HYPERBOLICITY CRITERIA FOR CERTAIN INVOLUTIONS

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Abstract. Using the ideas and techniques developed by Bayer-Fluckiger, Shapiro and Tignol about hyperbolic involutions of central simple algebras, criteria for the hyperbolicity of involutions of the form $\sigma \otimes \tau$ and $\sigma \otimes \rho$, where σ is an involution of a central simple algebra A, τ is the nontrivial automorphism of a quadratic extension of the center of A and ρ is an involution of a quaternion algebra are obtained.

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1. Introduction

Given a ring R with unity, a map $\sigma : R \to R$ is called an *involution* if $\sigma(x+y) = \sigma(x) + \sigma(y)$, $\sigma(xy) = \sigma(y)\sigma(x)$ and $\sigma(\sigma(x)) = x$ for all $x,\ y \in R$. The pair (R, σ) (or simply σ) is called *hyperbolic* if there exists an idempotent $e \in R$ such that $\sigma(e) = 1 - e$. This notion was introduced in [2]. An involution σ of R is said to be *isotropic* if there exists a nonzero element $a \in R$ such that $\sigma(a)a = 0$; otherwise, σ is called *anisotropic*.

Various hyperbolicity criteria have been obtained by several authors. It is helpful to recall some of these basic criteria here.

Let K be a field of characteristic different from 2. Let A be a K-central simple algebra with an involution σ and let k be the fixed field of $\sigma|_K$. When σ is of the first kind, i.e., $K = k$, it was shown in [2, Thm. 3.3] that if $L = K(\sqrt{d})$ is a quadratic extension of K, then $(A \otimes L, \sigma \otimes \mathrm{id}_L)$ is hyperbolic if and only if there exists $r \in A$ such that $r^2 = d$ and $\sigma(r) = -r$. This criterion excludes the exceptional case, where A is split, σ is of orthogonal type and the Witt index of the quadratic form, to which σ is adjoint, is odd. This result was generalized in [13] and [10], in various ways, to the case where σ is of the second kind.

When L/k is an extension of odd degree, it was shown in [1] that if $\sigma \otimes \mathrm{id}_L$ is hyperbolic, then so is σ. This result was originally stated in [1] in terms of hermitian forms (see [5, pp. 79-80]).

If L is the function field of a quadratic form, defined over k, the behavior of $\sigma \otimes \mathrm{id}_L$ (in other words, the behavior of σ under the field extension L/k) has been studied by various authors. If σ becomes hyperbolic over $L = K[t]/(\pi(t))$, where $\pi(t) \in K[t]$ is a separable monic polynomial of degree $2n$, it was shown in [13] that (A, σ) is homomorphic image of a certain universal free K-algebra H_π on $2n$ indeterminates with an involution σ_π. A quadratic form-theoretic formulation of this criterion was given in [4] by showing that H_π has a homomorphic image, which is the Clifford algebra of some quadratic form over a certain polynomial ring. A necessary and sufficient condition for a quaternion or a biquaternion algebra to become hyperbolic over a field extension was obtained in [3]. An interesting question about hyperbolicity of a central simple algebra with involution (A, σ) (of low dimension) over the function field of a quadratic form q and its relationship with the existence of some homomorphic images of the even Clifford algebra $C_0(q)$

in A was studied in [7]. The hyperbolicity of (A,σ) (of low dimension) over the function field of three-dimensional forms was studied in [12].

Let A be a K-central simple algebra with an involution σ and let k be the fixed field of $\sigma|_K$. Let τ be the nontrivial automorphism of a quadratic extension L/k. Let Q be a quaternion algebra over K with an involution ρ such that $\sigma|_K=\rho|_K$. In this article we provide criteria for hyperbolicity of $\sigma\otimes\tau$ and $\sigma\otimes\rho$ (see Theorem 3.2, Corollary 4.2 and Corollary 4.6). Our main motivation to search for such criteria arises from our previous work [11], which investigates the hyperbolicity of canonical involutions of Clifford algebra (or the even Clifford algebra) of a quadratic form q and its connection with the existence of particular subforms of q. In fact, the standard isomorphism theorems of Clifford algebras like $C(q'\perp q)\simeq C(q')\otimes C(d\cdot q)$ and $C_0(q''\perp q)\simeq C_0(q'')\otimes C(-d\cdot q)$, where q' is a form of even dimension, q'' is a form of odd dimension and d is the discriminant of q (see [8, Ch. V, §2]), induce isomorphisms of algebras with involution (see [9]). If we choose, in the first (resp. second) isomorphism, q to be a two (resp. one) dimensional form, the canonical involutions of $C(q'\perp q)$ (resp. $C_0(q''\perp q)$) are decomposed as $\sigma\otimes\rho$ (resp. $\sigma\otimes\tau$). Finding criteria for the hyperbolicity of $\sigma\otimes\tau$ and $\sigma\otimes\rho$ are therefore useful in this regard.

The criterion for hyperbolicity of $\sigma\otimes\rho$ is deduced from a general criterion for the hyperbolicity of a central simple algebra with involution in terms of certain subalgebra of codimension two. This criterion is given in Theorem 4.1 (see also Theorem 4.5).

As an application, a corollary about the hyperbolicity of the tensor product of two quaternion algebras with involution is drawn (c.f. Proposition 4.10, compare with [11, Prop. 3.9]).

2. Preliminaries

All fields considered in this paper are supposed to be of characteristic different from 2.

Let A be a K-central simple algebra with an involution σ. If $\sigma|_K$ is the identity map, σ is called of the *first kind*. Otherwise $\sigma|_K$ is a nontrivial automorphism of K. In this case, σ is called of the *second kind*. If k is the fixed field of $\sigma|_K$, in both cases, we say, in short, that σ is a *K/k-involution*. Let $\varepsilon\in K$ be an element with $\sigma(\varepsilon)\varepsilon=1$. An element $a\in A$ which satisfies $\sigma(a)=\varepsilon a$ is called *ε-hermitian*. A (1)-hermitian element is usually called a *symmetric* element and a (-1)-hermitian element is usually called a *skew-symmetric* element.

We define

$$A^+=\{x\in A:\sigma(x)=x\},$$
$$A^-=\{x\in A:\sigma(x)=-x\}.$$

Note that A^+ and A^- are k-vector spaces.

The involution σ is said to be of *orthogonal* type if $\dim A^+>\dim A^-$. It is said to be of *symplectic* type if $\dim A^+<\dim A^-$.

If σ is of the first kind, it is known that the square class of the reduced norm of any skew-symmetric invertible element a of A with respect to σ is independent of the choice of a (cf. [6]). The *discriminant* of σ is so defined in [5] as the square class of $(-1)^m\operatorname{Nrd}(a)$, where a is a skew-symmetric element of A and $m=\frac{1}{2}\deg(A)$. Here $\deg(A)$ is the degree of A (the *degree* of a central simple algebra is square root of its dimension, as a vector space, over its center).

Let (A,σ) be a K-central simple algebra with involution. Let $\varepsilon\in K$ be an element with $\varepsilon\sigma(\varepsilon)=1$ and let V be a right A-module of finite rank. An *ε-hermitian* form over V with respect to σ, is a biadditive map $h:V\times V\to A$ such that

1) $h(x\alpha,y\beta)=\sigma(\alpha)h(x,y)\beta$ for all $x,\ y\in V$ and all $\alpha,\ \beta\in A$,

2) $\varepsilon h(y,x) = \sigma(h(x,y))$ for all $x,\ y \in V$.

If $\varepsilon = 1$, h is called a *hermitian form* and if $\varepsilon = -1$, h is called a *skew-hermitian form.*

For a non-degenerate ε-hermitian space (V,h) over (A,σ), the *adjoint involution* of $\mathrm{End}_A(V)$ with respect to (V,h), is the unique involution I_h of $\mathrm{End}_A(V)$ such that

1) $I_h(\alpha) = \sigma(\alpha)$ for every $\alpha \in K$,
2) $h(x, f(y)) = h(I_h(f)(x), y)$ for all $x,\ y \in V$ and all $f \in \mathrm{End}_A(V)$.

When the role of $\mathrm{End}_A(V)$ is clear in the context, we simply say that I_h is the involution, which is adjoint to h.

3. Quadratic extensions and hyperbolic involutions

Lemma 3.1. *Let A be a K-central simple algebra with an anisotropic K/k-involution σ. For $d \in k$, let $L = k(\sqrt{d})$ be a quadratic extension of k with the nontrivial k-automorphism τ. If $\sigma \otimes \tau$ is hyperbolic, then there exists an element $r \in A$ such that $r^2 = d$ and $\sigma(r) = r$.*

Proof. If $\sigma \otimes \tau$ is hyperbolic, there exists $e \in A \otimes_k L$ such that $e^2 = e$ and $(\sigma \otimes \tau)(e) = 1 - e$. We can write $e = e_1 \otimes 1 + e_2 \otimes \sqrt{d}$, where $e_1,\ e_2 \in A$. The following systems of equations are obtained:

$$(1) \qquad \begin{cases} e_1^2 + de_2^2 = e_1 \\ e_1e_2 + e_2e_1 = e_2, \end{cases} \qquad \begin{cases} \sigma(e_1) = 1 - e_1 \\ \sigma(e_2) = e_2. \end{cases}$$

We first show that e_2 is invertible. As σ is anisotropic, the right ideal $I = \{x \in A : e_2x = 0\}$ is generated by a symmetric idempotent f (cf. [2, Cor. 1.8]). The relation $e_2f = 0$ implies $e_1e_2f = 0$ and $e_2^2f = 0$. Using the previous system of equations and $e_2^2f = 0$ we obtain $(e_1 - e_1^2)f = 0$. We deduce that $(fe_1)\sigma(fe_1) = fe_1\sigma(e_1)f = fe_1(1-e_1)f = f(e_1 - e_1^2)f = 0$. As σ is anisotropic, we obtain $fe_1 = 0$. A similar argument shows that $\sigma(e_1f)e_1f = f(1-e_1)e_1f = f(e_1 - e_1^2)f = 0$. Thus $e_1f = 0$.

We now have: $0 = \sigma(fe_1) = (1-e_1)f = f - e_1f = f$. Therefore $I = \{0\}$. This implies that e_2 is invertible. Now consider the element $r = e_1e_2^{-1}$. According to (1), we have $e_1 + e_2e_1e_2^{-1} = 1$ thus $e_2e_1e_2^{-1} = 1 - e_1$.

On the other hand $\sigma(r) = \sigma(e_2)^{-1}\sigma(e_1) = e_2^{-1}(1-e_1) = e_1e_2^{-1} = r$. Similarly $r^2 = e_1{e_2}^{-1}e_1{e_2}^{-1} = e_1{e_2}^{-1}(1 - e_2e_1{e_2}^{-1}){e_2}^{-1} = e_1{e_2}^{-2} - {e_1}^2e_2^{-2} = (e_1 - e_1^2)e_2^{-2} = de_2^2e_2^{-2} = d$. This completes the proof. □

Theorem 3.2. *Let A be a K-central simple algebra with a K/k-involution σ. We exclude the case where A is split, σ is symplectic and $\deg(A) = 2m$, where m is an odd integer. Let $L = k(\sqrt{d})$ be a quadratic extension of k and let τ be the nontrivial automorphism of L/k. Then there exists an element $r \in A$ with the properties $r^2 = d$ and $\sigma(r) = r$ if and only if $(A \otimes_k L, \sigma \otimes \tau)$ is hyperbolic.*

Proof. First, suppose that the element r with the indicated properties exists. Take $t = d^{-1}r \otimes \sqrt{d} \in A \otimes_k L$. We have: $t^2 = (d^{-1}r \otimes \sqrt{d})^2 = d^{-2}r^2 \otimes d = 1$ and $(\sigma \otimes \tau)(t) = d^{-1}r \otimes (-\sqrt{d}) = -t$. Take $e = \frac{1}{2}(1+t)$. We obtain $e^2 = e$ and $\sigma(e) = 1 - e$. Thus, $(A \otimes_k L, \sigma \otimes \tau)$ is hyperbolic.

Conversely, suppose that $(A \otimes_k L, \sigma \otimes \tau)$ is hyperbolic. Let $A = \mathrm{End}_D(V)$, where D is a division algebra which is Brauer-equivalent to A. Let σ' be an involution of D of the same kind as σ. Finally let (V,h) be a ε-hermitian space over (D,σ') such that σ is the adjoint involution with respect to (V,h) (we take $\varepsilon = 1$ when σ is of the second kind).

Consider the Witt decomposition $(V,h) = (V_0,h_0) \perp (V_1,h_1)$, where (V_0,h_0) is hyperbolic and (V_1,h_1) is anisotropic. Let σ_0 and σ_1 be the adjoint involutions of $\mathrm{End}_D(V_0)$ and $\mathrm{End}_D(V_1)$ with respect to h_0 and h_1, respectively.

The hermitian space (V_1, h_1) becomes hyperbolic over (L, τ). The previous lemma implies the existence of an element $r_1 \in \mathrm{End}_D(V_1)$ such that $r_1^2 = d$ and $\sigma_1(r_1) = r_1$.

Suppose that σ is of the second kind or of the first kind and of orthogonal type. According to [2, Thm. 2.2], $\mathrm{End}_D(V_0)$ contains a σ_0-invariant subalgebra M such that

$$(M_2(K), \Theta_1) \simeq (M, \sigma_0|_M), \tag{2}$$

where Θ_1 is the involution of $M_2(K)$ defined by

$$\Theta_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \sigma_0(d) & \sigma_0(b) \\ \sigma_0(c) & \sigma_0(a) \end{pmatrix}.$$

Let $r_0 \in M \subset \mathrm{End}_D(V_0)$ be the image of $\begin{pmatrix} 0 & d \\ 1 & 0 \end{pmatrix}$ under the isomorphism given in (2). We have $\sigma_1(r_0) = r_0$ and $r_0^2 = d$. Let $r \in A = \mathrm{End}(V)$ be the element defined by

$$r(x, y) = (r_0(x), r_1(y)). \tag{3}$$

We obtain $\sigma(r) = r$ and $r^2 = d$.

If σ is of the first kind and of symplectic type and $\deg(\mathrm{End}_D(V_0))$ is divisible by 4 (this is always the case except when D is split and $\deg(A) = 2m$, where m is an odd integer), then again using [2, Thm. 2.2], there exists a σ_0-invariant subalgebra M such that

$$(M, \sigma_0|_M) \simeq (M_2(K), \Theta_1). \tag{4}$$

Let r_0 be the image of $\begin{pmatrix} 0 & d \\ 1 & 0 \end{pmatrix}$ under the isomorphism given in (4). We have $\sigma_0(r_0) = r_0$ and $r_0^2 = d$. The element r defined in (3) satisfies $\sigma(r) = r$ and $r^2 = d$. □

4. Quadratic extensions of algebras and hyperbolic involutions

Theorem 4.1. *Let (A, σ) be a K-central simple algebra with involution. Suppose that there exist $\lambda,\ \mu \in A^\times$ such that $\lambda\mu = -\mu\lambda$, $\sigma(\lambda) = -\lambda$, $\sigma(\mu) = -\mu$ and $K(\lambda)/K$ is a quadratic extension. Let $\widetilde{A} = C_A(\lambda)$ be the centralizer of λ in A. Suppose that $\sigma|_{\widetilde{A}}$ is anisotropic. Then σ is hyperbolic if and only if there exists $r \in \widetilde{A}$ such that $\sigma(r)\mu = \mu r$ and $\sigma(r)r = \mu^2$.*

Proof. As σ is hyperbolic, there exists an idempotent $e \in A$ such that $\sigma(e) = 1 - e$. We can write $e = x + \mu y$, where $x,\ y \in \widetilde{A}$. We obtain the following systems of equations:

$$\begin{cases} x^2 + \mu y \mu y = x \\ x\mu y + \mu y x = \mu y, \end{cases} \qquad \begin{cases} \sigma(x) = 1 - x \\ \sigma(y) = \mu y \mu^{-1}. \end{cases} \tag{5}$$

We first show that y is invertible. We observe that the right ideal $I = \{z \in \widetilde{A} : yz = 0\}$ is generated by a symmetric idempotent f (cf. [2, Cor. 1.8]). The relation $yf = 0$ implies $\mu y \mu y f = 0$. Therefore $(x - x^2)f = 0$. We now have: $(fx)\sigma(fx) = fx(1-x)f = f(x - x^2)f = 0$. In the same way $\sigma(xf)(xf) = f(1-x)xf = f(x - x^2)f = 0$. As $\sigma|_{\widetilde{A}}$ is anisotropic we deduce that $fx = xf = 0$. On the other hand $0 = \sigma(fx) = (1-x)f = f - xf = f$. This implies $I = \{0\}$ and thus y is invertible.

Now take $r = xy^{-1}$. Using (5), we obtain:

$$\begin{aligned} \sigma(r)r &= \sigma(xy^{-1})xy^{-1} = \sigma(y^{-1})\sigma(x)xy^{-1} = \mu y^{-1}\mu^{-1}(1-x)xy^{-1} \\ &= \mu y^{-1}\mu^{-1}(x - x^2)y^{-1} = \mu y^{-1}\mu^{-1}(\mu y \mu y)y^{-1} \\ &= \mu^2, \end{aligned}$$

$$\begin{aligned}\sigma(r)\mu &= \sigma(xy^{-1})\mu = \sigma(y)^{-1}\sigma(x)\mu = \mu y^{-1}\mu^{-1}(1-x)\mu\\ &= \mu y^{-1} - \mu y^{-1}\mu^{-1}x\mu\\ &= \mu y^{-1} - \mu y^{-1}\mu^{-1}(\mu - \mu yxy^{-1}) = \mu xy^{-1}\\ &= \mu r.\end{aligned}$$

Conversely, suppose that there exists $r \in \widetilde{A}$ such that $\sigma(r)\mu = \mu r$ and $\sigma(r)r = \mu^2$. Take $t = r\mu^{-1}$. We have $t^2 = \mu^{-1}\sigma(r)r\mu^{-1} = 1$ and $\sigma(t) = -\mu^{-1}\sigma(r) = -r\mu^{-1} = -t$. Now, if $e = \frac{1}{2}(1+t)$, then $e^2 = e$ and $\sigma(e) = 1 - e$. Therefore σ is hyperbolic. □

Corollary 4.2. *Let A be a K-central simple algebra with an involution σ. Let Q be a quaternion algebra over K endowed with an involution ρ. Suppose that ρ and σ have the same restriction to K. Let $\lambda,\ \mu$ be elements in Q such that $\rho(\lambda) = -\lambda$, $\rho(\mu) = -\mu$, $\mu^2 \in K$, $\lambda\mu = -\mu\lambda$ and let $L = K(\lambda)$ be a quadratic extension of K. Suppose that $\sigma \otimes \rho|_L$ is anisotropic. Then the involution $\sigma \otimes \rho$ of $A \otimes_K Q$ is hyperbolic if and only if there exist $a,\ b \in A$ with $\sigma(a) = a$, $\sigma(b) = b$, $ab = ba$ and $\sigma(a)a - \lambda^2\sigma(b)b = \mu^2$.*

Proof. First, suppose that $\sigma \otimes \rho$ is hyperbolic. According to Theorem 4.1, there exists an element $r \in A \otimes L$ such that $(\sigma\otimes\rho)(r)r = 1\otimes\mu^2$ and $\sigma(r)(1\otimes\mu) = (1\otimes\mu)r$. We can write $r = a \otimes 1 + b \otimes \lambda$ for some $a,\ b \in A$. We have $(\sigma \otimes \rho)(r) = \sigma(a) \otimes 1 - \sigma(b) \otimes \lambda$.

The condition $(\sigma \otimes \rho)(r)(1 \otimes \mu) = (1 \otimes \mu)r$ implies $\sigma(a) \otimes \mu - \sigma(b) \otimes \lambda\mu = a\otimes\mu + b\otimes\mu\lambda$. Therefore $\sigma(a) = a$ and $\sigma(b) = b$. The condition $(\sigma\otimes\rho)(r)r = 1\otimes\mu^2$ implies $\sigma(a)a \otimes 1 - \sigma(b)b \otimes \lambda^2 + \sigma(a)b \otimes \lambda - \sigma(b)a \otimes \lambda = 1 \otimes \mu^2$. We deduce that $\sigma(a)a - \lambda^2\sigma(b)b = \mu^2$ and $\sigma(a)b - \sigma(b)a = 0$, which implies $ab = ba$.

Conversely, suppose that a and b with the indicated properties exist. Take $r = a\otimes 1 + b\otimes\lambda$. We have: $(\sigma\otimes\rho)(r)r = 1\otimes\mu^2$ and $(\sigma\otimes\rho)(r)(1\otimes\mu) = (1\otimes\mu)r$. From Theorem 4.1 we conclude that $\sigma \otimes \rho$ is hyperbolic. □

Remark 4.3. In Corollary 4.2, the condition concerning the anisotropy of $\sigma \otimes \rho|_L$ can also be stated as follows: If $a,\ b \in A$ satisfy $\sigma(a)a - \lambda^2\sigma(b)b = 0$ and $\sigma(a)b = \sigma(b)a$, then $a = b = 0$.

Remark 4.4. If $(A, \sigma) = (K, \mathrm{id})$, Corollary 4.2 states that the canonical involution of the quaternion algebra $Q = (\alpha, \beta)_K$ is hyperbolic if and only if the quadratic form $\langle 1, -\alpha, -\beta, \alpha\beta\rangle$ is isotropic over K if and only if Q is split. This is well known (see [4, Prop. 18]).

Theorem 4.1 can be established under more general hypotheses. In fact, we have:

Theorem 4.5. *Let (A, σ) be a K-central simple algebra with involution. Suppose that there exist $\lambda,\ \mu \in A^\times$ such that $\lambda\mu = -\mu\lambda$, $\sigma(\lambda) = \varepsilon_\lambda\lambda$, $\sigma(\mu) = \varepsilon_\mu\mu$ and $K(\lambda)/K$ is a quadratic extension, where $\varepsilon_\lambda,\ \varepsilon_\mu \in K$ satisfy $\sigma(\varepsilon_\lambda)\varepsilon_\lambda = 1$ and $\sigma(\varepsilon_\mu)\varepsilon_\mu = 1$. Let $\widetilde{A} = C_A(\lambda)$ be the centralizer of λ in A. Suppose that $\sigma|_{\widetilde{A}}$ is anisotropic. Then σ is hyperbolic if and only if there exists $r \in \widetilde{A}$ such that $\sigma(r)\mu = -\varepsilon_\mu\mu r$ and $\sigma(r)r = -\varepsilon_\mu\mu^2$.*

Proof. Using an argument similar to the one given in the proof of Theorem 4.1, we find the following systems of equations:

$$(6) \qquad \begin{cases} x^2 + \mu y\mu y = x \\ x\mu y + \mu yx = \mu y, \end{cases} \qquad \begin{cases} \sigma(x) = 1 - x \\ \sigma(y) = -\sigma(\varepsilon_\mu)\mu y\mu^{-1}. \end{cases}$$

For $r = xy^{-1}$ we have: $\sigma(r)r = -\varepsilon_\mu\mu^2$ and $\sigma(r)\mu = -\varepsilon_\mu\mu r$.

Conversely, suppose that there exists $r \in \widetilde{A}$ such that $\sigma(r)\mu = -\varepsilon_\mu\mu r$ and $\sigma(r)r = -\varepsilon_\mu\mu^2$. Take $t = r\mu^{-1}$. We have: $t^2 = r\mu^{-1}\cdot r\mu^{-1} = -\sigma(\varepsilon_\mu)\mu^{-1}\sigma(r)\cdot$

$r\mu^{-1} = \sigma(\varepsilon_\mu)\varepsilon_\mu = 1$ and $\sigma(t) = \sigma(\mu^{-1})\sigma(r) = (\varepsilon_\mu\mu)^{-1}(-\varepsilon_\mu\mu r\mu^{-1}) = -r\mu^{-1} = -t$. Now, if $e = \frac{1}{2}(1+t)$, then $e^2 = e$ and $\sigma(e) = 1-e$. Therefore σ is hyperbolic. $\square$

Corollary 4.6. *Let (A,σ) be a K-central simple algebra with involution. Let (Q,ρ) be a K-quaternion algebra with involution. Suppose that ρ and σ have the same restriction to K. Let λ, μ be elements of Q such that $\rho(\lambda) = \varepsilon_\lambda\lambda$, $\rho(\mu) = \varepsilon_\mu\mu$, $\mu^2 \in K$, $\lambda\mu = -\mu\lambda$ and let $L = K(\lambda)$ be a quadratic extension of K, where ε_λ, $\varepsilon_\mu \in K$ satisfy $\rho(\varepsilon_\lambda)\varepsilon_\lambda = 1$ and $\rho(\varepsilon_\mu)\varepsilon_\mu = 1$. Suppose that $\sigma\otimes\rho|_L$ is anisotropic. Then the involution $\sigma\otimes\rho$ of $A\otimes Q$ is hyperbolic if and only if there exist a, $b \in A$ with $\sigma(a) = -\varepsilon_\mu a$, $\sigma(b) = \varepsilon_\lambda^{-1}\varepsilon_\mu b$, $ab = ba$ and $\sigma(a)a + \varepsilon_\lambda\lambda^2\sigma(b)b + \varepsilon_\mu\mu^2 = 0$.*

Proof. According to Theorem 4.5, there exists an element $r \in A\otimes L$ such that $(\sigma\otimes\rho)(r)r = -\varepsilon_\mu(1\otimes\mu^2)$ and $\sigma(r)(1\otimes\mu) = -\varepsilon_\mu(1\otimes\mu)r$. We can write $r = a\otimes 1 + b\otimes\lambda$ for some a, $b \in A$. We have $(\sigma\otimes\rho)(r) = \sigma(a)\otimes 1 + \varepsilon_\lambda\sigma(b)\otimes\lambda$.

The condition $\sigma(r)(1\otimes\mu) = -\varepsilon_\mu(1\otimes\mu)r$ implies that $\sigma(a)\otimes\mu + \varepsilon_\lambda\sigma(b)\otimes\lambda\mu = -\varepsilon_\mu(a\otimes\mu - b\otimes\mu\lambda)$. Therefore $\sigma(a) = -\varepsilon_\mu a$ and $\varepsilon_\lambda\sigma(b) = \varepsilon_\mu b$. The condition $(\sigma\otimes\rho)(r)r = -\varepsilon_\mu(1\otimes\mu^2)$ implies $\sigma(a)a\otimes 1 + \varepsilon_\lambda\sigma(b)b\otimes\lambda^2 + \sigma(a)b\otimes\lambda + \varepsilon_\lambda\sigma(b)a\otimes\lambda = -\varepsilon_\lambda(1\otimes\mu^2)$. We deduce that $\sigma(a)a + \varepsilon_\lambda\lambda^2\sigma(b)b = -\varepsilon_\mu\mu^2$ and $\sigma(a)b + \varepsilon_\lambda\sigma(b)a = 0$, which implies $ab = ba$. $\square$

Remark 4.7. In Corollary 4.6, the condition concerning the anisotropy of $\sigma\otimes\rho|_L$ can also be stated as follows: If a, $b \in A$ satisfy $\sigma(a)a + \varepsilon_\lambda\lambda^2\sigma(b)b = 0$ and $\sigma(a)b + \varepsilon_\lambda\sigma(b)a$, then $a = b = 0$.

Remark 4.8. Let $(A,\sigma) = (K, \mathrm{id})$. Suppose that $Q = (a,b)_K$ is the quaternion algebra with the orthogonal involution ρ defined by $\rho(i) = i$, $\rho(j) = j$. Then Corollary 4.6 implies that the involution ρ is hyperbolic if and only if $-ab \in K^{\times 2}$. This is well known (see [3, Thm. 2.1]).

As an application of above results, we present an alternative proof for the following result stated in [3, Prop. 3.1]. As it is mentioned in [3], this result is an immediate consequence of [2, Cor. 2.5].

Corollary 4.9. *Let A be a biquaternion algebra over a field K with an orthogonal involution σ such that* $\mathrm{disc}(\sigma) = 1$. *We can write the following decomposition $(A,\sigma) = (Q_1,\sigma_1)\otimes_K(Q_2,\sigma_2)$, where σ_1 and σ_2 are the symplectic involutions. Then σ is hyperbolic if and only if Q_1 or Q_2 is split.*

Proof. If one of the algebras Q_1 or Q_2 is split, then one of the involutions σ_1 or σ_2 is hyperbolic. Therefore σ is hyperbolic.

Conversely, suppose that σ is hyperbolic. We can write $Q_2 = (c,d)_K$, where c, $d \in K^\times$. If Q_2 is not split, we have in particular, $c \notin K^{\times 2}$. If the involution $\sigma_1\otimes\sigma_2|_{K(\sqrt{c})}$ of $Q_1\otimes K(\sqrt{c})$ is isotropic, then [3, Lem. 2.3] implies that Q_1 is split. If $\sigma_1\otimes\sigma_2|_{K(\sqrt{c})}$ is anisotropic, Corollary 4.6 implies the existence of x, $y \in K$ such that $x^2 - cy^2 - d = 0$. Thus the quadratic form $\langle 1, -c, -d\rangle$ is isotropic. Therefore Q_2 is split. $\square$

Proposition 4.10. *Let $Q_1 = (a,b)_K$ and $Q_2 = (c,d)_K$ be two K-quaternion algebras. Let σ_1 be the orthogonal involution of Q_1 defined by $\sigma_1(\lambda) = \lambda$, $\sigma_1(\mu) = \mu$, where $\{\lambda,\mu\}$ is a standard basis of Q_1 with $\lambda\mu = -\mu\lambda$, $\lambda^2 = a$, $\mu^2 = b$ and let σ_2 be the canonical involution of Q_2. Then $\sigma = \sigma_1\otimes\sigma_2$ is hyperbolic if and only if Q_2 is split or at least one of the quadratic forms $\langle a, b, -ac, -bc, -d\rangle$ or $\langle a, b, -c\rangle$ is isotropic over K.*

Proof. Suppose that σ is hyperbolic. If $c \in K^{\times 2}$, then Q_2 is split. Now suppose that $c \notin K^{\times 2}$. Consider the quadratic extension L/K, where $L = K(\sqrt{c})$. The restriction $\sigma_2|_L$ is the nontrivial automorphism of L/K.

If $\sigma_1 \otimes \sigma_2|_L$ is isotropic, then $Q_1 \otimes L$ is split. Therefore $\sigma_1 \otimes \sigma_2|_L$ is adjoint to an isotropic hermitian form of dimension 2 over a field. In particular, $\sigma_1 \otimes \sigma_2|_L$ is hyperbolic. According to Theorem 3.2, there exists $r \in Q_1$ with the properties $r^2 = c$ and $\sigma_1(r) = r$. Thanks to these last properties one can write $r = \gamma_1 1 + \gamma_2\lambda + \gamma_3\mu$ for some $\gamma_1,\ \gamma_2,\ \gamma_3 \in K$. However, $r^2 = c$ implies that $\gamma_1^2 + a\gamma_2^2 + b\gamma_3^2 = c$ and $\gamma_1\gamma_2 = \gamma_1\gamma_3 = 0$. If $\gamma_1 = 0$, we deduce that the quadratic form $\langle a, b, -c\rangle$ is isotropic. If $\gamma_1 \neq 0$, we deduce that $\gamma_2 = \gamma_3 = 0$. Therefore c is a square, which is a contradiction.

Now consider the case where $\sigma_1 \otimes \sigma_2|_L$ is anisotropic. In this case, Corollary 4.6 implies the existence of $x,\ y \in Q_1$ such that $\sigma_1(x) = x$, $\sigma_1(y) = y$, $xy = yx$ and

$$\sigma_1(x)x - c\sigma_1(y)y - d = 0. \tag{7}$$

We can write $x = \gamma_1 1 + \gamma_2\lambda + \gamma_3\mu$ and $y = \gamma_1' 1 + \gamma_2'\lambda + \gamma_3'\mu$ for some γ_1, γ_2, γ_3, γ_1', γ_2', $\gamma_3' \in K$. The condition $xy = yx$ implies that the elements $w = \gamma_2\lambda + \gamma_3\mu$ and $w' = \gamma_2'\lambda + \gamma_3'\mu$ are linearly dependent over K. So there exists a nonzero element $v \in K\lambda \oplus K\mu \subset Q_1$ such that $w = \theta v$ and $w' = \theta' v$ for some $\theta,\ \theta' \in K$. Now (7) implies that $\lambda_1^2 - c\lambda_1'^2 + q(v)\theta^2 - cq(v)\theta'^2 - d = 0$, where q is the quadratic form $\langle a, b\rangle$. It follows that the quadratic form $\langle 1, -c\rangle \otimes \langle a, b\rangle \perp \langle -d\rangle \simeq \langle a, b, -ac, -bc, -d\rangle$ is isotropic over K.

Conversely, suppose that one of the following conditions holds:

Q_2 is split or
$\langle a, b, -c\rangle$ is isotropic or
$\langle a, b, -ab, -ac, -d\rangle$ is isotropic.

In the first case, σ is hyperbolic because σ_2 is hyperbolic too. In the second case, σ is hyperbolic because $\sigma_1 \otimes \sigma_2|_L$ is hyperbolic. In the third case, we obtain a system of the form of (7). Therefore σ is hyperbolic. □

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