ESSENTIAL p-DIMENSION OF $\mathbf{PGL}(p^2)$

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Abstract. Let p be a prime integer and let F be a field of characteristic different from p. We prove that the essential p-dimension of the group $\mathbf{PGL}_F(p^2)$ is equal to p^2+1.

1. Introduction

The essential dimension of an algebraic structure over a field F is the smallest number of algebraically independent parameters required to define this structure over a field extension of F (cf. [1]).

Let p be a prime integer. Essential p-dimension of an algebraic structure measures the complexity of the structure modulo the "effects of degree prime to p".

Let p denote either a prime integer or 0. An integer k is said to be *prime to* p when k is prime to p if $p>0$ and $k=1$ if $p=0$ and let F be a field. Consider the category Fields/F of field extensions of F and field homomorphisms over F. Let $\mathcal{F}:\mathit{Fields}/F\to\mathit{Sets}$ be a functor (an "algebraic structure") and $K,E\in\mathit{Fields}/F$. An element $\alpha\in\mathcal{F}(E)$ is said to be *p-defined over K* and *K is a field of p-definition of α* if there exist a finite field extension E'/E of degree prime to p, a field homomorphism $K\to E'$ over F and an element $\beta\in\mathcal{F}(K)$ such that the image of α under $\mathcal{F}(E)\to\mathcal{F}(E')$ coincides with the image of β under $\mathcal{F}(K)\to\mathcal{F}(E')$. The *essential p-dimension of α*, denoted $\operatorname{ed}_p^{\mathcal{F}}(\alpha)$, is the least integer $\operatorname{tr.deg}_F(K)$ over all fields of p-definition K of α. The *essential p-dimension of the functor T* is the integer

$$\operatorname{ed}_p(\mathcal{F})=\max\{\operatorname{ed}_p^{\mathcal{F}}(\alpha)\}$$

where the maximum is taken over all $\alpha\in\mathcal{F}(E)$ and fields $E\in\mathit{Fields}/F$.

We write $\operatorname{ed}(\mathcal{F})$ for $\operatorname{ed}_0(\mathcal{F})$ and simply call $\operatorname{ed}(\mathcal{F})$ the *essential dimension of $\mathcal{F}$*. Clearly, $\operatorname{ed}(\mathcal{F})\geq\operatorname{ed}_p(\mathcal{F})$ for all p.

Let G be an algebraic group over F. The *essential p-dimension of G* is the essential p-dimension of the functor $\mathcal{F}_G$ taking a field E to the set of isomorphism classes of G-torsors over $\operatorname{Spec}(E)$.

If $G=\mathbf{PGL}_n$ over F, the functor $\mathcal{F}_G$ is isomorphic to the functor taking a field E to the set of isomorphism classes of central simple E-algebras of degree n. If $n=p$ is prime, every central simple E-algebra of degree p is cyclic over a finite field extension of degree prime to p. It follows that $\operatorname{ed}_p\bigl(\mathbf{PGL}_F(p)\bigr)=2$ [11, Lemma 8.5].

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We prove the following:

Theorem 1.1. *Let p be a prime integer and F a field of characteristic different from p. Then*

$$\operatorname{ed}_p\big(\mathbf{PGL}_F(p^2)\big) = p^2 + 1.$$

Corollary 1.2. (Rost) *If F a field of characteristic different from 2, then* $\operatorname{ed}\big(\mathbf{PGL}_F(4)\big) = \operatorname{ed}_2\big(\mathbf{PGL}_F(4)\big) = 5$.

Proof. By theorem, $\operatorname{ed}\big(\mathbf{PGL}_F(4)\big) \geq \operatorname{ed}_2\big(\mathbf{PGL}_F(4)\big) = 5$ and $\operatorname{ed}\big(\mathbf{PGL}_F(4)\big) \leq 5$ by [8]. □

We use the following notation:

$\mathsf{X}(F)$ is the character group of the absolute Galois group $\operatorname{Gal}(F_{sep}/F)$ of a field F;

$\operatorname{Br}(F)$ is the Brauer group of F. For a field extension L/F, we write $\operatorname{Br}(L/F)$ for the relative Brauer group $\operatorname{Ker}(\operatorname{Br}(F) \to \operatorname{Br}(L))$.

$\mathbb{G}_m$ denoted the multiplicative group $\operatorname{Spec} F[t, t^{-1}]$ over F.

For a finite separable field extension L/F we write $R_{L/F}$ for the corestriction operation (cf. [7, §20.5]). In particular, $R_{L/F}(\mathbb{G}_{m,L})$ is the multiplicative group of L considered as an algebraic group (torus) over F. We write $R^{(1)}_{L/F}(\mathbb{G}_{m,L})$ for the torus of norm 1 elements in L.

If A is a central simple algebra over F, then $\operatorname{SB}(A)$ denotes the Severi-Brauer variety of A [7, §1.C].

In the present paper, the word "scheme" over a field F means a separated scheme of finite type over F and a "variety" over F is an integral scheme over F. If X is a scheme over F and E/F is a field extension, then $X(E) = \operatorname{Mor}_F(\operatorname{Spec}(E), X)$ is the set of points of X over E.

2. Degree of points of the norm 1 torus for a bicyclic field extension

2.1. **Chow groups and push-forward homomorphism.** (cf. [3])

Let X be a scheme over a field F. We write $\mathrm{Z}(X)$ for the group of algebraic cycles on X, i.e., the free abelian group generated by points of X. The group $\mathrm{Z}(X)$ is graded by the dimension of points. We write $\mathrm{CH}_i(X)$ for the factor group of $\mathrm{Z}_i(X)$ by the subgroup of cycles rationally equivalent to 0 (cf. [3, §1.3]). If $x \in X$ is a point of dimension i, $[x]$ denotes the class of x in $\mathrm{CH}_i(X)$.

If X is a variety of dimension d, then the group $\mathrm{CH}_d(X)$ is infinite cyclic generated by the class of the generic point of X.

Let $f : X \to Y$ be a morphism of schemes over F. The push-forward homomorphism $f_* : \mathrm{Z}(X) \to \mathrm{Z}(Y)$ is a graded homomorphism defined by

$$f_*(x) = \begin{cases} [F(x) : F(y)] \cdot y, & \text{if } [F(x) : F(y)] \text{ is finite;} \\ 0, & \text{otherwise,} \end{cases}$$

where $x \in X$ and $y = f(x)$. If f is a proper morphism, then f_* factors through the rational equivalence, providing the push-forward homomorphism $\mathrm{CH}(X) \to \mathrm{CH}(Y)$ still denoted by f_* (cf. [3, §1.4]).

2.2. **Degree of a point.** Let X be a scheme over a field F, $a \in X(E)$ a point over a field extension E/F and $\{x\}$ the image of $a : \mathrm{Spec}(E) \to X$. The *dimension of* a is the integer $\dim(a) := \dim(x)$. If $d = \dim(a)$, we define the *class* $[a]$ *of* a in $\mathrm{CH}_d(X)$ as follows:

$$[a] := \begin{cases} [E : F(x)] \cdot [x], & \text{if } [E : F(x)] \text{ is finite;} \\ 0, & \text{otherwise.} \end{cases}$$

In addition, if X is a variety, the *degree of* a is the integer $\deg(a)$ satisfying $[a] = \deg(a) \cdot [X]$ if $\dim(a) = \dim(X)$ and $\deg(a) = 0$ otherwise.

If E'/E is a field extension and $a \in X(E)$, we write $a_{E'}$ for the image of a in $X(E')$. If E'/E is finite, we have $\deg(a_{E'}) = [E' : E] \cdot \deg(a)$.

If $E = F(X)$ the function field of X and $a \in X(E)$ is the generic point, then $\deg(a) = 1$.

If $f : X \to Y$ is a morphism of varieties over F and $a \in X(E)$ for a field extension E/F, we have $\dim(a) \geq \dim f(a)$.

Proposition 2.1. *Let $f : X \to Y$ be a proper morphism of varieties over F and let $a \in X(E)$ be a point over a field extension E/F. Then $[f(a)] = f_*([a])$ in* $\mathrm{CH}(Y)$.

Proof. Let $\{x\}$ be the image of a in X and $y = f(x)$. If one of the field extensions $E/F(x)$ and $F(x)/F(y)$ is infinite, then $[f(a)] = 0$ and $f_*([a]) = 0$. We may assume that E is a finite extension of $F(y)$. Then

$$\begin{aligned} [f(a)] &= [E : F(y)] \cdot [y] \\ &= [E : F(x)]\big([F(x) : F(y)] \cdot [y]\big) \\ &= [E : F(x)] \cdot f_*([x]) \\ &= f_*([a]). \end{aligned}$$

□

If Z is a scheme over F, we write $n(Z)$ for $\gcd[F(z) : F]$ over all closed points $z \in Z$.

Example 2.2. Let T be an algebraic torus over F. We write $i(T)$ for the greatest common divisor of the integers $[E : F]$ over all finite filed extensions E/F such that T is isotropic over E. If X is a smooth complete geometrically irreducible variety containing T as an open set, then $n(X \setminus T) = i(T)$ by [2, Lemme 12] (see also [9, Lemma 5.1]).

We shall need a variant of a push-forward homomorphism for morphisms that are not proper.

Proposition 2.3. *Let X be a complete variety over F, $U \subset X$ an open subvariety, $Z = X \setminus U$ and $f : U \to Y$ a morphism over F, where Y is a variety of dimension d over F. If $n = n(Z_{F(Y)})$, then the push-forward homomorphism on cycles $\mathrm{Z}(U) \to \mathrm{Z}(Y)$, forwarded by the projection $\mathrm{Z}(Y) \to \mathrm{Z}_d(Y) = \mathbb{Z}$, gives rise to a well defined homomorphism*

$$f_\star : \mathrm{CH}(U) \to \mathbb{Z}/n\mathbb{Z}.$$

Moreover, for any point $a \in U(E)$ over a field extension E/F, one has $f_\star([a]) = \deg\big(f(a)\big)$ modulo n.

Proof. We define the map $f_\star$ to be trivial on all homogeneous components $\mathrm{CH}_i(U)$ except $i = d$, so we just need to define $f_\star$ on $\mathrm{CH}_d(U)$.

We claim that the image of the push-forward homomorphism

$$s_* : \mathrm{CH}_d(Z \times Y) \to \mathrm{CH}_d(Y) = \mathbb{Z}$$

for the projection $s : Z \times Y \to Y$ is contained in $n\mathbb{Z}$. Let $u \in Z \times Y$ be a point of dimension d. If $s(u)$ is not the generic point of Y, then $s_*([u]) = 0$. Otherwise, u is a closed point in $Z_{F(Y)}$ and $s_*([u])$ coincides with the degree of this closed point and hence is divisible by n. The claim is proven.

The map s_* factors as $s_* = q_* \circ i_*$, where $i : Z \times Y \to X \times Y$ is the closed embedding and $q : X \times Y \to Y$ is the projection. By localization [3, §1.8], $\mathrm{CH}_d(U \times Y)$ is canonically isomorphic to the cokernel of i_*. Hence, q_* gives rise to a homomorphism $\mathrm{CH}_d(U \times Y) \to \mathbb{Z}/n\mathbb{Z}$. Composing it with the push-forward homomorphism for the morphism $(1_U, f) : U \to U \times Y$, we get the required homomorphism $f_\star : \mathrm{CH}_d(U) \to \mathbb{Z}/n\mathbb{Z}$. The last equality in the statement follows from Proposition 2.1 applied to q. □

Example 2.4. Let T be an algebraic torus over F and $n = i(T)$ (see Example 2.2). Then the structure morphism $T \to \mathrm{Spec}(F)$ gives rise to a homomorphism $\mathrm{CH}_0(T) \to \mathbb{Z}/n\mathbb{Z}$ that takes the class of a closed point $t \in T$ to $[F(t) : F]$ modulo n.

2.3. **Key proposition.** Let p be a prime integer, L/F a bicyclic field extension of degree p^2, $G = \mathrm{Gal}(L/F)$, σ and τ generators of G. Consider the tori $T = R^{(1)}_{L/F}(\mathbb{G}_{m,L})$ and $P = R_{L/F}(\mathbb{G}_{m,L})/\,\mathbb{G}_m$, both of dimension $d := p^2 - 1$. The tori T and P become isotropic only over field extensions E/F such that $E \otimes_F L$ is not a field. It follows that $i(T) = i(P) = i(T \times P) = p$.

Consider the morphisms f and g from $T \times P$ to T defined by $f(t, v) = t$ and $g(t, v) = t\sigma(v)/v$. By Proposition 2.3 and Example 2.2, f and g give rise to a well defined homomorphisms $f_\star$ and $g_\star$ from $\mathrm{CH}_d(T \times P)$ to $\mathbb{Z}/p\mathbb{Z}$.

Proposition 2.5. *The maps $f_\star$ and $g_\star$ coincide.*

Proof. The torus P is an open subset in the projective space $\mathbb{P}_F(L)$, hence the ring $\mathrm{CH}(P)$ is generated by the restriction to P of the class e of a hyperplane in $\mathbb{P}_F(L)$. Moreover, by the Projective Bundle Theorem [3, Th. 3.3], $\mathrm{CH}_d(T \times P)$ coincides with the direct sum of subgroups $\mathrm{CH}_i(T) \times e^i$ over all $i = 0, 1, \dots, d$.

Let $\beta \in \mathrm{CH}_i(T)$. It suffices to show that $f_\star(\beta \times e^i) = g_\star(\beta \times e^i)$ for any $i = 0, 1, \dots, d$. If $i = d$, the class e^i is represented by the identity point 1 of P. The equality follows from the fact that f and g coincide on $T \times \{1\}$.

Now assume that $i < d$. In this case $f_\star(\beta \times e^i) = 0$ and we need to show that $g_\star(\beta \times e^i) = 0$.

Let K be the subfield of σ-invariant elements in L of degree p over F. We have $pk + 1 \le p^2 - i \le p(k + 1)$ for some integer $k = 0, \dots, p - 1$. Consider a

K-linear subspace W of L of K-dimension k such that $K \cap W = 0$. Let V be an F-subspace of L of dimension $p^2 - i$ over F such that

$$F \oplus W \subset V \subset K \oplus W.$$

The class of $P \cap \mathbb{P}(V)$ in $\mathrm{CH}^i(P)$ is equal to e^i.

Let $S = R_{K/F}\big(R^{(1)}_{L/K}(\mathbb{G}_{m,L})\big) \simeq R_{K/F}\big(R_{L/K}(\mathbb{G}_{m,L})/\,\mathbb{G}_{m,K}\big)$. We view S as an open subscheme of $R_{K/F}\big(\mathbb{P}_K(L)\big)$. The map g factors as follows:

$$T \times P \xrightarrow{1_T \times l} T \times S \xrightarrow{r} T,$$

where $l : P \to S$ is defined by $l(v) = v/\sigma(v)$ and $r(t,s) = ts^{-1}$. The image of $P \cap \mathbb{P}_F(K \oplus W)$ under l is the variety $S \cap R_{K/F}\big(\mathbb{P}_K(K \oplus W)\big)$ of dimension pk. Hence, if $p^2 - i > pk + 1$, then $\dim\big(P \cap \mathbb{P}(V)\big) > pk$, but dimension of the image of $P \cap \mathbb{P}(V)$ under l is at most pk, so $P \cap \mathbb{P}(V)$ loses dimension under l, therefore, $g_\star(\beta \times e^i) = 0$.

It remains to consider the case $p^2 - i = pk + 1$, $k = 1, \dots, p-1$, i.e., $V = F \oplus W$. Since the map $P \cap \mathbb{P}(V) \to R_{K/F}\big(\mathbb{P}_K(K \oplus W)\big)$ given by l is a birational isomorphism, and the class of $R_{K/F}\big(\mathbb{P}_K(K \oplus W)\big)$ in $\mathrm{CH}(S)$ is equal to h^{p-k-1}, where $h \in \mathrm{CH}^p(S)$ is the class given by a K-hyperplane in L, it suffices to show that $r_\star(\beta \times h^{p-k-1}) = 0$.

Let S_t be the fiber of the norm homomorphism $T \to T_1 := R^{(1)}_{K/F}(\mathbb{G}_{m,K})$ over the generic point t of T_1, so S_t is a principal homogeneous space of S over the function field $F(T_1)$. Denote by

$$r' : S_t \times S \to S_t$$

the morphism given by $r'(x,s) = xs^{-1}$. Thus we have a commutative diagram

$$\begin{tikzcd}
S_t \times S \arrow[r, "r'"] \arrow[d, "q"'] & S_t \arrow[d, "m"] \\
T \times S \arrow[r, "r"] & T
\end{tikzcd}$$

where m is the canonical morphism and $q = m \times 1_S$. It follows that $r_\star$ factors as the composition

$$\mathrm{CH}_d(T \times S) \xrightarrow{q^*} \mathrm{CH}_{p(p-1)}(S_t \times S) \xrightarrow{r'_\star} \mathbb{Z}/p\mathbb{Z}.$$

Thus, it suffices to show that $r'_\star(\alpha \times h^{p-k-1}) = 0$ for any $\alpha \in \mathrm{CH}^{pk}(S_t)$. This follows from Proposition 4.5 in the Appendix applied to the torus S over the field $F(T_1)$ (with $j = p - k - 1$) if we show that S_t is a nontrivial principal homogeneous space of S. Suppose that S_t has a point over $F(T_1)$. It follows that the exact sequence

$$1 \to S \to T \to T_1 \to 1$$

splits rationally, i.e., the torus T is birationally isomorphic to the product $S \times T_1$ and hence is a rational variety. But , T is not rational, a contradiction. $\square$

2.4. **Invariance of the degree under R-equivalence.**

Theorem 2.6. *Let p be a prime integer, L/F a bicyclic field extension of degree p^2 and $T = R^{(1)}_{L/F}(\mathbb{G}_{m,L})$. Let E/F be a field extension and let t and t' be R-equivalent points in $T(E)$. Then $\deg(t) \equiv \deg(t')$ modulo p.*

Proof. We have $t' = t \cdot \sigma(u)u^{-1} \cdot \tau(v)v^{-1}$ for some $u, v \in (EL)^\times$. Let $t'' = t \cdot \sigma(u)u^{-1}$. It suffices to prove that $\deg(t) = \deg(t'')$ and $\deg(t') = \deg(t'')$. We shall prove the first equality (the second being similar). So replacing t' by t'' we may assume that $t' = t \cdot \sigma(u)u^{-1}$.

Consider the point $w = (t, u)$ in $(T \times P)(E)$ and two morphisms f and g from $T \times P$ to T as in Section 2.3. We have $f(w) = t$ and $g(w) = t'$. By Propositions 2.3 and 2.5, we have in $\mathbb{Z}/p\mathbb{Z}$:

$$\deg(t) = \deg f(w) = f_\star([w]) = g_\star([w]) = \deg g(w) = \deg(t'). \qquad \square$$

3. Essential p-dimension of $\mathbf{PGL}(p^2)$

Let F be a field and p a prime integer different from $\mathrm{char}(F)$.

3.1. **The functors $\mathcal{F}_1$ and $\mathcal{F}_2$.** We define functors from the category *Fields*$/F$ of field extensions of F to the category *Sets*. Let E/F be a field extension. Then $\mathcal{F}_1(E)$ is the set of isomorphism classes of central simple E-algebras of degree p^2. Thus, $\mathrm{ed}_p(\mathcal{F}_1) = \mathrm{ed}_p\big(\mathbf{PGL}_F(p^2)\big)$.

Let $\mathcal{S}_2(E)$ be the class of pairs (B, K), where B is a central simple algebra of degree p^2 over E and K is a cyclic étale E-algebra of degree p such that $\mathrm{ind}(B_K) \le p$. We say that the pairs (B_1, K_1) and (B_2, K_2) are equivalent if $K_1 \simeq K_2$ over E and $[B_1] - [B_2] \in \mathrm{Br}(K_1/E) = \mathrm{Br}(K_2/E)$. Let $\mathcal{F}_2(E)$ be the set of equivalence classes in $\mathcal{S}_2(E)$. We write $[B, K]$ for the class in $\mathcal{F}_2(E)$ of a pair (B, K).

We say that the class $[B, K]$ is *decomposable* if $[B, K] = [B', K]$ with B' a split algebra.

Let $(B, K) \in \mathcal{S}_2(E)$ with K a field and let $\chi \in X(E)$ be a character (of order p) such that $K = E(\chi)$. As $\mathrm{ind}(B_K) \le p$, there is a central simple algebra C over the function field $E(y)$ (y is a variable) of degree p^2 such that

$$[C] = [B_{E(y)}] + \big(\chi_{E(y)} \cup (y)\big) \tag{1}$$

in $\mathrm{Br}\big(E(y)\big)$. We have $[C] \in \mathcal{F}_1(E(y))$ and $\partial([C]) = \chi$, where ∂ is taken with respect to the discrete valuation v on $E(y)$ associated to y (see Section 4.3).

Consider the following condition $(*)$ on the pair (B, K) in $\mathcal{S}_2(E)$ and the character χ:

For any finite field extension N/E of degree prime to p, the class $[B, K]_N$ is not decomposable and the class of the algebra B_N in $\mathrm{Br}(N)$ cannot be written in the form $[B_N] = \rho \cup (s)$ for some $s \in N^\times$ and a character $\rho \in X(N)$ of order p^2 such that $p \cdot \rho$ is a multiple of χ_N.

Proposition 3.1. *Let $\chi \in X(E)$ be a character of order p, $K = E(\chi)$, B a central simple algebra of degree p^2 over E such that $(B,K) \in \mathcal{S}_2(E)$ and (B,K) together with χ satisfy the condition $(*)$. Then $\operatorname{ed}_p^{\mathcal{F}_1}([C]) \geq \operatorname{ed}_p^{\mathcal{F}_2}([B,K]) + 1$ for the algebra C defined by* (1).

Proof. Let $M/E(y)$ be a finite field extension of degree prime to p, $M_0 \subset M$ a subfield over F and $[C_0] \in \mathcal{F}_1(M_0)$ such that

$$[C_0]_M = [C]_M \tag{2}$$

in $\mathcal{F}_1(M)$ and $\operatorname{ed}_p^{\mathcal{F}_1}([C]) = \operatorname{tr.deg}_F(M_0)$. We extend the discrete valuation v on $E(y)$ to a discrete valuation v' on M with ramification index e' and inertia degree prime to p (cf. [6, Lemma 1.1]). Thus, the residue field N of v' is a finite extension of E of degree prime to p. Let v_0 be the restriction of v' to M_0 and N_0 its residue field. As $[N : E]$ is not divisible by p, it follows from (24) that $\partial([C]_M) = e' \cdot \chi_N \neq 0$. Hence the algebra C_M is ramified, i.e., the class of C_M does not belong to the image of the map $\operatorname{Br}(O) \to \operatorname{Br}(M)$, where O is the valuation ring of v'. It follows that C_0 is also ramified, therefore v_0 is nontrivial and hence v_0 is a discrete valuation.

Let $\chi_0 = \partial([C_0]) \in X(N_0)\{p\}$ and $K_0 = N_0(\chi_0)$. Choose a prime element π_0 in M_0 and write

$$[C_0]_{\widehat{M_0}} = [\widehat{B}_0] + \big(\widehat{\chi}_0 \cup (\pi_0)\big) \tag{3}$$

in $\operatorname{Br}(\widehat{M_0})$, where B_0 is a central simple algebra over N_0 (see Section 4.3). By (23),

$$\operatorname{ind}(C_0) = \operatorname{ord}(\chi_0) \cdot \operatorname{ind}(B_0)_{K_0}. \tag{4}$$

Let e be the ramification index of M/M_0 and let π be a prime element in M. Write $\pi_0 = u\pi^e$ and $y = v\pi^{e'}$ with u and v units in M.

It follows from (2) and (24) that

$$e' \cdot \chi_N = \partial([C]_M) = \partial([C_0]_M) = e \cdot \partial([C_0])_N = e \cdot (\chi_0)_N. \tag{5}$$

Recall that e' is relatively prime to p. It follows that χ_N is a multiple of $(\chi_0)_N$. In particular, $\operatorname{ord}(\chi_0)$ is divisible by p.

It follows from (2),(3) and (5) that

$$[\widehat{(B_0)_N}] + \big(\widehat{(\chi_0)}_N \cup (u)\big) = [\widehat{B}_N] + \big(\widehat{\chi}_N \cup (v)\big) \tag{6}$$

in $\operatorname{Br}(\widehat{M})$, hence

$$[(B_0)_N] + \big((\chi_0)_N \cup (\bar{u})\big) = [B_N] + \big(\chi_N \cup (\bar{v})\big) \tag{7}$$

in $\operatorname{Br}(N)$.

Since $\operatorname{ind}(C_0) \leq p^2$, it follows from (23) and (4) that $\operatorname{ord}(\chi_0)$ divides p^2.

Case 1: $\operatorname{ord}(\chi_0) = p^2$. By (4), $\operatorname{ind}(B_0)_{K_0} = 1$, i.e., B_0 is split over K_0, hence $[B_0] = \chi_0 \cup (s_0)$ for some $s_0 \in N_0^\times$. It follows from (7) that $[B_N] = (\chi_0)_N \cup (s)$ for some $s \in N^\times$. If $\operatorname{ord}(\chi_0)_N = p$, then $(\chi_0)_N$ is a multiple of χ_N and hence $[B,K]_N$ is decomposable. If $\operatorname{ord}(\chi_0)_N = p^2$, the character $p \cdot (\chi_0)_N$ is a multiple of χ_N. In both cases, (B,K) and χ do not satisfy the condition $(*)$, a contradiction.

Case 2: $\mathrm{ord}(\chi_0)=p$. Then the characters χ_N and $(\chi_0)_N$ generate the same subgroup in $X(N)$. It follows that

$$K_0\otimes_{N_0} N\simeq N\big((\chi_0)_N\big)=N(\chi_N)\simeq K\otimes_E N. \tag{8}$$

It follows from (4) that $\mathrm{ind}(B_0)_{K_0}\le p$. Therefore, we may assume that $\deg(B_0)=p^2$ and hence $(B_0,K_0)\in\mathcal{S}_2(N_0)$. It follows from (7) that

$$[B]_N-[B_0]_N\in \mathrm{Br}(K\otimes_E N/N).$$

By (8), the pairs $(B_N,K\otimes_E N)$ and $((B_0)_N,K_0\otimes_{N_0}N)=(B_0,K_0)_N$ are equivalent in $\mathcal{S}_2(N)$. It follows that the class of $[B,K]$ in $\mathcal{F}_2(E)$ is p-defined over N_0, therefore,

$$\mathrm{ed}_p^{\mathcal{F}_1}([C])=\mathrm{tr.deg}_F(M_0)\ge \mathrm{tr.deg}_F(N_0)+1\ge \mathrm{ed}_p^{\mathcal{F}_2}([B,K])+1. \qquad \square$$

3.2. **The functor $\mathcal{F}_3$.** Let E/F be a field extension and let $\mathcal{S}_3(E)$ be the class of pairs (A,L), where A is a csa of degree p^2 over E and L is a bicyclic étale E-algebra of dimension p^2 such that L splits A, i.e., $[A]\in\mathrm{Br}(L/E)$. We say that the pairs (A_1,L_1) and (A_2,L_2) in $\mathcal{S}_3(E)$ are equivalent if $L_1\simeq L_2$ and $[A_1]-[A_2]\in\mathrm{Br}_{dec}(L_1/E)=\mathrm{Br}_{dec}(L_2/E)$ (see Section 4.2). Let $\mathcal{F}_3(E)$ be the set of equivalence classes in $\mathcal{S}_3(E)$. We write $[A,L]$ for the equivalence class of (A,L) in $\mathcal{F}_3(E)$.

We say that a class $[A,L]$ is *decomposable* if $[A,L]=[A',L]$ with A' a split algebra.

Let $(A,L)\in S_3(E)$. Choose the characters χ and η in $X(E)$ such that $L=E(\chi,\eta):=E(\chi)E(\eta)$. Let $K=E(\chi)$ and $K'=E(\eta)$. As $\mathrm{ind}(A_K)\le p$, there is a csa B over the function field $E(x)$ (x is a variable) of degree p^2 such that

$$[B]=[A_{E(x)}]+\big(\eta_{E(x)}\cup(x)\big) \tag{9}$$

in $\mathrm{Br}\big(E(x)\big)$. We have $\big(B,K(x)\big)\in\mathcal{S}_2\big(E(x)\big)$ and $\partial([B])=\eta$, where ∂ is taken with respect to the discrete valuation v on $E(x)$ associated to x.

Consider the following condition $(**)$ on the pair (A,L) in $\mathcal{S}_3(E)$ and the characters χ and η:

For any finite field extension N/E of degree prime to p, the class $[A,L]_N$ is not decomposable and the class of the algebra A_N in $\mathrm{Br}(N)$ cannot be written in the form $[A_N]=\big(\rho\cup(s)\big)+\big(\varepsilon\cup(t)\big)$ for some $s,t\in N^\times$ and characters $\varepsilon\in X(N)$ of order p and $\rho\in X(N)$ of order p^2 such that $\langle p\cdot\rho,\varepsilon\rangle=\langle\chi_N,\eta_N\rangle$.

Proposition 3.2. *Let $\chi,\eta\in X(E)$ be linearly independent characters of order p, $K=E(\chi)$, $L=E(\chi,\eta)$, A a central simple algebra of degree p^2 over E such that $(A,L)\in\mathcal{S}_3(E)$ and (A,L) with with the characters χ and η satisfy the condition $(**)$. Then $\mathrm{ed}_p^{\mathcal{F}_2}\big([B,K(x)]\big)\ge\mathrm{ed}_p^{\mathcal{F}_3}\big([A,L]\big)+1$ for the algebra B defined by* (9).

Proof. Let $M/E(x)$ be a finite field extension of degree prime to p, $M_0 \subset M$ a subfield over F and $[B_0, R_0] \in \mathcal{F}_2(M_0)$ such that

$$[B_0, R_0]_M = [B, K(x)]_M$$

in $\mathcal{F}_2(M)$ and $\mathrm{ed}_p^{\mathcal{F}_2}([B, K(x)]) = \mathrm{tr.deg}_F(M_0)$. This equality means that

$$R := K(x) \otimes_{E(x)} M \simeq R_0 \otimes_{M_0} M \quad \text{and} \tag{10}$$

$$[B]_M = [B_0]_M + \big(\chi_M \cup (f)\big) \tag{11}$$

for some $f \in M^\times$.

We extend the discrete valuation v on $E(x)$ to a discrete valuation v' on M with ramification index e' and inertia degree prime to p (cf. [6, Lemma 1.1]). Thus, the residue field N of v' is a finite extension of E of degree prime to p. Let v_0 be the restriction of v' to M_0 and N_0 its residue field. As $[N : E]$ is not divisible by p, it follows from (24) that $\partial([B]_M) = e' \cdot \chi_N \neq 0$. Hence the algebra B_M is ramified. It follows that B_0 is also ramified, therefore v_0 is nontrivial and hence v_0 is a discrete valuation.

As $R = KM$, the valuation v' on M extends uniquely to a discrete valuation on R and R/M is unramified.

Let $\eta_0 = \partial([B_0]) \in X(N_0)\{p\}$ and $K_0' = N_0(\eta_0)$. Choose a prime element π_0 in M_0 and write

$$[B_0]_{\widehat{M_0}} = [\widehat{A}_0] + \big(\widehat{\eta}_0 \cup (\pi_0)\big) \tag{12}$$

in $\mathrm{Br}(\widehat{M_0})$, where A_0 is a central simple algebra over N_0. By (23),

$$\mathrm{ind}(B_0) = \mathrm{ord}(\eta_0) \cdot \mathrm{ind}(A_0)_{K_0'}. \tag{13}$$

Let e be the ramification index of M/M_0 and let π be a prime in M. Write $\pi_0 = u\pi^e$, $x = v\pi^{e'}$ and $f = w\pi^k$ with u, v and w units in M.

It follows from (11) and (24) that

$$e' \cdot \eta_N = \partial([B]_M) = e \cdot \partial([B_0])_N + \partial\big(\chi_M \cup (f)\big) = e \cdot (\eta_0)_N + k \cdot \chi_N. \tag{14}$$

Note that the characters χ_N and η_N are linearly independent in $X(N)$ since $[N : E]$ is not divisible by p.

As e' is relatively prime to p, η_N belongs to the subgroup of $X(N)$ generated by $(\eta_0)_N$ and χ_N, and $\eta_0 \neq 0$ since χ_N and η_N are linearly independent. In particular, p divides $\mathrm{ord}(\eta_0)$.

It follows from (11), (12) and (14) that

$$[\widehat{(A_0)}_N] + \big(\widehat{(\eta_0)}_N \cup (u)\big) + \big(\widehat{\chi}_M \cup (w)\big) = [\widehat{A}_N] + \big(\widehat{\eta}_N \cup (v)\big) \tag{15}$$

in $\mathrm{Br}(\widehat{M})$, hence

$$[(A_0)_N] + \big((\eta_0)_N \cup (\bar{u})\big) + \big(\chi_N \cup (\bar{w})\big) = [A_N] + \big(\eta_N \cup (\bar{v})\big) \tag{16}$$

in $\mathrm{Br}(N)$.

Since $\mathrm{ind}(B_0) \leq p^2$, it follows from (13) that $\mathrm{ord}(\eta_0) \leq p^2$.

Case 1: $\mathrm{ord}(\eta_0) = p^2$. By (13), A_0 is split over $N_0(\eta_0)$, hence $[A_0] = \eta_0 \cup (s_0)$ for some $s_0 \in N_0^\times$. It follows from (16) that $[A_N] = \big((\eta_0)_N \cup (s)\big) + \big(\chi_N \cup (t)\big)$

for some $s, t \in N^\times$. If $\operatorname{ord}(\eta_0)_N = p$, by (14), $(\eta_0)_N$ is contained in $\langle \chi_N, \eta_N \rangle$ and hence $[A, L]_N$ is decomposable. If $\operatorname{ord}(\eta_0)_N = p^2$, then again by (14), $\langle p \cdot (\eta_0)_N, \chi_N \rangle = \langle \chi_N, \eta_N \rangle$. In both cases, (A, L) with the characters χ and η do not satisfy the condition $(**)$, a contradiction.

Case 2: $\operatorname{ord}(\eta_0) = p$. It follows from (14) that $(e, p) = 1$ and η_0 belongs to the subgroup generated by χ and η. Hence, by (10), the cyclic extension R_0/M_0 is unramified. Thus, there exists a character $\chi_0 \in X(N_0)$ with $\widehat{R}_0 = \widehat{M}_0(\widehat{\chi}_0)$ and $(\chi_0)_N = \chi_N$.

It follows from (14) that

$$\langle (\chi_0)_N, (\eta_0)_N \rangle = \langle \chi_N, \eta_N \rangle$$

in $X(N)$. Let $L_0 = N_0(\chi_0, \eta_0)$. It follows from (14) that

$$L_0 \otimes_{N_0} N = N\big((\chi_0)_N, (\eta_0)_N\big) = N(\chi_N, \eta_N) = L \otimes_E N \tag{17}$$

is a bicyclic field extension of degree p^2, hence so is the extension L_0/N_0. In particular, χ_0 and η_0 generate a subgroup of order p^2 in $X(N_0)$.

Let $K_0 = N_0(\chi_0)$. It follows from (12) that

$$[(B_0)_{\widehat{R}_0}] = [\widehat{(A_0)_{K_0}}] + \big(\widehat{(\eta_0)_{K_0}} \cup (\pi_0)\big).$$

As $(B_0, R_0) \in \mathcal{S}_2(M_0)$, we have $\operatorname{ind}(B_0)_{R_0} \le p$. Since the character $(\eta_0)_{K_0}$ is nontrivial, it follows from (23) that A_0 is split by $K_0((\eta_0)_{K_0}) = L_0$. We may then assume that $\deg(A_0) = p^2$ and hence $(A_0, L_0) \in \mathcal{S}_3(N_0)$.

It follows from (16) that $[A_N] - [(A_0)_N] \in \operatorname{Br}_{dec}(L \otimes_E N/N)$. By (17), the pairs $(A_N, L \otimes_E N)$ and $\big((A_0)_N, L_0 \otimes_{N_0} N\big) = (A_0, L_0)_N$ are equivalent in $\mathcal{S}_3(N)$. It follows that the class $[A, L]$ in $\mathcal{F}_3(E)$ is p-defined over N_0, therefore,

$$\operatorname{ed}_p^{\mathcal{F}_2}([B, K(x)]) = \operatorname{tr.deg}_F(M_0) \ge \operatorname{tr.deg}_F(N_0) + 1 \ge \operatorname{ed}_p^{\mathcal{F}_3}([A, L]) + 1. \quad \square$$

Let E be a field extension of F and L/E a bicyclic field extension of degree p^2. Let T be the torus over E of norm 1 elements for the field extension L/E. Let $t \in T\big(E(T)\big)$ be the generic point and let $[A, L(T)]$ be the corresponding element in $\mathcal{F}_3\big(E(T)\big)$ via the isomorphism in Proposition 4.1.

Proposition 3.3. $\operatorname{ed}_p^{\mathcal{F}_3}([A, L(T)]) \ge p^2 - 1$.

Proof. Let $M/E(T)$ be a field extension of degree prime to p, $M_0 \subset M$ a subfield over F and $[A_0, L_0] \in \mathcal{F}_3(M_0)$ such that $[A_0, L_0]_M = [A, L(T)]_M$. We need to prove that $\operatorname{tr.deg}_F(M_0) \ge p^2 - 1$. Set $LM = L \otimes_E M$. As $L_0 \otimes_{M_0} M \simeq LM$, we may assume that $L_0 \subset LM$.

Let T_0 be the torus over M_0 of norm 1 elements for the extension L_0/M_0. Consider the commutative diagram via the isomorphisms in Proposition 4.1:

$$\begin{array}{ccc} T_0(M_0)/R & \longrightarrow & T(M)/R \\ \wr\downarrow & & \wr\downarrow \\ \mathcal{F}_3(M_0) & \longrightarrow & \mathcal{F}_3(M) \end{array}$$

There exists an element $t_0 \in T_0(M_0)$ such that $(t_0)_M$ is R-equivalent to t_M in $T(M)$. We have $\deg(t) = 1$, hence $\deg(t_M) \neq 0$. By Theorem 2.6, $\deg((t_0)_M) \equiv \deg(t_M)$ modulo p, hence $\deg((t_0)_M) \neq 0$. It follows that $(t_0)_M$, considered as a morphism $\operatorname{Spec}(M) \to T$ is dominant. Therefore, there is a homomorphism $E(T) \to M$ over E taking t to $(t_0)_M$. The elements $\rho(t)$ over all $\rho \in G := \operatorname{Gal}(L/E)$ generate the field $L(T)$ over L. Hence the elements $\rho(t_0)_M$ generate a subfield in LM over L of transcendence degree $p^2 - 1$. As $t_0 \in L_0$ and L_0 is normal over M_0 and hence is G-invariant, the elements $\rho(t_0)$ generate a subfield in L_0 over F of transcendence degree p^2-1. It follows that $\operatorname{tr.deg}_F(L_0) \geq p^2-1$, hence $\operatorname{tr.deg}_F(M_0) \geq p^2 - 1$. $\square$

Remark 3.4. Let L be a bicyclic field extension of degree p^2 of a field F of arbitrary characteristic and let $T = R^{(1)}_{L/F}(\mathbb{G}_{m,L})$. A similar argument as the one in the proof of Proposition 3.3 shows that $\operatorname{ed}_p(T/R) = p^2 + 1$, where T/R is the functor taking a field E to $T(E)/R$.

3.3. Main theorem.

Theorem 3.5. *Let p be a prime integer and F a field of characteristic different from p. Then*
$$\operatorname{ed}_p\big(\mathbf{PGL}_F(p^2)\big) = p^2 + 1.$$

Proof. Recall that $\operatorname{ed}_p\big(\mathbf{PGL}_F(p^2)\big) = \operatorname{ed}_p(\mathcal{F}_1)$. First we prove the inequality $\operatorname{ed}_p(\mathcal{F}_1) \geq p^2 + 1$. We may replace F by any field extension. In particular, we may assume that there are linearly independent characters $\chi, \eta \in X(F)$ of order p, hence $L := F(\chi, \eta)/F$ is a field extension of degree p^2. Set $K = F(\chi)$ and $K' = F(\eta)$. Let T be the norm 1 torus for the extension L/F and set $E := F(T)$. Let $[A, LE]$ be the element of $\mathcal{F}_3(E)$ corresponding to the generic point $t \in T(E)$ via the isomorphism in Proposition 4.1. Consider the pair $(B, KE(x)) \in \mathcal{S}_2(E(x))$ with $[B] = [A_{E(x)}] + \big(\eta_{E(x)} \cup (x)\big)$ in $\operatorname{Br}(E(x))$ and the algebra C of degree p^2 over $E(x, y)$ with $[C] = [B_{E(x,y)}] + \big(\chi_{E(x,y)} \cup (y)\big)$ in $\operatorname{Br}\big(E(x, y)\big)$.

We claim that the pair (A, LE) in $\mathcal{S}_3(E)$ and the characters χ_E and η_E satisfy the condition $(**)$. Indeed, as $t \neq 1$ in $T(E)/R$ (see Section 4.2) we have $t_N \neq 1$ since $[N : E]$ is prime to p and hence $[A, LE]_N$ is not decomposable. Now suppose that $[A_N] = \big(\rho \cup (s)\big) + \big(\varepsilon \cup (t)\big)$ for a field extension N/E of degree prime to p, elements $s, t \in N^\times$ and characters $\varepsilon \in X(N)$ of order p and $\rho \in X(N)$ of order p^2 such that $\langle p \cdot \rho, \varepsilon\rangle = \langle \chi_N, \eta_N\rangle$. Let T_1 be the norm 1 torus for the field extension $L_1 = N(\rho, \varepsilon)$ over N. By Proposition 4.6, the image of t under the natural homomorphism $T(E)/R \to T(N)/R \to T_1(N)/R$ is not trivial. By Proposition 4.1, $[A_N]$ does not belong to the kernel of the homomorphism $\operatorname{Br}(LN/N)/\operatorname{Br}_{dec}(LN/N) \to \operatorname{Br}(L_1/N)/\operatorname{Br}_{dec}(L_1/N)$, a contradiction. The claim is proved.

We claim that the pair $(B, KE(x))$ in $\mathcal{S}_2(E(x))$ and the character $\chi_{E(x)}$ satisfy the condition $(*)$. The same argument as in the previous claim applied to the field $E(x)$ shows that $(A_{E(x)}, LE(x))$ in $\mathcal{S}_3(E(x))$ and the characters

$\chi_{E(x)}$ and $\eta_{E(x)}$ satisfy the condition $(**)$. Let $N/E(x)$ be a finite field extension of degree prime to p. As $[A_{E(x)}] = [B] - \big(\eta_{E(x)} \cup (x)\big)$, the class $[B, KE(x)]_N$ is not decomposable. Suppose that $[B_N] = \rho \cup (s)$ for some $s \in N^\times$ and a character $\rho \in X(N)$ of order p^2 such that $p \cdot \rho$ is a multiple of χ_N. Then $[A_N] = \big(\rho \cup (s)\big) - \big(\eta_N \cup (x)\big)$ and we have $\langle p \cdot \rho, \eta_N \rangle = \langle \chi_N, \eta_N \rangle$, a contradiction proving the claim.

By Propositions 3.1, 3.2 and 3.3,

$$\mathrm{ed}_p\big(\mathbf{PGL}_F(p^2)\big) = \mathrm{ed}_p(\mathcal{F}_1) \geq \mathrm{ed}_p^{\mathcal{F}_1}([C]) \geq \mathrm{ed}_p^{\mathcal{F}_2}([B, KE(x)]) + 1 \geq$$

$$\mathrm{ed}_p^{\mathcal{F}_3}\big([A, LE]\big) + 2 \geq (p^2 - 1) + 2 = p^2 + 1.$$

We shall show that $\mathrm{ed}_p(\mathcal{F}) \leq p^2 + 1$. In fact, this was shown in [8, Cor. 3.10(a)]. For completeness, we give the argument here.

Let $\mathcal{F}_1'(E)$ be the set of isomorphism classes of central simple E-algebras of degree p^2 that are crossed products with the group $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$. So $\mathcal{F}_1'$ is a subfunctor of $\mathcal{F}_1$. By [12, Th. 1.2], for every $[A] \in \mathcal{F}_1(E)$ there is a finite field extension E'/E of degree prime to p such that $[A_{E'}] \in \mathcal{F}_1'(E')$. Hence the inclusion of $\mathcal{F}_1'$ into $\mathcal{F}_1$ is p-surjective. It follows that $\mathrm{ed}_p(\mathcal{F}_1) \leq \mathrm{ed}_p(\mathcal{F}_1')$ [10, Prop. 1.3]. So it suffices to show that $\mathrm{ed}(\mathcal{F}_1') \leq p^2 + 1$.

Let E/F be a field extension and $[A] \in \mathcal{F}_1'(E)$. Then $[A] \in \mathrm{Br}(L/E)$ for a bicyclic field extension L/F of degree p^2 with Galois group G generated by σ and τ. The exact sequence (19) yields an epimorphism

$$\mathrm{Hom}_G(M, L^\times) \to \mathrm{Br}(L/E).$$

Choose a G-homomorphism $\varphi : M \to L^\times$ corresponding to $[A]$ in $\mathrm{Br}(L/E)$. Since $\mathrm{rank}(M) = p^2 + 1$, the image of φ is contained in $L_0^\times$, where L_0 is a G-invariant subfield of L with $\mathrm{tr.deg}_F(L_0) \leq p^2 + 1$. Note that G acts faithfully on M. Modifying φ by an element in the image of the map $\mathrm{Hom}_G(\Lambda^2, L^\times) \to \mathrm{Hom}_G(M, L^\times)$, we may assume that G acts faithfully on the image of φ and hence on L_0. Thus L_0 is a Galois extension of $E_0 := (L_0)^G$ with Galois group G and φ defines a central simple E_0-algebra A_0 with $[A_0] \in \mathrm{Br}(L_0/E_0)$ such that $A_0 \otimes_{F_0} E \simeq A$. Thus, A is defined over E_0, hence

$$\mathrm{ed}^{\mathcal{F}_1'}([A]) \leq \mathrm{tr.deg}_F(E_0) = \mathrm{tr.deg}_F(L_0) \leq p^2 + 1. \qquad \square$$

4. Appendix

We collect auxiliary results in the appendix.

4.1. **Characters, cyclic algebras and tori.** Let F be a field and $\Gamma = \mathrm{Gal}(F_{sep}/F)$ the absolute Galois group. The character group $X(F)$ of Γ is equal to

$$\mathrm{Hom}_c(\Gamma, \mathbb{Q}/\mathbb{Z}) = H^1(F, \mathbb{Q}/\mathbb{Z}) \simeq H^2(F, \mathbb{Z}).$$

For a character $\chi \in X(F)$ set $F(\chi) = (F_{sep})^{\mathrm{Ker}(\chi)}$. Then $F(\chi)/F$ is a cyclic field extension of degree $\mathrm{ord}(\chi)$. The Galois group $\mathrm{Gal}(F(\chi)/F)$ has a canonical generator σ such that $\chi(\tilde{\sigma}) = \mathrm{ord}(\chi)^{-1} + \mathbb{Z}$ for any lifting $\tilde{\sigma}$ of σ to Γ.

Let K/F be cyclic field extension. Choose a character $\chi \in X(F)$ such that $K = F(\chi)$. The cup-product

$$X(F) \otimes F^\times = H^2(F, \mathbb{Z}) \otimes H^0(F, F_{sep}^\times) \to H^2(F, F_{sep}^\times) = \mathrm{Br}(F)$$

takes $\chi \otimes a$ to the class $\chi \cup (a)$ of a cyclic algebra split by K. In fact, every element of $\mathrm{Br}(K/F)$ is of the form $\chi \otimes a$ for some $a \in F^\times$.

Let L be an étale F-algebra of dimension n and $S = R_{L/F}(\mathbb{G}_{m,L})/\,\mathbb{G}_m$. The exact sequence

$$1 \to \mathbb{G}_m \to \mathbb{G}_{m,L} \to S \to 1$$

and Hilbert Theorem 90 yield an isomorphism $\theta : H^1(F, S) \xrightarrow{\sim} \mathrm{Br}(L/F)$. Let $\alpha \in H^1(F, S)$ and let S_α be the corresponding principal homogeneous space of S. As S is an open subscheme of the projective space $\mathbb{P}_F(L)$, the variety S_α is an open subset of the Severi-Brauer variety $SB(A)$ of a central simple F-algebra A_α of degree n such that $[A_\alpha] = \theta(\alpha)$ in $\mathrm{Br}(L/F)$. Moreover, S_α is trivial if and only if A_α is split.

Let $\chi \in X(F)$ and $L = F(\chi)$. Then $S \simeq R^{(1)}_{L/F}(\mathbb{G}_{m,L})$ by Hilbert Theorem 90 and $[A] = \chi \cup a$ for some $a \in F^\times$. Moreover, the principal homogeneous space S_α coincides with the fiber S_a of the norm homomorphism $\mathbb{G}_{m,L} \to \mathbb{G}_m$ over a.

4.2. **Bicyclic algebras and tori.** Let χ and η be two characters in $X(F)$ of order n and m respectively. Then the fields $K = F(\chi)$ and $K' = F(\eta)$ are cyclic extensions of F of degree n and m respectively. Set $L = K \otimes_F K'$, so L is a bicyclic extension of F of degree nm with Galois group G generated by σ and τ such that $L^\sigma = K'$ and $L^\tau = K$.

Let I_G be the augmentation ideal in the group ring $\Lambda := \mathbb{Z}[G]$, i.e., $I_G = \mathrm{Ker}(\varepsilon)$, where $\varepsilon : \Lambda \to \mathbb{Z}$ is defined by $\varepsilon(\rho) = 1$ for all $\rho \in G$. We have:

$$\mathrm{Br}(L/F) = H^2(G, L^\times) = \mathrm{Ext}^2_G(\mathbb{Z}, L^\times) \simeq \mathrm{Ext}^1_G(I_G, L^\times). \tag{18}$$

Consider the exact sequences

$$0 \to M \to \Lambda^2 \xrightarrow{f} I_G \to 0, \tag{19}$$

where $f(x, y) = (\sigma - 1)x + (\tau - 1)y$ and $M = \mathrm{Ker}(f)$ and

$$0 \to \Lambda/\mathbb{Z}N_G \xrightarrow{g} M \xrightarrow{h} \mathbb{Z}^2 \to 0, \tag{20}$$

where $N_G = \sum_{\rho \in G} \rho$, $g(x + \mathbb{Z}N_G) = \big((\tau - 1)x, (1 - \sigma)x\big)$ and $h(x, y) = \big(\varepsilon(x)/n, \varepsilon(y)/m\big)$.

Let T be the torus of norm 1 elements for the extension L/F. We have

$$T(F) = \mathrm{Hom}_G\big(\Lambda/\mathbb{Z}N_G, L^\times\big). \tag{21}$$

The exact sequences (19) (20) and the isomorphisms (18) and (21) yield a commutative diagram:

$$\begin{tikzcd}
 & \mathrm{Hom}_G(\mathbb{Z}^2, L^\times) \arrow[d, "h^*"'] \arrow[dr, "\alpha"] & & \\
\mathrm{Hom}_G(\Lambda^2, L^\times) \arrow[r] \arrow[dr, "\beta"'] & \mathrm{Hom}_G(M, L^\times) \arrow[r] \arrow[d, "g^*"'] & \mathrm{Br}(L/F) \arrow[r] & 0 \\
 & T(F) \arrow[d] & & \\
 & 0 & &
\end{tikzcd}$$

It follows that the cokernels of α and β are naturally isomorphic. The image of $\alpha : (F^\times)^2 \to \mathrm{Br}(L/F)$ is the subgroup of decomposable elements $\mathrm{Br}_{dec}(L/F)$ of $\mathrm{Br}(L/F)$ generated by $\chi \cup (a)$ and $\eta \cup (b)$ with $a, b \in F^\times$.

The cokernel of $\beta : (L^\times)^2 \to T(F)$ is the group of R-equivalence classes $T(F)/R$, i.e., the factor group of $T(F)$ by the subgroup generated by $\sigma(x)/x$ and $\tau(y)/y$ for all $x \in K^\times$ and $y \in K'^\times$. We have proved:

Proposition 4.1. *Let L/F is a bicyclic extension and $T = R^{(1)}_{L/F}(\mathbb{G}_{m,L})$. Then there is a natural isomorphism*

$$\varphi : \mathrm{Br}(L/F)/\,\mathrm{Br}_{dec}(L/F) \simeq T(F)/R.$$

The torus $T = R^{(1)}_{L/F}(\mathbb{G}_{m,L})$ is not rational if L/F is a bicyclic field extension of degree p^2 by [14, §4.8]. Moreover, T is not R-trivial generically, i.e., there is a field extension E/F such that $T(E)/R \neq 1$. It follows that the image of the generic point of T in $T\big(F(T)\big)/R$ is not trivial.

4.3. **Central simple algebras and discrete valuations.** Let v be a discrete valuation on a field extension E over F, N the residue field, $\widehat{E}$ the completion of E. The field N is a filed extension of F. Let $\chi \in X(F)$. Then $F(\chi)/F$ is a cyclic field extension of degree $\mathrm{ord}(\chi)$ with the choice of a generator of $\mathrm{Gal}(F(\chi)/F)$. The group $X(F)$ is identified with the character group of the maximal unramified field extension of $\widehat{E}$. For a character $\chi \in X(F)$, we write $\widehat{\chi}$ for the corresponding character in $X(\widehat{E})$.

Let p be a prime integer different from $\mathrm{char}(F)$. There is an exact sequence of p-groups [4, Prop. 7.7]

$$0 \to \mathrm{Br}(N)\{p\} \xrightarrow{i} \mathrm{Br}(\widehat{E})\{p\} \xrightarrow{\partial} X(N)\{p\} \to 0. \tag{22}$$

The first map preserves indices of algebras. For a central simple algebra C over N with $C \in \mathrm{Br}(N)\{p\}$ let $\widehat{C}$ be a central simple algebra over $\widehat{E}$ of the same degree representing the image of $[C]$ under i. For example, if $[C] = \chi \cup (\bar{u})$ for some $\chi \in X(N)\{p\}$ and a unit $u \in \widehat{E}$, then $[\widehat{C}] = \widehat{\chi} \cup (u)$.

The choice of a prime element π in $\widehat{E}$ provides with a splitting of the sequence (22) by sending a character χ to the class of the cyclic algebra $\widehat{\chi} \cup (\pi)$. Thus for every central simple algebra A over $\widehat{E}$ we can write

$$[A] = [\widehat{C}] + \big(\widehat{\chi} \cup (\pi)\big)$$

in $\mathrm{Br}(\widehat{E})$ for unique $[C] \in \mathrm{Br}(N)\{p\}$ and $\chi = \partial([A])$. Moreover (cf. [5, Th. 5.15(a)] or [13, Prop. 2.4]),

$$\mathrm{ind}(A) = \mathrm{ord}(\chi) \cdot \mathrm{ind}(C_{N(\chi)}). \tag{23}$$

Let E'/E be a finite field extension and v' a discrete valuation on E' extending v with residue field N'. Then for any $[A] \in \mathrm{Br}(E)\{p\}$ one has

$$\partial_{v'}\big([A]_{E'}\big) = e \cdot \partial_v([A])_{N'}, \tag{24}$$

where e is the ramification index [4, Prop. 8.2].

4.4. **Chow groups of tori and Severi-Brauer varieties.** Let p be a prime integer and let Z be the product of r copies of the projective space $\mathbb{P}_F(W)$, where W is a vector space of dimension $n > 0$ over F. Then

$$\mathrm{CH}(Z) = \mathbb{Z}[\mathbf{h}] := \mathbb{Z}[h_1, h_2, \dots, h_r],$$

with $h_i^n = 0$ for all i, where h_i is pull-back on Z of the class of a hyperplane on the ith factor of Z. Moreover, $\mathbb{Z}[\mathbf{h}]$ is the factor ring of the polynomial ring of the variables $t_1, t_2, \dots, t_r$ by the ideal generated by $t_1^n, t_2^n, \dots, t_r^n$. Note that the homogeneous ith component $\mathbb{Z}[\mathbf{h}]_i$ is trivial if $i > r(n-1)$ and $\mathbb{Z}[\mathbf{h}]_{r(n-1)} = \mathbb{Z}h^{n-1}$.

Let K/F be a Galois field extension with cyclic Galois group H of prime order p and let σ be a generator of H. Let V be a vector space of dimension $n > 0$ over K. Consider the variety $X = R_{K/F}\big(\mathbb{P}_K(V)\big)$ over F. Then X_K is the product of p copies of $\mathbb{P}_K(V)$. We assume that the factors are numbered so that H acts on the product by cyclic permutation of the factors. We have the graded ring homomorphism

$$\mathrm{CH}(X) \to \mathrm{CH}(X_K) = \mathbb{Z}[\mathbf{h}],$$

where $\mathbf{h} = (h_1, h_2, \dots, h_p)$.

The group H acts on $\mathbb{Z}[\mathbf{h}]$ permuting cyclically the h_i's. Hence the image of the map $\mathrm{CH}(X) \to \mathbb{Z}[\mathbf{h}]$ is contained in the subring $\mathbb{Z}[\mathbf{h}]^H$ of H-invariant elements, so we have the graded ring homomorphism

$$\mathrm{CH}(X) \to \mathbb{Z}[\mathbf{h}]^H$$

(which is in fact an isomorphism). The image of an element $\alpha \in \mathrm{CH}(X)$ in $\mathbb{Z}[\mathbf{h}]^H$ is denoted by $\bar{\alpha}$. For example, if α is the class of the subscheme $R_{K/F}\big(\mathbb{P}_K(W)\big)$ of X, where W is a K-subspace of V of codimension $i = 0, 1, \dots, n-1$, then $\bar{\alpha} = h^i$, where $h := h_1 h_2 \cdots h_p$.

Consider the trace homomorphism

$$\mathrm{tr} : \mathbb{Z}[\mathbf{h}] \to \mathbb{Z}[\mathbf{h}]^H$$

defined by $\mathrm{tr}(x) = \sum_{i=0}^{p-1} \sigma^i(x)$. We write I for the image of tr. Clearly, I is a graded ideal in $\mathbb{Z}[\mathbf{h}]^H$. Note that

$$(25) \qquad \big(\mathbb{Z}[\mathbf{h}]^H\big)_j = \begin{cases} I_j, & \text{if } p \text{ does not divide } j; \\ \mathbb{Z}h^i + I_j, & \text{if } j = pi. \end{cases}$$

It follows that $\mathbb{Z}[\mathbf{h}]^H$ is generated by I and h^i, $i = 0, 1, \dots, n-1$ as an abelian group. Moreover, $ph^j \in I$ for all j and $I_{p(n-1)} = p\mathbb{Z}h^{n-1}$.

Let A be a central simple algebra over K of degree n and let $Y = R_{K/F}\big(\mathrm{SB}(A)\big)$, where $\mathrm{SB}(A)$ is the Severi-Brauer variety of A over K. The function field E of Y splits A and is linearly disjoint with K/F. Therefore, $Y_E \simeq X_E$ and we have the ring homomorphism

$$\mathrm{CH}(Y) \to \mathrm{CH}(Y_E) \simeq \mathrm{CH}(X_E) \to \mathbb{Z}[\mathbf{h}]^H.$$

The image of an element $\alpha \in \mathrm{CH}(Y)$ in $\mathbb{Z}[\mathbf{h}]^H$ is denoted by $\bar{\alpha}$.

Proposition 4.2. *Let K/F be a cyclic field extension of a prime degree p, let A be a nonsplit central simple K-algebra of degree p and $Y = R_{K/F}\big(\mathrm{SB}(A)\big)$. Then the image of the map $\mathrm{CH}(Y) \to \mathbb{Z}[\mathbf{h}]^H$ is contained in $\mathbb{Z} + I$.*

Proof. Consider a more general situation: A is a central simple K-algebra of index p and degree n. Let $\alpha \in \mathrm{CH}(Y)$. We shall prove in the cases 1 and 2 below that $\bar{\alpha} \in \mathbb{Z} + I$. By (25), we may assume that $\alpha \in \mathrm{CH}^{pi}(Y)$ for $i = 1, 2, \dots, n-1$. Let $a \in \mathbb{Z}$ be such that $\alpha \equiv ah^i$ modulo I. It suffices to prove that a is divisible by p.

Case 1: $i = n-1$. We have $\alpha = bh^{n-1}$ for some $b \equiv a$ modulo p. As h^{n-1} is the class of a rational point of Y over a splitting field and the degree of every closed point of Y is divisible by p, we have $b \in p\mathbb{Z}$. Therefore, $a \in p\mathbb{Z}$ as $I_{p(n-1)} = p\mathbb{Z}h^{n-1}$.

Case 2: i divides $n-1$. Write $n-1 = ij$. We have $a^j \in \mathrm{CH}^{p(n-1)}(Y)$ and $\alpha^j \equiv a^j h^{n-1}$ modulo I. By Case 1, a^j and hence a is divisible by p.

Now assume that A is a central division K-algebra of degree p and $\alpha \in \mathrm{CH}^{pi}(Y)$ with $i = 1, 2, \dots, p-1$. We shall prove that $\bar{\alpha} \in I$. Write $ik + pm = 1$ for some integers k and $m > 0$. Consider the varieties $Y' = R_{K/F}\big(\mathrm{SB}(M_m(A))\big)$ and $Y'' = R_{K/F}\big(\mathrm{SB}(M_{m-1}(A))\big)$. Then Y'' can be identified with a closed subvariety of Y' so that there is a vector bundle $Y' \setminus Y'' \to Y$. Therefore, we have a surjective homomorphism

$$\mathrm{CH}(Y') \to \mathrm{CH}(Y' \setminus Y'') \simeq \mathrm{CH}(Y).$$

Moreover, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{CH}(Y') & \longrightarrow & \mathrm{CH}(Y) \\ \big\downarrow & & \big\downarrow \\ \mathbb{Z}[\mathbf{h}']^H & \longrightarrow & \mathbb{Z}[\mathbf{h}]^H \end{array},$$

where the bottom map takes a monomial $\mathbf{h}'^{\alpha}$ to $\mathbf{h}^{\alpha}$ if $\alpha_i < p$ for all i and to 0 otherwise. Lift α to an element $\alpha' \in \mathrm{CH}^{pi}(Y')$. As i divides $pm-1$, by Case 2 applied to the algebra $M_m(A)$, we have $\bar{\alpha}' \in I'$. Since the bottom map in the diagram takes I' to I, we have $\bar{\alpha} \in I$. $\square$

Let K'/F be a cyclic field extension of degree p and

$$S = \big(R^{(1)}_{K'/F}(\mathbb{G}_{m,K'})\big)^r \simeq \big(R_{K'/F}(\mathbb{G}_{m,K'})/\,\mathbb{G}_m\big)^r$$

for some $r > 0$. We view the variety of the group S as an open subset of $Z := \mathbb{P}_F(K')^r$. Hence the restriction gives a surjective ring homomorphism

$$(\mathbb{Z}/p\mathbb{Z})[\mathbf{h}] = \mathrm{Ch}(Z) \to \mathrm{Ch}(S),$$

where $\mathbf{h} = (h_1, h_2, \dots, h_r)$, $h_i^p = 0$ for all i, and we write Ch for the Chow groups modulo p. We shall also write $\tilde{h}_i$ for the image of h_i in $\mathrm{Ch}^1(S)$. The class in $\mathrm{Ch}^{r(p-1)}(S)$ of a rational point of S is equal to $\tilde{h}^{p-1}$, where $\tilde{h} = \tilde{h}_1\tilde{h}_2\cdots\tilde{h}_r \in \mathrm{Ch}^p(S)$. As $i(S) = p$, we have $\tilde{h}^{p-1} \neq 0$ by Example 2.4.

Proposition 4.3. *The map $(\mathbb{Z}/p\mathbb{Z})[\mathbf{h}] \to \mathrm{Ch}(S)$ is a ring isomorphism.*

Proof. Suppose that $f(\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_r) = 0$ for a nonzero homogeneous polynomial f over $\mathbb{Z}/p\mathbb{Z}$. Suppose that a monomial $h_1^{\alpha_1}\cdots h_r^{\alpha_r}$ enters f with a nonzero coefficient. Multiplying the equality by $\tilde{h}_1^{\beta_1}\cdots\tilde{h}_r^{\alpha_r}$ with $\beta_i = p-1-\alpha_i$, we get $\tilde{h}^{p-1} = 0$, a contradiction. $\square$

For an element α in $\mathrm{Ch}(S)$ we shall write $\bar{\alpha}$ for the corresponding element in $(\mathbb{Z}/p\mathbb{Z})[\mathbf{h}]$.

Consider the homomorphism $f : S \times S \to S$ defined by $f(x, y) = xy^{-1}$. Recall that as $i(S) = p$, by Example 2.2 and Proposition 2.3, we have the homomorphism

$$f_\star : \mathrm{CH}_{r(p-1)}(S \times S) \to \mathbb{Z}/p\mathbb{Z}. \tag{26}$$

Lemma 4.4. *For any $\alpha \in \mathrm{Ch}^i(S)$ and $\beta \in \mathrm{Ch}^j(S)$ with $i + j = r(p-1)$ one has*

$$\bar{\alpha}\cdot\bar{\beta} = f_\star(\alpha \times \beta)h^{p-1}$$

in $(\mathbb{Z}/p\mathbb{Z})[\mathbf{h}]$.

Proof. It suffices to consider the case when α and β are monomials in $\tilde{h}_i$. As both sides of the equality commute with products, we may assume that $r = 1$, i.e., $S = R_{H/F}(\mathbb{G}_{m,K'})/\,\mathbb{G}_m$, and $\alpha = \tilde{h}^i$, $\beta = \tilde{h}^j$. The cycles α and β are represented by $\mathbb{P}(U) \cap S$ and $\mathbb{P}(W) \cap S$, where U and W are F-subspaces of K' of codimension i and j respectively. The fiber of the restriction $f' : (\mathbb{P}(U) \cap S) \times (\mathbb{P}(W) \cap S) \to S$ of f over a point s of S is isomorphic to $\mathbb{P}(U \cap sW) \cap S$. The vector space $U \cap sW$ is one-dimensional for generic s, hence f' is a birational isomorphism and $f_\star(\alpha \times \beta) = 1 + p\mathbb{Z}$. On the other hand, $\bar{\alpha}\cdot\bar{\beta} = h^i \cdot h^j = h^{p-1}$. $\square$

Let L/F be a bicyclic field extension of degree p^2 and $T = R^{(1)}_{L/F}(\mathbb{G}_{m,L})$. Choose a subfield K of L of degree p over F and let $t \in K^\times$ be an element with $N_{K/F}(t) = 1$, i.e., t is an F-point of the torus $R^{(1)}_{K/F}(\mathbb{G}_{m,K})$. Write S_t

for the fiber of the norm homomorphism $T \to R^{(1)}_{K/F}(\mathbb{G}_m)$ over t. The variety S_t is a principle homogeneous space of the torus $S = R_{K/F}\big(R^{(1)}_{L/K}(\mathbb{G}_{m,L})\big) \simeq R_{K/F}\big(R_{L/K}(\mathbb{G}_{m,L})/\,\mathbb{G}_{m,K}\big)$.

The variety S_t is canonically isomorphic to an open subscheme of the variety $Y := R_{K/F}\big(\mathrm{SB}(A_t)\big)$ for a central simple K-algebra A_t of degree p (see Section 4.1). Over the function field E of $\mathrm{SB}(A_t)$ over K, the varieties S_t and S become isomorphic to the torus $\big(R^{(1)}_{LE/E}(\mathbb{G}_{m,LE})\big)^p$, where $LE = L \otimes_K E$, so we can apply the constructions considered above to the torus S_E over E. In particular, we have the element $\bar{\alpha} \in (\mathbb{Z}/p\mathbb{Z})[\mathbf{h}]$ well defined for any cycle α on S_t and S.

Consider the morphism

$$f : S_t \times S \to S_t, \qquad f(x, y) = xy^{-1}.$$

We have defined the homomorphism (see (26))

$$f_\star : \mathrm{CH}_{p(p-1)}(S_t \times S) \to \mathrm{CH}_{p(p-1)}\big((S_t)_E \times S_E\big) \to \mathbb{Z}/p\mathbb{Z}.$$

Proposition 4.5. *Suppose that the principal homogeneous space S_t is not trivial. Then $f_\star(\alpha \times \tilde{h}^j) = 0$ for any $\alpha \in \mathrm{Ch}^{p(p-j-1)}(S_t)$ and $j = 0, 1, \ldots, p-2$.*

Proof. As S_t is not trivial, the algebra A_t is not split. We can lift α to a cycle β in $\mathrm{Ch}(Y)$. By Proposition 4.2, $\bar{\beta}$ belongs to the image $\tilde{I}$ of the ideal I in $(\mathbb{Z}/p\mathbb{Z})[\mathbf{h}]^H$. It follows that $\bar{\alpha} \cdot h^j = \bar{\beta} \cdot h^j \in \tilde{I}_{p(p-1)} = 0$. Lemma 4.4 (applied to the field extension E of F and $r = p$) shows that $f_\star(\alpha \times \tilde{h}^j) = 0$. ☐

4.5. **One technical result.** Let $(K_1/F, \sigma)$ and $(K'/F, \tau)$ be cyclic field extensions of degree p^2 and p respectively, $L_1 = K_1 \otimes_F K'$ and $G_1 = \mathrm{Gal}(L_1/F)$. We assume that L_1 is a field.

Let $K \subset K_1$ be the subfield of degree p and set $L = K \otimes_F K'$. We write G for the Galois group of L/F.

Let T_1 and T be the norm 1 tori for the extensions L_1/F and L/F respectively. We have a natural embedding of T into T_1 and an exact sequence of G_1-modules

$$0 \to \mathbb{Z} \to \mathbb{Z}[G_1] \to T_1^* \to 0. \tag{27}$$

and an exact sequence of G-modules

$$0 \to \mathbb{Z} \to \mathbb{Z}[G] \to T^* \to 0. \tag{28}$$

Proposition 4.6. *The image of the generic point t of T under the homomorphism $T\big(F(T)\big)/R \to T_1\big(F(T)\big)/R$ is not trivial.*

Proof. If M is a G_1-lattice and X a G_1-module, we write $\mathrm{Ext}^i_{G_1}(M, X)$ for the Tate cohomology group $\widehat{H}^i(G_1, M^* \otimes_{\mathbb{Z}} X)$.

It follows from (27) and (28) that

$$\widehat{\mathrm{Ext}}^0_{G_1}(T_1^*, T^*) \simeq \widehat{H}^{-1}(G_1, T^*) = \widehat{H}^{-1}(G, T^*) \simeq \widehat{H}^0(G, \mathbb{Z}) \simeq \mathbb{Z}/p^2\mathbb{Z}.$$

Moreover, the class of the embedding $i : T_1^* \to T^*$ corresponds to $1 + p^2\mathbb{Z}$, so i has order p^2 in $\widehat{\mathrm{Ext}}^0_{G_1}(T_1^*, T^*)$.

Let P_1 be the torus $R_{L_1/F}(\mathbb{G}_{m,L_1})$ and $\alpha : P_1 \to T_1$ the homomorphism taking (u, v) to $\sigma(u)\tau(v)/uv$. For a field extension E/F, the image of $P_1(E)$ in $T_1(E)$ coincides with the subgroup of R-trivial elements in $T_1(E)$.

Set $S_1 := \mathrm{Ker}(\alpha)$, so we have an exact sequence of G_1-modules

$$0 \to T_1^* \to P_1^* \to S_1^* \to 0. \tag{29}$$

It follows that for every field extension E/F, we have an exact sequence

$$P_1(E) \to T_1(E) \to H^1\big(G_1, S_1(L_1 \otimes E)\big) \to 0,$$

hence

$$H^1\big(G_1, S_1(L_1 \otimes E)\big) \simeq T_1(E)/R. \tag{30}$$

We write $\widetilde{L}_1(T)^\times$ for the factor group $L_1(T)^\times/L_1^\times$ and $D(T_{L_1})$ for the divisor group of the torus T over L_1. There is an exact sequence of G_1-modules

$$0 \to T^* \to \widetilde{L}_1(T)^\times \to D(T_{L_1}) \to 0. \tag{31}$$

The exact sequences 29 and 31 yield the commutative diagram

$$\begin{tikzcd}
\widehat{\mathrm{Ext}}^{-1}_{G_1}\big(T_1^*, D(T_{L_1})\big) \arrow[r, "\beta"] \arrow[d, "\wr"] & \widehat{\mathrm{Ext}}^0_{G_1}(T_1^*, T^*) \arrow[d, "\wr"] & \\
\widehat{\mathrm{Ext}}^0_{G_1}\big(S_1^*, D(T_{L_1})\big) \arrow[r] & \widehat{\mathrm{Ext}}^1_{G_1}(S_1^*, T^*) \arrow[r, "\gamma"] & \widehat{\mathrm{Ext}}^1_{G_1}\big(S_1^*, \widetilde{L}_1(T)^\times\big)
\end{tikzcd}$$

By (30), the map γ in the diagram factors through the group

$$\widehat{\mathrm{Ext}}^1_{G_1}\big(S_1^*, L_1(T)^\times\big) = H^1\big(G_1, S_1(L_1(T))\big) = T_1\big(F(T)\big)/R.$$

Note that the image of the canonical map i from $\widehat{\mathrm{Ext}}^0_{G_1}(T_1^*, T^*)$ in $T_1\big(F(T)\big)/R$ coincides with the image of the generic point t of T in this group. It suffices to show that i does not belong to the image of β. To prove this we shall show that the image of β is p-torsion.

The group $\widehat{\mathrm{Ext}}^{-1}_{G_1}\big(T_1^*, D(T_{L_1})\big)$ is isomorphic to the direct sum of the groups $\widehat{H}^{-1}(G_x, T_{1*})$ over all points $x \in T$ of codimension 1, where G_x is the decomposition subgroup of x in G_1. Let $x \in T$ be of codimension 1 and assume first that $G_x = G_1$. The commutativity of the diagram

$$\begin{tikzcd}
H^0(G_1, \mathbb{Z}) \otimes H^{-1}(G_1, T_{1*}) \arrow[r] \arrow[d] & H^{-1}(G_1, T_{1*}) \arrow[d] & \\
H^1(G_1, T^*) \otimes H^{-1}(G_1, T_{1*}) \arrow[r] & H^0(G_1, T_{1*} \otimes T^*) \arrow[r, equal] & \widehat{\mathrm{Ext}}^0_{G_1}(T_1^*, T^*)
\end{tikzcd}$$

where the horizontal homomorphisms are cup-products and the vertical ones are connecting maps with respect to the pull back of the exact sequence (31) induced by the inclusion $\mathbb{Z} \to D(T_{L_1})$ given by the point x, implies that $\beta(\rho) =$

$[x] \cup \rho$ for any $\rho \in H^{-1}(G_1, T_{1*})$, where $[x]$ is the class of x in $H^1(G_1, T^*) = H^1(G, T^*) = \mathrm{CH}^1(T)$.

Now let $x \in T$ be an arbitrary point of codimension 1. Write F_x for the field $(L_1)^{G_x}$ and apply the computation above to to the canonical point $x' \in T_{F_x}$ above x. Using compatibility with the corestriction map, we have

$$\beta(\rho) = \mathrm{cor}_{F_x/F}\big([x'] \cup \rho\big)$$

for any $\rho \in H^{-1}(G_x, T_{1*})$, where $[x']$ is the class of x' in $H^1(G_x, T^*) = \mathrm{CH}^1(T_{F_x})$. Let $\overline{G}_x$ be the image of G_x under the epimorphism $G_1 \to G$. It follows from (28) that

$$H^1(G_x, T^*) = H^1(\overline{G}_x, T^*) = H^2(\overline{G}_x, \mathbb{Z}) = (\overline{G}_x)^*.$$

The character group $(\overline{G}_x)^*$ is p-torsion as so is $\overline{G}_x$. Hence the image of β is p-torsion. $\square$

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