{4},$$

we see, that $\frac{4\eta}{(\eta-1)^2} + 1 \in L^{\times 2}$.
Denote $\tau = \frac{4\eta}{(\eta-1)^2}$. We have $\ell_L(\tau + \sigma) < 2^n$, since $\tau + \sigma = \tau + 1 + (\sigma - 1)$, where $\tau + 1 \in L^{\times 2}$ and $\ell_L(\sigma - 1) = \ell_L(\sigma) - 1$.
Now for $\nu \in L$ with $L = k(\nu)$, we know that $\rho_x = (\nu + x)^2 + \tau \in \sum L^2$ generates L/k for all but finitely many $x \in k$, by Proposition 3.1 (i).
And there are infinitely many $x \in k$ such that $\tilde{\rho}_x = \frac{\tau^2}{(\nu+x)^2} + \tau$ is a generator for L/k, also by Proposition 3.1 (i).

Hence, for infinitely many $x \in k$, both $k(\rho_x) = L$ and $k(\tilde{\rho}_x) = L$. For any of those x, note that

$$\tilde{\rho_x} = \frac{\tau}{(\nu + x)^2}\rho_x,$$

and since $\ell_L\left(\frac{\tau}{(\nu+x)^2}\right) > 2^{n+1}$, we see, thet either $\ell_L(\rho_x) > 2^{n+1}$ or $\ell_L(\tilde{\rho}_x) > 2^{n+1}$, as $D_L(2^n)$ is a multiplicative subgroup of $\sum L^2$. For one $x \in k$ with $k(\rho_x) = k(\tilde{\rho}_x) = L$, we set either $\rho := \rho_x$ or $\rho := \tilde{\rho}_x$, so that $\ell_L(\rho) > 2^{n+1}$. Note that in either case, we have

$$\ell_L(\rho + \sigma) \leq 1 + \ell_L(\tau + 1) + \ell_L(\sigma - 1) \leq 1 + 1 + (2^n - 2) = 2^n.$$

Now consider the nonreal extension $L(\sqrt{-\rho})/L$. By the choice of ρ we have $k(\sqrt{-\rho}) = L(\sqrt{-\rho})$. Hence we can consider $L(\sqrt{-\rho})$ as the residue field of the k-valuation v_p on $k(X)$ associated to the minimal polynomial $p \in k[X]$ of $\sqrt{-\rho}$. When we extend this valuation to $F = k(X)(\sqrt{-X^2 + \sigma})$, we obtain $M = k(\sqrt{-\rho})(\sqrt{\rho + \sigma})$ as the residue field extension. We can write $M = L(\sqrt{\rho + \sigma})(\sqrt{-\rho})$.

We show $s(M) \geq 2^n$ by showing, that $\ell_{L(\sqrt{\rho+\sigma})}(\rho) \geq 2^n$. Assume that on the contrary $\ell_{L(\sqrt{\rho+\sigma})}(\rho) < 2^n$. By case (ii) of Lemma 4.1, choosing $r = n$ and thus we get $m_2 = \ell_L(\sqrt{\rho + \sigma})(\rho)$. This yields the contradiction $2^{n+1} < \ell_L(\rho) \leq \ell_L(\sqrt{\rho + \sigma})(\rho) + 2^n \leq 2^{n+1}$.

Hence $s(M) \geq 2^n$ and consequently $p(F) > 2^n$ by Proposition 2.1. Contradiciton. □

Having stated these explicit 'Going Down' results for the Pythagoras numbers of function fields of conics that are defined by a certain kind of ternary forms, we widen our focus to arbitrary conics now. While we cannot give a general 'Going Down' result for the finiteness of the Pythagoras number, we show that in the special case where the Pythagoras number of the function field of the conic is assumed to be 2, the ground field k was necessarily herditarily pythagorean. This is the statement of Theorem 1.4, which we now reformulate and prove.

Theorem 4.7. *Let k be a real field that is not hereditarily pythagorean. Let C be a conic over k and $F = k(C)$. Then $p(F) \geq 3$.*

Proof. Let L be a finite real extension of k that is not pythagorean. C is given by 3-dimensional regular quadratic form φ over k.

We first consider the case, where φ is indefinite with respect to some ordering $\preceq$ of L. We can assume, that $\varphi = \langle 1, -a, -b\rangle$ for some $a, b \in k^\times$ with $a \succ 0$ and $b \succ 0$. Furthermore, we can assume, that $a, b \in L^{\times 2}$, since we could replace L by the real extension $L(\sqrt{a}, \sqrt{b})$, which is also not pythagorean by Theorem 1.1.

In the case where φ is totally definite over L, we can assume, that $\varphi = \langle 1, -a, -b\rangle$ for some $a, b \in k^\times$ that are totally negative in L. We can assume that $a, b \in -L^{\times 2}$, (otherwise we replace L by $L(\sqrt{-a}, \sqrt{-b})$).

In any case, according to Corollary 3.3, one can find a suitable $\eta \in \sum L^2 \setminus L^{\times 2}$, such that $k(\eta) = L$ and such that the nonreal field $M = L(\sqrt{-\eta})(\sqrt{-a\eta + b})$ has level at least 2.

We show that M is the residue field of a discrete rank one valuation of F. Let $p \in k[X]$ be the minimal polynomial of the algebraic element $\sqrt{-\eta}$ and let v_p be the corresponding k-valuation of $k(X)$. The residue field of this valuation is $k(\sqrt{-\eta})$. By the choice of η, we have $k(\eta) = L$, hence $k(\sqrt{-\eta}) = L(\sqrt{-\eta})$. Thus we can write $M = k(\sqrt{-\eta})(\sqrt{-a\eta + b})$. Let w be a extension of v_p to $F \cong k(X)(\sqrt{aX^2 + b})$. Then $M = k(\sqrt{-\eta})(\sqrt{-a\eta + b})$ is the residue field of w according to Lemma 2.3. With Proposition 2.1 we get $p(F) > s(M) \geq 2$. □

Corollary 4.8. *Let Q be a quadric of dimension at least 2 over the real field k. Then $p\,(k(Q)) \geq 3$.*

Proof. Let φ be a regular quadratic form over k with $Q = Q_\varphi$. Then $\dim(\varphi) \geq 4$, i.e. $\varphi(X_0, \ldots, X_n) = a_0 X_0^2 + \cdots + a_n X_n^2$ for some $n \geq 3$ and some $a_i \in k^\times$. Then

$$k(Q_\varphi) \cong k(X_1, \ldots, X_{n-1})\left(\sqrt{-a_n(a_{n-1}X_{n-1}^2 + a_{n-2}X_{n-2}^2 + \cdots + a_1 X_1^2 + a_0)}\right)$$

can be considered as the function field of the conic defined by the 3-dimensional form

$$\langle 1, a_n a_{n-1}, a_n(a_{n-2}x_{n-2}^2 + \cdots + a_1 x_1^2 + a_0)\rangle$$

over $k(X_1, \ldots, X_{n-2})$. By the *Cassels-Pfister Theorem* ([Lam, Chap. IX, Corollary 2.3]), the rational function field in at least one variable $k(X_1, \ldots, X_{n-2})$ is nonpythagorean, hence by Theorem 4.7, we get $p(k(Q_\varphi)) \geq 3$. □

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