SYMBOLS AND CYCLICITY OF ALGEBRAS AFTER A SCALAR EXTENSION

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Abstract. 1. For a field F and a family of central simple F-algebras we prove that there exists a regular field extension E/F preserving indices of F-algebras such that all the algebras from the family are cyclic after scalar extension by E.

2. Let $\mathcal{A}$ be a central simple algebra over a field F of degree n with a primitive n-th root of unity ρ_n. We construct a quasi-affine F-variety $\mathrm{Symb}(\mathcal{A})$ such that, for a field extension L/F, the variety $\mathrm{Symb}(\mathcal{A})$ has an L-rational point iff $\mathcal{A} \otimes_F L$ is a symbol algebra.

3. Let $\mathcal{A}$ be a central simple algebra over a field F of degree n and K/F a cyclic field extension of degree n. We construct a quasi-affine F-variety $C(\mathcal{A}, K)$ such that, for a field extension L/F with the property $[KL : L] = [K : F]$,the variety $C(\mathcal{A}, K)$ has an L-rational point iff KL is a subfield of $\mathcal{A} \otimes_F L$.

0. Introduction

Let $\mathcal{A}$ be a finite dimensional central simple algebra over a field F. By Wedderburn's theorem, there is a unique integer $m \geq 1$ and a central division F-algebra $\mathcal{D}$ which is unique up to F-isomorphism such that $\mathcal{A} \cong M_m(\mathcal{D})$. The degree of $\mathcal{A}$ is defined by $\deg(\mathcal{A}) = \sqrt{\dim_F \mathcal{A}}$, the index of $\mathcal{A}$ is said to be $\mathrm{ind}(\mathcal{A}) = \deg(\mathcal{D})$, and the exponent $\exp(\mathcal{A})$ of $\mathcal{A}$ is the order of the equivalence class $[\mathcal{A}]$ in the Brauer group $\mathrm{Br}(F)$. For a field extension E/F, $\mathrm{res} : \mathrm{Br}(F) \longrightarrow \mathrm{Br}(E)$ denotes the restriction homomorphism and $\mathcal{A}_E$ denotes the tensor product $\mathcal{A} \otimes_F E$.

In the paper we consider the following problems related to properties of central simple algebras over scalar extensions.

Problem 0.1. Let $\{\mathcal{A}_\alpha\}_{\alpha \in I}$ be some family of central simple F-algebras. Fix some central simple algebra property $\mathcal{P}$. Does there exist a field extension E/F (if possible, regular) such that each member of the family $\{\mathcal{A}_\alpha \otimes_F E\}_{\alpha \in I}$ has the property $\mathcal{P}$?

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Problem 0.2. Let $\mathcal{A}/F$ be a central simple algebra. Fix some central simple algebra property $\mathcal{P}$. Does there exists an algebraic F-variety V (affine, quasi-affine, projective or quasi-projective) such that $\mathcal{A} \otimes_F L$ has the property $\mathcal{P}$ iff V has an L-rational point?

Of course, the above problems in their all generality are vague, so below we explain (and motivate) which kind of properties $\mathcal{P}$ we mean.

Example 0.3. $\mathcal{P}$ is a property to have a given index.

Example 0.4. $\mathcal{P}$ is a property to have a given exponent.

Example 0.5. $\mathcal{P}$ is a property to have given both exponent and index.

In [12, Lemma 1.3.] it is proved that given a central simple algebra $\mathcal{A}$ over F of exponent e and index d with prime power decomposition $d = \prod p^{v_p(d)}$, then for any divisor δ of d there exists a field extension $E^{(\delta)}$ such that

$$\exp(\mathcal{A} \otimes_F E^{(\delta)}) = \gcd(e, \delta) \ \text{ and } \operatorname{ind}(\mathcal{A} \otimes_F E^{(\delta)}) = \prod_{p|\delta} p^{v_p(d)}.$$

Example 0.6. $\mathcal{P}$ is a property for an algebra to have coincided exponent and index (index-exponent property).

Remark 0.7. The latter Example leads us to the index-exponent problem.

Problem 0.8. Describe a class of fields F for which $\operatorname{ind}(\mathcal{A}) = \exp(\mathcal{A})$ for any central simple F-algebra.

It is an old open question whether this problem has a positive solution in the class of C_2-fields. The recent results of A. J. de Jong ([11]) and M. Lieblich ([14]) show that for the class of function fields of surfaces over algebraically closed fields this question has an affirmative answer.

Another kind of properties $\mathcal{P}$ is related to presentation of algebras by generators and defining relations. The most popular and simple classes of algebras in this sense are the following.

(i) *Matrix algebras* $M_n(F)$.

(ii) *Crossed products* $(L/F, \operatorname{Gal}(L/F), f)$. Let L/F be a Galois field extension, $\operatorname{Gal}(L/F)$ its Galois group and f a 2-cocycle of G with values in L^*. Then $(L/F, \operatorname{Gal}(L/F), f)$ is a left L-module with L-base $\{u_\tau\}_{\tau \in \operatorname{Gal}(L/F)}$ and multiplication table:

$$u_s l = l^s u_s, \ u_s u_t = f(s,t) u_{st}$$

for any $s, t \in \operatorname{Gal}(L/F)$ and $l \in L$.

(iii) *Cyclic algebras* $(Z/F, s, a)$. They are a special form of crossed products. Let Z/F be a cyclic field extension of degree n, s a generator of $\mathrm{Gal}(Z/F)$ and $a \in F^*$. Then $(Z/F, s, a)$ is a left Z-module with Z-base $\{u_s^i\}_{i=1,\dots,n}$ and multiplication table:

$$u_s^i c = c^{s^i} u_s^i$$

and

$$u_s^n = a$$

for any $i = 1, \dots, n$ and $c \in Z$.

(iv) *Symbol algebras* $(a, b)_n$. These algebras also have a simple set of generators and defining relations. Let $\rho_n \in F$ be a primitive root of unity of degree n and $a, b \in F^*$. Then $(a, b)_n$ is an n^2-dimensional vector F-space with an F-base

$$\{A^i B^j\}_{i,j=1,\dots,n}$$

and multiplication table

$$A^j B^i = \rho_n^j B^i A^j, \ A^n = a, B^n = b.$$

Remark 0.9. Because of the simplicity of sets of the above generators and defining relations it is interesting to study the properties $\mathcal{P}$ of algebras to be one of the kinds above four types (i)-(iv), especially to be cyclic or symbol (more generally to be a crossed product with a Galois group of simple structure, for instance abelian).

Remark 0.10. Over an arbitrary field F an arbitrary central simple algebra $\mathcal{A}/F$ does not necessarily belong to one of the classes (i)-(iv). Hamilton in 1843 proved that over $\mathbb{R}$ not every central simple algebra is a matrix algebra. Albert in 1932 ([1]) gave an example of a non-cyclic division algebra, and later Amitsur in 1972 ([2]) proved that there do exist noncrossed product algebras. It is easily derived from the latter result that Amitsur's algebras are not symbol algebras.

Before we formulate our main results let us list some known facts with different properties $\mathcal{P}$.

(i) If $\mathcal{P}$ is a property to be a matrix algebra, then for a single finite dimensional central simple algebra $\mathcal{A}$, problem 0.1 always has a solution, given by a so-called splitting field for $\mathcal{A}$. There are a lot of results about splitting fields, which are regular extensions of the ground field, obtained by Witt, Brauer, Roquette, Chatelet, Kovach, Heuser and others. In this case also problem 0.2 has a solution: The Severi-Brauer variety associated to $\mathcal{A}$ has the required properties.

(ii) If $\mathcal{P}$ is a property to be a crossed product algebra with a given finite group G and with injective restriction homomorphism $\mathrm{Br}(F) \longrightarrow$

$\mathrm{Br}(E)$, then the answer is also positive. In the appendix to [5, Th. 4] a proof of an unpublished result of D. Saltman is given:

Theorem 0.11. *Let F be a field, $\mathcal{A}$ a central simple algebra of degree n over F and G a finite group of order n. Then there exists a finitely generated field extension E/F such that:*
$\mathcal{A}_E$ is isomorphic to a G-crossed product;
the restriction homomorphism $\mathrm{res} : \mathrm{Br}(F) \longrightarrow \mathrm{Br}(E)$ *is an injection.*

Remark 0.12. For G cyclic and $\rho_n \in F$, a simple proof of this fact is given in [8, Th. 5.5.1].

Remark 0.13. The first mentioning about existence of positive solutions in the latter two cases is due to M. van den Bergh and A. Schofield. They also showed how one could be able to prove corresponding statements ([3]). In [3] it was noted that given a central division algebra $\mathcal{A}$ over a field F, there exists a regular field extension E/F such that $\mathcal{A}_E = \mathcal{A} \otimes_F E$ is a cyclic division algebra (see the discussion after Theorem 2.6 in [3]).

In our paper we extend this result and prove that for a field F there exists a regular field extension E/F preserving indices of all central simple F-algebras such that all E-algebras are cyclic.

As for Problem 0.2, we construct a quasi-affine variety such that the existence of L-rational point of that variety is responsible roughly speaking for $\mathcal{A} \otimes_F L$ to be an algebra with a maximal cyclic subfield coming from a given cyclic extension Z/F of degree $\deg(\mathcal{A})$. (The fact that for a given cyclic extension Z/F of degree $\deg(\mathcal{A})$ there exists at least one regular extension L/F such that $[LZ : L] = \deg(\mathcal{A})$ and LZ is a maximal subfield of $\mathcal{A} \otimes_F L$ follows from [4, Th. 3.7].)

More precisely, in Section 1 we prove the following

Theorem 0.14. *Let F be a field. Then there exists a regular field extension E/F with the following properties:*
(i) any central simple E-algebra is cyclic,
(ii) for any central simple F-algebra $\mathcal{C}$, $\mathrm{ind}(\mathcal{C}_E) = \mathrm{ind}(\mathcal{C})$*,*
(iii) for any central simple F-algebra $\mathcal{C}$, $\exp(\mathcal{C}_E) = \exp(\mathcal{C})$*,*
(iv) the restriction homomorphism $\mathrm{res} : \mathrm{Br}(F) \longrightarrow \mathrm{Br}(E)$ *is an injection.*

In Sections 2 and 3 we prove the following

Theorem 0.15. *Let $\rho_n \in F$ and $\mathcal{A}$ be a central simple F-algebra of degree n. Then there exists a quasi-affine F-variety* $\mathrm{Symb}(\mathcal{A})$ *such that* *for a field extension L/F* Symb$(\mathcal{A})$ *has an L-rational point iff $\mathcal{A}_L$ is a symbol algebra.*

Theorem 0.16. *Let $\mathcal{A}$ be a central simple algebra over a field F of degree n and K/F a cyclic field extension of degree n. Then there exists a quasi-affine F-variety $C(\mathcal{A}, K)$ such that for a field extension L/F with the property $[KL : L] = [K : F]$ $C(\mathcal{A}, K)$ has an L-rational point iff KL is a maximal subfield of $\mathcal{A}_L$.*

To illustrate the usefulness of our considerations we will show how one can reduce a well known conjecture of Suslin related to generic elements of reduced Whitehead groups to the case of central division algebras of a special type.

First recall a few definitions. For any central simple algebra $\mathcal{A}/F$ a reduced norm mapping $Nrd : \mathcal{A} \longrightarrow F$ is defined. Its restriction to the multiplicative group $\mathcal{A}^*$ of $\mathcal{A}$ gives a homomorphism $Nrd : \mathcal{A}^* \longrightarrow F^*$. Then the factor group $SK_1(\mathcal{A})$ of the kernel of this homomorphism by the commutator subgroup of $\mathcal{A}^*$ is usually called the reduced Whitehead group of $\mathcal{A}$. At least two observations make this group of a special interest. The first is its well known relation to Kneser-Tits conjecture for algebraic groups ([7]) and second its connection with a problem of rationality for such groups via group of R-equivalences classes for a group algebraic varieties ([7]).

In this context A.A. Suslin assumed that

Conjecture 0.17. (Suslin, 1991, [20], [21]). Let $\mathcal{A}/F$ be a central simple algebra of ind$(\mathcal{A})$ prime to $char(F)$, $\mathbb{G}$ the algebraic group defined by $SL(1, \mathcal{A})$, $F(\mathbb{G})$ its function field, $\mathbb{G}(\mathbb{F}(\mathbb{G}))$ the group of $F(\mathbb{G})$-rational points of $\mathbb{G}$. If ind$(\mathcal{A})$ is not square-free then a generic point $\xi \in \mathbb{G}(\mathbb{F}(\mathbb{G}))$ leads to a nontrivial generic element of $SK_1(\mathcal{A} \otimes_F F(\mathbb{G}))$.

Nowadays this conjecture is proved only in case where ind$(\mathcal{A})$ is divisible by 4 ([15]).

Remark 0.18. It is easy to see that to prove Suslin's conjecture it is sufficient to prove the following statement: for any central simple algebra $\mathcal{A}/F$ with ind$(\mathcal{A})$ which is not square-free there exists a field extension E/F such that $SK_1(\mathcal{A} \otimes_F E) \neq \{0\}$.

Using above results we prove immediately the following.

Theorem 0.19. *Suslin's conjecture is true iff it is true for all cyclic division algebras.*

Combining the above theorems with the main result of [18] one can prove the stronger

Theorem 0.20. *Suslin's conjecture is true iff it is true for all cyclic division algebras of the form* $(a,c)_p \otimes (b,d)_p$.

1. Cyclicity after a scalar extension

In order to prove the Theorem 0.14 we need a few preliminary statements.

Proposition 1.1. ([3, Th. 1.3], [19, Th. 13.10]) *Let* $\mathcal{D}$, $\mathcal{E}$ *be central division algebras over* F *of indices* m *and* n *respectively. Let* $BS(\mathcal{E})$ *be the Severi-Brauer variety of* $\mathcal{E}$ *and let* K *be its function field. Then*

$$\operatorname{ind}(\mathcal{D} \otimes_F K) = \gcd\{\operatorname{ind}(\mathcal{D} \otimes_F \mathcal{E}^i)\}$$

where i *ranges from 1 to* n.

Remark 1.2. In the literature the latter formula is called the index reduction formula.

Corollary 1.3. *Let* $\mathcal{D}$, $\mathcal{E}$ *be central division algebras over* F. *Let* K *be the function field of the Severi-Brauer variety* $BS(\mathcal{E})$. *Assume that* $\operatorname{ind}(\mathcal{D})$ *is coprime to* $\operatorname{ind}(\mathcal{E})$. *Then* $\operatorname{ind}(\mathcal{D} \otimes_F K) = \operatorname{ind}(\mathcal{D})$.

Proof. Use the index reduction formula. □

Lemma 1.4. ([6, p. 176, ex. 7]) *Let* E/F *be a cyclic field extension of degree* q^{l-1} *(*q *is prime). Let* $\operatorname{Gal}(E/F) =< \sigma >$. *Let also* $(\operatorname{char}(F), q) = 1$. *Assume that there exists an element* $\beta \in E$ *such that* $N_{E/F}(\beta) = \rho_q$ *(*ρ_q= *primitive* q*-th root of unity). Let* $a \in E$ *be such that* $a^\sigma/a = \beta^q$. *Then*

(i) for any $\lambda \in F^*$ *the polynomial*

$$x^q - \lambda a$$

is irreducible over E,

(ii) if θ *is a root of that polynomial, then* $E(\theta)$ *is a cyclic extension of* F *of degree* q^l.

The following result we will use only in the case $p = 2$, but for the sake of generality, we will prove it for any p.

Lemma 1.5. *Assume* $\rho_p \in F$. *Then for any* $m \in \mathbb{N}$ *there exists a tower of field extensions*

$$F \subset K \subset E$$

such that

(i) *E/F is a regular extension;*
(ii) *E/K is a cyclic extension of degree p^m;*
(iii) *for any central simple F-algebra $\mathcal{C}$, $\operatorname{ind}(\mathcal{C}_E) = \operatorname{ind}(\mathcal{C})$.*

Proof.
We will prove the statement by induction on m. Let $\mathcal{C}$ be a central simple F-algebra. If $m = 1$, set $K = F(x)$ and $E = K(\sqrt[p]{x})$ where x is a transcendental variable.

Suppose that the statement of lemma is true for $m = m_0$, i.e. there exists a tower of field extensions $F \subset K_{m_0} \subset E_{m_0}$ such that E_{m_0}/K_{m_0} is cyclic with $|E_{m_0} : K_{m_0}| = p^{m_0}$, E_{m_0}/F is regular and

$$\operatorname{ind}(\mathcal{C}_{E_{m_0}}) = \operatorname{ind}(\mathcal{C}).$$

Consider the case $m = m_0 + 1$. Set

$$\mathcal{B} = (E_{m_0}/K_{m_0}, \sigma, \rho_p)$$

be a cyclic algebra over K_{m_0}. Let M be the function field of the corresponding Severi-Brauer variety $\mathrm{SB}(\mathcal{B})$.

Note that the compositum ME_{m_0}/M is a cyclic extension of degree p^{m_0}. Since E_{m_0} is a splitting field of $\mathcal{B}$, then ME_{m_0} is a purely transcendental extension of E_{m_0}. Hence ME_{m_0}/F is a regular extension. Moreover,

$$\operatorname{ind}(\mathcal{C}_{ME_{m_0}}) = \operatorname{ind}(\mathcal{C}_{E_{m_0}}) = \operatorname{ind}(\mathcal{C}).$$

Besides, $\mathcal{B}_M \sim 1$. Then there exists $\beta \in ME_{m_0}$ such that

$$N_{ME_{m_0}/M}(\beta) = \rho_p.$$

Let $a \in ME_{m_0}$ be such that $a^\sigma/a = \beta^p$. Let also y be a new transcendental variable. Then $ME_{m_0}(y)/M(y)$ is cyclic of degree p^{m_0}. Moreover, using Lemma 1.4 we conclude that

$$ME_{m_0}(\sqrt[p]{ay})/M(y)$$

is cyclic of degree p^{m_0+1}. It is clear that $ME_{m_0}(\sqrt[p]{ay})/F$ is regular. Finally,

$$\operatorname{ind}(\mathcal{C}_{ME_{m_0}(\sqrt[p]{ay})}) = \operatorname{ind}(\mathcal{C}_{ME_{m_0}}) = \operatorname{ind}(\mathcal{C})$$

□

Lemma 1.6. *Let F be a field. Then for any $n \in \mathbb{N}$ there exists a tower of field extensions $F \subset K \subset E$ such that E/F is regular, E/K is cyclic of degree n and*

$$\operatorname{ind}(\mathcal{C}_E) = \operatorname{ind}(\mathcal{C})$$

for any central simple F-algebra $\mathcal{C}$.

Proof. At first, consider the case where $\mathrm{char}(F) = 2$ or $4 \nmid n$. Let x be a transcendental variable and v_x the valuation of $F(x)$ corresponding to the polynomial x. Then by [16, Th. 5], there exists a cyclic field extension $E/F(x)$ of degree n such that the completion E_{v_x} coincides with $F(x)_{v_x}$. Note that $F(x)_{v_x} = F((x))$. Hence for any central simple F-algebra $\mathcal{C}$, $\mathrm{ind}(\mathcal{C}) = \mathrm{ind}(\mathcal{C}_{F((x))}) = \mathrm{ind}(\mathcal{C}_{E_{v_x}})$. Then $\mathrm{ind}(\mathcal{C}) = \mathrm{ind}(\mathcal{C}_E)$.

Moreover, since F is algebraically closed in E_{v_x}, then E/F is a regular extension.

Now consider the case $\mathrm{char}(F) \neq 2$ and $4|n$. Let $n = 2^s m$ where $2 \nmid m$. In view of the case considered above, there exists a tower of field extensions $F \subset K_1 \subset E_1$ such that E_1/F is regular, E/K is cyclic of degree m and E_1 preserves indices of F-algebras. Besides, by Lemma 1.5, there exists a tower of field extensions $F \subset K_2 \subset E_2$ such that E_2/F is a regular extension, E_2/K_2 is cyclic of degree 2^s and $\mathrm{ind}(\mathcal{C}_{E_2}) = \mathrm{ind}(\mathcal{C})$ for any central simple F-algebra $\mathcal{C}$. Set

$$K = K_1K_2, \quad E = E_1E_2$$

be respectively the free composita over F of K_1, K_2 and E_1, E_2. Then E/K is a cyclic extension of degree n. Furthermore, since E_i does not change the index of $\mathcal{C}$, then

$$\mathrm{ind}(\mathcal{C}) = \mathrm{ind}(\mathcal{C}_{E_1}) = \mathrm{ind}(\mathcal{C}_{E_1E_2}) = \mathrm{ind}(\mathcal{C}_E).$$

□

Lemma 1.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be central simple F-algebras. Assume $\mathrm{ind}(\mathcal{A}) = p^m$, $\mathrm{ind}(\mathcal{B}) = p^n$ and $m \geq n$. Then $\mathrm{ind}(\mathcal{A} \otimes_F \mathcal{B}) \geq p^{m-n}$.*

Proof. Let E/F be a field extension of degree p^n which splits $\mathcal{B}$. Let also $\mathrm{ind}(\mathcal{A} \otimes_F \mathcal{B}) = p^s$. Assume $p^s < p^{m-n}$. Then there exists a field extension L/F of degree p^s splitting $\mathcal{A} \otimes_F \mathcal{B}$. Hence

$$1 \sim (\mathcal{A} \otimes_F \mathcal{B})_{EL} \sim \mathcal{A}_{EL} \otimes_{EL} \mathcal{B}_{EL} \sim \mathcal{A}_{EL}.$$

Thus EL is a splitting field of $\mathcal{A}$. Since $|EL : F| < p^m$, then $\mathrm{ind}(\mathcal{A}) < p^m$. Contradiction. □

Lemma 1.8. *Let $\mathcal{A}$ be a central simple F-algebra with $\mathrm{ind}(\mathcal{A}) = p^m$. Then $\mathrm{ind}(\mathcal{A}^{\otimes p}) < \mathrm{ind}(\mathcal{A})$.*

Proof. Without loss of generality we can assume that there exists a splitting field L of $\mathcal{A}$ such that $|L : F| = \mathrm{ind}(\mathcal{A})$ and L contains a subfield K with $|L : K| = p$. Then $\mathrm{ind}(\mathcal{A}_K) = p$. Hence $1 = \mathrm{ind}(\mathcal{A}_K^{\otimes p})$. Thus $\mathrm{ind}(\mathcal{A}^{\otimes p}) \leq |K : F| < |L : F| = \mathrm{ind}(\mathcal{A})$. □

Lemma 1.9. *Let K/F be a cyclic field extension, $\langle\sigma_i\rangle = \mathrm{Gal}(K(z)/F(z))$ and z is a transcendental variable. Let also $\mathcal{C}$ be a central division F-algebra such that $\mathcal{C}_K$ is a division algebra. Then*

$$(K(z)/F(z), \sigma, z) \otimes \mathcal{C}_{F(z)}$$

is a division $F(z)$-algebra.

Proof: Analogous to that of Proposition 1.3 from [13]. □

In the notations of the previous lemma we have immediately the following

Corollary 1.10. *Let $\mathcal{A}$ be a central simple F-algebra such that* $\mathrm{ind}(\mathcal{A}_K) = \mathrm{ind}(\mathcal{A})$. *Then*

$$\mathrm{ind}\left((K(z)/F(z), \sigma, z) \otimes \mathcal{A}_{F(z)}\right) = \mathrm{ind}((K(z)/F(z), \sigma, z))\mathrm{ind}(\mathcal{A}).$$

Theorem 1.11. *Let $\mathcal{A}$ be a central simple algebra over a field F. Then there exists a regular field extension M/F such that*
(i) $\mathcal{A}_M$ is cyclic,
(ii) for any central simple F-algebra $\mathcal{C}$, $\mathrm{ind}(\mathcal{C}_M) = \mathrm{ind}(\mathcal{C})$,
(iii) for any central simple F-algebra $\mathcal{C}$, $\exp(\mathcal{C}_M) = \exp(\mathcal{C})$,
(iv) the restriction homomorphism $\mathrm{res} : \mathrm{Br}(F) \longrightarrow \mathrm{Br}(M)$ *is an injection.*

Moreover, for any field extension L/F, the free compositum ML over F preserves indices of L-algebras.

Proof. Let $\deg(\mathcal{A}) = n = p_1^{n_1} \dots p_s^{n_s}$ (p_i-s are distinct primes) and $\mathcal{A} = \otimes_{i=1}^s \mathcal{A}_i$, where $\mathrm{ind}(\mathcal{A}_i) = p_i^{l_i}$, $l_i \leq n_i$. By Lemma 1.6, there exists a tower of field extensions $F \subset K \subset E$ such that E/F is regular, E/K is a cyclic extension of degree n and E preserves indices of F-algebras. Let E_i/K be a cyclic subextension of degree $p_i^{n_i}$.

Consider cyclic algebras

$$\mathcal{D}_i = (E_i(z)/K(z), \sigma_i, z),\ i = 1, \dots, s,$$

where $< \sigma_i >= \mathrm{Gal}(E_i(z)/K(z))$ and z is a transcendental variable.

Set

$$\mathcal{D} = \otimes_{i=1}^s \mathcal{D}_i.$$

Since

$$\mathcal{D}_i \sim (E(z)/K(z), \sigma, z^{n/p^{n_i}}),$$

where $< \sigma >= \mathrm{Gal}(E(z)/K(z))$, then

$$\mathcal{D} \cong (E(z)/K(z), \sigma, z^{\Sigma_{i=1}^s n/p^{n_i}}).$$

Then $\mathcal{D}$ is cyclic of index n with a maximal subfield $E(z)$.

One has

$$\mathcal{D} \sim \mathcal{D} \otimes_{K(z)} \mathcal{A}^{op}_{K(z)} \otimes_{K(z)} \mathcal{A}_{K(z)}.$$

Let M be the function field of the Severi-Brauer variety $\mathrm{SB}(\mathcal{D} \otimes_{K(z)} \mathcal{A}^{op}_{K(z)})$. Then $\mathcal{A}_M \sim \mathcal{D}_M$. Since $\deg(\mathcal{A}_M) = \deg(\mathcal{D}_M)$, then $\mathcal{A}_M \cong \mathcal{D}_M$

Let $\mathcal{C}$ be a central simple F-algebra and $\mathcal{C} = \otimes_{i=1}^m \mathcal{C}_i$ the decomposition of $\mathcal{C}$ as a tensor product of algebras of primary indices. Since $\mathrm{ind}(\mathcal{C}_M) = \prod_{i=1}^m \mathrm{ind}(\mathcal{C}_{iM})$, then it is enough to consider the case where $\mathcal{C}$ has a primary index. Moreover, if $p_i \nmid \mathrm{ind}(\mathcal{C})$, $1 \le i \le s$, then $\mathrm{ind}(\mathcal{C}_M) = \mathrm{ind}(\mathcal{C})$ by Corollary 1.3. Thus we will assume that $\mathrm{ind}(\mathcal{C}) = p_i^{m_i}$ is a power of p_i for some $1 \le i \le s$.

Using the index reduction formula we obtain

$$\mathrm{ind}(\mathcal{C}_M) = \gcd\{\mathrm{ind}(\mathcal{D}^{\otimes j} \otimes_{K(z)} {\mathcal{A}^{op}_{K(z)}}^{\otimes j} \otimes_{K(z)} \mathcal{C}_{K(z)})\}$$

where j ranges from 1 to n.

Thus since $\mathrm{ind}(\mathcal{C})$ is a power of some p_i, then

$$\mathrm{ind}(\mathcal{C}_M) = \min_{j=1}^{n_i}\{\mathrm{ind}(\mathcal{D}_i^{\otimes j} \otimes_{K(z)} {\mathcal{A}_{i}{}^{op}_{K(z)}}^{\otimes j} \otimes_{K(z)} \mathcal{C}_{K(z)})\}.$$

Consider the algebra $\mathcal{B}_{i,j} = \mathcal{D}_i^{\otimes j} \otimes_{K(z)} {\mathcal{A}_{i}{}^{op}_{K(z)}}^{\otimes j} \otimes_{K(z)} \mathcal{C}_{K(z)}$. Note that by Corollary 1.10,

$$\mathrm{ind}(\mathcal{B}_{i,j}) = \mathrm{ind}(\mathcal{D}_i^{\otimes j})\mathrm{ind}(\mathcal{A}_{iK}^{op \otimes j} \otimes_K \mathcal{C}_K).$$

Fix some j. Let $j = {p_i}^t j_1$, where $p_i \nmid j_1$. Then $\mathrm{ind}(\mathcal{D}_i^{\otimes j}) = p_i^{n_i - t}$. Let $\mathrm{ind}(\mathcal{A}^{op\otimes j}) = p_i^{s_i}$. Then $s_i \le l_i - t$ by Lemma 1.8. Hence

$$\mathrm{ind}(\mathcal{B}_{i,j}) = p_i^{n_i - t} p_i^{|s_i - m_i|} = p_i^{n_i - t + |s_i - m_i|}$$

in view of Lemma 1.7.

Finally consider two cases. If $s_i \ge m_i$, then $n_i - t \ge m_i$ and $n_i - t + |s_i - m_i| \ge m_i$. If $s_i < m_i$, then $n_i - t + |s_i - m_i| = n_i - t - s_i + m_i \ge m_i$. Therefore, $\mathrm{ind}(\mathcal{B}_{i,j}) \ge p^{m_i} = \mathrm{ind}(\mathcal{C})$ for any j. Thus $\mathrm{ind}(\mathcal{C}_M) = \mathrm{ind}(\mathcal{C})$.

Note that, for a field extension M/F, preserving indices for all F-algebras implies also preserving exponents of F-algebras. Indeed, assume $\mathcal{C}_M^{\otimes m} \sim 1$ for some central simple F-algebra $\mathcal{C}$. Since

$$1 = \mathrm{ind}(\mathcal{C}_M^{\otimes m}) = \mathrm{ind}(\mathcal{C}^{\otimes m}),$$

then $\mathcal{C}^{\otimes m} \sim 1$. Thus $\exp(\mathcal{C}_M) = \exp(\mathcal{C})$. Moreover, preserving exponents implies, in turn, that the restriction homomorphism

$$\mathrm{res} : \mathrm{Br}(F) \longrightarrow \mathrm{Br}(M)$$

is an embedding.

Now consider the last statement about free composita. The free compositum LM can be constructed using the same procedure as the field M. We just replace the constant field F by L. Now the statement about preserving indices of L-algebras is obtained automatically. $\square$

Remark 1.12. Note that preserving exponents for all F-algebras does not imply preserving indices as is easily seen.

Now we are in a position to prove the main result.

Proof of Theorem 0.14.

Note that in view of Remark after Theorem 1.11, preserving indices implies preserving exponents and injectivity of the restriction map.

First, we will prove that for any field K there exists a regular field extension making all K-algebras cyclic and preserving their indices. If the set I of central simple K-algebras is finite, use Theorem 1.11 and induction.

If I is countable, we can construct a sequence of field extensions

$$E_1 \subset E_2 \subset \ldots E_i \subset E_{i+1} \subset \ldots$$

such that E_{i+1} is a regular extension of E_i making ${\mathcal{A}_{i+1}}_{E_i}$ cyclic and preserving indices of E_i-algebras. Then the field $E = \cup_i E_i$ has the required properties.

Finally, if I is not countable, then the statement can be proved using Zorn's lemma. Indeed, consider the set

$$\mathcal{M} = \left\{ \begin{array}{l|l} \begin{array}{l} \text{regular field} \\ \text{extensions } K/F \end{array} & \begin{array}{l} \text{for any field extension } L/F\text{, the free} \\ \text{compositum } LK \text{ over } F \text{ preserves} \\ \text{indices of } L\text{ -algebras and there exists} \\ \text{an } F\text{-algebra } \mathcal{A} \text{ s.t. } \mathcal{A}_K \text{ is cyclic} \end{array} \end{array} \right\}$$

All the fields in $\mathcal{M}$ are assumed to be in some universal domain. Inclusions of fields define partial order on this set. If we have a totally ordered subset $\mathcal{S} = \{K_\alpha\} \subset \mathcal{M}$, then $\cup_\alpha K_\alpha$ is an upper bound for this subset. Hence, by Zorn's lemma, there exists a maximal element E in $\mathcal{M}$.

We will show that E has the required properties by the rule of contraries. Assume that there exists an F-algebra $\mathcal{A}$ such that $\mathcal{A}_E$ is not cyclic. By Theorem 1.11, there exists a field extension $E_1 \in \mathcal{M}$ such that $\mathcal{A}_{E_1}$ is cyclic. Then $E_1 E \in \mathcal{M}$ and contains E. Contradiction. Thus, the field E has the required properties.

Now we will finish the proof of the theorem. Let K_0 be the field making all F-algebras cyclic and preserving their indices. Let also K_1 be the field making all K_0-algebras cyclic and preserving their indices, and so on. Then the field $\cup_i K_i$ has the required properties.

□

2. Symbol algebra variety $\mathrm{Symb}(\mathcal{A})$

Recall firstly how the Severi-Brauer variety corresponding to a central simple algebra $\mathcal{A}$ can be defined by polynomial equations. For a given n-dimensional F-vector space V, let $\mathrm{Grass}_F(m, V)$ be the set of its m-dimensional subspaces. $\mathrm{Grass}_F(m, V)$ has the structure of a projective variety via the Plücker embedding

$$\mathrm{Grass}_F(m, V) \longrightarrow \mathbb{P}(\wedge^m V),$$

$$Fw_1 \oplus Fw_2 \oplus \cdots \oplus Fw_m \mapsto F(w_1 \wedge w_2 \wedge \cdots \wedge w_m),$$

where $\wedge^m V$ is the m-th wedge power of V. Fixing a base $e_1, e_2, \ldots, e_n$ for V we obtain a base $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_m}$, $1 \leq i_1 < i_2 < \cdots < i_m \leq n$ for $\wedge^m V$. This gives in turn homogeneous coordinates for $F(w_1 \wedge w_2 \wedge \cdots \wedge w_m)$.

Now let $\mathcal{A}$ be a central simple F-algebra with F-base $e_1, e_2, \ldots, e_{n^2}$. Then the Severi-Brauer variety $\mathrm{SB}(\mathcal{A})$ is a subvariety of the Grassmanian $\mathrm{Grass}_F(n, \mathcal{A})$ consisting of those points which correspond to right ideals of $\mathcal{A}$. This condition can be expressed by polynomial relations as follows (see [10, p.112]).)

Let $T \subset \mathcal{A}^*$ be a subset such that the multiplicative group generated by T contains a base of $\mathcal{A}$. Let $I = F(w_1 \wedge w_2 \wedge \cdots \wedge w_n) \in \mathrm{Grass}_F(n, \mathcal{A})$, i.e., I corresponds to a vector space $W = Fw_1 \oplus Fw_2 \oplus \cdots \oplus Fw_n \subset \mathcal{A}$. Then W is a right ideal of $\mathcal{A}$ iff $I = F(w_1 t \wedge w_2 t \wedge \cdots \wedge w_n t)$ for all $t \in T$.

For $t \in T$, write

$$e_i t = \sum a_{ij} e_j. \tag{2.1}$$

Then

$$e_{i_1} t \wedge e_{i_2} t \wedge \cdots \wedge e_{i_n} t = \sum_{1 \leq j_1 < \cdots < j_n \leq n^2} t_{j_1, \ldots, j_n, i_1, \ldots, i_n} e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_n},$$

where

$$t_{j_1, \ldots, j_n, i_1, \ldots, i_n} = \begin{vmatrix} a_{i_1 j_1} & \ldots & a_{i_1 j_n} \\ \vdots & & \vdots \\ a_{i_n j_1} & \ldots & a_{i_n j_n} \end{vmatrix}.$$

Then for

$$w_1 \wedge w_2 \wedge \cdots \wedge w_n = \sum_{1 \leq i_1 < \cdots < i_n \leq n^2} p_{i_1, \ldots, i_n} e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n}$$

we have

$$w_1 t \wedge w_2 t \wedge \cdots \wedge w_n t = \sum_{1 \le i_1 < \cdots < i_n \le n^2} q_{i_1,\dots,i_n} e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n},$$

where

$$q_{i_1,\dots,i_n} = \sum_{1 \le j_1 < \cdots < j_n \le n^2} t_{i_1,\dots,i_n,j_1,\dots,j_n} p_{j_1,\dots,j_n}.$$

Thus $F(w_1 \wedge w_2 \wedge \cdots \wedge w_n) = F(w_1 t \wedge w_2 t \wedge \cdots \wedge w_n t)$ iff their Plücker coordinates are proportional, that is

$$q_{i_1,\dots,i_n} p_{j_1,\dots,j_n} = q_{j_1,\dots,j_n} p_{i_1,\dots,i_n}.$$

Hence each $t \in T$ defines the following set of polynomial equations for $\mathrm{SB}(\mathcal{A})$

$$\left(\sum_{1 \le k_1 < \cdots < k_n \le n^2} t_{i_1,\dots,i_n,k_1,\dots,k_n} \xi_{k_1,\dots,k_n} \right) \xi_{j_1,\dots,j_n}$$
$$- \left(\sum_{1 \le k_1 < \cdots < k_n \le n^2} t_{j_1,\dots,j_n,k_1,\dots,k_n} \xi_{k_1,\dots,k_n} \right) \xi_{i_1,\dots,i_n} = 0$$

where $1 \le i_1 < \cdots < i_n \le n^2$, $1 \le j_1 < \cdots < j_n \le n^2$.

Thus the Severi-Brauer variety of $\mathcal{A}$ is defined in $\mathrm{Grass}_F(n, \mathcal{A})$ by this system of equations for all $t \in T$.

Let $\rho_n \in F$ and $\mathcal{B}$ be a central simple F-algebra of degree m. Set

$$\mathcal{D} = (x, y)_n \otimes \mathcal{B}_{F(x,y)}$$

where $F(x, y)$ is a purely transcendental extension of F. Let

$$\sqrt[n]{x}^{\,i} \sqrt[n]{y}^{\,j},\ 0 \le i, j \le n-1,$$

be a standard base for the algebra $(x, y)_n$ and $v_1, \dots, v_{m^2}$ a base for $\mathcal{B}$ over F consisting from invertible elements. Then

$$\sqrt[n]{x}^{\,i} \sqrt[n]{y}^{\,j} v_l,\ 0 \le i, j \le n-1,\ 1 \le l \le m^2,$$

is a base for $\mathcal{D}$ over $F(x, y)$.

Now using the procedure above we will construct equations for the Severi-Brauer variety $\mathrm{SB}(\mathcal{D})$. First of all take standard polynomials $G_j \in F(x, y)[\xi_0, \dots, \xi_N]$, $j \in J$, defining the Grassmanian

$$\mathrm{Grass}(nm, \mathcal{D}) \subset \mathbb{P}^N,$$

where $N = C^{nm}_{n^2 m^2} - 1$. Note that coefficients of G_j-s, $j \in J$, belong to the set $\{\pm 1, 0\}$.

Further, set

$$T_{\mathcal{D}} = \{ \sqrt[n]{x}, \sqrt[n]{y}, v_1, \dots, v_{m^2} \}.$$

Then the multiplicative group generated by $T_{\mathcal{D}}$ contains a base of $\mathcal{D}$ over $F(x,y)$. Then each $t \in T_{\mathcal{D}}$ defines a family of polynomial equations $\{F_i^{(t)}\}$, $i \in I_t$, of $\mathrm{SB}(\mathcal{D})$. Note that even $\{F_i^{(t)}\} \in F[x,y][\xi_0,\dots,\xi_N]$, $i \in I_t$. To see this it is enough to show that, for each $t \in T_{\mathcal{D}}$, the coefficients a_{ij} from (2.1) belong to $F[x,y]$. Indeed, if $t = v_l$, then

$$(\sqrt[n]{x}^i \sqrt[n]{y}^j v_k)v_l = \sqrt[n]{x}^i \sqrt[n]{y}^j (\sum a_s v_s)$$

for some $a_s \in F$. If $t = \sqrt[n]{x}$, then

$$(\sqrt[n]{x}^i \sqrt[n]{y}^j v_k)\sqrt[n]{x} = \begin{cases} \rho_n^j \sqrt[n]{x}^{i+1} \sqrt[n]{y}^j v_k & \text{if } i < n-1; \\ x\rho_n^j \sqrt[n]{y}^j v_k & \text{if } i = n-1. \end{cases} \tag{2.2}$$

Similar relations hold for $\sqrt[n]{y}$.

Now let L/F be a field extension and $a, b \in L^*$. Consider the algebra

$$\overline{\mathcal{D}} = (a,b)_n \otimes_L \mathcal{B}_L.$$

Then

$$\sqrt[n]{a}^i \sqrt[n]{b}^j v_l,\ 0 \le i,j \le n-1,\ 1 \le l \le m^2,$$

is a base for $\overline{\mathcal{D}}$ over L. Moreover, the multiplicative group generated by $\{\sqrt[n]{a}, \sqrt[n]{b}, v_l |\ 1 \le l \le m^2\}$ contains a base of $\overline{\mathcal{D}}$ over L. Hence this set can be used in order to construct equations of the Severi-Brauer variety of $\mathrm{SB}(\overline{\mathcal{D}})$.

Note that

$$(\sqrt[n]{a}^i \sqrt[n]{b}^j v_k)\sqrt[n]{a} = \begin{cases} \rho_n^j \sqrt[n]{a}^{i+1} \sqrt[n]{b}^j v_k & \text{if } i < n-1; \\ a\rho_n^j \sqrt[n]{b}^j v_k & \text{if } i = n-1. \end{cases}$$

Thus the coefficients of this expansion are obtained as the specialization $x \mapsto a$, $y \mapsto b$ of the coefficients of the expansion 2.2. Similar relations take place for $\sqrt[n]{b}$ and v_l, $1 \le l \le m^2$.

Hence polynomials from $L[\xi_1,\dots,\xi_N]$ defining $\mathrm{SB}(\overline{\mathcal{D}})$ are obtained as the specialization

$$x \mapsto a,\ y \mapsto b$$

of polynomials in $F[x,y][\xi_1,\dots,\xi_N]$ defining $\mathrm{SB}(\mathcal{D})$.

Thus we have proved the following

Lemma 2.1. *Let $\rho_n \in F$, $\mathcal{B}$ a central simple F-algebra of degree m and $a, b \in L^*$. Let also $\mathcal{D} = (x,y)_n \otimes \mathcal{B}_{F(x,y)}$. Then the Severi-Brauer variety* $\mathrm{SB}(\mathcal{D})$ *of $\mathcal{D}$ can be defined by polynomials in $F[x,y][\xi_0,\dots,\xi_N]$ ($N = C_{n^2m^2}^{nm} - 1$) such that for any field extension L/F and $a, b \in L^*$, their specialization $x \mapsto a$, $y \mapsto b$ gives polynomials in $L[\xi_0,\dots,\xi_N]$ defining the Severi-Brauer variety* $\mathrm{SB}(\overline{\mathcal{D}})$ *of $\overline{\mathcal{D}} = (a,b)_n \otimes \mathcal{B}_L$.*

Now we are in a position to prove the main result of this section.
Proof of Theorem 0.15. Set

$$\mathcal{D} = (x, y)_n \otimes \mathcal{A}^{op}_{F(x,y)}$$

where $F(x, y)$ is a purely transcendental extension of F.

The Severi-Brauer variety of $\mathcal{D}$ is a subvariety of the projective space $\mathbb{P}^N_{F(x,y)}$, where $N = C^n_{n^2} - 1$. Let $F_j \in F[x, y][\xi_0, \dots, \xi_N]$, $j \in J$, be polynomials defining $\mathrm{SB}(\mathcal{D})$ constructed in the proof of Lemma 2.1. Now consider the F_j as polynomials in $F[x, y, \xi_0, \dots, \xi_N]$. Then these polynomials define the affine variety $X \subset \mathbb{A}^{N+3}_F$. For $H \in F[x, y, \xi_0, \dots, \xi_N]$, denote by $D(H)$ the open complement of the variety defined by H in $\mathbb{A}^{N+3}_F$. Set

$$\mathrm{Symb}(\mathcal{A}) = X \cap D(x) \cap D(y) \cap \left(\cup_{i=0}^N D(\xi_i)\right).$$

Now we will show that $\mathrm{Symb}(\mathcal{A})$ has the required properties. Assume $(x_0, y_0, c_0, \dots, c_N) \in \mathrm{Symb}(\mathcal{A})(L)$ for a field extension L/F. Hence

$$F_j(x_0, y_0, c_0, \dots, c_N) = 0 \tag{2.3}$$

for any $j \in J$. Note that the specialization $x \mapsto x_0$, $y \mapsto y_0$ of polynomials $F_j \in F[x, y, \xi_0, \dots, \xi_N]$ gives the polynomials defining the Severi-Brauer variety of the algebra $(x_0, y_0)_n \otimes \mathcal{A}^{op}_L$. The condition (2.3) shows that $\mathrm{SB}((x_0, y_0)_n \otimes \mathcal{A}^{op}_L)$ has an L-rational point. Then $(x_0, y_0)_n \otimes \mathcal{A}^{op}_L$ is a matrix algebra, that is $(x_0, y_0)_n$ is Brauer equivalent to $\mathcal{A}_L$. Since $\deg(\mathcal{A}_L) = \deg((x_0, y_0)_n)$, we have $(x_0, y_0) \cong \mathcal{A}_L$.

Now assume that $(x_0, y_0)_n \cong \mathcal{A}_L$ for some $x_0, y_0 \in L^*$. Then

$$\mathrm{SB}((x_0, y_0)_n \otimes \mathcal{A}^{op}_L)$$

has an L-rational point, say, $(c_0, \dots, c_N)$. That is, $F_j(x_0, y_0, c_0, \dots, c_N) = 0$ for any $j \in J$. Hence $(x_0, y_0, c_0, \dots, c_N) \in X$. Since $x_0, y_0 \in L^*$ and some of c_i-s is not zero, then $(x_0, y_0, c_0, \dots, c_N) \in \mathrm{Symb}(\mathcal{A})$.

□

Finally, one can easy prove the following property of $\mathrm{Symb}(\mathcal{A})$.

Proposition 2.2. *Let $\mathcal{A}/F$ be a central simple algebra. Then, for any field extension K/F,*

$$\mathrm{Symb}(\mathcal{A} \otimes_F K) = \mathrm{Symb}(\mathcal{A}) \times_F K.$$

3. The variety $C(\mathcal{A}, K)$

Let $\mathcal{A}$ be a central simple algebra over a field F of degree n and K/F a cyclic field extension of degree n.

Consider a cyclic $F(x)$-algebra

$$\mathcal{C} = (K(x)/F(x), \sigma, x),$$

where $\mathrm{Gal}(K(x)/F(x)) = \langle\sigma\rangle$. One has $\mathcal{C} = \sum_{0\le i\le n-1} u_{\sigma^i} K(x)$ and

$$u_{\sigma^i} u_{\sigma^j} = \begin{cases} u_{\sigma^{i+j}} & \text{if } i+j<n, \\ x u_{\sigma^{i+j-n}} & \text{if } i+j\ge n. \end{cases}$$

Let $b_1, \dots, b_n$ be a base of an F-vector space K and $v_1, \dots, v_{n^2}$ a base of a central simple F-algebra $\mathcal{B}$ of degree n consisting of invertible elements.

Then the multiplicative group generated by the set

$$\{u_\sigma, b_1, \dots, b_n, v_1, \dots, v_{n^2}\}$$

contains a base of

$$\mathcal{D} = \mathcal{C} \otimes_{F(x)} \mathcal{B}_{F(x)}.$$

Then, as in the section 2, we can construct polynomials

$$F_j \in F(x)[\xi_0, \dots, \xi_N],$$

$j \in J$, defining the Severi-Brauer variety $\mathrm{SB}(\mathcal{D})$, where $N = C_{n^4}^{n^2} - 1$. Moreover, one can prove that these polynomials have coefficients not only in $F(x)$, but in $F[x]$.

Let L/F be a field extension such that $[LK : L] = [K : F]$. For $a \in L^*$, consider the algebra

$$\overline{\mathcal{D}} = (LK/L, \tau, a) \otimes_L \mathcal{B}_L,$$

where $\mathrm{Gal}(LK/L) = \langle\tau\rangle$. One has $(LK/L, \tau, a) = \sum_{0\le i\le n-1} w_{\tau^i} LK$ and

$$w_{\tau^i} w_{\tau^j} = \begin{cases} w_{\tau^{i+j}} & \text{if } i+j<n, \\ a w_{\tau^{i+j-n}} & \text{if } i+j\ge n. \end{cases}$$

Since $[LK : L] = [K : F]$, then $b_1, \dots, b_n$ is a base of an L-vector space LK. Then the multiplicative group generated by the set

$$\{w_\tau, b_1, \dots, b_n, v_1, \dots, v_{n^2}\}$$

contains a base of $\overline{\mathcal{D}}$.

Proceeding as in the proof of Lemma 2.1 we can prove

Lemma 3.1. *Let $\mathcal{B}$ be a central simple F-algebra of degree n, K/F a cyclic field extension of degree n. Let also*

$$\mathcal{D} = (K(x)/F(x), \sigma, x) \otimes_{F(x)} \mathcal{B}_{F(x)}.$$

Then the Severi-Brauer variety $\mathrm{SB}(\mathcal{D})$ of $\mathcal{D}$ can be defined by polynomials in $F[x][\xi_0, \dots, \xi_N]$ such that for any field extension L/F and $a \in L^*$ their specialization $x \mapsto a$ gives polynomials in $L[\xi_0, \dots, \xi_N]$ defining the Severi-Brauer variety $\mathrm{SB}(\overline{\mathcal{D}})$ of $\overline{\mathcal{D}} = (LK/L, \tau, a) \otimes_L \mathcal{B}_L$.

Now we can construct the variety $C(\mathcal{A}, K)$.

Proof of Theorem 0.16. Set

$$\mathcal{D} = (K(x)/F(x), \sigma, x) \otimes_{F(x)} \mathcal{A}_{F(x)},$$

where $F(x)$ is a purely transcendental extension of F.

The Severi-Brauer variety of $\mathcal{D}$ is a closed subvariety of the projective space $\mathbb{P}^N_{F(x)}$, where $N = C^{n^2}_{n^4} - 1$. Let $F_j \in F[x][\xi_0, \ldots, \xi_N]$, $j \in J$, be polynomials defining $\mathrm{SB}(\mathcal{D})$ constructed in the proof of Lemma 3.1. Now consider F_j as polynomials in $F[x, \xi_0, \ldots, \xi_N]$. Then these polynomials define an affine variety $X \subset \mathbb{A}^{N+2}_F$. For $H \in F[x, \xi_0, \ldots, \xi_N]$, denote by $D(H)$ the open complement of the variety defined by H in $\mathbb{A}^{N+2}_F$. Set

$$C(\mathcal{A}, K) = X \cap D(x) \cap \left(\cup_{i=0}^{N} D(\xi_i)\right).$$

The rest of the proof is the same as in Theorem 1.11.

□

As in the case of $\mathrm{Symb}(\mathcal{A})$, we have the following property of $C(\mathcal{A}, K)$.

Proposition 3.2. *Let $\mathcal{A}$ be a central simple algebra over a field F of degree n and K/F a cyclic field extension of degree n. Let also L/F be a field extension such that $|K : F| = |LK : L|$.*

Then

$$C(\mathcal{A} \otimes_F L, LK) = C(\mathcal{A}, K) \times_F L.$$

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