VALUATIONS ON ALGEBRAS WITH INVOLUTION

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Valuations are a major tool for the study of the structure of division algebras. The purpose of this work is to introduce a notion that plays a similar role for central simple algebras with involution, and to prove analogues for this notion to fundamental results on valuations on division algebras.

Since the definition of a (Schilling) valuation implies an absence of zero divisors, the only central simple algebras that can have valuations are division algebras. Given a division algebra D finite-dimensional over its center F, it is natural to view valuations on a D as extensions of valuations on F, since valuations on fields are abundant and their theory is well-developed. But not every valuation v on F extends to D. In the extension question, Henselian valuations play a special role. Schilling proved in [Sch, pp. 53–54] that if v on F is Henselian, then v has an extension to a valuation on D, and this extension is unique. Much later it was proved by Ershov [Er_1] and Wadsworth [W] that for any valuation v on F, v extends to D if and only if it satisfies a Henselian-like condition with respect to the field extensions of F within D; they also proved that when v extends to D the extension is unique. Another fundamental criterion was proved by Morandi [M]: v on F extends to a valuation on D if and only if D remains a division algebra after scalar extension to the Henselization F_h of F for v. We will prove analogues for central simple algebras with involutions to these theorems of Schilling, Ershov-Wadsworth, and Morandi.

An involution on a central simple algebra A is a ring-anti-automorphism σ such that $\sigma^2 = id_A$. As Weil suggested in [We], the theory of central simple algebras with involution is a natural sibling to the theory of central simple algebras, since the associated automorphism groups are the basic types of classical groups. In each setting there is a notion of anisotropic object, corresponding to when the associated automorphism group is anisotropic as an algebraic group. The anisotropic central simple algebras are the division algebras. An involution σ on a central simple algebra A is anisotropic just when the equation $\sigma(x)x = 0$ holds only for $x = 0$. In earlier work [TW] we have developed the theory of gauges, which are a kind of value functions for central simple algebras. (The definition of a gauge is recalled at the end of this introduction.) For a central simple algebra A with involution σ, we define a *σ-special gauge* to be a gauge φ on A satisfying the condition[1] that $\varphi(\sigma(x)x) = 2\varphi(x)$ for all $x \in A$. A σ-special gauge for an algebra with involution is our analogue to a valuation on a division algebra. If A has a σ-special gauge, then σ is easily seen to be anisotropic. If v is a Henselian valuation on the σ-invariant part F of $Z(A)$ and σ is anisotropic, we show in Th. 2.2 that there is a unique σ-invariant gauge φ on A extending v, and φ is a σ-special gauge. When v on F is not Henselian, we show in Th. 6.1 that there is a σ-special gauge φ on A extending v if and only if the anisotropic involution σ remains anisotropic after scalar extension to the Henselization of F with respect to v; furthermore, there is only one such φ. Our results require tame ramification and exclude orthogonal involutions if the residue characteristic is 2; see the statements of Theorems 2.2 and 6.1 for the precise conditions required.

A gauge φ on a central simple algebra A induces a filtration on A which yields an associated graded ring $\mathsf{gr}(A)$, analogous to what one has with a valuation on a field or a division ring. The graded structure is intrinsic to the definition of a gauge, and is used heavily throughout this paper. The degree 0 part of $\mathsf{gr}(A)$, denoted A_0, is the residue ring of the "valuation ring" of A determined by the gauge φ;

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[1] Notice the similarity with the definition of C^*-algebras, cf. [Dix, Déf. 1.3.1].

A_0 is always a semisimple $Z(A)_0$-algebra, but not simple in general. If σ is an involution on A and φ is invariant under σ, then σ induces involutions $\widetilde{\sigma}$ on $\mathsf{gr}(A)$ and σ_0 on A_0. We show in Prop. 1.1 and Remark 2.5(1) that a σ-invariant gauge φ is σ-special if and only if $\widetilde{\sigma}$ is anisotropic, if and only if σ_0 is anisotropic. We also prove an analogue of a theorem of Springer: when the base field is Henselian, an involution σ is isotropic if and only if its residue involution σ_0 is isotropic (Cor. 2.3). This criterion is applied to show that under specified valuation-theoretic conditions, an anisotropic involution remains anisotropic after certain scalar extensions (Cor. 3.6).

An outline of this paper is as follows: In Section 1, we discuss in general terms the compatibility of a value function with an involution, relating that notion to a compatibility condition between norms and hermitian forms defined in [RTW]. In Section 2, we restrict to the case of Henselian valuations and give the proofs of Th. 2.2 and Cor. 2.3. Some applications to scalar extensions (in particular Cor. 3.6) are given in Section 3. Sections 4 and 5 prepare the ground for the extension of our results to the non-Henselian case in Section 6. The main problem is to analyze how the condition for the existence of a splitting base of a value function (which is a critical part of the definition of a gauge) behaves under restriction of scalars; this is done in Section 5. In Section 4, we investigate this condition for the composition of value functions. This is used in Section 6 in the proof of Th. 6.1 by induction on the rank of valuations.

For the convenience of the reader, we now review the basic notions of value functions, norms, and gauges introduced in [RTW] and [TW]. Throughout the paper, we fix a divisible totally ordered abelian group Γ, which will contain the values of all the valuations and the degrees of all the gradings we consider. Thus, a valued field (F, v) is a pair consisting of a field F and a valuation $v\colon F \to \Gamma \cup \{\infty\}$. The group $v(F^\times)$ of values of F is denoted by Γ_F, and the residue field by $\overline{F}$. We use analogous notation for valuations on division rings.

Let (F, v) be a valued field. A *v-value function* on an F-vector space V is a map $\alpha\colon V \to \Gamma \cup \{\infty\}$ such that

(i) $\alpha(x) = \infty$ if and only if $x = 0$;
(ii) $\alpha(x + y) \geq min\big(\alpha(x), \alpha(y)\big)$ for $x, y \in V$;
(iii) $\alpha(xc) = \alpha(x) + v(c)$ for all $x \in V$ and $c \in F$.

The v-value function α is called a *norm* if V is finite-dimensional and contains a base $(e_i)_{i=1}^n$ such that

$$\alpha\Big(\sum_{i=1}^n e_i c_i\Big) \;=\; \min_{1\leq i\leq n}\big(\alpha(e_i c_i)\big) \qquad \text{for } c_1, \ldots, c_n \in F.$$

Such a base is called a *splitting base* of V for α. A v-value function φ on an F-algebra A is *surmultiplicative* if $\varphi(1) = 0$ and $\varphi(xy) \geq \varphi(x) + \varphi(y)$ for $x, y \in A$.

The valuation v defines a filtration on F: for $\gamma \in \Gamma$ we set

$$F^{\geq\gamma} \;=\; \{x \in F \mid v(x) \geq \gamma\}, \quad F^{>\gamma} \;=\; \{x \in F \mid v(x) > \gamma\}, \quad \text{and } F_\gamma \;=\; F^{\geq\gamma}/F^{>\gamma}.$$

The associated graded ring is

$$\mathsf{gr}(F) \;=\; \bigoplus_{\gamma\in\Gamma} F_\gamma.$$

It is called a *graded field* because every nonzero homogeneous element in $\mathsf{gr}(F)$ is invertible. Likewise, every v-value function α on an F-vector space V defines a filtration, and the associated graded structure $\mathsf{gr}_\alpha(V)$ is a graded module over $\mathsf{gr}(F)$, which we call a *graded vector space*. It is a free module, whose rank is called its dimension. The value function is a norm if and only if $dim_{\mathsf{gr}(F)}(\mathsf{gr}_\alpha(V)) = dim_F(V) < \infty$, see [RTW, Cor. 2.3]. Every nonzero element $x \in V$ has an image $\widetilde{x}$ in $\mathsf{gr}_\alpha(V)$ defined by

$$\widetilde{x} \;=\; x + V^{>\alpha(x)} \;\in V_{\alpha(x)}.$$

We also set $\widetilde{0} = 0 \in \mathsf{gr}_\alpha(V)$. If φ is a surmultiplicative v-value function on an F-algebra A, then $\mathsf{gr}_\varphi(A)$ is an algebra over $\mathsf{gr}(F)$, in which multiplication is defined by

$$\widetilde{a}\widetilde{b} \;=\; ab + V^{>\varphi(a)+\varphi(b)} \;=\; \begin{cases} \widetilde{ab} & \text{if } \varphi(ab) = \varphi(a)+\varphi(b), \\ 0 & \text{if } \varphi(ab) > \varphi(a)+\varphi(b), \end{cases} \quad \text{for } a,\, b \in A.$$

Now, suppose A is a finite-dimensional simple F-algebra. We denote by $[A{:}F]$ its dimension and by $Z(A)$ its center. A surmultiplicative v-value function φ on A is called a *v-gauge* if it satisfies the following conditions:

(i) φ is a v-norm, i.e., $[A{:}F] = [\mathsf{gr}_\varphi(A){:}\,\mathsf{gr}(F)]$;
(ii) $\mathsf{gr}_\varphi(A)$ is a graded semisimple $\mathsf{gr}(F)$-algebra, i.e., it does not contain any nonzero nilpotent homogeneous two-sided ideal.

The v-gauge φ is said to be *tame* if $Z\big(\mathsf{gr}_\varphi(A)\big) = \mathsf{gr}_\varphi\big(Z(A)\big)$ and $Z\big(\mathsf{gr}_\varphi(A)\big)$ is separable over $\mathsf{gr}(F)$. If the residue characteristic is 0, then every v-gauge is tame, see [TW, Cor. 3.6].

1. Special gauges

Let (F, v) be a valued field and let A be an F-algebra. An F-linear involution on A is an F-linear map $\sigma\colon A \to A$ such that

(i) $\sigma(x+y) = \sigma(x)+\sigma(y)$ for $x,\, y \in A$;
(ii) $\sigma(xy) = \sigma(y)\sigma(x)$ for $x,\, y \in A$;
(iii) $\sigma^2(x) = x$ for $x \in A$.

(The F-linearity implies that $\sigma|_F = id_F$.) A surmultiplicative v-value function $\varphi\colon A \to \Gamma \cup \{\infty\}$ is said to be *invariant under* σ if

$$\varphi\big(\sigma(x)\big) \;=\; \varphi(x) \qquad \text{for all } x \in A. \tag{1.1}$$

The involution then preserves the filtration on A defined by φ. Therefore, it induces an involution $\widetilde{\sigma}$ on $\mathsf{gr}_\varphi(A)$ such that

$$\widetilde{\sigma}(\widetilde{x}) \;=\; \widetilde{\sigma(x)} \qquad \text{for all } x \in A.$$

As in [KMRT, §6.A], we say that the involution σ is *anisotropic* if there is no nonzero element $x \in A$ such that $\sigma(x)x = 0$. Likewise, $\widetilde{\sigma}$ is said to be anisotropic if there is no nonzero homogeneous element $\xi \in \mathsf{gr}_\varphi(A)$ such that $\widetilde{\sigma}(\xi)\xi = 0$. Clearly, if $\widetilde{\sigma}$ is anisotropic, then σ is anisotropic.

Proposition 1.1. *Let φ be a surmultiplicative v-value function and σ an F-linear involution on A. The following conditions are equivalent:*

(a) $\varphi(\sigma(x)x) = 2\varphi(x)$ *for all $x \in A$;*
(b) φ *is invariant under σ, and $\widetilde{\sigma}$ is anisotropic.*

They imply that if $x,\, y \in A$ satisfy $\sigma(x)y = 0$ or $x\sigma(y) = 0$, then

$$\varphi(x+y) \;=\; min\big(\varphi(x), \varphi(y)\big). \tag{1.2}$$

Moreover, when these equivalent conditions hold, σ is anisotropic and the $\mathsf{gr}(F)$-algebra $\mathsf{gr}_\varphi(A)$ contains no nonzero homogeneous nil left or right ideal.

Proof. (a) $\Rightarrow$ (b): If $\sigma(x)x = 0$, then condition (a) implies that $\varphi(x) = \infty$, so $x = 0$. Thus, σ is anisotropic. By surmultiplicativity, we have

$$\varphi(\sigma(x)x) \;\ge\; \varphi(\sigma(x)) + \varphi(x) \qquad \text{for all } x \in A.$$

Therefore, (a) implies $\varphi(x) \ge \varphi(\sigma(x))$ for all $x \in A$. Substituting $\sigma(x)$ for x in this inequality, we obtain $\varphi(\sigma(x)) \ge \varphi(x)$ for all $x \in A$. Therefore, φ is invariant under σ, and condition (a) can be reformulated as $\varphi(\sigma(x)x) = \varphi\big(\sigma(x)\big) + \varphi(x)$ for all $x \in A$. Thus, it implies

$$\widetilde{\sigma(x)}\widetilde{x} \;=\; (\sigma(x)x)^{\sim} \qquad \text{for all } x \in A,$$

whence $\widetilde{\sigma}$ is anisotropic, as σ is anisotropic.

(b) $\Rightarrow$ (a): For all $x \in A$ we have

$$\widetilde{\sigma}(\widetilde{x})\widetilde{x} \;=\; \begin{cases} (\sigma(x)x)^{\sim} & \text{if } \varphi(\sigma(x)x) = \varphi\big(\sigma(x)\big) + \varphi(x), \\ 0 & \text{if } \varphi(\sigma(x)x) > \varphi\big(\sigma(x)\big) + \varphi(x). \end{cases}$$

Condition (b) implies that the first case always occurs. Hence, for all x, $\varphi(\sigma(x)x) = \varphi(\sigma(x)) + \varphi(x) = 2\varphi(x)$.

For the rest of the proof, assume (a) and (b) hold. Then clearly σ is anisotropic. Also, for x, $y \in A$ we have by surmultiplicativity

$$\varphi\big(\sigma(x)\cdot(x+y)\big) \;\geq\; \varphi\big(\sigma(x)\big) + \varphi(x+y) \;=\; \varphi(x) + \varphi(x+y). \tag{1.3}$$

If $\sigma(x)y = 0$, then

$$\varphi\big(\sigma(x)\cdot(x+y)\big) \;=\; \varphi(\sigma(x)x) \;=\; 2\varphi(x). \tag{1.4}$$

By combining (1.3) and (1.4), we obtain $\varphi(x) \geq \varphi(x+y)$. Similarly, interchanging x and y we get $\varphi(y) \geq \varphi(x+y)$, hence

$$min\big(\varphi(x), \varphi(y)\big) \;\geq\; \varphi(x+y).$$

The reverse inequality holds by definition of a value function, hence (1.2) is proved when $\sigma(x)y = 0$. If $x\sigma(y) = 0$, we substitute $\sigma(x)$ for x and $\sigma(y)$ for y in the arguments above, obtaining

$$\varphi\big(\sigma(x) + \sigma(y)\big) \;=\; min\big(\varphi(\sigma(x)), \varphi(\sigma(y))\big).$$

Equation (1.2) follows since $\varphi \circ \sigma = \varphi$.

To complete the proof, suppose $\mathsf{I} \subset \mathsf{gr}_\varphi(A)$ is a homogeneous nil left (resp. right) ideal and $\xi \in \mathsf{I}$ is a nonzero homogeneous element. Let $\eta = \widetilde{\sigma}(\xi)\xi$ (resp. $\eta = \xi\widetilde{\sigma}(\xi)$). Then $\eta \in \mathsf{I}$ is $\widetilde{\sigma}$-symmetric, homogeneous, and nonzero since $\widetilde{\sigma}$ is anisotropic. Since I is nil, we may find $k \geq 1$ such that $\eta^k \neq 0$ and $\eta^{k+1} = 0$. For $\zeta = \eta^k$ we have

$$\widetilde{\sigma}(\zeta)\zeta \;=\; \zeta^2 \;=\; \eta^{2k} \;=\; 0,$$

so $\zeta = 0$, a contradiction. $\square$

Definition 1.2. A surmultiplicative v-value function φ on a central simple algebra A with involution σ is called *σ-special* if it satisfies the conditions (a) and (b) of Prop. 1.1.

For use in §§3 and 6, we record how involution invariance of value functions behaves with respect to tensor products. Recall from [TW, Prop. 1.23, (1.25)] that if V is a finite-dimensional F-vector space with a v-norm α and W is an F-vector space with v-value function β, then there is a v-value function $\alpha \otimes \beta$ on $V \otimes_F W$ uniquely determined by the condition that the map $(x \otimes y)^{\sim} \mapsto \widetilde{x} \otimes \widetilde{y}$ (for $x \in V$ and $y \in W$) defines an isomorphism of graded vector spaces

$$\Omega\colon\; \mathsf{gr}_{\alpha\otimes\beta}(V \otimes_F W) \xrightarrow{\sim} \mathsf{gr}_\alpha(V) \otimes_{\mathsf{gr}(F)} \mathsf{gr}_\beta(W). \tag{1.5}$$

In particular,

$$(\alpha \otimes \beta)(x \otimes y) = \alpha(x) + \beta(y) \quad \text{for all } x \in V \text{ and } y \in W. \tag{1.6}$$

The value function $\alpha \otimes \beta$ can be defined as follows: take any splitting base $(e_i)_{i=1}^n$ for α on V; then,

$$(\alpha \otimes \beta)\Big(\sum_{i=1}^n e_i \otimes y_i\Big) \;=\; \min_{1\leq i\leq n}\big(\alpha(e_i) + \beta(y_i)\big) \quad \text{for any } y_1, \ldots, y_n \in W.$$

Furthermore, analogous to [TW, Cor. 1.26], if (W, β) is a valued field extending (F, v), then $\alpha \otimes \beta$ is a β-norm on $V \otimes_F W$ and (1.5) is a $\mathsf{gr}_\beta(W)$-vector space isomorphism.

Proposition 1.3. *Let σ and τ be F-linear involutions on F-algebras A and B respectively, and let φ (resp. ψ) be a surmultiplicative v-value function on A (resp. B) invariant under σ (resp. τ). Suppose A is finite-dimensional and φ is a v-norm. Then, $\varphi \otimes \psi$ is a surmultiplicative v-value function on $A \otimes_F B$ invariant under the involution $\sigma \otimes \tau$, and the canonical isomorphism Ω of (1.5) is an isomorphism of graded $\mathsf{gr}(F)$-algebras with involution,*

$$\big(\mathsf{gr}_{\varphi\otimes\psi}(A \otimes_F B), \widetilde{\sigma \otimes \tau}\big) \xrightarrow{\sim} \big(\mathsf{gr}_\varphi(A) \otimes_{\mathsf{gr}(F)} \mathsf{gr}_\psi(B), \widetilde{\sigma} \otimes \widetilde{\tau}\big).$$

Proof. Let $(e_i)_{i=1}^n$ be a splitting base of A for φ. For $x, y \in A \otimes_F B$ we may write

$$x = \sum_{i=1}^n e_i \otimes x_i \qquad \text{and} \qquad y = \sum_{j=1}^n e_j \otimes y_j \qquad \text{for some } x_1, \ldots, y_n \in B.$$

Then,

$$(\varphi \otimes \psi)(xy) = (\varphi \otimes \psi)\big(\sum_{i,j} e_i e_j \otimes x_i x_j\big) \geq \min_{1\leq i,j\leq n}\big((\varphi \otimes \psi)(e_i e_j \otimes x_i y_j)\big) = \min_{i,j}\big(\varphi(e_i e_j) + \psi(x_i y_j)\big).$$

Since φ and ψ are surmultiplicative, we have $\varphi(e_i e_j) \geq \varphi(e_i) + \varphi(e_j)$ and $\psi(x_i y_j) \geq \psi(x_i) + \psi(y_j)$, hence

$$(\varphi \otimes \psi)(xy) \geq \min_{i,j}\big(\varphi(e_i) + \varphi(e_j) + \psi(x_i) + \psi(y_j)\big) = \min_i\big(\varphi(e_i) + \psi(x_i)\big) + \min_j\big(\varphi(e_j) + \psi(y_j)\big).$$

The right side is $(\varphi \otimes \psi)(x) + (\varphi \otimes \psi)(y)$, so

$$(\varphi \otimes \psi)(xy) \geq (\varphi \otimes \psi)(x) + (\varphi \otimes \psi)(y).$$

Since moreover $(\varphi \otimes \psi)(1 \otimes 1) = \varphi(1) + \psi(1) = 0$, surmultiplicativity of $\varphi \otimes \psi$ is proved.

To show that the gr_F-vector space isomorphism Ω is a ring isomorphism, we check this for Ω^{-1}. The canonical F-algebra homomorphisms $\iota_A \colon A \to A \otimes_F B$, $a \mapsto a \otimes 1$ and $\iota_B \colon B \to A \otimes_F B$, $b \mapsto 1 \otimes b$, are value-preserving. Hence, they induce $\mathsf{gr}(F)$-algebra homomorphisms $\widetilde{\iota}_A \colon \mathsf{gr}_\alpha(A) \to \mathsf{gr}_{\alpha\otimes\beta}(A \otimes_F B)$, $\widetilde{a} \mapsto \widetilde{a \otimes 1}$, and $\widetilde{\iota}_B \colon \mathsf{gr}_\beta(B) \to \mathsf{gr}_{\alpha\otimes\beta}(A \otimes_F B)$, $\widetilde{b} \mapsto \widetilde{1 \otimes b}$. For any $a \in A$ and $b \in B$, we have from (1.6),

$$\widetilde{a \otimes 1} \cdot \widetilde{1 \otimes b} = [(a \otimes 1) \cdot (1 \otimes b)]^\sim = \widetilde{a \otimes b} = [(1 \otimes b) \cdot (a \otimes 1)]^\sim = \widetilde{1 \otimes b} \cdot \widetilde{a \otimes 1}.$$

Thus, $im(\widetilde{\iota}_B)$ centralizes $im(\widetilde{\iota}_A)$ in $\mathsf{gr}_{\alpha\otimes\beta}(A \otimes_F B)$. So, there is an induced $\mathsf{gr}(F)$-algebra homomorphism $\mathsf{gr}_\alpha(A) \otimes_{\mathsf{gr}(F)} \mathsf{gr}_\beta(B) \to \mathsf{gr}_{\alpha\otimes\beta}(A \otimes_F B)$ given by $\widetilde{a} \otimes \widetilde{b} \mapsto \widetilde{a \otimes 1} \cdot \widetilde{1 \otimes b} = \widetilde{a \otimes b}$. The description of Ω preceding (1.5) shows that this algebra homomorphism is Ω^{-1}.

To prove $\varphi \otimes \psi$ is invariant under $\sigma \otimes \tau$, we first show that $(\sigma(e_i))_{i=1}^n$ also is a splitting base of A for φ. Take any $c_1, \ldots, c_n \in F$. Then, as φ is invariant under σ and the c_i are central in A and fixed under σ,

$$\varphi\big(\sum_{i=1}^n \sigma(e_i)c_i\big) = \varphi\big(\sigma\big(\sum_{i=1}^n \sigma(e_i)c_i\big)\big) = \varphi\big(\sum_{i=1}^n e_i c_i\big) = \min_{1\leq i\leq n}\big(\varphi(e_i) + v(c_i)\big) = \min_{1\leq i\leq n}\big(\varphi(\sigma(e_i)) + v(c_i)\big).$$

Thus, $(\sigma(e_i))_{i=1}^n$ is a splitting base for φ. With the notation above, we then have

$$(\varphi \otimes \psi)\big((\sigma \otimes \tau)(x)\big) = (\varphi \otimes \psi)\big(\sum_{i=1}^n \sigma(e_i) \otimes \tau(x_i)\big) = \min_{1\leq i\leq n}\big(\varphi(\sigma(e_i)) + \psi(\tau(x_i))\big).$$

Since φ is invariant under σ and ψ under τ, we have

$$\min_{1\leq i\leq n}\big(\varphi(\sigma(e_i)) + \psi(\tau(x_i))\big) = \min_{1\leq i\leq n}\big(\varphi(e_i) + \psi(x_i)\big) = (\varphi \otimes \psi)(x).$$

Therefore, $\varphi \otimes \psi$ is invariant under $\sigma \otimes \tau$. To complete the proof, observe that for $a \in A$ and $b \in B$ we have

$$\widetilde{\sigma \otimes \tau}(\widetilde{a \otimes b}) = (\sigma(a) \otimes \tau(b))^\sim = (\widetilde{\sigma} \otimes \widetilde{\tau})(\widetilde{a} \otimes \widetilde{b}),$$

hence the involution $\widetilde{\sigma} \otimes \widetilde{\tau}$ corresponds to $\widetilde{\sigma \otimes \tau}$ under the canonical isomorphism (1.5). □

The following special case will be particularly useful:

Corollary 1.4. *Let A be a finite-dimensional F-algebra with an F-linear involution σ and let (K, v_K) be any valued field extension of (F, v). If φ is a surmultiplicative v-norm on A which is invariant under σ, then $\varphi \otimes v_K$ is a surmultiplicative v_K-norm on $A \otimes_F K$ invariant under the involution $\sigma \otimes \mathrm{id}_K$, and $(\sigma \otimes \mathrm{id}_K)^\sim \cong \widetilde{\sigma} \otimes \mathrm{id}_{\mathsf{gr}(K)}$ under the canonical isomorphism* (1.5).

Proof. It suffices to note that $\widetilde{id_K} = id_{\mathsf{gr}(K)}$ and that the canonical isomorphism is an isomorphism of $\mathsf{gr}(K)$-algebras, see [TW, Cor. 1.26]. □

Now, assume A is simple and finite-dimensional, and let $n = deg\, A$, so $[A{:}Z(A)] = n^2$. Recall from [KMRT] that involutions on A are classified into two kinds and three types: an involution σ is of the *first kind* if $\sigma|_{Z(A)} = id_{Z(A)}$; otherwise it is of the *second kind*. Involutions of the second kind are also said to be *of unitary type* (or simply *unitary*). To define the type of an involution σ of the first kind we consider the subspaces of symmetric and of symmetrized elements in A, defined by

$$Sym(A,\sigma) \,=\, \{x \in A \mid \sigma(x) = x\} \qquad \text{and} \qquad Symd(A,\sigma) \,=\, \{x + \sigma(x) \mid x \in A\}.$$

The involution σ is *of symplectic type* (or simply *symplectic*) if either $char(F) \neq 2$ and $dim_{Z(A)} Sym(A,\sigma) = \frac{1}{2}n(n-1)$ or $char(F) = 2$ and $1 \in Symd(A,\sigma)$. Involutions of the first kind that are not symplectic are said to be *of orthogonal type* (or simply *orthogonal*). If σ is orthogonal, then $dim_{Z(A)} Sym(A,\sigma) = \frac{1}{2}n(n+1)$. The same terminology is used for involutions on graded simple algebras.

Proposition 1.5. *Let σ be an F-linear involution on a finite-dimensional simple F-algebra A and let g be a tame v-gauge on A that is invariant under σ. Suppose F is the subfield of $Z(A)$ fixed under σ.*

If σ is unitary, two cases may arise:

- *if the valuation v extends uniquely from F to $Z(A)$, then $\mathsf{gr}_g(A)$ is a graded simple $\mathsf{gr}(F)$-algebra and $\widetilde{\sigma}$ is a unitary involution;*
- *if the valuation v has two different extensions to $Z(A)$, then $\mathsf{gr}_g(A)$ is a direct product of two graded central simple $\mathsf{gr}(F)$-algebras, which are exchanged under $\widetilde{\sigma}$.*

If σ is symplectic, then $\widetilde{\sigma}$ is a symplectic involution on the graded central simple $\mathsf{gr}(F)$-algebra $\mathsf{gr}_g(A)$.

If σ is orthogonal and $\mathrm{char}(\overline{F}) \neq 2$, then $\widetilde{\sigma}$ is an orthogonal involution on the graded central simple $\mathsf{gr}(F)$-algebra $\mathsf{gr}_g(A)$.

Proof. Suppose first that σ is unitary, so $Z(A)/F$ is a quadratic extension. By [TW, Cor. 2.5], the number of simple components of $\mathsf{gr}(A)$ equals the number of extensions of v to $Z(A)$. Therefore, to complete the description of $\widetilde{\sigma}$ it suffices to show that $\widetilde{\sigma}$ does not identically fix $Z\big(\mathsf{gr}(A)\big) = \mathsf{gr}\big(Z(A)\big)$. Since the Galois group $\mathcal{G}(Z(A)/F)$ acts transitively on the set of extensions of v to $Z(A)$, see [EP, Th. 3.2.15, p. 64], if there are two such extensions, then $\sigma|_{Z(A)}$ must permute them; then $\widetilde{\sigma}$ permutes the corresponding components of $\mathsf{gr}(Z(A))$. So, we may assume that v has a unique extension to $Z(A)$. Then, $\mathsf{gr}\,\big(Z(A)\big)$ is a graded field separable over $\mathsf{gr}(F)$, and $[\mathsf{gr}\,\big(Z(A)\big){:}\,\mathsf{gr}(F)] = [Z(A){:}F] = 2$ since $g|_{Z(A)}$ is a norm, by [RTW, Prop. 2.5]. If $char(\overline{F}) \neq 2$ we can find $z \in Z(A)$ nonzero such that $\sigma(z) = -z$, hence $\widetilde{\sigma}(\widetilde{z}) = -\widetilde{z} \neq \widetilde{z}$. If $char(\overline{F}) = 2$ the separability of $\mathsf{gr}\big(Z(A)\big)$ over $\mathsf{gr}(F)$ implies by [HW$_1$, Th. 3.11, Def. 3.4] that $\Gamma_{Z(A)} = \Gamma_F$ and $Z(A)_0$ is separable over F_0; furthermore, $[Z(A)_0{:}F_0] = [Z(A){:}F] = 2$ since $g|_{Z(A)}$ is a norm, by [RTW, Prop. 2.5]. So $Z(A)$ is unramified Galois over F, hence the non-trivial automorphism $\sigma|_{Z(A)}$ induces a non-trivial automorphism of the residue algebra $Z(A)_0$, by [E, Th. 19.6, p. 124], showing that $\widetilde{\sigma}|_{\mathsf{gr}(Z(A))}$ is nontrivial.

Suppose next that σ is of the first kind, so $Z(A) = F$. For $x \in A$ we have $\widetilde{x} + \widetilde{\sigma}(\widetilde{x}) = \big(x + \sigma(x)\big)^{\sim}$ or 0. On the other hand, $\sigma(x) = x$ implies $\widetilde{\sigma}(\widetilde{x}) = \widetilde{x}$. Therefore, the following inclusions are clear:

$$\mathsf{gr}\big(Sym(A,\sigma)\big) \,\subseteq\, Sym(\mathsf{gr}(A),\widetilde{\sigma}), \qquad Symd(\mathsf{gr}(A),\widetilde{\sigma}) \,\subseteq\, \mathsf{gr}\big(Symd(A,\sigma)\big). \tag{1.7}$$

If $char(\overline{F}) \neq 2$ (hence $char(F) \neq 2$) we have

$$Sym(A,\sigma) \,=\, Symd(A,\sigma) \qquad \text{and} \qquad Sym(\mathsf{gr}(A),\widetilde{\sigma}) \,=\, Symd(\mathsf{gr}(A),\widetilde{\sigma}),$$

so the inclusions in (1.7) above yield $\mathsf{gr}\big(Sym(A,\sigma)\big) = Sym(\mathsf{gr}(A),\widetilde{\sigma})$. Since the type of an involution can be determined from the dimension of the space of symmetric elements, it follows that $\widetilde{\sigma}$ has the same type as σ.

To complete the proof, suppose $char(\overline{F}) = 2$ and σ is symplectic, and let $n = deg\, A$. Since $\widetilde{\sigma}$ is of the first kind we have

$$dim_{\mathsf{gr}(F)} Symd(\mathsf{gr}(A),\widetilde{\sigma}) \,=\, \tfrac{1}{2}n(n-1).$$

On the other hand, since σ is symplectic we have

$$\mathit{dim}_F\, \mathit{Symd}(A,\sigma) \;=\; \tfrac{1}{2}n(n-1)$$

(independently of whether $\mathit{char}(F)=2$). Since g is a norm we have

$$\mathit{dim}_F\, \mathit{Symd}(A,\sigma) \;=\; \mathit{dim}_{\mathsf{gr}(F)}\, \mathsf{gr}\big(\mathit{Symd}(A,\sigma)\big),$$

hence $\mathit{Symd}(\mathsf{gr}(A),\widetilde{\sigma}) = \mathsf{gr}\big(\mathit{Symd}(A,\sigma)\big)$. Since $1\in \mathit{Symd}(A,\sigma)$, it follows that $\widetilde{1}\in \mathit{Symd}(\mathsf{gr}(A),\widetilde{\sigma})$, hence $\widetilde{\sigma}$ is symplectic. □

Remark. If σ is orthogonal and $\mathit{char}(\overline{F})=2$, the involution $\widetilde{\sigma}$ may be symplectic, as the following example shows: let (F,v) be a valued field with $\mathit{char}(F)=0$ and $\mathit{char}(\overline{F})=2$, and let $A=M_2(F)$. Define an orthogonal involution σ on A by

$$\sigma\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \;=\; \left(\begin{smallmatrix} d & b\\ c & a\end{smallmatrix}\right)$$

and a v-gauge g by

$$g\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \;=\; \mathit{min}\big(v(a),v(b),v(c),v(d)\big).$$

This gauge is clearly invariant under σ. We have $\mathsf{gr}_g(A) = M_2\big(\mathsf{gr}(F)\big)$ with the entrywise grading, and

$$\left(\begin{smallmatrix} 1 & 0\\ 0 & 1\end{smallmatrix}\right) \;=\; \left(\begin{smallmatrix} 1 & 0\\ 0 & 0\end{smallmatrix}\right) + \widetilde{\sigma}\left(\begin{smallmatrix} 1 & 0\\ 0 & 0\end{smallmatrix}\right) \in \mathit{Symd}(\mathsf{gr}_g(A),\widetilde{\sigma}).$$

Therefore, $\widetilde{\sigma}$ is symplectic.

Henceforth, we systematically avoid orthogonal involutions in characteristic 2.

In [RTW, §3], a notion of compatibility is defined between norms and hermitian forms. In the rest of this section, we relate that notion of compatibility with the invariance of value functions under involutions.

Let D be a finite-dimensional division F-algebra with an F-linear involution τ. Suppose v extends to a valuation w on D invariant under τ and let V be a finite-dimensional right D-vector space. Consider a nondegenerate hermitian form $h\colon V\times V\to D$ with respect to τ, and a w-norm α on V. The dual norm $\alpha^\sharp$ is defined by

$$\alpha^\sharp(x) \;=\; \mathit{min}\{\, w\big(h(x,y)\big) - \alpha(y) \mid y\in V,\ y\neq 0\} \qquad \text{for } x\in V, \tag{1.8}$$

see [RTW, §3]. The norm α is said to be *compatible with* h if and only if $\alpha^\sharp=\alpha$ (see [RTW, Prop. 3.5]). This is the condition needed in order for h to induce a nondegenerate graded hermitian form on $\mathsf{gr}_\alpha(V)$. On the simple algebra $\mathit{End}_D(V)$ there is the involution ad_h adjoint to h, defined by

$$h(\mathrm{ad}_h(f)(x),y) \;=\; h(x,f(y)) \qquad \text{for all } x,y\in V.$$

There is also the well-defined surmultiplicative v-value function $\mathit{End}(\alpha)$ on $\mathit{End}_D(V)$ defined by

$$\mathit{End}(\alpha)(f) \;=\; \mathit{min}\{\alpha\big(f(x)\big)-\alpha(x) \mid x\in V,\ x\neq 0\}.$$

Recall that $\mathit{End}(\alpha)$ is a v-gauge if and only if w on D is defectless over v, see [TW, Prop. 1.19].

Proposition 1.6. *The value functions $\mathit{End}(\alpha)$ and $\mathit{End}(\alpha^\sharp)$ are related by*

$$\mathit{End}(\alpha)\circ \mathrm{ad}_h \;=\; \mathit{End}(\alpha^\sharp). \tag{1.9}$$

Moreover, the following conditions are equivalent:

(a) *$\mathit{End}(\alpha)$ is invariant under ad_h;*
(b) *$\mathit{End}(\alpha^\sharp) = \mathit{End}(\alpha)$;*
(c) *$\alpha-\alpha^\sharp$ is constant on V;*
(d) *there is a constant $\gamma\in\Gamma$ such that $\alpha-\gamma$ is compatible with h.*

Proof. Let $(e_i)_{i=1}^n$ be a splitting base of V for α. The h-dual base $(e_i^\sharp)_{i=1}^n$ for V is a splitting base for $\alpha^\sharp$, by [RTW, Lemma 3.4]. Fix some $f\in \mathit{End}_D V$, and let

$$f(e_j^\sharp) \;=\; \sum_{i=1}^n e_i^\sharp d_{ij} \qquad \text{for some } d_{ij}\in D.$$

Then, computation yields

$$ad_h(f)(e_j) \;=\; \sum_{i=1}^{n} e_i\tau(d_{ji}).$$

We may compute $End(\alpha^\sharp)(f)$ using the splitting base $(e_i^\sharp)_{i=1}^n$, and $End(\alpha)\big(ad_h(f)\big)$ using the splitting base $(e_i)_{i=1}^n$, obtaining

$$\begin{aligned} End(\alpha^\sharp)(f) &\;=\; \min_{1\le i,j\le n}\big(\alpha^\sharp(e_i^\sharp)+w(d_{ij})-\alpha^\sharp(e_j^\sharp)\big),\\ End(\alpha)\big(ad_h(f)\big) &\;=\; \min_{1\le i,j\le n}\big(\alpha(e_i)+w\big(\tau(d_{ji})\big)-\alpha(e_j)\big). \end{aligned}$$

Equation (1.9) follows since $\alpha^\sharp(e_i^\sharp)=-\alpha(e_i)$, see [RTW, Lemma 3.4].

The equivalence of (a) and (b) readily follows from (1.9), and the equivalence of (b) and (c) from [TW, Prop. 1.22].

(c) $\Leftrightarrow$ (d): By the definition of the dual norm in (1.8), for any constant γ in the divisible group Γ, $(\alpha-\gamma)^\sharp=\alpha^\sharp+\gamma$. Therefore, $\alpha-\gamma$ is compatible with h if and only if $(\alpha-\gamma)^\sharp=\alpha-\gamma$, which holds if and only if $\alpha-\alpha^\sharp=2\gamma$. □

Suppose the equivalent conditions of Prop. 1.6 hold, and write simply g_α for $End(\alpha)$. Recall from [TW, Prop. 1.19] that the graded algebra $\mathsf{gr}_{g_\alpha}(End_D V)$ may be identified with $End_{\mathsf{gr}(D)}\big(\mathsf{gr}_\alpha(V)\big)$ so that for $f\in End_D V$ the element $\widetilde f\in\mathsf{gr}_{g_\alpha}(End_D V)$ is viewed as the map $\widetilde f\colon\ \mathsf{gr}_\alpha(V)\to\mathsf{gr}_\alpha(V)$ defined by

$$\widetilde f(\widetilde x) \;=\; \begin{cases}\widetilde{f(x)} & \text{if } \alpha\big(f(x)\big)=\alpha(x)+g_\alpha(f),\\ 0 & \text{if } \alpha\big(f(x)\big)>\alpha(x)+g_\alpha(f).\end{cases}$$

On the other hand, after adding a constant if necessary, we may assume α is compatible with h; hence we may define a graded hermitian form

$$\widetilde h\colon\ \mathsf{gr}_\alpha(V)\times\mathsf{gr}_\alpha(V)\;\to\;\mathsf{gr}_w(D)$$

(with respect to the involution $\widetilde\tau$) as follows: for $x,\,y\in V$,

$$\widetilde h(\widetilde x,\widetilde y) \;=\; h(x,y)+D^{>\alpha(x)+\alpha(y)} \;=\; \begin{cases}\widetilde{h(x,y)} & \text{if } w\big(h(x,y)\big)=\alpha(x)+\alpha(y),\\ 0 & \text{if } w\big(h(x,y)\big)>\alpha(x)+\alpha(y).\end{cases}$$

This hermitian form is well-defined and nondegenerate (see [RTW, Remark 3.2]), so we may consider the adjoint involution $ad_{\widetilde h}$ on $End_{\mathsf{gr}_w(D)}\big(\mathsf{gr}_\alpha(V)\big)=\mathsf{gr}_{g_\alpha}(End_D V)$.

Proposition 1.7. *Assuming α is compatible with h, the involution $\widetilde{\mathrm{ad}}_h$ on $\mathsf{gr}_{g_\alpha}(End_D V)$ is the adjoint involution of $\widetilde h$ under the identification above; i.e.,*

$$\widetilde{\mathrm{ad}}_h \;=\; \mathrm{ad}_{\widetilde h}\,.$$

In particular, the value function $g_\alpha=End(\alpha)$ is σ-special (see Def. 1.2) if and only if $\widetilde h$ is anisotropic.

Proof. To verify the equality of graded involutions, it suffices to show, for all $x,\,y\in V$ and $f\in End_D V$,

$$\widetilde h\big(\mathrm{ad}_{\widetilde h}(\widetilde f)(\widetilde x),\widetilde y\big) \;=\; \widetilde h\big(\widetilde{\mathrm{ad}}_h(\widetilde f)(\widetilde x),\widetilde y\big).$$

From the definition of $ad_{\widetilde h}$, it is equivalent to prove

$$\widetilde h\big(\widetilde x,\widetilde f(\widetilde y)\big) \;=\; \widetilde h\big(\widetilde{\mathrm{ad}}_h(\widetilde f)(\widetilde x),\widetilde y\big). \tag{1.10}$$

Since α is compatible with h, Prop. 1.6 shows g_α is invariant under ad_h; hence, $g_\alpha(ad_h(f))=g_\alpha(f)$. Therefore, each side of (1.10) lies in D_ϵ, where $\epsilon=\alpha(x)+\alpha(y)+g_\alpha(f)$. Suppose $w\big(h(x,f(y))\big)=\epsilon$. Then, necessarily $\alpha(f(y))=g_\alpha(f)+\alpha(y)$, and the left side of (1.10) equals $h(x,f(y))^\sim$. But since $h(x,f(y))=h\big(ad_h(f)(x),y\big)$, we then also have $g_\alpha\big(\,ad_h(f)(x)\big)=g_\alpha\big(\,ad_h(f)\big)+\alpha(x)$, and the right side of (1.10) becomes $h\big(ad_h(f)(x),y\big)^{\sim}$. So, (1.10) then holds. But, if $w\big(h(x,f(y))\big) > \epsilon$, then each side of (1.10) is 0. Thus, the equality (1.10) holds in all cases, so that $\widetilde{ad_h} = ad_{\widetilde{h}}$.

Since g_α is invariant under ad_h, Prop. 1.1(b) holds if and only if $\widetilde{ad_h}$ is anisotropic. But, the involution $\widetilde{ad_h} = ad_{\widetilde{h}}$ is anisotropic if and only if its associated graded hermitian form $\widetilde{h}$ is anisotropic. This is proved analogously to the ungraded case [KMRT, §6.A], using the fact that $\widetilde{h}$ is anisotropic if and only if $\widetilde{h}(\widetilde{x},\widetilde{x}) \neq \widetilde{0}$ for all nonzero $x \in V$, as remarked in [RTW, p. 101]. □

2. Henselian valuations

Throughout this section, (F,v) is a Henselian valued field and A is a finite-dimensional simple F-algebra with an involution σ. We let $K = Z(A)$ and assume F is the subfield of K fixed by σ. (Thus, A is central over F if $\sigma|_{Z(A)} = id_{Z(A)}$). We assume A is tame over F, which means that A is split by the maximal tamely ramified extension of K, and that K is tame over F. Moreover, if $char(\overline{F}) = 2$ we assume σ is not an orthogonal involution.

Proposition 2.1. *With the hypotheses above, every v-gauge on A is tame. Furthermore, there exist v-gauges on A that are invariant under σ.*

Proof. We may represent $A = End_D V$ for some finite-dimensional right vector space V over a central division K-algebra D. Since v is Henselian, it extends uniquely to a valuation w on D (see for instance [Sch, p. 53, Th. 9] or [W, Th.]). Since A is tame over F, by [TW, Prop. 1.19] D must also be tame over F; hence by [TW, Prop. 1.12 and 1.13] w is a tame v-gauge. Therefore, every v-gauge on A is tame, by [TW, Th. 3.1].

If A is split and σ is symplectic, then $\sigma = ad_b$ for some alternating bilinear form b on V, see [KMRT, §4.A]. Choose a symplectic base $\mathcal{B} = (e_i, f_i)_{i=1}^n$ of V for b and define a v-norm α on V by

$$\alpha\big(\sum_{i=1}^{n} e_i\lambda_i + f_i\mu_i\big) \;=\; \min_{1\le i\le n}\big(v(\lambda_i), v(\mu_i)\big) \qquad \text{for } \lambda_1, \ldots, \mu_n \in F,$$

i.e., $\mathcal{B}$ is a splitting base for α on V, and each $\alpha(e_i) = \alpha(f_j) = 0$. The v-norm α on V induces the v-gauge $End(\alpha)$ on $End_D(V)$. For $g \in End_D(V)$, if g has matrix (c_{ij}) relative to $\mathcal{B}$, then $End(\alpha)(g) = min_{1\le i,j\le n}\, v(c_{ij})$. The matrix for $\sigma(g)$ has the same set of entries up to sign as (c_{ij}), though the entries are relocated. Hence, $End(\alpha)$ is invariant under σ. We exclude this case of A split and σ symplectic for the rest of the proof. We may then choose an F-linear involution θ on D of the same type as σ and an even hermitian form h on V with respect to θ such that $\sigma = ad_h$, see [KMRT, (4.2)]. By [RTW, Cor. 3.6], there exists a w-norm α on V that is compatible with h. By [TW, Prop. 1.19], $End(\alpha)$ is a v-gauge; by Prop. 1.6, this gauge is invariant under σ. □

Theorem 2.2. *With the hypotheses of this section, if σ is anisotropic, then for the Henselian valuation v on F there is a unique σ-special value function φ on A for v. This φ is a tame v-gauge and its value set Γ_A lies in the divisible hull of Γ_F. It is the unique v-gauge on A invariant under σ.*

Proof. We use the same notation as in the proof of Prop. 2.1, representing $A = End_D(V)$ as in that proof. Since σ is anisotropic, it is not a symplectic involution on a split algebra. Therefore, it is the adjoint involution of some even Hermitian form h on V with respect to an involution θ on D of the same type as σ, see [KMRT, Th. (4.2)]. The form h is anisotropic since σ is anisotropic. By [RTW, Th. 4.6 and Prop. 4.2], the map $\alpha\colon V \to \frac{1}{2}\Gamma_D \cup \{\infty\}$ defined by

$$\alpha(x) \;=\; \tfrac{1}{2}w\big(h(x,x)\big) \tag{2.1}$$

is a w-norm on V that is compatible with h, and the residue form $\widetilde{h}$ is anisotropic. Prop. 1.6 then shows that $\varphi = End(\alpha)$ is a surmultiplicative v-value function on A that is invariant under σ, and Prop. 1.7 shows that φ Is σ-special. Since A is tame over F, the valuation w is a v-gauge on D by [TW, Prop. 1.13], hence φ is a tame v-gauge by [TW, Prop. 1.19]. Its value set obviously lies in the divisible hull of Γ_D, which is also the divisible hull of Γ_F.

To prove uniqueness, suppose φ_1 and φ_2 are each σ-special value functions on A for v. To show that $\varphi_1 = \varphi_2$, we argue by induction on the matrix size $ms(A)$, which is defined as the dimension of V in the representation $A = End_D(V)$.

Suppose first that A is a division algebra. For any subfield $L \subseteq A$ fixed elementwise under σ we have

$$\varphi_i(x^2) = \varphi_i(\sigma(x)x) = 2\varphi_i(x) \qquad \text{for all } x \in L \text{ and } i = 1, 2,$$

hence for nonzero $\xi = \widetilde{x}^{\varphi_i} \in \mathsf{gr}_{\varphi_i}(L)$ we have $\xi^2 = \widetilde{x^2}^{\varphi_i} \neq 0$. Therefore, $\mathsf{gr}_{\varphi_i}(L)$ is semisimple. By [TW, Prop. 1.8], it follows that φ_1 and φ_2 coincide with the unique valuation on L extending v. (The extension of v to L is unique because (F, v) is Henselian.) For any $x \in A$, the product $\sigma(x)x$ lies in a subfield of A fixed under σ, so $\varphi_1(\sigma(x)x) = \varphi_2(\sigma(x)x)$. Therefore,

$$\varphi_1(x) = \tfrac{1}{2}\varphi_1(\sigma(x)x) = \tfrac{1}{2}\varphi_2(\sigma(x)x) = \varphi_2(x).$$

The claim is thus proved if $ms(A) = 1$.

Suppose next that $ms(A) > 1$. We may then find in A a symmetric idempotent $e \neq 0, 1$. (Representing $A = End_D(V)$ as above, we have $dim_D V > 1$ and we may take for e the orthogonal projection onto any nonzero proper subspace of V.) Let $f = 1 - e$. The involution σ restricts to eAe and fAf, and $ms(eAe)$, $ms(fAf) < ms(A)$. By the induction hypothesis, the restrictions of φ_1 and φ_2 coincide on eAe and fAf. For any $x \in A$, we have $\sigma(xe)xe \in eAe$ and $\sigma(xf)xf \in fAf$, hence

$$\varphi_1(\sigma(xe)xe) = \varphi_2(\sigma(xe)xe) \quad \text{and} \quad \varphi_1(\sigma(xf)xf) = \varphi_2(\sigma(xf)xf).$$

Since φ_1 and φ_2 are σ-special value functions, Prop. 1.1 shows that

$$\varphi_1(xe) = \varphi_2(xe) \quad \text{and} \quad \varphi_1(xf) = \varphi_2(xf). \tag{2.2}$$

On the other hand, we have $xe\sigma(xf) = 0$ and $xe + xf = x$, hence Prop. 1.1 also yields

$$\varphi_1(x) = min\big(\varphi_1(xe), \varphi_1(xf)\big) \quad \text{and} \quad \varphi_2(x) = min\big(\varphi_2(xe), \varphi_2(xf)\big).$$

By (2.2), it follows that $\varphi_1(x) = \varphi_2(x)$.

Now, suppose g is a gauge on A that is invariant under σ. By [TW, Th. 3.1] we may find a w-norm β on V such that $g = End(\beta)$. Up to the addition of a constant, we may assume β is compatible with h in view of Prop. 1.6. But the norm α of (2.1) is the only w-norm on V that is compatible with h by [RTW, Prop. 4.2], so $\beta = \alpha$ and $g = End(\alpha)$. $\square$

If g is a v-gauge on A that is invariant under σ, we denote by σ_0 the 0-component of $\widetilde{\sigma}$. Thus, σ_0 is an involution on the $\overline{F}$-algebra $A_0 = A^{\geq 0}/A^{>0}$, which may be viewed as the residue algebra of A. The algebra A_0 is semisimple, but not necessarily simple, see [TW, §2]. Note that if $A_0 = B_1 \times \ldots \times B_k$ with the B_i simple, then an involution τ on A_0 is anisotropic if and only if $\tau(B_i) = B_i$ and $\tau|_{B_i}$ is anisotropic for $1 \leq i \leq k$.

Corollary 2.3. *With the hypotheses of this section, if g is a v-gauge on A that is invariant under σ, the following conditions are equivalent:*

(a) *σ is anisotropic;*
(b) *$\widetilde{\sigma}$ is anisotropic;*
(c) *σ_0 is anisotropic.*

Proof. The implication (a) $\Rightarrow$ (b) readily follows from Th. 2.2, and the implications (b) $\Rightarrow$ (a) and (b) $\Rightarrow$ (c) are clear. To prove (c) $\Rightarrow$ (b), suppose $\xi \in \mathsf{gr}_g(A)$ is a nonzero homogeneous element such that $\widetilde{\sigma}(\xi)\xi = 0$. Every element $\eta \in \big(\xi\,\mathsf{gr}_g(A)\big) \cap A_0$ satisfies $\widetilde{\sigma}(\eta)\eta = \sigma_0(\eta)\eta = 0$. Therefore, σ_0 is isotropic if $\big(\xi\,\mathsf{gr}_g(A)\big) \cap A_0 \neq \{0\}$. The corollary thus follows from the following general result:

Lemma 2.4. *Let* A *be a graded simple algebra finite-dimensional over a graded field* K*, and let* $\mathsf{I} \subseteq \mathsf{A}$ *be a homogeneous right ideal. Then, there is a homogeneous idempotent* $e \in \mathsf{A}$ *of degree* 0 *such that* $\mathsf{I} = e\mathsf{A}$.

Proof. By [HW$_2$, Prop. 1.3], we may identify $\mathsf{A} = \mathsf{End}_{\mathsf{D}}(\mathsf{V})$ for some graded division K-algebra D and some finite-dimensional graded D-vector space V. Let $\mathsf{W} = \sum im(f)$, with the sum taken over all homogeneous $f \in \mathsf{I}$. Then, W is a graded D-subspace of V and, just as in the ungraded case, $\mathsf{I} = \mathsf{Hom}_{\mathsf{D}}(\mathsf{V}, \mathsf{W})$. Take any graded D-subspace Y of V, such that Y is complementary to W, and let $e\colon \mathsf{V} \to \mathsf{W}$ be the projection of V onto W along Y. Then, the idempotent e is a degree-preserving graded homomorphism, so $e \in \mathsf{A}_0$. Clearly, $\mathsf{I} = e\mathsf{A}$. □

Remarks 2.5. (1) In Cor. 2.3, the hypothesis that (F, v) is Henselian is used only to prove that (a) implies (b) and (c); the implications (c) $\iff$ (b) $\Rightarrow$ (a) hold without this hypothesis (nor any tameness assumption).

(2) Corollary 2.3 may be regarded as a version of Springer's theorem for involutions. In a slightly different form, it has already been proved by Larmour [L, Th. 4.5]: to see this, observe that the residue involutions defined by Larmour are the direct summands of our residue involution σ_0 for a suitable gauge.

If the involution σ is isotropic, we may still define up to isomorphism an anisotropic kernel $(A, \sigma)_{an}$ in such a way that if $A = End_D\, V$ and $\sigma = ad_h$, then $(A, \sigma)_{an} \cong (End_D\, V_0, ad_{h_0})$ where (V_0, h_0) is an anisotropic kernel of (V, h), see [DLT], and [D] for involutions of the second kind. The same construction holds for graded simple algebras with involution.

Theorem 2.6. *Let* σ_1*,* σ_2 *be* F*-linear involutions on* A *such that* $\sigma_1|_{Z(A)} = \sigma_2|_{Z(A)}$*. For the Henselian valuation* v *on* F*, let* g_1*,* g_2 *be* v*-gauges on* A *invariant under* σ_1 *and* σ_2 *respectively. If* $char(\overline{F}) = 2$*, assume neither* σ_1 *nor* σ_2 *is orthogonal. The following conditions are equivalent:*

(a) *the algebras with involution* (A, σ_1) *and* (A, σ_2) *are isomorphic;*
(b) *the graded algebras with anisotropic involution* $(\mathsf{gr}_{g_1}(A), \widetilde{\sigma}_1)_{an}$ *and* $(\mathsf{gr}_{g_2}(A), \widetilde{\sigma}_2)_{an}$ *are isomorphic.*

Proof. It follows from Prop. 1.5 that σ_i and $\widetilde{\sigma}_i$ are of the same type. Therefore, (a) and (b) each imply that σ_1 and σ_2 are of the same type. If A is split and σ_1, σ_2 are symplectic, then $\mathsf{gr}_{g_1}(A)$ and $\mathsf{gr}_{g_2}(A)$ are split and $\widetilde{\sigma}_1$, $\widetilde{\sigma}_2$ are symplectic, hence hyperbolic. In this case, (a) and (b) both hold trivially. For the rest of the proof, we exclude this case and fix a representation $A = End_D\, V$ where V is a right vector space over a central division K-algebra D. We also fix an involution θ on D of the same type as σ_1 and σ_2, and non-degenerate even hermitian forms h_1, h_2 on V with respect to θ such that

$$\sigma_1 = \mathrm{ad}_{h_1} \qquad \text{and} \qquad \sigma_2 = \mathrm{ad}_{h_2}\,.$$

As observed in the proof of Prop. 2.1, the valuation v extends uniquely to a valuation w on D. By [TW, Th. 3.1] and Prop. 1.6 we may also find norms α_1 and α_2 on V that are compatible with h_1 and h_2 respectively, such that

$$g_1 = End(\alpha_1) \qquad \text{and} \qquad g_2 = End(\alpha_2),$$

hence

$$\mathsf{gr}_{g_1}(A) = \mathsf{End}_{\mathsf{gr}(D)}\big(\mathsf{gr}_{\alpha_1}(V)\big) \qquad \text{and} \qquad \mathsf{gr}_{g_2}(A) = \mathsf{End}_{\mathsf{gr}(D)}\big(\mathsf{gr}_{\alpha_2}(V)\big).$$

It then follows from Prop. 1.7 that

$$\widetilde{\sigma}_1 = \mathrm{ad}_{\widetilde{h_1}} \qquad \text{and} \qquad \widetilde{\sigma}_2 = \mathrm{ad}_{\widetilde{h_2}};$$

hence, denoting by (V_1, k_1) and (V_2, k_2) the anisotropic kernels of $(\mathsf{gr}_{\alpha_1}(V), \widetilde{(h_1)})$ and $(\mathsf{gr}_{\alpha_2}(V), \widetilde{(h_2)})$ respectively,

$$(\mathsf{gr}_{g_1}(A), \widetilde{\sigma}_1)_{an} \cong (\mathsf{End}_{\mathsf{gr}(D)}(\mathsf{V}_1), ad_{k_1}) \quad \text{and} \quad (\mathsf{gr}_{g_2}(A), \widetilde{\sigma}_2)_{an} \cong (\mathsf{End}_{\mathsf{gr}(D)}(\mathsf{V}_2), ad_{k_2}).$$

If (a) holds, then h_1 and h_2 are similar. Scaling h_2 by a factor in $F^\times$, we may assume $h_1 \cong h_2$. By [RTW, Th. 3.11], the anisotropic kernels of $\widetilde{h_1}$ and $\widetilde{h_2}$ are isometric, hence (b) holds.

Conversely, if (b) holds, then the anisotropic kernels of $\widetilde{h_1}$ and $\widetilde{h_2}$ are similar. Scaling h_2 by a factor in $F^\times$, we may assume that they are isometric. By [RTW, Th. 4.6], it follows that h_1 and h_2 are isometric, hence (a) holds. $\square$

Corollary 2.7. *With the hypotheses of this section, up to Witt-equivalence the graded algebra with involution $(\mathsf{gr}_g(A), \widetilde{\sigma})$ depends only on the Witt-equivalence class of (A, σ), and not on the choice of the invariant v-gauge g.*

3. Scalar extensions of involutions

As an application of the results of §2, we consider a basic case of the problem of determining when an anisotropic involution remains anisotropic over a scalar extension.

Let σ be an F-linear involution on a finite-dimensional simple algebra A over a field F. Assume v is a valuation on F and A carries a v-gauge g invariant under σ. For any extension (L, v_L) of (F, v), we may consider the v_L-gauge $g \otimes v_L$ on the L-algebra $A_L = A \otimes_F L$. By Prop. 1.3, this v_L-gauge is invariant under the involution $\sigma \otimes id_L$ on A_L. If the "residue" involution $(\sigma \otimes id_L)_0$ is anisotropic, then $\sigma \otimes id_L$ is anisotropic by Cor. 2.3 and Remark 2.5(1), and the converse holds if v_L is Henselian and A_L is tame over L, unless σ is orthogonal and $char(\overline{F}) = 2$. We consider below a case where this residue can be explicitly calculated.

We first recall some facts which will be used repeatedly below. Let α be a surmultiplicative v-norm on a finite-dimensional algebra A over a field F with valuation v. If e is an idempotent of A with $\alpha(e) = 0$ and N is any F-subspace of A, then $\widetilde{e}^2 = \widetilde{e}$ in $\mathsf{gr}(A)$ and by [TW, Lemma 1.7],

$$\mathsf{gr}(eN) = \widetilde{e}\,\mathsf{gr}(N) \quad \text{and} \quad \mathsf{gr}(Ne) = \mathsf{gr}(N)\widetilde{e} \quad \text{in } \mathsf{gr}(A). \tag{3.1}$$

If $e \neq 1$, let $f = 1 - e$. Then, $\alpha(f) \geq min\big(\alpha(1), \alpha(e)\big) = 0$, but since $f^2 = f$, $\alpha(f) \leq 0$. So, $\alpha(f) = 0$, hence $\widetilde{f} = \widetilde{1} - \widetilde{e}$ in $\mathsf{gr}(A)$, and hence

$$\mathsf{gr}(A) \;=\; \widetilde{e}\,\mathsf{gr}(A) \oplus \widetilde{f}\,\mathsf{gr}(A) \;=\; \mathsf{gr}(eA) \oplus \mathsf{gr}(fA).$$

Therefore, by [RTW, Remark 2.6], the direct sum decomposition $A = eA \oplus fA$ is a splitting decomposition, i.e., $\alpha(a) = min\big(\alpha(ea), \alpha(fa)\big)$ for any $a \in A$. Likewise $A = Ae \oplus Af$ is a splitting decomposition.

Recall also that an element $s \in A^\times$ is said to be *α-stable* if $\alpha(s^{-1}) = -\alpha(s)$. For such an s we have by [TW, Lemma 1.3 and (1.5)],

$$\alpha(as) \;=\; \alpha(sa) \;=\; \alpha(a) + \alpha(s), \quad \text{hence} \quad \widetilde{a}\,\widetilde{s} \;=\; \widetilde{as} \quad \text{and} \quad \widetilde{s}\,\widetilde{a} \;=\; \widetilde{sa} \quad \text{for every } a \in A. \tag{3.2}$$

We now make some general observations on the tensor product of valuations. Let L/F be a finite separable field extension. Recall that the separability idempotent of L is the idempotent $e \in L \otimes_F L$ determined uniquely by the conditions that

$$e \cdot (x \otimes 1) \;=\; e \cdot (1 \otimes x) \qquad \text{for all } x \in L \tag{3.3}$$

and the multiplication map $L \otimes_F L \to L$ carries e to 1, see for instance [KMRT, Prop. (18.10)]. The separability of L/F implies that the bilinear trace form

$$T\colon L \times L \to F, \qquad T(x, y) \;=\; Tr_{L/F}(xy)$$

is nondegenerate.

Proposition 3.1. *Suppose $v\colon F \to \Gamma \cup \{\infty\}$ is a valuation that extends uniquely to a valuation v_L on L, and that the valued field extension (L, v_L) of (F, v) is tame. Then v_L is a v-norm on L which is compatible with the bilinear trace form T, and $(v_L \otimes v_L)(e) = 0$.*

Proof. Since v_L is the unique valuation extending v to L and since the extension is defectless, it follows that v_L is a v-norm (indeed, a v-gauge) on L, see [TW, Cor. 1.9].

We claim that $v\big(Tr_{L/F}(x)\big) \geq v_L(x)$ for all $x \in L^\times$. To see this, consider a Galois closure M of L over F and an extension v_M of v to M. For every F-linear embedding $\iota\colon L \hookrightarrow M$ the composition $v_M \circ \iota$ is a valuation on L extending v, hence $v_M \circ \iota = v_L$. Since $Tr_{L/F}(x) = \sum \iota(x)$, where the sum extends over all embeddings $\iota\colon L \hookrightarrow M$, we have

$$v\big(Tr_{L/F}(x)\big) \ = \ v_M\big(\textstyle\sum_\iota \iota(x)\big) \ \geq \ \min_\iota\big(v_M \circ \iota(x)\big) \ = \ v_L(x),$$

proving the claim. It follows that for all $x,\, y \in L^\times$,

$$v\big(T(x,y)\big) \ \geq \ v_L(x) + v_L(y). \tag{3.4}$$

To show that v_L is compatible with T, it remains to show that for any $x \in L^\times$ there exists $y \in L^\times$ for which equality holds in (3.4). For this, it suffices to show that there exists $\ell \in L^\times$ such that $v\big(Tr_{L/F}(\ell)\big) = v_L(\ell)$, since equality then holds in (3.4) with $y = \ell x^{-1}$. For every $\ell \in L^\times$ with $v_L(\ell) = 0$ we have

$$\overline{Tr_{L/F}(\ell)} \ = \ |\Gamma_L{:}\Gamma_F| \cdot Tr_{\overline{L}/\overline{F}}(\overline{\ell}) \tag{3.5}$$

by [Er$_2$, p. 65, Cor. 1]. (Ershov assumes his valuation is Henselian; but the result carrries over to the situation here: Let F_h be the Henselization of F with respect to v. Since the unique extension of v to L is defectless, for any compositum of L with F_h we have $[L \cdot F_h{:}F_h] \geq [\overline{L}{:}\overline{F}]|\Gamma_L{:}\Gamma_F| = [L{:}F]$. Hence, $L \otimes_F F_h$ is a field, and (3.5) holds for L/F because it holds for $(L \otimes_F F_h)/F_h$.)

Since L/F is tame, the residue extension $\overline{L}/\overline{F}$ is separable and $char(\overline{F})$ does not divide $|\Gamma_L{:}\Gamma_F|$. Therefore, we may find $\ell \in L$ such that $v_L(\ell) = 0$ and $Tr_{\overline{L}/\overline{F}}(\overline{\ell}) \neq 0$. Then (3.5) shows that $\overline{Tr_{L/F}(\ell)} \neq 0$, hence

$$v\big(Tr_{L/F}(\ell)\big) \ = \ 0 \ = \ v_L(\ell).$$

Therefore, v_L is compatible with T; it thus coincides with its dual norm $v_L^\sharp$.

To complete the proof, we compute $(v_L \otimes v_L)(e)$. Let $(\ell_i)_{i=1}^n$ be a splitting F-base of L for v_L, and let $(\ell_i^\sharp)_{i=1}^n$ be the dual base for the form T. By [KMRT, Prop. (18.12)] we have,

$$e \ = \ \sum_{i=1}^n \ell_i \otimes \ell_i^\sharp,$$

hence,

$$(v_L \otimes v_L)(e) \ = \ \min_{1 \leq i \leq n} \big(v_L(\ell_i) + v_L(\ell_i^\sharp)\big).$$

Now, for all $i = 1, \ldots, n$ we have $v_L(\ell_i^\sharp) = v_L^\sharp(\ell_i^\sharp) = -v_L(\ell_i)$ by [RTW, Lemma 3.4]. Therefore, $(v_L \otimes v_L)(e) = 0$. □

Continuing with the same notation and hypotheses as in Prop. 3.1, we now assume further that the extension L/F is Galois. Let G denote its Galois group. Since v_L is the unique extension of v to L, $v_L \circ \iota = v_L$ for any $\iota \in G$, and hence ι induces a graded $\mathsf{gr}(F)$-automorphism $\widetilde{\iota}$ of $\mathsf{gr}(L)$. For $\iota \in G$, let

$$e_\iota = (id \otimes \iota)(e) \in L \otimes_F L,$$

and let $\widetilde{e}_\iota$ be the image of e_ι in $\mathsf{gr}(L \otimes_F L)$, which is canonically identified with $\mathsf{gr}(L) \otimes_{\mathsf{gr}(F)} \mathsf{gr}(L)$ by Prop. 1.3.

Lemma 3.2. *The elements $(e_\iota)_{\iota \in G}$ form a family of orthogonal idempotents such that $\sum_{\iota \in G} e_\iota = 1$. They are the primitive idempotents of $L \otimes_F L$. They satisfy $(v_L \otimes v_L)(e_\iota) = 0$ and*

$$e_\iota \cdot (x \otimes 1) \ = \ e_\iota \cdot \big(1 \otimes \iota(x)\big) \qquad \textit{for } x \in L. \tag{3.6}$$

Likewise, for any $y \in \mathsf{gr}(L)$,

$$\widetilde{e}_\iota \cdot (y \otimes \widetilde{1}) \ = \ \widetilde{e}_\iota \cdot \big(\widetilde{1} \otimes \widetilde{\iota}(y)\big) \quad \textit{in} \ \ \mathsf{gr}(L) \otimes_{\mathsf{gr}(F)} \mathsf{gr}(L). \tag{3.7}$$

Moreover, $(\iota \otimes \iota)(e) = e$ for $\iota \in G$.

Proof. Equation (3.3) shows that $e\cdot(L\otimes_F L) = e\cdot(L\otimes 1)\cong L$. Since L is a field, e must be a primitive idempotent. Equation (3.6) readily follows by applying $id\otimes\iota$ to each side of (3.3). For equation (3.7), it suffices to verify the equality when y is homogeneous and nonzero. But then $y=\widetilde{x}$ for some nonzero $x\in L$. Both $x\otimes 1$ and $1\otimes\iota(x)$ are $v_L\otimes v_L$-stable in $L\otimes_F L$, as defined preceding (3.2) above. Hence, using equations (3.2) and (3.6),

$$\widetilde{e}_\iota\cdot\big(\widetilde{x}\otimes\widetilde{1}\big) = \widetilde{e}_\iota\cdot\big(\widetilde{x\otimes 1}\big) = \big[e_\iota\cdot(x\otimes 1)\big]^\sim = \big[e_\iota\cdot\big(1\otimes\iota(x)\big)\big]^\sim = \widetilde{e}_\iota\cdot\big(\widetilde{1\otimes\iota(x)}\big) = \widetilde{e}_\iota\cdot\big(\widetilde{1}\otimes\widetilde{\iota}(\widetilde{x})\big).$$

Since e is a primitive idempotent, it is clear that each e_ι is also a primitive idempotent. For ι, $\kappa\in G$ and $x\in L$, as $L\otimes_F L$ is commutative we have

$$e_\iota e_\kappa\cdot\big(1\otimes[\kappa(x)-\iota(x)]\big) = e_\iota e_\kappa\cdot\big(1\otimes\kappa(x)\big) - e_\kappa e_\iota\cdot\big(1\otimes\iota(x)\big) = \big(e_\iota e_\kappa - e_\kappa e_\iota\big)\cdot(x\otimes 1) = 0.$$

For $\iota\neq\kappa$, if we choose $x\in L$ with $\iota(x)\neq\kappa(x)$, then $1\otimes[\kappa(x)-\iota(x)]$ is a unit of $L\otimes_F L$; hence, $e_\iota e_\kappa=0$.

As observed in the proof of Prop. 3.1, we have $e=\sum_{i=1}^n \ell_i\otimes\ell_i^\sharp$ if $(\ell_i)_{i=1}^n$ is an F-base of L and $(\ell_i^\sharp)_{i=1}^n$ is the dual base for the bilinear form T. It follows that $e_\iota=\sum_{i=1}^n \ell_i\otimes\iota(\ell_i^\sharp)$ for $\iota\in G$, hence

$$\sum_{\iota\in G} e_\iota = \sum_{i=1}^n \ell_i\otimes Tr_{L/F}(\ell_i^\sharp). \tag{3.8}$$

Since $(\ell_i^\sharp)_{i=1}^n$ is the dual base of $(\ell_i)_{i=1}^n$, we have

$$x = \sum_{i=1}^n \ell_i\, Tr_{L/F}(\ell_i^\sharp x) \qquad \text{for } x\in L.$$

In particular, $\sum_{i=1}^n \ell_i\, Tr_{L/F}(\ell_i^\sharp)=1$, and equation (3.8) yields $\sum_{\iota\in G} e_\iota = 1$. So, the e_ι are all the primitive idempotents of $L\otimes_F L$.

Since v_L is the unique valuation extending v to L, we have $v_L\circ\iota=v_L$ for all $\iota\in G$, hence

$$(v_L\otimes v_L)(e_\iota) = (v_L\otimes v_L)(e) = 0 \qquad \text{for all } \iota\in G.$$

Finally, it is clear that $(\iota\otimes\iota)(e)$ satisfies the same equation (3.3) as e and is carried to 1 by the multiplication map $L\otimes_F L\to L$. Since these properties determine e uniquely, we have $(\iota\otimes\iota)(e)=e$ for all $\iota\in G$. □

It is well-known (cf. [P, pp. 256–257, Lemma b]) that the primitive idempotents of $L\otimes_F L$ are indexed by G and satisfy (3.6). The further properties of the e_ι given in Lemma 3.2 will be useful in what follows.

Now, assume further that $L\subseteq D$ for some finite-dimensional division F-algebra D, and that v extends to a valuation v_D on D such that D/F is defectless; i.e., v_D is a v-norm on D. The restriction of v_D to L is then the unique valuation v_L extending v. We will use the idempotents $(e_\iota)_{\iota\in G}$ to analyze extensions of involutions from D to $D\otimes_F L$. Let C be the centralizer $C_D(L)$. Viewing D as a right C-vector space, we have the canonical isomorphism

$$\eta\colon D\otimes_F L \xrightarrow{\sim} End_C(D), \tag{3.9}$$

which carries $d\otimes\ell$ to the map $x\mapsto dx\ell$ for d, $x\in D$ and $\ell\in L$. For $\iota\in G$, consider the following C-subspace of D:

$$D_\iota = \{d\in D \mid \ell d = d\,\iota(\ell) \text{ for all } \ell\in L\}.$$

Since ι on L is induced by an inner automorphism of D by Skolem-Noether, $D_\iota\neq\{0\}$. Since in addition, $D_{id}=C$ and $D_\kappa\cdot D_\iota\subseteq D_{\iota\kappa}$ for all $\iota,\kappa\in G$, we must have $dim_C(D_\iota)=1$ for each ι.

Lemma 3.3. *We have $D\otimes_F L=\bigoplus_{\iota\in G} e_\iota\,(D\otimes 1)$ and $D=\bigoplus_{\iota\in G} D_\iota$, and these direct sums are splitting decompositions of $D\otimes_F L$ and D with respect to $v_D\otimes v_L$ and v_D, respectively. More precisely, we have*

$$(v_D\otimes v_L)\big(\sum_{\iota\in G} e_\iota\cdot(x_\iota\otimes 1)\big) = \min_{\iota\in G}\big(v_D(x_\iota)\big) \qquad \textit{for } x_\iota\in D,$$

and

$$v_D\big(\sum_{\iota\in G} y_\iota\big) = \min_{\iota\in G}\big(v_D(y_\iota)\big) \qquad \textit{for } y_\iota\in D_\iota.$$

Furthermore, for all ι, $\kappa\in G$,

$$e_\iota(D\otimes_F L)e_\kappa = e_\iota(D_{\kappa^{-1}\iota}\otimes 1).$$

Proof. Let $A = D \otimes_F L$ and $\alpha = v_D \otimes v_L$. Since $(e_\iota)_{\iota\in G}$ is a family of orthogonal idempotents with $\sum_{\iota\in G} e_\iota = 1$ and $\alpha(e_\iota) = 0$ for each ι, the collection $(\widetilde{e}_\iota)_{\iota\in G}$ is a family of orthogonal idempotents in $\mathsf{gr}(A)$ with $\sum_{\iota\in G} \widetilde{e}_\iota = \widetilde{1}$. Hence, using (3.1),

$$A = \bigoplus_{\iota\in G} e_\iota A \quad \text{and} \quad \mathsf{gr}(A) = \bigoplus_{\iota\in G} \widetilde{e}_\iota\, \mathsf{gr}(A) = \bigoplus_{\iota\in G} \mathsf{gr}(e_\iota A). \tag{3.10}$$

Likewise, for any $\iota \in G$,

$$e_\iota A = \bigoplus_{\kappa\in G} e_\iota A e_\kappa \quad \text{and} \quad \mathsf{gr}(e_\iota A) = \bigoplus_{\kappa\in G} \mathsf{gr}(e_\iota A)\widetilde{e}_\kappa = \bigoplus_{\kappa\in G} \mathsf{gr}(e_\iota A e_\kappa). \tag{3.11}$$

In view of (3.6), we have $e_\iota\cdot(1\otimes L) = e_\iota\cdot(L\otimes 1)$, hence $e_\iota\cdot(D\otimes_F L) = e_\iota\cdot(D\otimes 1)$. For any nonzero $x \in D$, $\alpha(x\otimes 1) = v_D(x)$ and $\alpha\big((x\otimes 1)^{-1}\big) = \alpha(x^{-1}\otimes 1) = -v_D(x)$. So, $x\otimes 1$ is α-stable, and (3.2) applies. Since (3.10) shows that the direct sum $A = \bigoplus_{\iota\in G} e_\iota A$ is a splitting decomposition of A for α, it follows using (3.2) that for any $x_\iota \in D$,

$$\alpha\Big(\sum_{\iota\in G} e_\iota\cdot(x_\iota\otimes 1)\Big) = \min_{\iota\in G}\big(\alpha(e_\iota\cdot(x_\iota\otimes 1))\big) = \min_{\iota\in G}\big(\alpha(e_\iota)+\alpha(x_\iota\otimes 1)\big) = \min_{\iota\in G}\big(v_D(x_\iota)\big).$$

To prove the rest, we use the canonical isomorphism η of (3.9). For each $\iota \in G$, let $\pi_\iota = \eta(e_\iota)$, which is a projection in $End_C(D)$. By (3.6) and the commutativity of $L\otimes_F L$, for any $\ell \in L$ and $d\in D$,

$$\ell\cdot\pi_\iota(d) = \eta\big((\ell\otimes 1)e_\iota\big)(d) = \eta\big((1\otimes\iota(\ell))e_\iota\big)(d) = \pi_\iota(d)\cdot\iota(\ell).$$

Hence, $\pi_\iota(D) \subseteq D_\iota$. Since $im(\pi_\iota)$ is a nonzero C-subspace of the 1-dimensional C-vector space D_ι, it follows that $im(\pi_\iota) = D_\iota$. Because $(\pi_\iota)_{\iota\in G}$ is a family of orthogonal idempotents of $End_C(D)$ such that $\sum_{\iota\in G}\pi_\iota = id_D$, we have $D = \bigoplus_{\iota\in G} D_\iota$; furthermore, each π_ι is the projection of D onto D_ι parallel to $\bigoplus_{\kappa\neq\iota} D_\kappa$. Thus, for any $\iota,\kappa \in G$, $\pi_\iota\, End_C(D)\pi_\kappa$ consists of those endomorphisms sending D_κ to D_ι and D_λ to $\{0\}$ for $\lambda\neq\kappa$. For any $\lambda\in G$, since $D_{\kappa^{-1}\iota}D_\lambda \subseteq D_{\lambda\kappa^{-1}\iota}$, we have

$$[\pi_\iota\circ\eta(D_{\kappa^{-1}\iota}\otimes 1)](D_\lambda) = \pi_\iota(D_{\kappa^{-1}\iota}D_\lambda) \subseteq \pi_\iota(D_{\lambda\kappa^{-1}\iota}) \subseteq \begin{cases} D_\iota, & \text{if } \lambda=\kappa;\\ \{0\}, & \text{if } \lambda\neq\kappa.\end{cases}$$

Hence, $\pi_\iota\circ\eta(D_{\kappa^{-1}\iota}\otimes 1) \subseteq \pi_\iota\, End_C(D)\pi_\kappa$. By applying η^{-1}, this yields

$$e_\iota(D_{\kappa^{-1}\iota}\otimes 1) \subseteq e_\iota A e_\kappa \quad \text{for all } \iota,\kappa\in G. \tag{3.12}$$

Now, fix $\iota\in G$. We have seen that $e_\iota A = e_\iota(D\otimes 1)$. The F-epimorphism $\rho_\iota\colon D\to e_\iota A$ given by $d\mapsto e_\iota(d\otimes 1)$ is clearly injective; ρ_ι is also norm-preserving, as $\alpha(e_\iota)=0$ and $d\otimes 1$ is stable in A for each nonzero $d\in D$. Since $D = \bigoplus_{\kappa\in G} D_{\kappa^{-1}\iota}$, we have

$$\bigoplus_{\kappa\in G} e_\iota A e_\kappa = e_\iota A = \rho_\iota(D) = \bigoplus_{\kappa\in G}\rho_\iota(D_{\kappa^{-1}\iota}) = \bigoplus_{\kappa\in G} e_\iota(D_{\kappa^{-1}\iota}\otimes 1). \tag{3.13}$$

This shows that the inclusions in (3.12) must all be equalities. It follows from (3.11) above that the direct sum decomposition $\bigoplus_{\kappa\in G} e_\iota A e_\kappa$ is a splitting decomposition of $e_\iota A$. Therefore, by applying the norm-preserving map ρ_ι^{-1} to the terms in (3.13), it follows that $\bigoplus_{\kappa\in G} D_\kappa$ is a splitting decomposition of D. □

While $D\otimes_F L$ is simple, the degree 0 part $(D\otimes_F L)_0$ of $\mathsf{gr}(D\otimes_F L)$ is in general only semisimple. The value sets Γ_{D_ι} of the D_ι encode how $(D\otimes_F L)_0$ decomposes: Since each D_ι is a 1-dimensional C-subspace of D and $v_D|_{D_\iota}$ is a $v_D|_C$-norm on D_ι, each Γ_{D_ι} is a coset of Γ_C in Γ_D. Therefore, there is a well-defined map

$$\psi\colon G\to\Gamma_D\big/\Gamma_C \quad \text{given by} \quad \psi(\iota) = \Gamma_{D_\iota}.$$

Because $D_\iota\cdot D_\kappa \subseteq D_{\kappa\iota}$ and Γ_D is abelian, ψ is a group homomorphism, which is surjective since $D = \bigoplus_{\iota\in G} D_\iota$ is a splitting decomposition of D by Lemma 3.3. So, $|\,ker(\psi)| = |G|\big/|\Gamma_D{:}\Gamma_C| = [D{:}C]\big/|\Gamma_D{:}\Gamma_C|$, which shows that

$$\psi \text{ is injective if and only if } D \text{ is totally ramified over } C. \tag{3.14}$$

Lemma 3.4. *Let $A = D \otimes_F L$. Then, each $\widetilde{e}_\iota$ is a primitive idempotent of A_0, and $A_0 = \bigoplus_{\iota \in G} \widetilde{e}_\iota(A_0) = \bigoplus_{\iota \in G} \widetilde{e}_\iota(D_0 \otimes 1)$. For any $\iota, \kappa \in G$,*

$$\widetilde{e}_\iota A_0 \widetilde{e}_\kappa \;=\; \begin{cases} \widetilde{e}_\iota\big((D_{\kappa^{-1}\iota})_0 \otimes 1\big) \neq 0, & \text{if } \psi(\kappa) = \psi(\iota); \\ 0, & \text{if } \psi(\kappa) \neq \psi(\iota). \end{cases}$$

Proof. We saw in (3.10) that $\mathsf{gr}(A) = \bigoplus_{\iota \in G} \widetilde{e}_\iota \,\mathsf{gr}(A)$. Moreover, as $\mathsf{gr}(A) = \mathsf{gr}(D) \otimes_{\mathsf{gr}(F)} \mathsf{gr}(L)$ and $\widetilde{e}_\iota\big(1 \otimes \mathsf{gr}(L)\big) = \widetilde{e}_\iota\big(\mathsf{gr}(L) \otimes 1\big)$ by (3.7), we have $\widetilde{e}_\iota \,\mathsf{gr}(A) = \widetilde{e}_\iota\big(\mathsf{gr}(D) \otimes 1\big)$. So, for the degree 0 components we have $A_0 = \bigoplus_{\iota \in G} \widetilde{e}_\iota(A_0) = \bigoplus_{\iota \in G} \widetilde{e}_\iota\big(D_0 \otimes 1\big)$. Similarly, for $\iota, \kappa \in G$, by (3.1) and Lemma 3.3

$$\widetilde{e}_\iota \,\mathsf{gr}(A) \widetilde{e}_\kappa \;=\; \mathsf{gr}(e_\iota A e_\kappa) \;=\; \mathsf{gr}\left(e_\iota (D_{\kappa^{-1}\iota} \otimes 1)\right) \;=\; \widetilde{e}_\iota \,\mathsf{gr}(D_{\kappa^{-1}\iota} \otimes 1) \;=\; \widetilde{e}_\iota\big(\mathsf{gr}(D_{\kappa^{-1}\iota}) \otimes 1\big).$$

Hence, for the degree 0 components,

$$\widetilde{e}_\iota(A_0)\widetilde{e}_\kappa \;=\; \widetilde{e}_\iota\big((D_{\kappa^{-1}\iota})_0 \otimes 1\big).$$

If $\psi(\kappa) \neq \psi(\iota)$, then $\psi(\kappa^{-1}\iota)$ is a nonzero element of Γ_D/Γ_C, so $(D_{\kappa^{-1}\iota})_0 = \{0\}$. If $\psi(\kappa) = \psi(\iota)$, then $(D_{\kappa^{-1}\iota})_0 \neq 0$, and since nonzero elements of $D_{\kappa^{-1}\iota} \otimes 1$ are stable, (3.2) yields $\widetilde{e}_\iota\big((D_{\kappa^{-1}\iota})_0 \otimes 1\big) \neq \{0\}$. If $\kappa = \iota$, then $(D_{\kappa^{-1}\iota})_0 \otimes 1 = C_0 \otimes 1$, so $\widetilde{e}_\iota(A_0)\widetilde{e}_\iota = \widetilde{e}_\iota\big(C_0 \otimes 1\big)$. Since $C \otimes 1$ centralizes $e_\iota \in L \otimes_F L$, $C_0 \otimes 1$ centralizes $\widetilde{e}_\iota$. Hence, $\widetilde{e}_\iota(A_0)\widetilde{e}_\iota \cong C_0 \otimes 1 \cong C_0$. Since C_0 is a division ring, $\widetilde{e}_\iota$ is a primitive idempotent of A_0. □

Now, assume σ is an F-linear involution on D which stabilizes L, and therefore restricts to an automorphism σ_L of L, and let $\iota \in G$ be such that $\iota^2 = id$. Then $\sigma \otimes \iota$ is an involution on $D \otimes_F L$. Since the valuation v_D extending v to D is unique by [W, Th.], it is invariant under σ. Likewise, v_L is invariant under ι, hence $v_D \otimes v_L$ is invariant under $\sigma \otimes \iota$ by Prop. 1.3.

Proposition 3.5. *The involution $\sigma \otimes \iota$ on $D \otimes_F L$ is isotropic unless $\sigma_L = \iota$ and ι lies in the center $Z(G)$ of G. If $\sigma_L = \iota \in Z(G)$ and D is totally ramified over $C_D(L)$, then $\sigma \otimes \iota$ is anisotropic.*

Proof. For $\kappa \in G$ we have

$$(\sigma_L \otimes \iota)(e_\kappa) = (\sigma_L \otimes \iota\kappa)(e) = (id_L \otimes \iota\kappa\sigma_L) \circ (\sigma_L \otimes \sigma_L)(e).$$

Since $(\sigma_L \otimes \sigma_L)(e) = e$ by Lemma 3.2, it follows that

$$(\sigma_L \otimes \iota)(e_\kappa) = e_{\iota\kappa\sigma_L}.$$

If $\iota \neq \sigma_L$ or if $\iota = \sigma_L$ and $\iota \notin Z(G)$, we may find $\kappa \in G$ such that $\iota\kappa\sigma_L \neq \kappa$, hence

$$(\sigma \otimes \iota)(e_\kappa) \cdot e_\kappa = e_{\iota\kappa\sigma_L} \cdot e_\kappa = 0.$$

Therefore, $\sigma \otimes \iota$ is isotropic.

Now assume $\sigma_L = \iota$ and $\iota \in Z(G)$. So, $(\sigma \otimes \iota)(e_\kappa) = e_\kappa$ for all $\kappa \in G$; hence, in $(D \otimes_F L)_0$, $(\sigma \otimes \iota)_0(\widetilde{e}_\kappa) = \widetilde{e}_\kappa$. Assume further that D is totally ramified over $C = C_D(L)$. Then ψ is injective by (3.14), so by Lemma 3.4, $\widetilde{e}_\iota(D \otimes_F L)_0\, \widetilde{e}_\kappa = 0$ whenever $\kappa \neq \iota$. Hence, $(D \otimes_F L)_0 = \bigoplus_{\iota \in G} \widetilde{e}_\iota(D \otimes_F L)_0\, \widetilde{e}_\iota$. Since $(\sigma \otimes \iota)_0$ maps each direct summand to itself and each summand is a division ring, $(\sigma \otimes \iota)_0$ is anisotropic. It follows from Cor. 2.3 (see also Remark 2.5(1)) that $\sigma \otimes \iota$ is anisotropic. □

Corollary 3.6. *Let D be a central division algebra over a field F. Assume v is a valuation on F which extends to a valuation on D so that D is tame over F. Let σ be an involution of the first kind on D and let $L \subseteq D$ be a subfield Galois over F, consisting of σ-symmetric elements. If D is totally ramified over $C_D(L)$, then the involution $\sigma \otimes \mathrm{id}_L$ on $D \otimes_F L$ is anisotropic.*

Proof. This is immediate from Prop. 3.5. □

Remarks 3.7. (a) The assumption in Cor. 3.6 that D is totally ramified over $C_D(L)$ holds whenever D is totally ramified over F. In this case we do not have to assume that L is Galois over F. For, since v extends to D, it follows from a theorem of Morandi [M] that D remains a division ring after scalar extension to a Henselization F_h of F for v. Therefore, we may assume that F is Henselian. The extension L/F is then Galois, since it is tame and totally ramified.

(b) Another case in which D is totally ramified over $C_D(L)$ occurs whenever the subfield L of D is unramified over F and $\overline{L} \subseteq Z(\overline{D})$.

(c) Another way to obtain the information about $(D \otimes_F L)_0$ needed in the proof of Prop. 3.5 is to prove that if the F-central division ring D has a valuation tame over F and L is any subfield of D containing F, and $C = C_D(L)$, then the canonical isomorphism $D \otimes_F L \cong End_C(D)$ is norm-preserving; so this induces a graded isomorphism $\mathsf{gr}(D \otimes_F L) \cong \mathsf{gr}\big(End_C(D)\big) \cong \mathsf{End}_{\mathsf{gr}(C)}\big(\mathsf{gr}(D)\big)$.

4. Composition of value functions

Let $v\colon F \to \Gamma \cup \{\infty\}$ be a valuation on a field F, and let $\Delta \subset \Gamma$ be a convex subgroup, i.e., if $0 \le \gamma \le \delta$ with $\gamma \in \Gamma$ and $\delta \in \Delta$, then $\gamma \in \Delta$. Let $\Lambda = \Gamma/\Delta$, and let $\varepsilon\colon \Gamma \to \Lambda$ be the canonical map. The ordering on Γ induces a total ordering on Λ such that for $\gamma_1, \gamma_2 \in \Gamma$, if $\gamma_1 \le \gamma_2$, then $\varepsilon(\gamma_1) \le \varepsilon(\gamma_2)$. Consequently,

$$\text{if } \varepsilon(\gamma_2) < \varepsilon(\gamma_1), \quad \text{then } \gamma_2 < \gamma_1. \tag{4.1}$$

Because Γ is assumed to be divisible, Δ and Λ are also divisible. By composing v with ε, we obtain a coarser valuation on F,

$$w = \varepsilon \circ v\colon F \to \Lambda \cup \{\infty\}.$$

Let $\overline{F}^v$ (resp. $\overline{F}^w$) denote the residue field of F for the valuation v (resp. w). The valuation v induces a valuation

$$u\colon \overline{F}^w \to \Delta \cup \{\infty\},$$

with residue field

$$\overline{\overline{F}^w}^u = \overline{F}^v,$$

see [EP, pp. 44–45].

Now, let V be an F-vector space and let $\alpha\colon V \to \Gamma \cup \{\infty\}$ be a v-value function. Composition with ε yields a w-value function

$$\beta = \varepsilon \circ \alpha\colon V \to \Lambda \cup \{\infty\}.$$

Each $\lambda \in \Lambda = \Gamma/\Delta$ is a coset of Δ, and may therefore be viewed as a subset of Γ. For $x \in V$, we have by definition

$$\beta(x) = \lambda \in \Lambda \quad \text{if and only if} \quad \alpha(x) \in \lambda \subset \Gamma.$$

For $\lambda \in \Lambda$, let

$$V^{\beta \ge \lambda} = \{x \in V \mid \beta(x) \ge \lambda\}, \quad V^{\beta > \lambda} = \{x \in V \mid \beta(x) > \lambda\}, \quad V^\beta_\lambda = V^{\beta \ge \lambda}/V^{\beta > \lambda}.$$

The group V^β_λ is an $\overline{F}^w$-vector space.

Lemma 4.1. *If $x, y \in V^{\beta \ge \lambda}$ satisfy $x \equiv y \not\equiv 0 \bmod V^{\beta > \lambda}$, then $\alpha(x) = \alpha(y)$.*

Proof. We have $\beta(x-y) > \lambda = \beta(y)$. Since $\beta = \varepsilon \circ \alpha$, (4.1) shows that $\alpha(x-y) > \alpha(y)$. Hence, $\alpha(x) = min\big(\alpha(x-y), \alpha(y)\big) = \alpha(y)$. □

In view of this lemma, we may define

$$\alpha_\lambda\colon V^\beta_\lambda \to \lambda \cup \{\infty\} \quad \text{by} \quad x + V^{\beta > \lambda} \mapsto \begin{cases} \alpha(x) & \text{if } \beta(x) = \lambda, \\ \infty & \text{if } \beta(x) > \lambda. \end{cases}$$

Clearly, α_λ is a u-value function on V^β_λ. For $\gamma \in \lambda$ we have

$$(V^\beta_\lambda)^{\alpha_\lambda}_\gamma = V^\alpha_\gamma.$$

Therefore,

$$\mathsf{gr}_\alpha(V) \;=\; \bigoplus_{\lambda\in\Lambda} \mathsf{gr}_{\alpha_\lambda}(V_\lambda^\beta) \quad\text{where}\quad \mathsf{gr}_{\alpha_\lambda}(V_\lambda^\beta) \;=\; \bigoplus_{\gamma\in\lambda} V_\gamma^\alpha \qquad\text{while}\qquad \mathsf{gr}_\beta(V) \;=\; \bigoplus_{\lambda\in\Lambda} V_\lambda^\beta.$$

Now, let

$$\Gamma_F \,=\, v(F^\times) \,\subseteq\, \Gamma, \qquad \Delta_F \,=\, \Delta\cap\Gamma_F \,\subseteq\, \Delta, \qquad \Lambda_F \,=\, w(F^\times) \,=\, \Gamma_F/\Delta_F \,\subseteq\, \Lambda.$$

These groups are the value groups of, respectively, v, u, and w. Similarly, let

$$\Gamma_V \,=\, \alpha(V\setminus\{0\}) \,\subseteq\, \Gamma \quad\text{and}\quad \Lambda_V \,=\, \beta(V\setminus\{0\}) \,\subseteq\, \Lambda.$$

For each $\lambda\in\Lambda_V$, let also

$$\lambda_V \,=\, \alpha_\lambda(V_\lambda^\beta\setminus\{0\}) \,\subseteq\, \lambda.$$

Clearly, $\lambda_V = \lambda\cap\Gamma_V$. Note Γ_F (resp. Δ_F, resp. Λ_F) is a subgroup of Γ (resp. Δ, resp. Λ), while Γ_V (resp. Λ_V, resp. λ_V for $\lambda\in\Lambda_V$) is a union of cosets of Γ_F (resp. Λ_F, resp. Δ_F). We denote by $|\Gamma_V{:}\Gamma_F|$ the cardinality of the set of cosets of Γ_F in Γ_V, and define likewise $|\Lambda_V{:}\Lambda_F|$ and $|\lambda_V{:}\Delta_F|$ for $\lambda\in\Lambda_V$.

Lemma 4.2. *If $\dim_F V$ is finite, then $|\Gamma_V{:}\Gamma_F|$, $|\Lambda_V{:}\Lambda_F|$, and $|\lambda_V{:}\Delta_F|$ for $\lambda\in\Lambda_V$ are finite. If λ_1, …, $\lambda_r\in\Lambda_V$ are representatives of the various cosets of Λ_V modulo Λ_F, then*

$$|\Gamma_V{:}\Gamma_F| \;=\; \sum_{i=1}^r |(\lambda_i)_V{:}\Delta_F|.$$

Proof. By [RTW, Prop. 2.2] we have

$$|\Gamma_V{:}\Gamma_F| \,\le\, \dim_F V, \qquad |\Lambda_V{:}\Lambda_F| \,\le\, \dim_F V,$$

and also

$$|\lambda_V{:}\Delta_F| \,\le\, \dim_{\overline{F}^w} V_\lambda \,\le\, \dim_F V \qquad \text{for } \lambda\in\Lambda_V.$$

For $i=1$, …, r, let γ_{i1}, …, $\gamma_{is_i}\in(\lambda_i)_V\subseteq\Gamma$ be representatives of the various cosets of $(\lambda_i)_V$ modulo Δ_F. Thus,

$$(\lambda_i)_V \;=\; \coprod_{j=1}^{s_i}(\gamma_{ij}+\Delta_F),$$

where $\coprod$ denotes the disjoint union. For $\gamma\in\Gamma_V$, we have $\varepsilon(\gamma)\in\Lambda_V$, hence

$$\varepsilon(\gamma) \;=\; \lambda_i + w(a) \qquad \text{for some } i\in\{1,\,\dots,\,r\} \text{ and some } a\in F^\times.$$

It follows that $\gamma - v(a)\in(\lambda_i)_V$; hence,

$$\gamma - v(a) \;=\; \gamma_{ij}+v(b) \qquad \text{for some } j\in\{1,\,\dots,\,s_i\} \text{ and some } b\in F^\times.$$

This shows that $\gamma\equiv\gamma_{ij} \bmod \Gamma_F$, hence

$$\Gamma_V \;=\; \bigcup_{i=1}^r\bigcup_{j=1}^{s_i}(\gamma_{ij}+\Gamma_F). \tag{4.2}$$

To complete the proof, it suffices to show the union is disjoint. If $\gamma_{ij}\equiv\gamma_{k\ell} \bmod \Gamma_F$ for some i, j, k, ℓ, then $\varepsilon(\gamma_{ij})\equiv\varepsilon(\gamma_{k\ell}) \bmod \Lambda_F$, hence $i=k$ since $\varepsilon(\gamma_{ij})=\lambda_i$ and $\varepsilon(\gamma_{k\ell})=\lambda_k$. Moreover, from $\varepsilon(\gamma_{ij})=\varepsilon(\gamma_{k\ell})$ it follows that $\gamma_{ij}-\gamma_{k\ell}\in\Delta$, hence $\gamma_{ij}\equiv\gamma_{k\ell} \bmod \Gamma_F$ implies $\gamma_{ij}\equiv\gamma_{k\ell} \bmod \Delta_F$, hence also $j=\ell$. □

Proposition 4.3. *Suppose $\dim_F V$ is finite, and let λ_1, …, $\lambda_r\in\Lambda_V$ be representatives of the various cosets of Λ_V modulo Λ_F. The following conditions are equivalent:*

(i) *α is a norm;*
(ii) *β is a norm and α_λ is a norm for all $\lambda\in\Lambda_V$;*
(iii) *β is a norm and α_{λ_i} is a norm for $i=1$, …, r.*

Proof. Use the same notation as in the lemma. For simplicity, denote $\alpha_i = \alpha_{\lambda_i}$ and $V_i = V^{\beta}_{\lambda_i}$ for $i = 1, \ldots, r$, and $V_{ij} = V^{\alpha}_{\gamma_{ij}}$ for $i = 1, \ldots, r$ and $j = 1, \ldots, s_i$, and use the notation $[V{:}F]$ for $dim_F V$. From (4.2) it follows that

$$[\mathsf{gr}_\alpha(V){:}\, \mathsf{gr}_v(F)] \;=\; \sum_{i=1}^{r}\sum_{j=1}^{s_i}[V_{ij}{:}\overline{F}^v]. \tag{4.3}$$

Likewise,

$$[\mathsf{gr}_\beta(V){:}\, \mathsf{gr}_w(F)] \;=\; \sum_{i=1}^{r}[V_i{:}\overline{F}^w] \tag{4.4}$$

and

$$[\mathsf{gr}_{\alpha_i}(V_i){:}\, \mathsf{gr}_u(\overline{F}^w)] \;=\; \sum_{j=1}^{s_i}[V_{ij}{:}\overline{F}^v] \qquad \text{for } i = 1, \ldots, r. \tag{4.5}$$

If α_k is not a norm for some $k \in \{1, \ldots, r\}$, then

$$[V_k{:}\overline{F}^w] \;>\; [\mathsf{gr}_{\alpha_k}(V_k){:}\, \mathsf{gr}_u(\overline{F}^w)].$$

On the other hand, we have

$$[V_i{:}\overline{F}^w] \;\geq\; [\mathsf{gr}_{\alpha_i}(V_i){:}\, \mathsf{gr}_u(\overline{F}^w)] \quad \text{for all } i;$$

hence, by (4.4) and (4.5),

$$[\mathsf{gr}_\beta(V){:}\, \mathsf{gr}_w(F)] \;>\; \sum_{i=1}^{r}[\mathsf{gr}_{\alpha_i}(V_i){:}\, \mathsf{gr}_u(\overline{F}^w)] \;=\; \sum_{i=1}^{r}\sum_{j=1}^{s_i}[V_{ij}{:}\overline{F}^v].$$

In view of (4.3), it follows that $[\mathsf{gr}_\beta(V){:}\, \mathsf{gr}_w(F)] > [\mathsf{gr}_\alpha(V){:}\, \mathsf{gr}_v(F)]$. Since $[V{:}F] \geq [\mathsf{gr}_\beta(V){:}\, \mathsf{gr}_w(F)]$, we have $[V{:}F] > [\mathsf{gr}_\alpha(V){:}\, \mathsf{gr}_v(F)]$, hence α is not a norm.

If each α_i is a norm, then $[V_i{:}\overline{F}^w] = [\mathsf{gr}_{\alpha_i}(V_i){:}\, \mathsf{gr}_u(F)]$ for $i = 1, \ldots, r$, hence (4.4), (4.5), and (4.3) yield

$$[\mathsf{gr}_\beta(V){:}\, \mathsf{gr}_w(F)] \;=\; \sum_{i=1}^{r}\sum_{j=1}^{s_i}[V_{ij}{:}\overline{F}^v] \;=\; [\mathsf{gr}_\alpha(V){:}\, \mathsf{gr}_v(F)].$$

It follows that α is a norm if and only if β is a norm. We have thus proved (i) $\iff$ (iii). Since any $\lambda \in \Lambda$ can be chosen as a representative of its coset, the arguments above also show (i) $\Rightarrow$ (ii). Since (ii) $\Rightarrow$ (iii) is clear, the proof is complete. □

To set Prop. 4.3 in perspective, we relate the graded vector spaces $\mathsf{gr}_\alpha(V)$ and $\mathsf{gr}_\beta(V)$ by means of a value-function-like map

$$\alpha_*\colon\ \mathsf{gr}_\beta(V) \to \Gamma \cup \{\infty\}$$

defined as follows: for $\xi \in \mathsf{gr}_\beta(V)$, $\xi \neq 0$, let $\ell(\xi)$ be the homogeneous component of ξ of least degree, and let $\lambda = deg\big(\ell(\xi)\big)$, so $\ell(\xi) \in V^\beta_\lambda$; then let

$$\alpha_*(\xi) \;=\; \alpha_\lambda\big(\ell(\xi)\big) \in \lambda \;\subseteq\; \Gamma.$$

Let also $\alpha_*(0) = \infty$. For $x \in V$ we thus have

$$\alpha_*(\widetilde{x}^\beta) \;=\; \alpha(x), \tag{4.6}$$

where $\widetilde{x}^\beta$ denotes the image of x in $\mathsf{gr}_\beta(V)$.

A similar construction applies to the valuation v, and yields a map

$$v_*\colon\ \mathsf{gr}_w(F) \;\to\; \Gamma \cup \{\infty\},$$

which satisfies the same properties as a valuation, and such that the image $v_*(\rho)$ of any nonzero $\rho \in \mathsf{gr}_w(F)$ depends only on its homogeneous component of least degree. The map α_* deserves the name of a *graded v_*-value function* since it satisfies the following properties:

(i) $\alpha_*(\xi) = \infty$ if and only if $\xi = 0$; if $\xi \neq 0$, then $\alpha_*(\xi) = \alpha_*\big(\ell(\xi)\big)$ and $\varepsilon \circ \alpha_*(\xi) = deg\,\ell(\xi)$;
(ii) $\alpha_*(\xi + \eta) \geq min\big(\alpha_*(\xi), \alpha_*(\eta)\big)$ for ξ, $\eta \in \mathsf{gr}_\beta(V)$;
(iii) $\alpha_*(\xi\rho) = \alpha_*(\xi) + v_*(\rho)$ for $\xi \in \mathsf{gr}_\beta(V)$ and $\rho \in \mathsf{gr}_w(F)$.

We may thus consider the associated graded structure $\mathsf{gr}_{\alpha_*}\big(\mathsf{gr}_\beta(V)\big)$. If $x \in V$ satisfies $\beta(x) = \lambda$ and $\alpha(x) = \gamma$, we may identify

$$(x + V^{\beta > \lambda}) + \mathsf{gr}_\beta(V)^{\alpha_* > \gamma} \;=\; x + V^{\alpha > \gamma};$$

thus

$$\mathsf{gr}_{\alpha_*}\big(\mathsf{gr}_\beta(V)\big) \;=\; \mathsf{gr}_\alpha(V). \tag{4.7}$$

We define α_* to be a *graded v_*-norm* if $[\mathsf{gr}_{\alpha_*}\big(\mathsf{gr}_\beta(V)\big) \colon \mathsf{gr}_{v_*}\big(\mathsf{gr}_w(F)\big)] = [\mathsf{gr}_\beta(V) \colon \mathsf{gr}_w(F)]$. It is easy to check that this holds if and only if each α_λ is a u-norm. By an argument analogous to the one in [RTW, Prop. 2.5] for ungraded norms, one can check that if α_* is a graded norm, then for any graded subspace W of $\mathsf{gr}_\beta(V)$, $\alpha_*|_{\mathsf{W}}$ is a graded norm on W. Consequently, by dimension count, the functor $\mathsf{gr}_{\alpha_*}(\_)$ preserves strict inclusions of graded subspaces of $\mathsf{gr}_\beta(V)$. Prop. 4.3 may be rephrased as follows: α is a v-norm if and only if β is a w-norm and α_* is a graded v_*-norm. Indeed, if $(e_i)_{i=1}^n$ is a splitting base of V for α, then it is also a splitting base for β, and $(\widetilde{e}_i^{\beta})_{i=1}^n$ is a splitting base of $\mathsf{gr}_\beta(V)$ for α_*.

We now apply this construction to a finite-dimensional F-algebra A. If $\alpha \colon A \to \Gamma \cup \{\infty\}$ is a surmultiplicative v-value function, then the coarser w-value function $\beta = \varepsilon \circ \alpha$ is clearly surmultiplicative, and the map α_* is also surmultiplicative, by an easy calculation using (4.6). The notions of gauge and tame gauge for graded norms are defined analogously to the ungraded cases.

Proposition 4.4. *The map α is a v-gauge (resp. a tame v-gauge) if and only if β is a w-gauge (resp. a tame w-gauge) and α_* is a graded v_*-gauge (resp. a tame graded v_*-gauge).*

Proof. Prop. 4.3 already shows that α is a v-norm if and only if β is a w-norm and α_* is a graded v_*-norm. We noted above that α is surmultiplicative if and only if β and α_* are surmultiplicative.

Suppose α is a v-gauge. Since $\mathsf{gr}_{\alpha_*}\big(\mathsf{gr}_\beta(A)\big) = \mathsf{gr}_\alpha(A)$ and $\mathsf{gr}_\alpha(A)$ is semisimple, it follows that $\mathsf{gr}_\beta(A)$ is semisimple. For, if I is a nontrivial nilpotent homogeneous left ideal of $\mathsf{gr}_\beta(A)$, then $\mathsf{gr}_{\alpha_*}(\mathsf{I})$ is a nontrivial nilpotent homogeneous left ideal of $\mathsf{gr}_{\alpha_*}\big(\mathsf{gr}_\beta(A)\big)$. Thus, β is a w-gauge. Also, $\mathsf{gr}_{\alpha_*}\big(\mathsf{gr}_\beta(A)\big)$ is semisimple by hypothesis, hence α_* is a graded v_*-gauge. Conversely, if β is a w-gauge and α_* is a graded v_*-gauge, then α is a v-gauge since $\mathsf{gr}_\alpha(A) = \mathsf{gr}_{\alpha_*}\big(\mathsf{gr}_\beta(A)\big)$.

Assume now that α is a v-gauge. For the centers we have the obvious inclusions

$$\mathsf{gr}_\alpha\big(Z(A)\big) \;=\; \mathsf{gr}_{\alpha_*}\big(\mathsf{gr}_\beta(Z(A))\big) \;\subseteq\; \mathsf{gr}_{\alpha_*}\big(Z(\mathsf{gr}_\beta(A))\big) \;\subseteq\; Z\big(\mathsf{gr}_{\alpha_*}(\mathsf{gr}_\beta(A))\big) \;=\; Z\big(\mathsf{gr}_\alpha(A)\big). \tag{4.8}$$

Thus, $Z\big(\mathsf{gr}_\alpha(A)\big) = \mathsf{gr}_\alpha\big(Z(A)\big)$ if and only if we have equalities throughout (4.8); since $\mathsf{gr}_{\alpha_*}(\_)$ preserves strict inclusions, this holds if and only if

$$\mathsf{gr}_\beta\big(Z(A)\big) = Z\big(\mathsf{gr}_\beta(A)\big) \quad \text{and} \quad \mathsf{gr}_{\alpha_*}\big(Z(\mathsf{gr}_\beta(A))\big) = Z\big(\mathsf{gr}_{\alpha_*}(\mathsf{gr}_\beta(A))\big).$$

Assume we have these equalities. Let $Z = Z(A)$, which is a direct product of fields, as A is semisimple. The separability condition on the graded center required for tameness holds for α if and only if it holds for α_*, since they have the same graded rings. Suppose now that $\mathsf{gr}_\beta(Z)$ is not separable over $\mathsf{gr}_w(F)$. Because $\mathsf{gr}_\beta(A)$ is semisimple, its center $\mathsf{gr}_\beta(Z)$ is a direct product $\mathsf{C}_1 \times \ldots \times \mathsf{C}_k$ of graded fields, and some C_j must not be separable over $\mathsf{gr}_w(F)$. By [HW$_1$, Prop. 3.7, Prop. 3.5] there is a graded field T with $\mathsf{gr}_w(F) \subseteq \mathsf{T} \subsetneqq \mathsf{C}_j$ and C_j purely inseparable over T. So,

$$\mathsf{gr}_v(F) \;=\; \mathsf{gr}_{\alpha_*}\big(\mathsf{gr}_w(V)\big) \subseteq \mathsf{gr}_{\alpha_*}(\mathsf{T}) \;\subsetneqq\; \mathsf{gr}_{\alpha_*}(\mathsf{C}_j),$$

and $\mathsf{gr}_{\alpha_*}(\mathsf{C}_j)$ is purely inseparable over $\mathsf{gr}_{\alpha_*}(\mathsf{T})$. Now, $\mathsf{gr}_\alpha(Z) = \mathsf{gr}_{\alpha_*}\big(\mathsf{gr}_\beta(Z)\big) = \prod_{i=1}^k \mathsf{gr}_{\alpha_*}(\mathsf{C}_i)$. Since $\mathsf{gr}_{\alpha_*}(\mathsf{C}_j)$ is purely inseparable over $\mathsf{gr}_{\alpha_*}(\mathsf{T})$, it cannot be separable over $\mathsf{gr}_v(F)$, so $\mathsf{gr}_\alpha(Z)$ is not separable over $\mathsf{gr}_v(F)$. Thus, $\mathsf{gr}_\alpha(Z)$ is separable over $\mathsf{gr}_v(F)$ if and only $\mathsf{gr}_{\alpha_*}\big(\mathsf{gr}_\beta(Z)\big)$ is separable over $\mathsf{gr}_{v_*}\big(\mathsf{gr}_w(F)\big)$ and $\mathsf{gr}_\beta(Z)$ is separable over $\mathsf{gr}_w(F)$. Therefore, α is a tame v-gauge if and only if β is a tame w-gauge and α_* is a tame graded v_*-gauge. $\square$

5. Descent of norms

Throughout this section, we fix the following notation: V is a finite-dimensional vector space over a field F, and $v\colon F\to\Gamma\cup\{\infty\}$ is a valuation. Let (F_h,v_h) be a Henselization of (F,v). If $\alpha\colon V\otimes_F F_h\to\Gamma\cup\{\infty\}$ is a v_h-norm, then clearly $\alpha|_V\colon V\to\Gamma\cup\{\infty\}$ is a v-value function, but not necessarily a v-norm unless Γ has rank one, see Prop. 5.4 and Ex. 5.6. In this section, we give an inductive criterion for $\alpha|_V$ to be a v-norm when Γ is the divisible hull of Γ_F and the rank $rk(\Gamma)$ is finite, see Prop. 5.5.

We first discuss the descent problem in a general context: let (K,v_K) be an arbitrary valued field extension of (F,v), and let $\alpha\colon V\otimes_F K\to\Gamma\cup\{\infty\}$ be a v_K-norm. We identify V with its canonical image in $V\otimes_F K$. For any $x\in V$ and $c\in K$ we have

$$\alpha(x\otimes c)=\alpha((x\otimes 1)\cdot c)=\alpha|_V(x)+v_K(c).$$

Therefore, for any $\gamma\in\Gamma_{V,\alpha|_V}$ and $\delta\in\Gamma_K$ the usual F-bilinear map $V\times K\to V\otimes_F K$ sends $V^{\geq\gamma}\times K^{\geq\delta}$ into $(V\otimes_F K)^{\geq\gamma+\delta}$. Likewise, $V^{>\gamma}\times K^{\geq\delta}$ and $V^{\geq\gamma}\times K^{>\delta}$ map into $(V\otimes_F K)^{>\gamma+\delta}$. Consequently, there is a well-defined induced map $V_\gamma\times K_\delta\to(V\otimes_F K)_{\gamma+\delta}$ given by $(\widetilde{x},\widetilde{c})\mapsto\widetilde{x\otimes c}$. The direct sum of these maps over all such γ,δ yields a map $\mathsf{gr}_{\alpha|_V}(V)\times\mathsf{gr}_{v_K}(K)\to\mathsf{gr}_\alpha(V\otimes_F K)$, which is clearly $\mathsf{gr}_v(F)$-bilinear, hence there is a canonical map

$$\chi\colon\ \mathsf{gr}_{\alpha|_V}(V)\otimes_{\mathsf{gr}_v(F)}\mathsf{gr}_{v_K}(K)\to\mathsf{gr}_\alpha(V\otimes_F K)$$

which maps $\widetilde{x}\otimes\widetilde{c}$ to $\widetilde{x\otimes c}$ for $x\in V$ and $c\in K$.

On the other hand, recall from Sec. 1 (see (1.5)) that if $\alpha|_V$ is a v-norm on V, then there is a canonical isomorphism of $\mathsf{gr}_{v_K}(K)$-vector spaces

$$\rho\colon\ \mathsf{gr}_{\alpha|_V\otimes v_K}(V\otimes_F K)\xrightarrow{\sim}\mathsf{gr}_{\alpha|_V}(V)\otimes_{\mathsf{gr}_v(F)}\mathsf{gr}_{v_K}(K)$$

which maps $\widetilde{x\otimes c}$ to $\widetilde{x}\otimes\widetilde{c}$ for $x\in V$ and $c\in K$.

Lemma 5.1. *The following conditions are equivalent:*

(a) *$\alpha|_V$ is a v-norm on V and $\alpha=\alpha|_V\otimes v_K$.*
(b) *V contains a K-splitting base of the norm α on $V\otimes_F K$.*
(c) *$\alpha|_V$ is a v-norm and the canonical map χ is injective.*

When these conditions hold, the map χ is a graded isomorphism, which is the inverse of ρ, and $\Gamma_{V\otimes_F K,\alpha}=\Gamma_{V,\alpha|_V}+\Gamma_{K,v_K}$.

Proof. (b) $\Rightarrow$ (a) If $\mathcal{B}=(e_i)_{i=1}^n\subseteq V$ is a splitting base for α on $V\otimes_F K$, then $\mathcal{B}$ is clearly also a splitting base for $\alpha|_V$ on V. So, $\alpha|_V$ is a v-norm. Furthermore, by the definition of $\alpha|_V\otimes v_K$, we have for any $k_1,\dots,k_n\in K$,

$$\begin{aligned}(\alpha|_V\otimes v_K)\big(\sum_{i=1}^n e_i\otimes k_i\big)&=\min_{1\leq i\leq n}\big(\alpha|_V(e_i)+v_K(k_i)\big)=\min_{1\leq i\leq n}\big(\alpha(e_i\otimes 1)+v_K(k_i)\big)\\&=\alpha\big(\sum_{i=1}^n(e_i\otimes 1)\cdot k_i\big)=\alpha\big(\sum_{i=1}^n e_i\otimes k_i\big),\end{aligned}$$

showing that $\alpha|_V\otimes v_K=\alpha$.

(a) $\Rightarrow$ (c) When (a) holds, $\alpha|_V$ is a norm, and χ is clearly the inverse of ρ, so χ is injective.

(c) $\Rightarrow$ (b) Suppose (c) holds. Let $(e_i)_{i=1}^n$ be an F-splitting base for $\alpha|_V$ on V. Then, by [RTW, Cor. 2.3(ii)] $\widetilde{e_1},\dots,\widetilde{e_n}$ are $\mathsf{gr}_v(F)$-linearly independent in $\mathsf{gr}_{\alpha|_V}(V)$. Hence, $\widetilde{e_1}\otimes\widetilde{1},\dots,\widetilde{e_n}\otimes\widetilde{1}$ are $\mathsf{gr}_{v_K}(K)$-linearly independent in $\mathsf{gr}_{\alpha|_V}(V)\otimes_{\mathsf{gr}_v(F)}\mathsf{gr}_{v_K}(K)$. By the injectivity of χ the $\chi(\widetilde{e_i}\otimes\widetilde{1})=\widetilde{e_i\otimes 1}$ are $\mathsf{gr}_{v_K}(K)$-linearly independent in $\mathsf{gr}_\alpha(V\otimes_F K)$. But, since α and $\alpha|_V$ are norms,

$$[\mathsf{gr}_\alpha(V\otimes_F K)\colon\mathsf{gr}_{v_K}(K)]=[V\otimes_F K\colon K]=[V\colon F]=n.$$

So, $\big(\widetilde{e_i \otimes 1}\big)_{i=1}^n$ is a homogeneous $\mathsf{gr}_{v_K}(K)$-vector space base of $\mathsf{gr}_\alpha(V \otimes_F K)$, hence $(e_i \otimes 1)_{i=1}^n$ is a K-splitting base for α on $V \otimes_F K$ by [RTW, Cor. 2.3(ii)].

When the conditions (a) – (c) hold, we have

$$\Gamma_{V\otimes_F K,\alpha} \;=\; \Gamma_{V\otimes_F K,\alpha|_V\otimes v_K} \;=\; \Gamma_{V,\alpha|_V} + \Gamma_{K,v_K}$$

and the map χ is the inverse of ρ, so χ is an isomorphism. □

Note that under the hypotheses of Lemma 5.1 if $\alpha|_V$ is a norm then $\alpha \ge \alpha|_V \otimes v_K$. For, if $(e_i)_{i=1}^n$ is a v-splitting base for $\alpha|_V$ on V, then for any $k_1, \ldots, k_n \in K$,

$$\alpha\big(\sum_{i=1}^n e_i\otimes k_i\big) \;\ge\; \min_{1\le i\le n}\big(\alpha(e_i\otimes k_i)\big) \;=\; \min_{1\le i\le n}\big(\alpha(e_i)+v_K(k_i)\big) \;=\; (\alpha|_V\otimes v_K)\big(\sum_{i=1}^n e_i\otimes k_i\big).$$

We next show that the inequality $\alpha \ge \alpha|_V \otimes v_K$ is actually an equality when K is immediate over F, but not in general.

Corollary 5.2. *Let (K, v_K) be an immediate valued field extension of (F, v) and let $\alpha\colon V \otimes_F K \to \Gamma \cup \{\infty\}$ be a v_K-norm. If $\alpha|_V$ is a norm, then $\alpha = \alpha|_V \otimes v_K$. So, the canonical map $\mathsf{gr}_{\alpha|_V}(V) \to \mathsf{gr}_\alpha(V \otimes_F K)$ is an isomorphism $\mathsf{gr}_{\alpha|_V}(V) \cong \mathsf{gr}_\alpha(V \otimes_F K)$, and $\Gamma_V = \Gamma_{V\otimes_F K}$.*

Proof. Since v_K is immediate over v, we have $\mathsf{gr}_{v_K}(K) = \mathsf{gr}_v(F)$, so the canonical map χ of Lemma 5.1(c) is just the injection $\mathsf{gr}_{\alpha|_V}(V) \hookrightarrow \mathsf{gr}_\alpha(V \otimes_F K)$ arising from the canonical inclusion $V \hookrightarrow V \otimes_F K$. Thus, the corollary follows from Lemma 5.1, using $\Gamma_K = \Gamma_F$ for the last assertion. □

Example 5.3. Let (K, v_K) be an extension of (F, v) with $\overline{F} \subsetneqq \overline{K}$. Let $\xi \in K$ be such that $v_K(\xi) = 0$ and $\overline{\xi} \notin \overline{F}$, and let V be a 2-dimensional F-vector space with base (e_1, e_2). Let $f = e_1\otimes 1+e_2\otimes\xi \in V\otimes_F K$, and consider the v_K-norm α on $V \otimes_F K$ with splitting base $(e_1 \otimes 1, f)$ such that

$$\alpha(e_1\otimes 1) = 0 \qquad\text{and}\qquad \alpha(f) > 0.$$

Then, as $e_2 = (f - e_1)\xi^{-1}$, we have for $c_1, c_2 \in F$

$$\begin{aligned}\alpha|_V(e_1c_1+e_2c_2) \;&=\; \alpha\big((e_1(c_1-\xi^{-1}c_2)+fc_2\big) \;=\; \min\big(\alpha(e_1)+v_K(c_1-\xi^{-1}c_2), \alpha(f)+v_K(c_2)\big)\\ &=\; \min\big(v(c_1), v(c_2)\big).\end{aligned}$$

Hence, (e_1, e_2) is a v-splitting base of V for $\alpha|_V$, showing that $\alpha|_V$ is a v-norm on V. However, $\alpha|_V \otimes v_K < \alpha$ since $(\alpha|_V \otimes v_K)(f) = \min\big(v_K(1), v_K(\xi)\big) = 0 < \alpha(f)$. Thus, the first condition in Lemma 5.1(a) holds, but not the second. The Lemma shows that V does not contain any splitting base for the v_K-norm α on $V \otimes_F K$. Also, the second condition in part (c) of the Lemma fails, since the canonical map χ satisfies

$$\chi(\widetilde{e_1}\otimes 1+\widetilde{e_2}\otimes\widetilde{\xi}) \;=\; \widetilde{e_1\otimes 1}+\widetilde{e_2\otimes\xi} \;=\; 0.$$

We now turn to the descent problem posed at the beginning of this section, for $(K, v_K) = (F_h, v_h)$ a Henselization of (F, v). The rank one case is easy:

Proposition 5.4. *Let $\alpha\colon V\otimes_F F_h \to \Gamma\cup\{\infty\}$ be a v_h-norm, let $\gamma \in \Gamma$, and suppose $im(\alpha) \subseteq \gamma + (\Gamma_F \otimes_{\mathbb{Z}} \mathbb{Q})$. If $rk(\Gamma_F) = 1$, then $\alpha|_V$ is a v-norm and $\alpha = \alpha|_V \otimes v_h$.*

Proof. Let $(e_i)_{i=1}^n$ be an arbitrary F-base of V and let $x \in V \otimes_F F_h$,

$$x \;=\; \sum_{i=1}^n e_i\otimes k_i \qquad \text{for some } k_i \in F_h.$$

Since $rk(\Gamma_F) = 1$, the field F is dense in F_h for the topology of the valuation v_h: see [Er$_3$, §1.6] or use the fact that F is dense in its completion $\widehat{F}$ and that F_h embeds in $\widehat{F}$ by [E, Th. 17.18]. Furthermore, Γ_F is dense in its divisible hull $\Gamma_F \otimes_{\mathbb{Z}} \mathbb{Q}$. For each i, $1 \le i \le n$, since $\alpha(x) - \alpha(e_i \otimes 1) \in \Gamma_F \otimes_{\mathbb{Z}} \mathbb{Q}$, we may therefore find an element $f_i \in F$ such that

$$v_h(k_i - f_i) \;>\; \alpha(x) - \alpha(e_i\otimes 1).$$

Let $y = \sum_{i=1}^n e_i \otimes f_i = \sum_{i=1}^n e_i f_i \otimes 1 \in V$. Then,

$$\alpha(x-y) \;=\; \alpha\big(\sum_{i=1}^n e_i \otimes (k_i - f_i)\big) \;\geq\; \min_{1\leq i\leq n}\big(\alpha(e_i \otimes (k_i - f_i))\big) \;=\; \min_{1\leq i\leq n}\big(\alpha(e_i) + v_h(k_i - f_i)\big) >\; \alpha(x).$$

Hence,

$$\widetilde{x} \;=\; \widetilde{y} \,\in\, \mathsf{gr}_{\alpha|_V}(V).$$

This proves that the monomorphism $\mathsf{gr}_{\alpha|_V}(V) \hookrightarrow \mathsf{gr}_\alpha(V \otimes_F F_h)$ is an isomorphism. Hence, as α is a norm,

$$[\mathsf{gr}_{\alpha|_V}(V) : \mathsf{gr}_v(F)] \;=\; [\mathsf{gr}_\alpha(V \otimes_F F_h) : \mathsf{gr}_{v_h}(F_h)] \;=\; [(V \otimes_F F_h) {:} F_h] \;=\; [V {:} F],$$

which shows that $\alpha|_V$ is a v-norm. $\square$

Now, suppose $\Gamma = \Gamma_F \otimes_{\mathbb{Z}} \mathbb{Q}$, with $rk(\Gamma) > 1$, and suppose Γ contains a convex subgroup Δ of rank 1. As in §4, we consider the canonical map $\varepsilon\colon \Gamma \to \Gamma/\Delta = \Lambda$ and the coarser valuation

$$w \;=\; \varepsilon \circ v\colon F \to \Lambda \cup \{\infty\}.$$

Let $(F_{h,v}, v_h)$ be a Henselization of (F, v) and $(F_{h,w}, w_h)$ a Henselization of (F, w). Let also

$$y \;=\; \varepsilon \circ v_h\colon F_{h,v} \to \Lambda \cup \{\infty\}.$$

By [EP, Cor. 4.1.4, p. 90], the valuation y is Henselian, hence we may assume $(F_{h,w}, w_h) \subseteq (F_{h,v}, y)$.

Let $\alpha\colon V \otimes_F F_{h,v} \to \Gamma \cup \{\infty\}$ be a v_h-norm, and let

$$\beta \;=\; \varepsilon \circ \alpha\colon V \otimes_F F_{h,v} \to \Lambda \cup \{\infty\}.$$

By Prop. 4.3, the map β is a y-norm.

Proposition 5.5. *If $\beta|_V$ is a w-norm and $\beta = \beta|_V \otimes y$, then $\alpha|_V$ is a v-norm on V and $\alpha = \alpha|_V \otimes v_h$.*

Proof. As observed in §4, the valuation v induces a valuation u on the residue field $\overline{F}^w$,

$$u\colon \overline{F}^w \to \Delta \cup \{\infty\}.$$

Note that the value group of u is $\Delta_F = \Gamma_F \cap \Delta$ and, as Δ is divisible and torsion-free,

$$\Delta_F \otimes_{\mathbb{Z}} \mathbb{Q} \;=\; \big(\Gamma_F \otimes_{\mathbb{Z}} \mathbb{Q}\big) \cap \big(\Delta \otimes_{\mathbb{Z}} \mathbb{Q}\big) \;=\; \Gamma \cap \Delta \;=\; \Delta.$$

Let $\Lambda_V = \beta|_V(V \setminus \{0\}) \subseteq \Lambda$. Clearly, $\beta|_V = \varepsilon \circ (\alpha|_V)$. In order to show $\alpha|_V$ is a norm, it therefore suffices, by Prop. 4.3, to show that each map

$$(\alpha|_V)_\lambda\colon V_\lambda^{\beta|_V} \to \lambda \cup \{\infty\}, \qquad \text{for } \lambda \in \Lambda_V, \tag{5.1}$$

is a u-norm. To simplify notation, we write V_λ for $V_\lambda^{\beta|_V}$. Note that the canonical inclusion $V \hookrightarrow V \otimes_F F_{h,v}$ is compatible with the respective value functions $\beta|_V$ and β so yields an injection $V_\lambda \hookrightarrow (V \otimes_F F_{h,v})_\lambda^\beta$; let V'_λ denote the image of V_λ. Then, clearly $\alpha_\lambda|_{V'_\lambda} \cong (\alpha|_V)_\lambda$.

Let $u_h\colon \overline{F_{h,v}}^y \to \Delta \cup \{\infty\}$ be the valuation induced by v_h. As observed by Morandi [M, p. 239], $(\overline{F_{h,v}}^y, u_h)$ is a Henselization of $(\overline{F}^w, u)$. Since α is a v_h-norm, Prop. 4.3 shows that

$$\alpha_\lambda\colon (V \otimes_F F_{h,v})_\lambda^\beta \to \lambda \cup \{\infty\}$$

is a u_h-norm for every $\lambda \in \Lambda_V$. Since $(F_{h,v}, y)$ is an inertial extension of $(F_{h,w}, w_h)$ by [M, p. 239], we have

$$\mathsf{gr}_y(F_{h,v}) \;\cong\; \mathsf{gr}_{w_h}(F_{h,w}) \otimes_{\mathsf{gr}_{w_h}(F_{h,w})_0} \mathsf{gr}_y(F_{h,v})_0 \;=\; \mathsf{gr}_w(F) \otimes_{\mathsf{gr}_w(F)_0} \mathsf{gr}_y(F_{h,v})_0.$$

Because $\beta = \beta|_V \otimes y$, this yields graded isomorphisms

$$\begin{aligned}\mathsf{gr}_\beta(V \otimes_F F_{h,v}) \;&\cong\; \mathsf{gr}_{\beta|_V}(V) \otimes_{\mathsf{gr}_w(F)} \mathsf{gr}_y(F_{h,v}) \;\cong\; \mathsf{gr}_{\beta|_V}(V) \otimes_{\mathsf{gr}_w(F)_0} \mathsf{gr}_y(F_{h,v})_0 \\ &\cong\; \mathsf{gr}_{\beta|_V}(V) \otimes_{\overline{F}^w} \overline{F_{h,v}}^y.\end{aligned}$$

For any $\lambda \in \Lambda_F$, when we restrict these graded isomorphisms to the λ-component we obtain the $\overline{F_{h,v}}^y$-vector space isomorphism

$$\psi\colon (V \otimes_F F_{h,v})^\beta_\lambda \xrightarrow{\sim} V_\lambda \otimes_{\overline{F}^w} \overline{F_{h,v}}^y.$$

Let $\widehat{\alpha} = \alpha_\lambda \circ \psi^{-1}\colon V_\lambda \otimes_{\overline{F}^w} \overline{F_{h,v}}^y \to \lambda \cup \{\infty\}$, which is the u_h-value function on $im(\psi)$ corresponding to α_λ on the domain of ψ. Since α_λ is a u_h-norm, so is $\widehat{\alpha}$. Because $(\overline{F_{h,v}}^y, u_h)$ is a Henselization of $(\overline{F}^w, u)$ and λ is a coset of $\Delta = \Delta_F \otimes_{\mathbb{Z}} \mathbb{Q}$, which has rank 1, with Δ_F the value group of u, Prop. 5.4 applies to $\widehat{\alpha}$, and shows that $\widehat{\alpha}|_{V_\lambda}$ is a u-norm. Note that ψ maps the V'_λ defined above after (5.1) to the copy of V_λ in $im(\psi)$. So, $\alpha_\lambda|_{V'_\lambda} \cong \widehat{\alpha}|_{V_\lambda}$. But, we saw above that $(\alpha|_V)_\lambda \cong \alpha_\lambda|_{V'_\lambda}$. Since $\widehat{\alpha}|_{V_\lambda}$ is a u-norm, these isomorphisms show that $(\alpha|_V)_\lambda$ is also a u-norm. Thus, by Prop. 4.3 $\alpha|_V$ is a v-norm; then $\alpha = \alpha|_V \otimes v_h$ by Cor. 5.2. □

The following is an example of a norm on a Henselization that does not descend to a norm.

Example 5.6. Let k be any field with $char(k) \neq 2$, and let $F = k(x, y)$ with x and y algebraically independent over k. Let v be the valuation on F obtained by restriction from the canonical Henselian valuation on $k((x))((y))$, so $\Gamma_F = \mathbb{Z} \times \mathbb{Z}$ and $\overline{F} = k$. Let (F_h, v_h) be a Henselization of (F, v). Let $A = \left(\frac{1+x,\, y}{F}\right)$, a quaternion division algebra over F, and let $A_h = A \otimes_F F_h$. The algebra A_h is split since $1 + x \in F_h^{\times 2}$. Therefore, we may find v_h-gauges on A_h that are unramified, in the sense that $\Gamma_{A_h} = \Gamma_F$. Fix such a v_h-gauge α. We claim that $\alpha|_A$ is not a v-norm on A.

Suppose the contrary. Then $\mathsf{gr}_{\alpha|_A}(A) = \mathsf{gr}_\alpha(A_h)$ by Lemma 5.2, so $\alpha|_A$ is a v-gauge. Consider the convex subgroup $\Delta = \mathbb{Z} \times \{0\} \subseteq \Gamma_F$ and the canonical epimorphism

$$\varepsilon\colon \Gamma_F \to \Gamma_F/\Delta = \mathbb{Z}.$$

Let $w = \varepsilon \circ v\colon F \to \mathbb{Z} \cup \{\infty\}$, which is the y-adic valuation on F, and $\beta = \varepsilon \circ \alpha\colon A_h \to \mathbb{Z} \cup \{\infty\}$. Proposition 4.4 shows that $\beta|_A$ is a tame w-gauge on A. However, the y-adic valuation w extends to A, so by [TW, Cor. 3.4] $\beta|_A$ is the (unique) valuation on A that extends w. In particular, if $j \in A$ satisfies $j^2 = y$ we must have $\beta|_A(j) = \frac{1}{2}$. This is a contradiction since $\beta|_A$ is unramified.

6. Non-Henselian valuations

Let (F, v) be a valued field and let A be a finite-dimensional simple F-algebra with an involution σ. Let $K = Z(A)$, and assume F is the subfield of K fixed under σ. Fix a Henselization (F_h, v_h) of (F, v).

Theorem 6.1. *Suppose A is split by the maximal tamely ramified extension of F_h. Moreover, if $char(\overline{F}) = 2$ suppose that σ is not an orthogonal involution. Then, the following conditions are equivalent:*

(a) *$\sigma \otimes id_{F_h}$ is an anisotropic involution on $A \otimes_F F_h$;*
(b) *there exists a σ-special v-gauge φ on A i.e., $\varphi(\sigma(x)x) = 2\varphi(x)$ for all $x \in A$.*

When they hold, φ is the unique v-gauge on A that is invariant under σ, it is tame, and its value group lies in the divisible hull of Γ_F.

Proof. Let $A_h = A \otimes_F F_h$ and $\sigma_h = \sigma \otimes id_{F_h}$. If φ is a σ-special v-gauge on A, then by Prop. 1.1 φ is invariant under σ and $\widetilde{\sigma}$ is anisotropic on $\mathsf{gr}_\varphi(A)$. By Cor. 1.4, $\varphi \otimes v_h$ is invariant under σ_h. Since $\mathsf{gr}_{\varphi \otimes v_h}(A_h) \cong \mathsf{gr}_\varphi(A) \otimes_{\mathsf{gr}_v(F)} \mathsf{gr}_{v_h}(F_h) \cong \mathsf{gr}_\varphi(A)$ and $\widetilde{\sigma_h} \cong \widetilde{\sigma}$, it follows that $\widetilde{\sigma_h}$ is anisotropic, hence σ_h must also be anisotropic, proving (b) ⇒ (a).

Now, suppose (a) holds. Let φ_1 and φ_2 be v-gauges on A that are each invariant under σ. Then, by Cor. 1.4 each $\varphi_i \otimes v_h$ is a surmultiplicative v_h-norm on A_h which is invariant under σ_h. Moreover, $\varphi_i \otimes v_h$ is a gauge on A_h since $\mathsf{gr}_{\varphi_i \otimes v_h}(A_h) \cong \mathsf{gr}_{\varphi_i}(A) \otimes_{\mathsf{gr}_v(F)} \mathsf{gr}_{v_h}(F_h) \cong \mathsf{gr}_{\varphi_i}(A)$ and φ_i is a gauge on A. Since σ_h is assumed anisotropic, the uniqueness part of Th. 2.2 (applied to σ_h on A_h) yields $\varphi_1 \otimes v_h = \varphi_2 \otimes v_h$, hence $\varphi_1 = \varphi_2$. Th. 2.2 also shows that $\varphi_1 \otimes v_h$ is tame and satisfies $\varphi_1\big(\sigma_h(x)x\big) = 2\varphi_1(x)$ for all $x \in A_h$, hence φ_1 is tame and satisfies condition (b). Furthermore, $\Gamma_{A,\varphi_1} = \Gamma_{A_h,\varphi_1 \otimes v_h}$ which lies in the divisible hull of $\Gamma_{F_h} = \Gamma_F$ by Th. 2.2.

Thus, it only remains to prove the existence of a v-gauge on A invariant under σ, assuming σ_h is anisotropic. Note first that $K \otimes_F F_h$ is a field. For, otherwise, as K is Galois over F with $[K{:}F] = 2$, $K \otimes_F F_h$ would be a direct sum of two fields, and the nontrivial F_h-automorphism $\sigma|_{K\otimes_F F_h}$ must permute the two primitive idempotents of $K \otimes_F F_h$, call them e_1 and e_2. Then, $\sigma_h(e_1)e_1 = e_2e_1 = 0$; but, this cannot happen as σ_h is anisotropic. Since $K \otimes_F F_h$ is a field and $K = Z(A)$, $A_h \cong A \otimes_K (K \otimes_F F_h)$ is a central simple $K \otimes_F F_h$-algebra.

Because A_h is simple, σ_h is anisotropic, and v_h is Henselian, Th. 2.2 yields a σ_h-invariant v_h-gauge φ_h on A_h whose value set lies in the divisible hull of $\Gamma_{F_h} = \Gamma_F$. The restriction $\varphi = \varphi_h|_A$ is clearly a σ-invariant v-value function whose value set lies in the divisible hull of Γ_F. Henceforth, we may thus assume $\Gamma = \Gamma_F \otimes_{\mathbb{Z}} \mathbb{Q}$. If we show that φ is a v-norm, then Cor. 5.2 yields $\varphi_h = \varphi \otimes v_h$, so $\mathsf{gr}_\varphi(A) = \mathsf{gr}_{\varphi\otimes v_h}(A_h) = \mathsf{gr}_{\varphi_h}(A_h)$, hence φ is a v-gauge, and the proof will be complete.

Suppose first that $rk(\Gamma_F) < \infty$. We then argue by induction on $rk(\Gamma_F)$. If $rk(\Gamma_F) = 1$, then Prop. 5.4 shows that φ is a v-norm. So, we may assume $rk(\Gamma_F) > 1$. Let $\Delta \subseteq \Gamma$ be a convex subgroup of rank 1 and let

$$\varepsilon\colon \Gamma \to \Lambda = \Gamma/\Delta$$

be the canonical epimorphism. Let $w = \varepsilon \circ v$ and $y = \varepsilon \circ v_h$, to agree with the notation of §5. So, w has value group $\Lambda_F = (\Gamma_F + \Delta)/\Delta$, and $\Lambda = \Lambda_F \otimes_{\mathbb{Z}} \mathbb{Q}$, which has rank $rk(\Gamma_F) - 1$. Let $(F_{h,w}, w_h) \subseteq (F_h, y)$ be a Henselization of (F, w). Since σ_h is anisotropic, its restriction $\sigma \otimes id_{F_{h,w}}$ is an anisotropic involution on the subring $A \otimes_F F_{h,w}$ of A_h. Since $A_h \cong (A \otimes_F F_{h,w}) \otimes_{F_{h,w}} F_h$ and A_h is simple, $A \otimes_F F_{h,w}$ must also be simple. Therefore, Th. 2.2 applies, yielding a w_h-gauge ψ_h on $A \otimes_F F_{h,w}$ invariant under $\sigma \otimes id_{F_{h,w}}$. By induction, $\psi_h|_A$ is a w-gauge on A invariant under σ. The same argument as for φ_1 above shows that the gauge $\psi|_A$ is tame. Therefore, by [TW, Cor. 1.26] $\psi_h|_A \otimes y$ is a y-gauge on A_h, which is σ_h-invariant by Cor. 1.4. But, $\varepsilon \circ \varphi_h$ is also a y-gauge on A_h, by Prop. 4.4 since φ is a gauge, and $\varepsilon \circ \varphi_h$ is invariant under σ_h because φ_h is. By the uniqueness given in Th. 2.2, it follows that $\varepsilon \circ \varphi_h = \psi_h|_A \otimes y$. Restricting to A, we also have $\varepsilon \circ \varphi = \psi_h|_A$, which is a w-gauge so a w-norm on A. Furthermore, $(\varepsilon \circ \varphi) \otimes y = \psi_h|_A \otimes y = \varepsilon \circ \varphi_h$. Prop. 5.5 with $\alpha = \varphi_h$ then shows that φ is a v-norm. The theorem is thus proved if $rk(\Gamma_F) < \infty$.

For the rest of the proof, assume that Γ_F has infinite rank. Let $(a_i)_{i=1}^n$ be an F-base of A. Write $a_ia_k = \sum_l c_{ikl}a_l$ for some $c_{ikl} \in F$ and $\sigma(a_i) = \sum_k d_{ik}a_k$ for some $d_{ik} \in F$. Let F_0 be the prime subfield of F, and let $F_1 = F_0(\{c_{ikl}, d_{ik} \mid 1 \le i, k, l \le n\}) \subseteq F$. Let A_1 be the F_1-span of the a_i, which is an F_1-algebra. We have $A_1 \otimes_{F_1} F = A$ and σ restricts to an involution σ_1 on A_1. Now, let $(e_i)_{i=1}^n$ be a splitting base of A_h for the v-norm φ_h. We need to enlarge F_1 to capture the e_i in the Henselization: let L be any field with $F_1 \subseteq L \subseteq F$ and L finitely generated over F_1, and let $v_L = v|_L$. Since F_h is Henselian, there is a unique Henselization $(L_h, v_{L,h})$ of (L, v_L) inside (F_h, v_h) by [EP, Th. 5.2.2(2), p. 121]. Because F is the direct limit of such fields L, the direct limit over such L of the $(L_h, v_{L,h})$ is a Henselian valued field (M, v_M) with $F \subseteq M \subseteq F_h$ and $v_h|_M = v_M$. Therefore, $(M, v_M) = (F_h, v_h)$ by the uniqueness of the Henselization. Since $A_h = A_1 \otimes_{F_1} M$, there is a field F_2 finitely generated over F_1 (hence also over F_0) such that $e_1, \ldots, e_n \in A_1 \otimes_{F_1} (F_2)_h$. Let

$$A_2 = A_1 \otimes_{F_1} F_2 \subseteq A, \qquad A_{2,h} = A_1 \otimes_{F_1} (F_2)_h = A_2 \otimes_{F_2} (F_2)_h \subseteq A_h.$$

Note that A_2 is a simple F_2-algebra since $A = A_{F_2} \otimes_{F_2} F$ and A is simple. Let $\sigma_2 = \sigma|_{A_2}$, which is an involution on A_2, and let $\sigma_{2,h} = \sigma_h|_{A_{2,h}} = \sigma_2 \otimes id_{(F_2)_h}$, which is an anisotropic involution on $A_{2,h}$. Let $\varphi_2 = \varphi_h|_{A_2}$ and $\varphi_{2,h} = \varphi_h|_{A_{2,h}}$. Since $A_h = A_{2,h} \otimes_{(F_2)_h} F_h$ and $e_1, \ldots, e_n \in A_{2,h}$, Lemma 5.1 says that $\varphi_{2,h}$ is a $v_{F_2,h}$-norm on $A_{2,h}$ and $\varphi_h = \varphi_{2,h} \otimes v_h$. Now, F_2 is finitely generated over the prime field F_0, so $rk(\Gamma_{F_2,v_{F_2}}) \le trdeg(F_2/F_0) < \infty$ by [B, §10.3, Cor. 2]. Since $\varphi_{2,h}$ is a $v_{F_2,h}$-norm, the finite rank case shows that φ_2 is a v_{F_2}-norm on A_2; then, $\varphi_{2,h} = \varphi_2 \otimes v_{F_2,h}$ by Cor. 5.2. Hence,

$$\varphi_2 \otimes v_h \;=\; (\varphi_2 \otimes v_{F_2,h}) \otimes v_h \;=\; \varphi_{2,h} \otimes v_h \;=\; \varphi_h.$$

Therefore, as φ_2 is a norm, $\varphi_h|_A = (\varphi_2 \otimes v_h)|_{A_2 \otimes_{F_2} F} = \varphi_2 \otimes v$, which is a norm since it is a scalar extension of the norm φ_2. □

Corollary 6.2. *With the hypotheses on A, σ, and v as in Th. 6.1, let φ be a v-gauge on A which is invariant under σ. Then,*

(a) *If the residue involution σ_0 is anisotropic, then φ is the unique σ-special v-gauge on A.*
(b) *If σ_0 is isotropic, then there is no σ-special v-gauge on A.*

Proof. Let (F_h, v_h) be a Henselization of (F, v), and let $A_h = A \otimes_F F_h$ and $\sigma_h = \sigma \otimes id_{F_h}$. Let $\varphi_h = \varphi \otimes v_h$, a surmultiplicative value function on A_h which is invariant under the involution σ_h, by Cor. 1.4. The graded isomorphisms $\mathsf{gr}_{\varphi_h}(A_h) \cong \mathsf{gr}_\varphi(A) \otimes_{\mathsf{gr}_v(F)} \mathsf{gr}_{v_h}(F_h) \cong \mathsf{gr}_\varphi(A)$ show that φ_h is a gauge on A_h, and $\widetilde{\sigma}_h \cong \widetilde{\sigma}$ and $(\sigma_h)_0 \cong \sigma_0$. (a) If σ_0 is anisotropic, then so is $(\sigma_h)_0$, and so also is σ_h by Cor. 2.3. Th. 6.1 then shows that φ is a σ-special v-gauge and is the unique such v-gauge on A, proving (a). For (b), we prove the contrapositive: If there were a σ-special v-gauge ψ for A then the uniqueness in Th. 6.1 shows that $\varphi = \psi$. Hence, σ_h is anisotropic by Th. 6.1, so $(\sigma_h)_0$ is anisotropic by Cor. 2.3, which implies σ_0 is anisotropic as well. □

Example 6.3. Even when there is no σ-special v gauge on A, there may still be tame v-gauges on A invariant under σ, but they need not be unique. For example, let A the quaternion division algebra $(-1, -1)_{\mathbb{Q}}$ over the field of rational numbers, and let v be the 3-adic valuation on $\mathbb{Q}$. Let $(1, i, j, k)$ be the quaternion base of A with $i^2 = j^2 = -1$ and $k = ij = -ji$. As shown in [TW, Ex. 1.16], a v-gauge φ can be defined on A by

$$\varphi(a_0 + a_1 i + a_2 j + a_3 k) \;=\; min\big(v(a_0), v(a_1), v(a_2), v(a_3)\big).$$

Clearly, the residue algebra of A for φ is $A_0 = (-1, -1)_{\mathbb{F}_3} \cong M_2(\mathbb{F}_3)$. The v-gauge φ is obviously invariant under the conjugation involution σ on A. (This is the involution with $\sigma(i) = -i$ and $\sigma(j) = -j$, which is the unique symplectic involution on A.) Since A is a division algebra, σ must be anisotropic. The residue involution σ_0 is the conjugation involution on A_0, which is isotropic, since $\sigma_0(t)t = Nrd_{A_0}(t)$ for any t in the split quaternion algebra A_0. So, by Cor. 6.2(b) there is no σ-special v-gauge on A. For any unit $u \in A^\times$ the map φ_u defined by

$$\varphi_u(x) \;=\; \varphi(uxu^{-1}) \qquad \text{for } x \in A$$

is a v-gauge on A, and Prop. 1.17 of [TW] shows that $\varphi_u = \varphi$ if and only if $\widetilde{u}$ is invertible in $\mathsf{gr}_\varphi(A)$, which is not a graded division ring. But, for *every* $u \in A^\times$, since φ is invariant under σ and $\sigma(u)u$ is central,

$$\varphi_u(\sigma(x)) \;=\; \varphi(\sigma[(\sigma(u^{-1})x\sigma(u)]) \;=\; \varphi(\sigma(u^{-1})x\sigma(u)) \;=\; \varphi_u([\sigma(u)u]^{-1}x[\sigma(u)u]) \;=\; \varphi_u(x),$$

showing that φ_u is invariant under σ.

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