A NOTE ON K-THEORY OF AZUMAYA ALGEBRAS

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Abstract. For an Azumaya algebra A which is free over its centre R, we prove that K-theory of A is isomorphic to K-theory of R up to its rank torsions. We conclude that $K_i(A,\mathbb{Z}/m) = K_i(R,\mathbb{Z}/m)$ for any m relatively prime to the rank and $i \geq 0$. This covers, for example, K-theory of division algebras, K-theory of Azumaya algebras over semi-local rings and K-theory of graded central simple algebras indexed by a totally ordered abelian group.

1. Introduction

Let R be a ring and $K_i(R)$, $i \geq 0$, be Quillen's K-groups. The construction of K-groups is functorial. Furthermore, K_i functors induce identity maps on inner-automorphisms of a ring and $K_i(R) \to K_i(M_tR) \to K_i(R)$ is multiplication by t, where $R \to M_tR$ is the diagonal homomorphism, $r \mapsto rI_t$.

For the class of division rings finite dimensional over their centres (which are fields), Green et. al. [8] proved that K-theory of a division algebra is essentially the same as K-theory of its centre, i.e., for a division algebra D over its centre F of index n,

$$K_i(D) \otimes \mathbb{Z}[1/n] \cong K_i(F) \otimes \mathbb{Z}[1/n]. \tag{1.1}$$

Their proof combines the above observations with the Skolem-Noether theorem which guarantees that algebra homomorphisms in the setting of central simple algebras are inner, and then uses the main result of [7] which states that

$$\lim_{i\to\infty} M_{n^{2i}}F \cong \lim_{i\to\infty} M_{n^{2(i+1)}}D.$$

This note is a continuation of [10] where it was observed that one can naturally deduce (1.1) by using the fact that an F-central simple algebra A is a twisted form of a matrix algebra, i.e., there is a finite field extension L/F such that $A \otimes_F L = M_kL$. However, using this takes us out of the category of F-algebras to the category of field extensions of F.

Another equivalent defining property for a central simple algebra A is that the opposite algebra is the inverse element in the Brauer group, i.e., $A \otimes_F A^{\mathrm{op}} \cong M_mF$, where m is the dimension of A over F as a vector space. In this note we will use this to prove (1.1) type properties (see also Remark 2). This enables us to extend our category from that of central simple algebras to the category of Azumaya algebras free over their centres. In particular we will prove that K-theory of an Azumaya algebra over a semi-local ring is essentially the

The first author acknowledges the support of EPSRC first grant scheme EP/D03695X/1. Part of this work has been done in the Winter of 2007 at KIAS, Seoul, Korea and the Summer of 2008 at ICTP, Trieste, Italy. The authors would like to thank Raymond Hoobler for several discussions regarding the subject.

same as K-theory of its centre. Along the same lines, another corollary of this approach is to conclude that K-theory of graded central simple algebras indexed by a totally ordered abelian group has (1.1) type property. However, it remains as a question whether the same can be said about any Azumaya algebra of constant rank:

Question 1. *Let A be an Azumaya algebra over its centre R of constant rank n. Then is it true that for any $i \geq 0$, $K_i(A) \otimes \mathbb{Z}[1/n] \cong K_i(R) \otimes \mathbb{Z}[1/n]$?*

Remark 2. The term Azumaya algebra originates from the work done by Azumaya in his 1951 paper [2]. The definition has developed since then, and an Azumaya algebra is now defined to be an R-algebra A such that A is faithfully projective as an R-module and the natural homomorphism $A \otimes_R A^{\mathrm{op}} \rightarrow \mathrm{End}_R(A)$ is an isomorphism (see [16], Chapter III, §5.1). An equivalent condition is that (i) A is a finitely generated R-module and (ii) for every maximal ideal m of R, $A/\mathrm{m}A$ is a central simple algebra over R/m (see [16], Chapter III, Theorem 5.1.1). This equivalence has been proven by Azumaya ([2], Theorem 15, page 130) under the additional assumption that A is free over R, and by Bass ([3], Chapter III, Theorem 4.1) under the assumption that A is projective over R, thanks to the work of Auslander and Goldman [1]. So for an Azumaya algebra A as originally defined by Azumaya with A free over its centre R, our Theorem 6 covers its K-theory.

2. $\mathcal{D}$-functors

In this section we set up an abstract functor, called a $\mathcal{D}$-functor, which will handle several quotient groups coming from K-groups. Some of the ideas behind this definition have already been employed in [9] to define a functor in the category of central simple algebras (compare this definition with the one in [9] and [10]). One main distinction of this definition, besides being defined over the category of Azumaya algebras, is that the domain is restricted to a fixed base ring. The aim is to show that such functors have bounded torsion abelian groups as their range.

Let $\mathrm{Az}(R)$ be the category of Azumaya algebras over the commutative ring R, $C(R)$ the category of all commutative R-algebras and $\mathcal{A}b$ the category of abelian groups. Consider a functor $\mathcal{F} : \mathrm{Az}(R) \rightarrow \mathcal{F}unc(C(R), \mathcal{A}b)$. Thus for any Azumaya R-algebra A, $\mathcal{F}_A : C(R) \rightarrow \mathcal{A}b$ is a functor from $C(R)$ to the category of abelian groups. Consider the following property for $\mathcal{F}$:

(1) $\mathcal{F}_R$ is a trivial functor, i.e., $\mathcal{F}_R(S) = 1$ for any R-algebra S.

Consider also the following properties at the base point R:

(2) For any faithfully projective R-module P, there is a homomorphism

$$d : \mathcal{F}_{A \otimes \mathrm{End}_R(P)}(R) \rightarrow \mathcal{F}_A(R)$$

such that the composition

$$\mathcal{F}_A(R) \rightarrow \mathcal{F}_{A \otimes \mathrm{End}_R(P)}(R) \rightarrow \mathcal{F}_A(R)$$

is $\eta_{[P:R]}$ if P has a constant rank $[P : R]$. Here $\eta_k(x) = x^k$.

(3) If P has a constant rank over R then $\ker(d)$ is $[P : R]$-torsion.

We call a functor $\mathcal{F}$ with the above three properties a $\mathcal{D}$-functor with respect to the class of constant faithfully projective modules.

Remark 3. When evaluating $\mathcal{F}$ at the base point, i.e., $\mathcal{F}_A(R)$, one can drop R and simply write $\mathcal{F}(A)$. With this simplification, when P is a free module of rank n, from the condition (2) it follows that the composition of $\mathcal{F}(A) \to \mathcal{F}(M_n A) \to \mathcal{F}(A)$ is η_n.

Remark 4. The goal is to show that $\mathcal{F}_A(R)$ is a torsion abelian group. However, one would like to prove the same statement for any $\mathcal{F}_A(S)$ where S is an R-algebra. This could be done if one further assumes:

(4) For any $S \in C(R)$, $\mathcal{F}_A(S) = \mathcal{F}_{A\otimes S}(S)$.

Here $\mathcal{F}_{A\otimes S}$ is a functor from $C(S)$ to $\mathcal{A}b$. There is no need to introduce an extra notation, say, $\mathcal{F}_A^R$ and $\mathcal{F}_{A\otimes S}^S$ here. The condition (4) simply states that evaluating $\mathcal{F}_A$ at S is the same as evaluating the functor $\mathcal{F}_{A\otimes S}$ at the base point.

Theorem 5. *Let A be an Azumaya algebra over R of constant rank n. Then $\mathcal{F}_A(R)$ is n^2-torsion, where $\mathcal{F}$ is a $\mathcal{D}$-functor.*

Proof. Since we will be working with the functor $\mathcal{F}$ on the base point, for an R-Azumaya algebra A, we simply write $\mathcal{F}(A)$ instead of $\mathcal{F}_A(R)$ (see Remark 3 above). The two R-algebra homomorphisms, $i : A \to A \otimes_R A^{\mathrm{op}}$ and $r : A^{\mathrm{op}} \to \mathrm{End}_R(A^{\mathrm{op}})$ induce the homomorphisms $\mathcal{F}(A) \to \mathcal{F}(A \otimes_R A^{\mathrm{op}})$ and $\mathcal{F}(A^{\mathrm{op}}) \to \mathcal{F}(\mathrm{End}_R(A^{\mathrm{op}}))$. Since A is an Azumaya algebra over R, we have $A \otimes_R A^{\mathrm{op}} = \mathrm{End}_R(A)$ and A is faithfully projective of rank n (see [16], Chapter III, §5.1 and Theorem 5.1.1). By (2) in the definition of a $\mathcal{D}$-functor, we have a homomorphism $d : \mathcal{F}(\mathrm{End}_R(A)) \to \mathcal{F}(R)$. But $\mathcal{F}(R)$ is trivial by (1) and thus the kernel of d is $\mathcal{F}(\mathrm{End}_R(A))$ which is, by (3), n-torsion. Consider the following diagram

$$\begin{tikzcd}
\mathcal{F}(A) \arrow[d, "i"] \arrow[ddr, bend left, "\eta_n"] & \\
\mathcal{F}(\mathrm{End}_R(A)) \cong \mathcal{F}(A \otimes_R A^{\mathrm{op}}) \arrow[d, "r"] & \\
\mathcal{F}(A \otimes_R \mathrm{End}_R(A^{\mathrm{op}})) \arrow[r, "d"] & \mathcal{F}(A)
\end{tikzcd} \tag{2.1}$$

By (2), $d \circ r \circ i = \eta_n$. But from the commutativity of the diagram and the fact that $\mathcal{F}(\mathrm{End}_R(A))$ is n-torsion, it follows that $\mathcal{F}(A)$ is n^2-torsion. $\square$

For a ring A with centre R, consider the inclusion $R \to A$. This induces the map $K_i(R) \to K_i(A)$ for $i \geq 0$. Consider the exact sequence

$$1 \to \mathrm{ZK}_i(A) \to K_i(R) \to K_i(A) \to \mathrm{CK}_i(A) \to 1 \tag{2.2}$$

where ZK_i and CK_i are the kernel and cokernel of the map $K_i(R) \to K_i(A)$ respectively. It is not difficult to see that one can consider CK_i as the following functor:

$$\begin{aligned} \mathrm{CK}_i : \mathrm{Az}(R) &\to \mathcal{F}unc(C(R), \mathcal{A}b) \\ A &\mapsto \mathrm{CK}_i(A \otimes_R -). \end{aligned}$$

Thus for any morphism $S \to T$ in $C(R)$, we have $\mathrm{CK}_i(A \otimes_R S) \to \mathrm{CK}_i(A \otimes_R T)$. We will check that CK_i (and in a similar manner ZK_i) is a $\mathcal{D}$-functor. Condition (1) in the definition of a $\mathcal{D}$-functor is straightforward to check. To prove the second condition, since $A \otimes_R \mathrm{End}_R(P) \cong \mathrm{End}_A(P \otimes_R A)$ (see [5], Chapter II, §5.3, Proposition 7) and $P \otimes_R A$ is faithfully projective over A, it is enough to work with faithfully projective modules over A. Let P be a faithfully projective A-(bi-)module. Let $\mathcal{P}(A)$ and $\mathcal{P}(\mathrm{End}_A(P))$ denote the categories of finitely generated projective (left) modules over A and $\mathrm{End}_A(P)$ respectively.

Consider the functors

$$\begin{aligned} \mathcal{P}(A) &\xrightarrow{\phi} \mathcal{P}(\mathrm{End}_A(P)) \\ X &\longmapsto \mathrm{End}_A(P) \otimes_A X \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} \mathcal{P}(\mathrm{End}_A(P)) &\xrightarrow{\psi} \mathcal{P}(A) \\ Y &\longmapsto \mathrm{Hom}_A(P, A) \otimes_{\mathrm{End}_A(P)} Y \end{aligned} \tag{2.4}$$

By the Morita theory, ψ is a natural equivalence (see [16], Chapter I, Theorem 9.1.3), thus it gives rise to the isomorphism from $K_i(\mathrm{End}_A(P))$ to $K_i(A)$. Furthermore, $\psi\phi(X) = \mathrm{Hom}_A(P, X)$. Clearly if $P = A^k$, then $\psi\phi(X) = X^k$, i.e., k copies of X. Since K_i are functors which respect direct sums ([17], §2), this composition induces a multiplication by k in the level of K-groups. This implies that Condition (2) holds for K-functors *with respect to free modules of finite ranks.*

If we could prove that ZK_i and CK_i are $\mathcal{D}$-functors in the generality that we set up our definition of a $\mathcal{D}$-functor, i.e., for all faithfully projective modules of constant rank, then we would be able to answer Question 1 positively. However, at this stage, we can show that CK_i and ZK_i are in fact $\mathcal{D}$-functors with respect to free modules of finite ranks and, consequently, for Theorem 5 to follow, we need to have Azumaya algebras free over their centres. Examples of Azumaya algebras free over their centres, apart from division algebras, are Azumaya algebras over semi-local rings and graded central simple algebras. Thus restricting to free modules, (2.3) and (2.4) give rise to the homomorphisms ϕ and ψ on the level of K-groups and we have the following commutative diagram,

$$\begin{tikzcd}
K_i(R) \arrow[r] \arrow[d, "="] & K_i(A) \arrow[r] \arrow[d, "\phi"] & \mathrm{CK}_i(A) \arrow[r] \arrow[d] & 1 \\
K_i(R) \arrow[r] \arrow[d, "\eta_k"] & K_i(M_kA) \arrow[r] \arrow[d, "\psi", "\cong"'] & \mathrm{CK}_i(M_kA) \arrow[r] \arrow[d, "d"] & 1 \\
K_i(R) \arrow[r] & K_i(A) \arrow[r] & \mathrm{CK}_i(A) \arrow[r] & 1
\end{tikzcd} \tag{2.5}$$

where compositions of columns are η_k, and thus Condition (2) holds. Now if $x \in \mathrm{CK}_i(M_kA)$ such that $d(x) = 1$, then a diagram chase shows that $x^k = 1$, satisfying Condition (3) of a $\mathcal{D}$-functor. In the same manner one can show that ZK_i is a $\mathcal{D}$-functor with respect to free modules of finite ranks.

We are now in a position to prove that the K-theory of Azumaya algebras over semi-local rings and K-theory of graded central simple algebras are isomorphic to K-theory of their centres up to (their ranks) torsions.

Theorem 6. *Let A be an Azumaya algebra free over its centre R of rank n. Then for any $i \geq 0$,*

$$K_i(A) \otimes \mathbb{Z}[1/n] \cong K_i(R) \otimes \mathbb{Z}[1/n].$$

Proof. The argument before the theorem shows that CK_i (and in the same manner ZK_i) is a $\mathcal{D}$-functor with respect to free modules of finite ranks, and thus by Theorem 5, $\mathrm{CK}_i(A)$ and $\mathrm{ZK}_i(A)$ are n^2-torsion abelian groups. Tensoring the exact sequence (2.2) by $\mathbb{Z}[1/n]$, since $\mathrm{CK}_i(A) \otimes \mathbb{Z}[1/n]$ and $\mathrm{ZK}_i(A) \otimes \mathbb{Z}[1/n]$ vanish, the result follows. $\square$

Recall that K-theory with coefficients is related to absolute K-theory by the following exact sequence (see [20])

$$1 \longrightarrow K_i(A) \otimes \mathbb{Z}[1/m] \longrightarrow K_i(A, \mathbb{Z}/m) \longrightarrow {}_mK_{i-1}(A) \longrightarrow 1 \tag{2.6}$$

for any ring A and $m \geq 0$. The following corollary is now immediate.

Corollary 7. *Let A be an Azumaya algebra free over its centre R of rank n. Then for any m relatively prime to n, and for any $i \geq 0$,*

$$K_i(A, \mathbb{Z}/m) \cong K_i(R, \mathbb{Z}/m).$$

Proof. Consider the diagram

$$\begin{tikzcd}
1 \arrow[r] & K_i(R) \otimes \mathbb{Z}[1/m] \arrow[r] \arrow[d] & K_i(R, \mathbb{Z}/m) \arrow[r] \arrow[d] & {}_mK_{i-1}(R) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & K_i(A) \otimes \mathbb{Z}[1/m] \arrow[r] & K_i(A, \mathbb{Z}/m) \arrow[r] & {}_mK_{i-1}(A) \arrow[r] & 1
\end{tikzcd} \tag{2.7}$$

Since $\mathrm{ZK}_i(A)$ and $\mathrm{CK}_i(A)$ are n^2-torsion and n and m are relatively prime, it is an easy exercise to show that the outer vertical maps are isomorphisms. Thus the middle vertical map is an isomorphism. $\square$

Corollary 8. *Let A be an Azumaya algebra over its centre R, a semi-local ring, of rank n. Then for any $i \geq 0$, $K_i(A) \otimes \mathbb{Z}[1/n] \cong K_i(R) \otimes \mathbb{Z}[1/n]$.*

Proof. Since A is finitely generated projective of constant rank and R is a semi-local ring, it follows that A is a free module over R (see [6], Chapter II, §5.3, Proposition 5), and thus the corollary follows from Theorem 6. $\square$

Let us note that, for an F-central division algebra D, and $i = 1$, $\mathrm{CK}_1(D)$ is the group D^*/F^*D' where D^* is the multiplicative group of D and D' is its commutator subgroup. This group has a direct application in solving the open problem whether a multiplicative group of a division algebra has a maximal subgroup [12]. Indeed, since $\mathrm{CK}_1(D)$ is torsion of bounded exponent (by Theorem 5), if it is not trivial, it has maximal subgroups and therefore D^* has (normal) maximal subgroups. Thus finding the maximal subgroups in D^* reduces to the case that $\mathrm{CK}_1(D)$ is trivial. It is a conjecture that $\mathrm{CK}_1(D)$ is trivial if and only if D is an ordinary quaternion division algebra over a Pythagorean field. It was shown in [12] that such division algebras do have (non-normal) maximal subgroups. Thus if the above conjecture is settled positively, one concludes that the multiplicative group of a division algebra does have a maximal subgroup.

The theory of graded division algebras, indexed by a totally ordered abelian group, has recently been given considerable attention as the theory is closely related to the theory of division algebras equipped with a valuation. Let D be a division algebra with a valuation. To this one associates a graded division algebra $gr D = \bigoplus_{\gamma\in\Gamma_D} gD_\gamma$ where Γ_D is the value group of D and gD_γ comes from the principal fractional ideals of the valuation ring (see §4 in [15] for details). As it is mentioned in [15], even though computations in the graded setting are easier (and discrete), it seems not so much is lost in passage from D to its corresponding graded division algebra $gr D$. This was a motivation to the systematic study of graded central simple algebras and their correspondences, notably by Boulagouaz [4], Hwang, Tignol and Wadsworth [13, 14, 15, 19] and to the comparison of certain functors defined on these objects, notably the Brauer group and the reduced Whitehead group.

Recall that a unital ring $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ is called a *graded ring* if Γ is a group, each R_γ is a subgroup of $(R, +)$ and $R_\gamma \cdot R_\delta \subseteq R_{\gamma+\delta}$ for all $\gamma, \delta \in \Gamma$. Throughout this paper, we will assume that Γ is a totally ordered abelian group (thus it is a torsion free abelian group). The elements of R_γ are called *homogeneous of degree* γ. A graded ring $E = \bigoplus_{\gamma\in\Gamma_E} E_\gamma$ is called a *graded division ring* if every non-zero homogeneous element has a multiplicative inverse. Note that Γ_E, the support of E, is a group and that E_0 is a division ring. Since Γ_E is totally ordered, it follows that E has no zero divisors and that E^*, the multiplicative group of E, is the set $E^h \setminus \{0\}$. Note that every graded E-module M is a *graded free* E-module, i.e., M is a graded E-module which is free as an E-module with a homogeneous basis. A *graded field* is a commutative graded division ring. A graded algebra A over a graded field R is said to be a *graded central simple algebra* over R if A is a simple graded ring (i.e., no non-trivial graded two sided ideals), finite dimensional over R with centre R.

Corollary 9. *Let A be a graded central simple algebra over its graded centre R of degree n. Then for any $i \geq 0$,*
$$K_i(A) \otimes \mathbb{Z}[1/n] \cong K_i(R) \otimes \mathbb{Z}[1/n].$$

Proof. By [4], Proposition 5.1 (see also [15], Corollary 1.2), a graded central simple algebra A over R is an Azumaya algebra. Since R is a graded field (indexed by a torsion free group), A is a free R-module. The corollary now follows from Theorem 6. □

The above result seems also to follow (in a hard way) by using Corollary 8, in combination with the fundamental theorem of K-theory, i.e., Theorem 8, its corollary and the exercise on p. 38 of Quillen's paper [17].

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