INCOMPRESSIBILITY OF GENERALIZED SEVERI-BRAUER VARIETIES

NIKITA A. KARPENKO

Abstract. Let F be an arbitrary field. Let A be a central simple F-algebra. Let G be the algebraic group $\operatorname{Aut} A$ of automorphisms of A. Let $\mathfrak{X}_A$ be the class of finite direct products of projective G-homogeneous F-varieties (the class $\mathfrak{X}_A$ includes the *generalized Severi-Brauer varieties* of the algebra A).

Let p be a positive prime integer. For any variety in $\mathfrak{X}_A$, we determine its *canonical dimension at* p. In particular, we find out which varieties in $\mathfrak{X}_A$ are *p-incompressible*. If A is a division algebra of degree p^n for some $n \geq 0$, then the list of p-incompressible varieties includes the generalized Severi-Brauer variety $X(p^m; A)$ of ideals of reduced dimension p^m for $m = 0, 1, \dots, n$.

We also determine the structure of the *Chow motives with coefficients in* $\mathbb{F}_p$ of the varieties in $\mathfrak{X}_A$. More precisely, it is known that the motive of any variety in $\mathfrak{X}_A$ decomposes (in a unique way) into a sum of indecomposable motives, and we describe the indecomposable summands which appear in the decompositions. An application of the above results is a proof of the *hyperbolicity conjecture* on orthogonal involutions.

1. Introduction

A smooth complete irreducible variety X over a field F is said to be *incompressible*, if any rational map $X \dashrightarrow X$ is dominant.

An important (at least by the amount of available applications, see, e.g., [16], [11], or [5]) example of an incompressible variety is as follows. Let p be a positive prime integer. Let D be a central division F-algebra of degree a power of p, say, p^n (where $n \geq 1$). For any integer i, the *generalized Severi-Brauer variety* $X(i; D)$ is the F-variety of right ideals in D of reduced dimension i. The variety $X(1; D)$ (the usual *Severi-Brauer variety* of D) is incompressible.

In fact, the variety $X = X(1; D)$ has a stronger property: it is *p-incompressible*, meaning that for any element α in the Chow group $\mathrm{CH}_{\dim X}(X \times X)$ of the product of two copies of X, the *multiplicity* of α over the first factor is divisible by p if and only if the multiplicity of α over the second factor is so. The original proof, given in [16] and [14], makes use of Quillen's computation of the K-theory of Severi-Brauer varieties. A recent proof, given in [21], makes use of Steenrod operations on the modulo p Chow groups (it better explains the reason of the p-incompressibility but works only over fields of characteristic $\neq p$). A third, particularly simple proof is given here (see Remark 2.19 and Corollary 2.22).

Date: March 2009.

Key words and phrases. Algebraic groups, projective homogeneous varieties, Chow groups.

2000 Mathematical Subject Classifications: 14L17; 14C25.

Supported by the Collaborative Research Centre 701 of the Bielefeld University.

The main result of the present paper is Theorem 3.7 which says that the variety $X(p^m; D)$ is p-incompressible also for $m = 1, \ldots, n-1$ (in the case of $p = 2$ and $m = n-1$ this was shown earlier by Bryant Mathews [19] using a different method). The remaining results of this paper are either obtained on the way to the main result or are quite immediate consequences of it.

We start by associating to each integer $m = 0, 1, \ldots, n$ an indecomposable motive $M_{m,D}$ in the category of Chow motives with coefficients in $\mathbb{F}_p$. This is the summand of the complete motivic decomposition of the variety $X(p^m; D)$ such that the 0-codimensional Chow group of $M_{m,D}$ is non-zero.

Then we show (see Theorem 3.3) that the motive of any finite product of generalized Severi-Brauer varieties of the algebra D decomposes into a sum of twists of the motives $M_{m,D}$ (with various m). In fact, we prove this for more general related to D varieties: for products of projective $(\operatorname{Aut} D)$-homogeneous varieties (that is, for products of varieties of *flags* of ideals in D).

Even more generally, for an arbitrary central simple F-algebra A, we consider the class $\mathfrak{X}_A$ of finite direct products of projective $(\operatorname{Aut} A)$-homogeneous F-varieties. Taking as D the p-primary component of a central division algebra Brauer-equivalent to A, we decompose the motive (still with coefficients in $\mathbb{F}_p$) of any variety in $\mathfrak{X}_A$ into a sum of twists of the motives $M_{m,D}$ (see Corollary 3.4). Therefore we solve the case of the inner type A_n of the following general problem.

Let G be a semisimple affine algebraic group over a field. According to [9] (see also §2a here), the motive (still with $\mathbb{F}_p$ coefficients, p a fixed prime) of any product of projective G-homogeneous F-varieties decomposes and in a unique way in a finite direct sum of indecomposable motives. For simple groups G of any given type, the problem is to describe the indecomposable summands which appear this way.

The situation with the types B_n and D_n however is very complicated: numerous examples of motivic decompositions of projective quadrics are produced (e.g., in [26]) which do not leave a hope of a simple general answer. Therefore a reasonable answer for the inner type A_n obtained here is a fortune.

With this in hand, we prove two structure results concerning the motives $M_{m,D}$ (Theorem 3.6). We show that the d-dimensional Chow group of $M_{m,D}$, where $d = \dim X(p^m; D)$, is also non-zero. This result is equivalent to the p-incompressibility of the variety $X(p^m; D)$, so that we get the main result of the paper (Theorem 3.7) at this point. The second structure result on the motive $M_{m,D}$ is a computation of the p-adic valuation of its rank. In fact, we can not separate the proofs of these two structure results: we prove them simultaneously by induction on $\deg D$ (and using Theorem 3.3).

Now we start getting consequences of the main result. We recall (in §2e) the notion of *canonical dimension at p* (or *canonical p-dimension*) $\operatorname{cd}_p(X)$ of a smooth complete irreducible algebraic variety X. This is a certain non-negative integer satisfying $\operatorname{cd}_p X \leq \dim X$; moreover, $\operatorname{cd}_p X = \dim X$ if and only if X is p-incompressible. In particular, by our main result, $\operatorname{cd}_p X(p^m; D) = \dim X(p^m; D) = p^m(p^n - p^m)$. The canonical dimension at p of any variety in $\mathfrak{X}_A$ (where A is an arbitrary central simple F-algebra) can be easily computed in terms of $\operatorname{cd}_p X(p^m; D)$ (where D is the p-primary part of a division algebra Brauer-equivalent to A), see Corollary 3.8.

In spite of a big number of obtained results, one may say that (the motivic part of) this paper raises more questions that it answers. Indeed, although we show that the motives of the varieties in $\mathfrak{X}_D$ decompose into sums of twists of $M_{m,D}$ (and find a restriction on m in terms of a given variety), we do not precisely determine this decomposition: neither we know how many copies of $M_{m,D}$ (for a given m and a given variety) do really appear in the decomposition, nor we determine the twisting numbers. Moreover, the understanding of the structure of the motives $M_{m,D}$ themselves is not satisfactory. It may happen that $M_{m,D}$ is always the whole motive of the variety $X(p^m, D)$ (that is, the motive of this variety probably is indecomposable): we do not possess a single counter-example. In fact, the variety $X(p^m, D)$ *is* indecomposable for certain values of p, n, and m. Two cases are known for a long time: $m = 0$ (the Severi-Brauer case, see Remark 2.19) and $m = 1$ with $p = 2 = n$ (reducing the exponent of D to 2, we come to the case of an Albert quadric here, where we can refer to [26], [10], or [12]). The Albert case is generalized in Example 3.5. The other values of p, n, m should be studied in this regard.

But the qualitative analysis is done (for instance, the properties of $M_{m,D}$ we establish show that this motive behaves essentially like the whole motive of the variety $X(p^m, D)$ even if it is "smaller"). And the proofs are not complicated. This is a study of generalized Severi-Brauer varieties which are twisted forms of grassmannians, and there is no single Young diagram in the text! Combinatorics or complicated formulas do not show up at all, in particular, because we neglect the twisting numbers of motivic summands. The results we are getting this way are less precise but, as we believe, they contain the essential piece of information. They can be (and are) applied (in [13]) to prove the hyperbolicity conjecture on orthogonal involutions, which was the main motivation to study the problems of this paper.

We conclude the introduction by some remarks on the motivic category we are using. First of all, the category of Chow motives with coefficients in $\mathbb{F}_p$, in which we are working in this paper, can be replaced by a simpler category. This simpler category is constructed in exactly the same way as the category of Chow motives with the only difference that one kills the elements of Chow groups which vanish over extensions of the base field (see Remark 2.7). Working with this simpler category, we do not need the nilpotence tricks (the nilpotence theorem and its standard consequences, cf. §2a) anymore. This simplification of the motivic category does not harm to any external application of our motivic results. So, this is more a question of taste than a question of necessity that we stay with the usual Chow motives.

On the other hand, somebody may think that our category of usual Chow motives is not honest or usual enough because these are Chow motives with coefficients in $\mathbb{F}_p$ and not in $\mathbb{Z}$. Well, there are at least three arguments here. First, decompositions into sums of indecomposables are not unique for coefficients in $\mathbb{Z}$, even in the case of projective homogenous varieties of inner type A_n (see [9, Example 32] or [7, Corollary 2.7]). Therefore the question of describing the indecomposables does not seem so reasonable for the integral motives. Second, any decomposition with coefficients in $\mathbb{F}_p$ lifts (and in a unique way) to the coefficients $\mathbb{Z}/p^n\mathbb{Z}$ for any $n \geq 2$, [23, Corollary 2.7]. Moreover, in the case of varieties in $\mathfrak{X}_A$, where A is a central simple algebra of a p-primary index, it also lifts to $\mathbb{Z}$ (non-uniquely this time), [23, Theorem 2.16]. And third, may be the most important argument is that the results on motives with coefficients in $\mathbb{F}_p$ are sufficient for the applications.

2. Preliminaries

This section is long because it also includes some non-standard (but simple) staff.

2a. **Chow motives with finite coefficients.** Our basic reference for Chow groups and Chow motives (including notation) is [10]. We fix an associative unital commutative ring Λ (most frequently Λ will be the finite field $\mathbb{F}_p$ of p elements, where p is a prime) and for a variety (i.e., a separated scheme of finite type over a field) X we write $\mathrm{CH}(X;\Lambda)$ for its Chow group with coefficients in Λ. We use a simplified notation $\mathrm{CH}(X)$ for the integral Chow group $\mathrm{CH}(X;\mathbb{Z})$, and a simplified notation $\mathrm{Ch}(X)$ for the modulo p Chow group $\mathrm{CH}(X;\mathbb{F}_p) = \mathrm{CH}(X)/p$. Our category of motives is the category $\mathrm{CM}(F,\Lambda)$ of *graded Chow motives with coefficients in* Λ, [10, definition of §64]. By a *sum* of motives we always mean the *direct* sum. We also write Λ for the motive $M(\operatorname{Spec} F) \in \mathrm{CM}(F,\Lambda)$. A *Tate motive* is the motive $\Lambda(i)$ with i an integer (which may differ from ± 1).

We shall often assume that our coefficient ring Λ is finite. This simplifies significantly the situation (and is sufficient for most applications). For instance, for a finite Λ, the endomorphism rings of finite sums of Tate motives are also finite and the following easy statement applies:

Lemma 2.1. *An appropriate power of any element of any* finite *associative (not necessarily commutative) ring is idempotent.*

Proof. Since the ring is finite, any its element x satisfies $x^a = x^{a+b}$ for some $a \geq 1$ and $b \geq 1$. It follows that x^{ab} is idempotent. $\square$

Let X be a smooth complete variety over F and let M be a motive. We call M *split*, if it is a finite sum of Tate motives. We call X *split*, if its *integral* motive $M(X) \in \mathrm{CM}(F,\mathbb{Z})$ (and therefore the motive of X with an arbitrary coefficient ring Λ) is split. We call M or X *geometrically split*, if it splits over a field extension of F. We say that X satisfies the *nilpotence principle*, if for any field extension E/F and any coefficient ring Λ, the kernel of the change of field homomorphism $\mathrm{End}(M(X)) \to \mathrm{End}(M(X)_E)$ consists of nilpotents. Any projective homogeneous variety is geometrically split and satisfies the nilpotence principle, [10, Theorem 92.4 with Remark 92.3].

Corollary 2.2. *Assume that the coefficient ring Λ is finite. Let X be a geometrically split variety satisfying the nilpotence principle. Then an appropriate power of any endomorphism of the motive of X is a projector.*

Proof. Let $\bar{F}/F$ be a splitting field of the motive $M(X)$, that is, $M(X)_{\bar{F}}$ is a sum of Tate motives. Let f be an endomorphism of $M(X)$. Since Λ is finite, the ring $\mathrm{End}(M(X)_{\bar{F}})$ is finite. Therefore a power of $f_{\bar{F}}$ is idempotent by Lemma 2.1, and (replacing f by an appropriate power of f) we may assume that $f_{\bar{F}}$ is idempotent. Since X satisfies the nilpotence principle, the element $\varepsilon := f^2 - f$ is nilpotent. Let n be a positive integer such that $\varepsilon^n = 0 = n\varepsilon$. Then $(f+\varepsilon)^{n^n} = f^{n^n}$ because the binomial coefficients $\binom{n^n}{i}$ for $i < n$ are divisible by n. Therefore f^{n^n} is a projector. $\square$

Lemma 2.3 (cf. [9, Theorem 28]). *Assume that the coefficient ring Λ is finite. Let X be a geometrically split variety satisfying the nilpotence principle and let $\pi \in \mathrm{End}(M(X))$ be a projector. Then the motive (X, π) decomposes into a finite sum of indecomposable motives.*

Proof. If (X, π) does not decompose this way, we get an infinite sequence

$$\pi_0 = \pi,\ \pi_1,\ \pi_2,\ \dots\ \in \mathrm{End}(M(X))$$

of pairwise distinct projectors such that $\pi_i \circ \pi_j = \pi_j = \pi_j \circ \pi_i$ for any $i < j$.

Let $\bar{F}/F$ be a splitting field of X. Since the ring $\mathrm{End}(M(X)_{\bar{F}})$ is finite, we have $(\pi_i)_{\bar{F}} = (\pi_j)_{\bar{F}}$ for some $i < j$. The difference $\pi_i - \pi_j$ is nilpotent and idempotent, therefore $\pi_i = \pi_j$. □

A (non necessarily commutative) ring is called *local*, if the sum of any two non-invertible elements differs from 1 in the ring. Since the sum of two nilpotents is never 1, we have

Lemma 2.4. *A ring, where each non-invertible element is nilpotent, is local. In particular, by Corollary 2.2, so is the ring* $\mathrm{End}(M(X))$*, if Λ is finite and X is a geometrically split variety satisfying the nilpotence principle and such that the motive $M(X)$ is indecomposable.* □

A *complete decomposition* of an object in an additive category is a finite direct sum decomposition with indecomposable summands.

Theorem 2.5 ([2, Theorem 3.6 of Chapter I]). *Let M be an object of a pseudo-abelian category which is a direct sum of a finite number of indecomposable objects having* local *endomorphism rings. Then any finite direct sum decomposition of M can be refined to a complete one, and there is only one (up to a permutation of the summands) complete decomposition of M.*

To be precise, the uniqueness part of Theorem 2.5 states that if

$$M = M_1 \oplus \cdots \oplus M_m = N_1 \oplus \cdots \oplus N_n$$

are two complete decompositions of M, then $m = n$ and there exists a permutation σ of the set $\{1, 2, \dots, n\}$ such that $M_i \simeq N_{\sigma(i)}$ for any i. The isomorphism here is meant to be an isomorphism of *abstract* objects: in general, there is no such isomorphism respecting the embeddings into M. Later on, when we speak of "isomorphism of summands" of a motive, we always mean an isomorphism of abstract motives between the summands.

We say that the *Krull-Schmidt principle* holds for a given object of a given additive category, if every direct sum decomposition of the object can be refined to a complete one (in particular, a complete decomposition exists) and there is only one (up to a permutation of the summands) complete decomposition of the object. In the sequel, we are constantly using the following statement which is an immediate consequence of Lemmas 2.3 and 2.4 and Theorem 2.5:

Corollary 2.6 (cf. [9, Corollary 35]). *Assume that the coefficient ring Λ is finite. The Krull-Schmidt principle holds for any twist of any summand of the motive of any geometrically split F-variety satisfying the nilpotence principle. In other words, the Krull-Schmidt principle holds for the objects of the pseudo-abelian Tate subcategory in* $\mathrm{CM}(F, \Lambda)$

generated by the motives of the geometrically split F-varieties satisfying the nilpotence principle. $\square$

Remark 2.7. Replacing the Chow groups $\mathrm{CH}(-;\Lambda)$ by the *reduced* Chow groups $\overline{\mathrm{CH}}(-;\Lambda)$ (cf. [10, §72]) in the definition of the category $\mathrm{CM}(F,\Lambda)$, we get a "simplified" motivic category $\overline{\mathrm{CM}}(F,\Lambda)$ (which is still sufficient for the main purpose of this paper). Working within this category, we do not need the nilpotence principle any more. In particular, the Krull-Schmidt principle holds (with a simpler proof) for the twists of the summands of the motives of the geometrically split F-varieties.

2b. **Outer summands.** We assume here that the coefficient ring Λ is connected. We shall often assume that Λ is finite.

The following definition is extending some terminology of [25].

Definition 2.8. Let $M \in \mathrm{CM}(F,\Lambda)$ be a summand of the motive of a smooth complete irreducible variety of dimension d. The summand M is called *left-outer*, if $\mathrm{CH}^0(M;\Lambda) \neq 0$. The summand M is called *right-outer*, if $\mathrm{CH}_d(M;\Lambda) \neq 0$. The summand M is called *outer*, if it is left-outer and right-outer simultaneously.

For instance, the whole motive of a smooth complete irreducible variety is an outer summand of itself.

Given a correspondence, an element $\alpha \in \mathrm{CH}_{\dim X}(X \times Y;\Lambda)$ of the Chow group of the product of smooth complete irreducible varieties X and Y, we write $\operatorname{mult}\alpha \in \Lambda$ for the *multiplicity* (or *multiplicity over the first factor*) of α, [10, definition of §75]. Multiplicity of a composition of two correspondences is the product of multiplicities of the composed correspondences (cf. [16, Corollary 1.7]). In particular, multiplicity of a projector is idempotent and therefore $\in \{0,1\}$ because the coefficient ring Λ is connected.

Lemma 2.9. *Let X be a smooth complete irreducible variety. The motive (X,π) given by a projector $\pi \in \mathrm{CH}_{\dim X}(X \times X;\Lambda)$ is left-outer if and only if $\operatorname{mult}\pi = 1$. The motive (X,π) is right-outer if and only if $\operatorname{mult}\pi^t = 1$, where π^t is the transpose of π.*

Proof. The group $\mathrm{CH}^0\big((X,\pi);\Lambda\big)$, defined as $\mathrm{Hom}\big((X,\pi),\Lambda\big)$, is the image of the endomorphism of the group $\mathrm{CH}^0(X;\Lambda) = \Lambda\cdot[X]$ given by the multiplication by $\operatorname{mult}\pi$. The group $\mathrm{CH}_d\big((X,\pi);\Lambda\big)$, defined as $\mathrm{Hom}\big(\Lambda(d),(X,\pi)\big)$, is the image of the endomorphism of the same group $\mathrm{CH}_d(X;\Lambda) = \mathrm{CH}^0(X;\Lambda) = \Lambda\cdot[X]$ given by the multiplication by $\operatorname{mult}\pi^t$. $\square$

Lemma 2.10. *Let d be a non-negative integer and let X be a smooth complete irreducible variety of dimension d such that the degree homomorphism $\deg : \mathrm{CH}^d(X;\Lambda) \to \Lambda$ is an isomorphism. The following three statements on a summand M of $M(X) \in \mathrm{CM}(F,\Lambda)$ are equivalent:*

- *M is right-outer;*
- *the subgroup $\mathrm{CH}^d(M)$ of the group $\mathrm{CH}^d(X;\Lambda)$ coincides with $\mathrm{CH}^d(X;\Lambda)$;*
- *$\mathrm{CH}^d(M) \neq 0$.*

Proof. Let $\pi \in \mathrm{CH}^d(X \times X;\Lambda)$ be the projector giving M. Let x be an element of $\mathrm{CH}^d(X;\Lambda)$ with $\deg x = 1 \in \Lambda$. In particular, x is a generator of the Λ-module $\mathrm{CH}^d(X;\Lambda)$.

Then the Λ-module $\mathrm{CH}^d(M) = \mathrm{Hom}(M, \Lambda(d))$ coincides with the submodule of $\mathrm{CH}^d(X;\Lambda)$ generated by $(\operatorname{mult}\pi^t)\cdot x$. $\square$

Lemma 2.11. *Assume that a summand M of the motive of a smooth complete irreducible variety of dimension d decomposes into a sum of Tate motives. Then M is left-outer if and only if the Tate motive Λ is present in the decomposition; it is right-outer if and only if the Tate motive $\Lambda(d)$ is present in the decomposition.*

Proof. For any $i \in \mathbb{Z}$ we have: $\mathrm{CH}^0(\Lambda(i);\Lambda) \neq 0$ if and only if $i = 0$; $\mathrm{CH}_d(\Lambda(i);\Lambda) \neq 0$ if and only if $i = d$. $\square$

Remark 2.12. Assume that the coefficient ring Λ is finite. Let X be an irreducible geometrically split variety satisfying the nilpotence principle. Then the complete motivic decomposition of X contains precisely one left-outer summand and it follows by Corollary 2.6 that a left-outer indecomposable summand of $M(X)$ is unique up to an isomorphism (of motives, not of summands). Of course, the same is true for the right-outer summands.

Lemma 2.13. *Assume that the coefficient ring is finite. Let X be an irreducible geometrically split variety satisfying the nilpotence principle. Let M be a motive. Assume that there exist morphisms $\alpha : M(X) \to M$ and $\beta : M \to M(X)$ such that* $\operatorname{mult}(\beta\circ\alpha) = 1$. *Then the irreducible left-outer summand of $M(X)$ is isomorphic to a summand of M.*

Proof. By Corollary 2.2, the composition $\pi := (\beta\circ\alpha)^{\circ n} \in \operatorname{End} M(X)$ is a projector for some integer $n \geq 1$. Therefore $\tau := (\alpha\circ\beta)^{\circ 2n} \in \operatorname{End} M$ is also a projector and the summand (X,π) of $M(X)$ (which is left-outer by Lemma 2.9) is isomorphic to the summand (M,τ) of M given by the image of τ: mutually inverse isomorphisms are, say,

$$\alpha\circ(\beta\circ\alpha)^{\circ(2n)} : (X,\pi) \to (M,\tau) \quad\text{and}\quad \beta\circ(\alpha\circ\beta)^{\circ(4n-1)} : (M,\tau) \to (X,\pi)\ .$$

Consequently, any (in particular, an irreducible left-outer) summand of (X,π) is isomorphic to a summand of M. $\square$

2c. **Rank of a motive.** We are still assuming that the coefficient ring Λ is connected.

Definition 2.14. Let M be a geometrically split motive. Over an extension of the base field the motive M becomes isomorphic to a finite sum of Tate motives. The *rank* $\operatorname{rk} M$ of M is defined as the number of summands in this decomposition.

Remark 2.15. The number of summands in the above definition does not depend on the choice of the extension or of the decomposition. This is simply the rank of the free Λ-module $\operatorname{colim}_{L/F}\mathrm{CH}(M_L;\Lambda)$, where the colimit is taken over all field extensions L/F.

Example 2.16. Let n be a positive integer and let i be an integer in the interval $[0,\ n]$. Let A be a central simple F-algebra of degree n. Since the variety $X = X(i,A)$ (see §2d) is a twisted form of the grassmannian of i-planes in an n-dimensional vector space, the rank of the motive of X coincides with the rank of the motive of the grassmannian and is equal to $\binom{n}{i}$.

Remark 2.17. Let M be a direct summand of a geometrically split variety X satisfying the nilpotence principle. Then we have: $\operatorname{rk} M = 0$ if and only if $M = 0$; $\operatorname{rk} M = \operatorname{rk} M(X)$ if and only if $M = M(X)$.

For any integer l, we write $v_p(l)$ for the exponent of the highest power of p dividing l.

Lemma 2.18. *Let M be a direct summand of the motive with coefficients in $\mathbb{F}_p$ of a geometrically split equi-dimensional variety X. Let d be the greatest common divisor* (gcd) *of the degrees of the closed points on X. Then $v_p(d) \le v_p(\operatorname{rk} M)$.*

Proof. Let $\pi \in \operatorname{End} M(X) = \operatorname{CH}_{\dim X}(X \times X)/p$ be the projector defining the summand M. Let $n = v_p(d)$. Let $\pi' \in \operatorname{CH}_{\dim X}(X\times X)/p^n$ be a lifting of π. By Lemma 2.1, replacing π' by its appropriate power, we may assume that π' is a projector. The rank of the motive $(X, \pi') \in \operatorname{CM}(F, \mathbb{Z}/p^n)$ coincides with $\operatorname{rk} M$. Let L/F be a splitting field of the motive (X, π'). Mutually inverse isomorphisms between $(X, \pi')_L$ and a sum of $m = \operatorname{rk} M$ Tate motives are given by two sequences of homogeneous elements $a_1, \dots, a_m$ and $b_1, \dots, b_m$ in $\operatorname{CH}(X_L; \mathbb{Z}/p^n)$ satisfying $\pi'_L = a_1 \times b_1 + \dots + a_m \times b_m$ and such that for any $i, j = 1, \dots, m$ the degree $\deg(a_i b_j)$ is 0 for $i \ne j$ and $1 \in \mathbb{Z}/p^n$ for $i = j$. The pull-back of π' via the diagonal morphism of X is therefore a 0-cycle class on X of degree m (modulo p^n). □

Remark 2.19. Lemma 2.18 gives a new, particularly simple proof of the fact that the motive with coefficients in $\mathbb{F}_p$ of the Severi-Brauer variety X of a central division algebra of degree p^n ($n \ge 0$ an integer) is indecomposable: since $\operatorname{rk} M(X) = p^n$ and the gcd of the degrees of the closed points on X is also p^n, the rank of any summand of $M(X)$ is 0 or p^n by Lemma 2.18. The original proof of indecomposability of $M(X)$, given in [14], makes use of Quillen's computation of the K-theory of X and actually gives a more precise statement: the reduced Chow group $\overline{\operatorname{Ch}}_{\dim X}(X \times X)$ is generated by the diagonal class (while the above argument only shows that the diagonal class is the only non-zero projector).

2d. **Varieties (of flags) of ideals.** Let A be a central simple F-algebra. The F-dimension of any right ideal in A is divisible by the degree $\deg A$ of A; the quotient is the *reduced dimension* of the ideal. For any integer i, we write $X(i; A)$ for the generalized Severi-Brauer variety of the right ideals in A of reduced dimension i. In particular, $X(0; A) = \operatorname{Spec} F = X(\deg A; A)$ and $X(i, A) = \emptyset$ for i outside of the interval $[0, \deg A]$. The variety $X(1, A)$ is the usual Severi-Brauer variety of A studied in [1].

For a finite sequence of integers $i_1, \dots, i_r$, we write $X(i_1, \dots, i_r; A)$ for the variety of flags of right ideals in A of reduced dimensions $i_1, \dots, i_r$ (non-empty if and only if $0 \le i_1 \le \dots \le i_r \le \deg A$).

By an index reduction formula of [22], we have

$$\operatorname{ind} A_{F(X(i_1,\dots,i_r;A))} = \gcd(i_1, \dots, i_r, \operatorname{ind} A).$$

Another classical property of the variety $X(i_1, \dots, i_r; A)$ which we are using frequently is that the greatest common divisor of the degrees of its closed points is equal to

$$\operatorname{ind} A / \gcd(i_1, \dots, i_r, \operatorname{ind} A).$$

The varieties introduced above are projective homogeneous under the natural action of the algebraic group $\operatorname{Aut} A$. As in Introduction, we write $\mathfrak{X}_A$ for the class of finite direct products of such varieties.

Lemma 2.20. *Let A be a central simple F-algebra. Let n be a positive integer. Let $M_n(A)$ be the F-algebra of $(n\times n)$-matrices with entries in A. Let X be a variety in $\mathfrak{X}_{M_n(A)}$. Let $m = v_p(\operatorname{ind} A_{F(X)})$.*

Then the motive $M(X)$ (with integral and therefore also with arbitrary coefficients) decomposes into a sum of twists of motives of projective $(\operatorname{Aut} A)$-homogeneous F-varieties Y satisfying $v_p(\operatorname{ind} A_{F(Y)}) \le m$.

Proof. By the assumption on X we have $X = X_1 \times \cdots \times X_r$, where $X_1, \dots, X_r$ are some projective $(\operatorname{Aut} M_n(A))$-homogeneous F-varieties. Since $m = \min_{i=1}^r\{v_p(\operatorname{ind} A_{F(X_i)})\}$, we may assume that $m = v_p(\operatorname{ind} A_{F(X_1)})$. According to [9], the motive of X decomposes into a sum of twists of motives of projective $(\operatorname{Aut} M_n(A))$-homogeneous varieties Y admitting a morphism to X_1. In particular, $X_1(F(Y)) \neq \emptyset$ and consequently $v_p(\operatorname{ind} A_{F(Y)}) \le m$. Therefore we may assume that X is a projective $(\operatorname{Aut} M_n(A))$-homogeneous F-variety, that is, $X = X(n_1, \dots, n_r; M_n(A))$ for some positive integer r and some integers $n_1, \dots, n_r$ satisfying $0 \le n_1 \le \cdots \le n_r \le \deg M_n(A)$.

According to [15] (see also [8]), the motive of X decomposes into a sum of twists of motives of products of n varieties

$$Y = X(i_{11}, \dots, i_{1r}; A) \times \cdots \times X(i_{n1}, \dots, i_{nr}; A)$$

where $i_{11}, \dots, i_{nr}$ run over the integers satisfying $0 \le i_{j1} \le \cdots \le i_{jr} \le \deg A$ for any $j = 1, \dots, n$ and $i_{1k} + \cdots + i_{nk} = n_k$ for any $k = 1, \dots, r$. Since $\min_{jk} v_p(i_{jk}) \le \min_k v_p(n_k)$, it follows that

$$v_p(\operatorname{ind} A_{F(Y)}) = \min\{\operatorname{ind} A,\ \min_{jk} v_p(i_{jk})\} \le \min\{\operatorname{ind} A,\ \min_k v_p(n_k)\} = v_p(\operatorname{ind} A_{F(X)}) = m,$$

and we are done. $\square$

2e. **Canonical dimension.** The notion of canonical dimension was introduced in [4], of canonical dimension at p in [17], of essential dimension in [6], and of essential dimension at p in [24]. We refer to [17] and [20] for proofs of the statements cited below.

Let X be a smooth complete variety over F. We associate to X a functor

$$\mathcal{F}_X : \mathit{Fields}/F \to \mathit{Sets},$$

where Fields/F is the category of field extensions of F and Sets is the category of sets. For a field L/F such that $X(L) = \emptyset$ the value $\mathcal{F}_X(L)$ is empty. For a field L/F such that $X(L) \neq \emptyset$ the value $\mathcal{F}_X(L)$ is the singleton $\{L\}$. *Canonical dimension* $\operatorname{cd} X$ of X is defined as the essential dimension (see [3, Definition 1.2] or [18, §1.3] or [20, §1.1]) of the functor $\mathcal{F}_X$. For a positive prime integer p, *canonical dimension at* p, or *canonical p-dimension* $\operatorname{cd}_p X$ of X is defined as the essential dimension at p (see [18, §1.3] or [20, §1.1]) of the functor $\mathcal{F}_X$.

If L/F is a finite field extension of degree prime to p, then $\operatorname{cd}_p X_L = \operatorname{cd}_p X$.

If two smooth complete irreducible F-varieties X_1 and X_2 are such that there exist rational maps $X_1 \dashrightarrow X_2$ and $X_2 \dashrightarrow X_1$, then $\operatorname{cd} X_1 = \operatorname{cd} X_2$ and $\operatorname{cd}_p X_1 = \operatorname{cd}_p X_2$ for any p.

Assume that X is irreducible.

One has $\operatorname{cd}_p X \leq \operatorname{cd} X \leq \dim X$.

The variety X is called *incompressible* (*minimal* in [20]) if $\operatorname{cd} X = \dim X$ and *p-incompressible* (*p-minimal* in [20]) if $\operatorname{cd}_p X = \dim X$. A p-incompressible (for some prime p) variety is incompressible.

The variety X is incompressible if and only if any rational map $X \dashrightarrow X$ is dominant.

The variety X is p-incompressible if and only if for any element $\alpha \in \mathrm{CH}_{\dim X}(X \times X)$ the multiplicity $\operatorname{mult}(\alpha)$ is divisible by p if and only if the multiplicity $\operatorname{mult}(\alpha^t)$ of the transpose α^t of α is divisible by p.

Lemma 2.21. *A smooth complete irreducible variety X is p-incompressible if and only if any left-outer summand of the motive of X with coefficients in $\mathbb{F}_p$ is outer.*

Proof. A left-outer summand of $M(X)$ is given by a projector

$$\pi \in \mathrm{CH}_{\dim X}(X \times X; \mathbb{F}_p).$$

Since the summand is left-outer, $\operatorname{mult}(\pi) = 1 \in \mathbb{F}_p$. If X is p-incompressible, then $\operatorname{mult}(\pi^t) \neq 0$; therefore $\operatorname{mult}(\pi^t) = 1$ and the summand is right-outer.

Conversely, let $\alpha \in \mathrm{CH}_{\dim X}(X \times X; \mathbb{F}_p)$ be an element with $\operatorname{mult}(\alpha) \neq 0$. A power (with respect to the composition of correspondences) of α is a projector which determines a left-outer summand of $M(X)$. If this summand is also right-outer, then $\operatorname{mult}(\alpha^t) \neq 0$. $\square$

Corollary 2.22. *If the motive with coefficients in $\mathbb{F}_p$ of a smooth complete variety X is indecomposable, then the variety X is p-incompressible.* $\square$

3. Basic Theorems

Let p be a positive prime integer. The coefficient ring Λ is $\mathbb{F}_p$ in this section.

Let n be a non-negative integer. Let F be a field. Let D be a central division F-algebra of degree p^n.

Definition 3.1. For any integer l satisfying $0 \leq l \leq n$, we write $M_{l,D}$ for the indecomposable left-outer (see Definition 2.8 and Remark 2.12) summand of the Chow motive (with coefficients in $\mathbb{F}_p$) of the generalized Severi-Brauer variety $X(p^l, D)$.

Remark 3.2. It can be seen using Lemma 2.13 that the motive $M_{l,D}$ is also the indecomposable left-outer motivic summand of the variety $X(i, D)$ for any integer i satisfying $0 \leq i \leq \deg D$ and $v_p(i) = l$.

The first basic theorem of the present paper is as follows:

Theorem 3.3 (First basic theorem)**.** *Any indecomposable summand of the Chow motive with coefficients in $\mathbb{F}_p$ of any variety X in the class $\mathfrak{X}_D$ (see §2d) is isomorphic to a twist of $M_{l,D}$ for some $l \leq v_p(\operatorname{ind} D_{F(X)})$.*

Proof. We proof Theorem 3.3 by induction on n. The base of the induction is $n = 0$ and trivial (the class $\mathfrak{X}_D$ consists of one variety $\operatorname{Spec} F$ whose motive is equal to $\mathbb{F}_p = M_{n,D}$).

From now on we are assuming that $n \geq 1$ and that Theorem 3.3 is already proven for p-primary central division algebras (over all fields) of degree less than $\deg D = p^n$.

Let X be a variety in $\mathfrak{X}_D$ and let $m = v_p(\operatorname{ind} D_{F(X)})$. We have $0 \le m \le n$. The statement of Theorem 3.3 is trivial in the case of $m = n$ (because $X = \operatorname{Spec} F$ and $M(X) = \mathbb{F}_p = M_{n,D}$ in this case). We assume that $m < n$ in the sequel.

Let M be an indecomposable summand of $M(X)$. We have to show that M is isomorphic to a twist of $M_{l,D}$ for some l satisfying $0 \le l \le m$.

Let L be the function field of the variety $X(p^{n-1}, D)$. Therefore $\operatorname{ind} D_L = p^{n-1}$. Let C be a central division L-algebra Brauer-equivalent to D_L. By Lemma 2.20, the motive $M(X)_L$ decomposes into a sum of twists of motives of projective $(\operatorname{Aut} C)$-homogeneous L-varieties Y satisfying $\operatorname{ind} C_{L(Y)} \le p^m$. It follows by the induction hypothesis (applied to C), that the summands of the complete decomposition of $M(X)_L$ are twists of $M_{l,C}$ with $0 \le l \le m$. The complete decomposition of M_L is a part of the above decomposition (in the sense of the Krull-Schmidt principle, see §2a).

Each summand of the complete decomposition of M_L decomposes over an algebraic closure $\bar{L}$ of L into a sum of Tate motives. This gives a decomposition of $\bar{M} = M_{\bar{L}}$ into a sum of Tate motives. Let us choose a Tate summand $\mathbb{F}_p(i)$ with the smallest i in the decomposition of $\bar{M}$. This summand comes from the decomposition of the ith twist of some $\bar{M}_{l,C}$ for some integer l with $0 \le l \le m$. We shall show that $M \simeq M_{l,D}(i)$ for these l and i.

By Lemma 2.13 and since M is indecomposable, it suffices to construct morphisms

$$\alpha : M\big(X(p^l, D)\big)(i) \to M \quad\text{and}\quad \beta : M \to M\big(X(p^l, D)\big)(i)$$

satisfying $\operatorname{mult}(\beta \circ \alpha) = 1$.

The fixed above summand $\mathbb{F}_p(i)$ of $\bar{M}$ is produced by two elements $b \in \operatorname{Ch}^i(\bar{X})$ and $a \in \operatorname{Ch}_i(\bar{X})$ such that $\deg(a \cdot b) = 1 \in \mathbb{F}_p$. Let $M^t_{l,C}$ be the summand of $M\big(X(p^l, C)\big)$ obtained by transposing the projector giving the summand $M_{l,C}(i)$. Since $M_{l,C}$ is the indecomposable left-outer motivic summand of the variety $X(p^l, C)$, a twist of $M^t_{l,C}$ is the indecomposable right-outer motivic summand of the same variety. (Later on we shall prove that the left-outer indecomposable motivic summand of $X(p^l, C)$ coincides with the right-outer one, but this fact not needed here.) The element a is the image under the embedding $\operatorname{Ch}(\bar{M}^t_{l,C}) \hookrightarrow \operatorname{Ch}(\bar{X})$ of the class of a rational point in $\operatorname{Ch}\big(\bar{X}(p^l, C)\big)$ (see Lemma 2.10). Therefore the element $a_{\bar{L}(X(p^l,D))}$ is $L(X(p^l, D))$-rational. The field extension $L(X(p^l, D))/F(X(p^l, D))$ is purely transcendental. Consequently, the element $a_{\bar{L}(X(p^l,D))}$ is $F(X(p^l, D))$-rational and lifts to an element $\alpha_1 \in \operatorname{Ch}\big(X(p^l, D) \times X\big)$. We mean here a lifting with respect to the composition

$$\operatorname{Ch}\big(X(p^l, D) \times X\big) \twoheadrightarrow \operatorname{Ch}(X_{F(X(p^l,D))}) \xrightarrow{\operatorname{res}_{\bar{L}(X(p^l,D))/F(X(p^l,D))}} \operatorname{Ch}(X_{\bar{L}(X(p^l,D))})$$

where the first map is the epimorphism given by the pull-back with respect to the morphism $X_{F(X(p^l,D))} \to X(p^l, D) \times X$ induced by the generic point of the variety $X(p^l, D)$.

We define the morphism α as the composition

$$M\big(X(p^l, D)\big)(i) \xrightarrow{\alpha_1} M(X) \longrightarrow M.$$

The element b is the image of the class $[X(p^l, C)]$ under the embedding $\operatorname{Ch}(\bar{M}_{l,C}) \hookrightarrow \operatorname{Ch}(\bar{X})$. Let $\beta_1 \in \operatorname{Ch}\big(X(p^l, C) \times X(p^l, D_L)\big)$ be the class of the closure of the graph of a rational map (of L-varieties) $X(p^l, C) \to X(p^l, D_L)$. Let β_2 be the image of β_1 under the composition of the homomorphisms

$$\mathrm{Ch}(X(p^l,C)\times X(p^l,D_L))\to \mathrm{Ch}(M_{l,C}\otimes M(X(p^l,D_L)))\to \mathrm{Ch}(X_L\times X(p^l,D_L)).$$

Since the field extension $L(X)/F(X)$ is purely transcendental, the element $(\beta_2)_{L(X)}$ is $F(X)$-rational. Consequently, it lifts to an element $\beta_3\in \mathrm{Ch}(X\times X\times X(p^l,D))$ (where we use the generic point of the second factor in this triple direct product). Let $\pi\in\mathrm{Ch}(X\times X)$ be the projector defining the summand M of $M(X)$. Considering β_3 as a correspondence from X to $X\times X(p^l,D)$, we define $\beta_4\in\mathrm{Ch}(X\times X\times X(p^l,D))$ as the composition $\beta_3\circ\pi$. We get $\beta_5\in\mathrm{Ch}(X\times X(p^l,D))$ as the image of β_4 under the pull-back with respect to the diagonal of X. Finally, we define the morphism β as the composition

$$M \longrightarrow M(X)\xrightarrow{\beta_5} M\big(X(p^l,D)\big)(i).$$

We finish the proof by checking that $\mathrm{mult}(\beta\circ\alpha)=1$. Since the multiplicity is not changed under extension of scalars, we may do the computation over the field $\bar L$. Decompositions of the motives of the varieties $\bar X(p^l,D)=X(p^l,D)_{\bar L}$ and $\bar X=X_{\bar L}$ into sums of Tate motives give certain homogeneous $\mathbb{F}_p$-bases of their Chow groups with coefficients in $\mathbb{F}_p$. Let us use an arbitrary motivic decomposition of $\bar X(p^l,D)$ and we use a decomposition of $\bar X$ which is a refinement of the decomposition into the sum of $\bar M$ and the complementary summand. Then the elements a and b are in the basis of $\mathrm{Ch}(\bar X)$. The basis of $\mathrm{Ch}\big(\bar X(p^l,D)\big)$ contains the class $1=[\bar X(p^l,D)]$ and the class x of a rational point.

We consider the Chow group of the products

$$\bar X(p^l,D)\times\bar X,\ \ \bar X\times\bar X(p^l,D),\ \text{and}\ \bar X(p^l,D)\times\bar X(p^l,D)$$

together with the bases given by the external products of the elements of the bases of the Chow groups of the factors. We have $\bar\alpha_1=1\times a+\dots$, where "$\dots$" stands for a linear combination of basis elements whose first factor has codimension $>\operatorname{codim}1=0$. The projector π which determines the summand M of the motive of X looks over $\bar L$ as $\bar\pi=b\times a+\dots$, where "$\dots$" stands for a linear combination of basis elements whose first factor has codimension $>\operatorname{codim}b=i$. Since $\alpha=\pi\circ\alpha_1$, it follows that $\bar\alpha=1\times a+\dots$, where "$\dots$" still stands for a linear combination of basis elements whose first factor is of positive codimension.

Now let us go through the construction of β. To describe $\bar\beta_1$, we fix a homogeneous basis of $\mathrm{Ch}\big(\bar X(p^l,C)\big)$ and use it to built up a basis of $\mathrm{Ch}\big(\bar X(p^l,C)\times\bar X(p^l,D)\big)$. Abusing notation we write 1 also for the unit class in $\mathrm{Ch}\big(\bar X(p^l,C)\big)$.

We have $\bar\beta_1=1\times x+\dots$ with the usual convention on the meaning of "$\dots$". Then we have $\bar\beta_2=b\times x+\dots$. For β_3 we have $\bar\beta_3=b\times 1\times x+\dots$, where 1 is the unit of $\mathrm{Ch}(\bar X)$ and "$\dots$" stands for a linear combination of basis elements of $\mathrm{Ch}\big(\bar X\times\bar X\times\bar X(p^l,D)\big)$ which have a second factor of positive codimension *or* the first factor of dimension $<\dim b=i$. The element $\bar\beta_4$ has the same shape with the additional property that the dimension of the first factor in each basis element appearing in the linear combination is $\le i$. By this reason $\bar\beta_5$ and also $\bar\beta$ look as $b\times x+\dots$. Therefore $\bar\beta\circ\bar\alpha=1\times x+\dots$, that is, $\mathrm{mult}(\bar\beta\circ\bar\alpha)=1$. □

Corollary 3.4. *For an arbitrary central simple F-algebra A, let D be the p-primary component of a central division F-algebra Brauer-equivalent to A. The motive with coefficients* *in $\mathbb{F}_p$ of any variety in $\mathfrak{X}_A$ decomposes into a sum of twists of motives $M_{l,D}$ (with various l).*

Proof. We may replace F by a finite field extension of prime to p degree over which A is Brauer-equivalent to D. Then the motive of any variety in $\mathfrak{X}_A$ decomposes into a sum of twists of motives of varieties in $\mathfrak{X}_D$ and we may apply Theorem 3.3. □

Using Theorem 3.3, we give a new example of a generalized Severi-Brauer variety with indecomposable motive:

Example 3.5. Let F be a field. Let D be a central division F-algebra of degree 2^n with $n \geq 1$. Then the motive with coefficients in $\mathbb{F}_2$ of the variety $X(2; D)$ is indecomposable (this is trivial for $n = 1$, well known for $n = 2$, recently proved using K-theory by Maksim Zykhovich for $n = 3$, new for $n \geq 4$).

Proof. Let us prove it by induction on n. Assume that $n \geq 2$ and that the statement is already proved for algebras (over all fields) of degree $< 2^n$. By Theorem 3.3 and Remark 2.19, if the motive of $X(2; D)$ is decomposable (for a given D), then some twist of the motive of the Severi-Brauer variety $X(1; D)$ is a summand of $M\big(X(2; D)\big)$. We can check however that no twist of $M\big(X(1; D)\big)_L$ is a summand of $M\big(X(2; D)\big)_L$ where L/F is a field extension such that $\operatorname{ind} D_L = 2^{n-1}$.

Indeed, let C be a central division L-algebra Brauer-equivalent to D_L. The complete decompositions of the motives of these two varieties over L are:

$$M\big(X(1; D)\big)_L = M\big(X(1; C)\big) \oplus M\big(X(1; C)\big)(2^{n-1}) = M_{0,C} \oplus M_{0,C}(2^{n-1})$$

and (we apply the induction hypothesis to C)

$$\begin{aligned} &M\big(X(2; D)\big)_L = \\ &\quad M\big(X(2; C)\big) \oplus \Big(M\big(X(1; C)\big) \otimes M\big(X(1; C)\big)\Big)(2^{n-1} - 1) \oplus M\big(X(2; C)\big)(2^n) = \\ &\qquad M_{1,C} \oplus M_{0,C}(2^{n-1} - 1) \oplus M_{0,C}(2^{n-1}) \oplus \cdots \oplus M_{0,C}(2^n - 2) \oplus M_{1,C}(2^n). \end{aligned}$$

Note that the motives $M_{0,C}$ and $M_{1,C}$ are not isomorphic because, for instance, they have different ranks (see Example 2.16): $\operatorname{rk} M_{0,C} = \operatorname{rk} M\big(X(1, C)\big) = \binom{\deg C}{1} = 2^{n-1}$ and

$$\operatorname{rk} M_{1,C} = \operatorname{rk} M\big(X(2, C)\big) = \binom{\deg C}{2} = 2^{n-2}(2^{n-1} - 1).$$ □

We come back to an arbitrary central division F-algebra of degree p^n. The second basic theorem describes some properties of the motives $M_{m,D}$, $0 \leq m \leq n$:

Theorem 3.6 (Second basic theorem)**.** *The summand $M_{m,D}$ of the motive of the variety $X(p^m, D)$ is outer and*

$$v_p(\operatorname{rk} M_{m,D}) = n - m.$$

Proof. We prove Theorem 3.6 by induction on n. The base of the induction is the case of $n = 0$ which is trivial. Below we are assuming that $n \geq 1$.

The statement is trivial for $m = n$. Below we are assuming that $m < n$.

We are proving Theorem 3.6 for a given n by induction on m. The base of the induction is the case of $m = 0$ where we know (see Remark 2.19) that $M_{0,D} = M\big(X(1, D)\big)$ and $\operatorname{rk} M\big(X(1, D)\big) = p^n$. Below we are assuming that $m \geq 1$.

We ask the reader to note that in this proof we do not pay attention to the twisting numbers of summands in motivic decompositions: when we say that a motive W is a summand of a motive V, we mean that a *twist* of W is a summand of V.

Let L be the function field of the variety $X(p^{n-1}, D)$. Let C be a central division L-algebra (of degree p^{n-1}) Brauer-equivalent to D_L. The motive $M\big(X(p^m, D)\big)_L$ decomposes into the sum (of some twists) of the motives

$$(i_1, \dots, i_p) := M\big(X(i_1, C) \times X(i_2, C) \times \cdots \times X(i_p, C)\big),$$

where $i_1, \dots, i_p$ run over the non-negative integers satisfying $i_1 + i_2 + \cdots + i_p = p^m$. The left-outer summand in this decomposition is $(p^m, 0, \dots, 0) = M\big(X(p^m, C)\big)$.

Let $M = M_{m,D}$. Since the summand M of $M\big(X(p^m, D)\big)$ is left-outer, the L-motive M_L contains the summand $M_{m,C}$ of the summand $(p^m, 0, \dots, 0)$. We claim that in fact M_L contains the p summands $M_{m,C}$ coming from each of the p summands

$$(p^m, 0, \dots, 0) = M\big(X(p^m, D)\big),\ (0, p^m, 0, \dots, 0) = M\big(X(p^m, D)\big),$$
$$\dots,\ (0, \dots, 0, p^m) = M\big(X(p^m, D)\big).$$

Indeed, since the degree of any closed point on the variety $X(p^m, D)$ is divisible by p^{n-m}, we have $v_p(\operatorname{rk} M) \geq n - m$ by Lemma 2.18. On the other hand, by the induction hypothesis, $v_p(\operatorname{rk} M_{m,C}) = n - 1 - m$. The remaining summands of the complete motivic decomposition of $X(p^m, D)_L$ are $M_{l,C}$ with $l \leq m - 1$; v_p of their ranks are at least $n - m$ (by the induction hypothesis once again).

We have in particular proved that M_L contains the summand $M_{m,C}$ coming from the right-outer summand $(0, \dots, 0, p^m)$ of $M\big(X(p^m, D_L)\big)$. Since $M_{m,C}$ is a right-outer summand of $X(p^m, C)$ (by the induction hypothesis), it follows that M is right-outer.

It remains to prove the statement on the rank of $M = M_{m,D}$.

To do this, we look at ranks of the summands in the complete decomposition of M_L. We have p summands $M_{m,C}$ with $v_p(\operatorname{rk} M_{m,C}) = n - 1 - m$. So, we have $v_p(\operatorname{rk} M_{m,C}^{\oplus p}) = n - m$ for this part of the complete decomposition of M_L.

The summands $M_{l,C}$ with $l \leq m - 2$ have $v_p(\operatorname{rk} M_{l,C}) \geq n - m + 1$; so, we do not care about the number of such summands.

In order to show that $v_p(\operatorname{rk} M) = n - m$, it suffices to show that the number of the summands $M_{m-1,C}$ (which have $v_p(\operatorname{rk} M_{m-1,C}) = n - m$) in the complete decomposition of M_L is divisible by p.

Let us first count the number of summands $M_{m-1,C}$ in the complete motivic decomposition of $X(p^m, D)_L$. We use the complete decomposition of $X(p^m, D)_L$ which is a refinement of the decomposition into a sum of $(i_1, \dots, i_p)$ considered above. There are two types of such summands $M_{m-1,C}$: those which appear as summands of $M\big(X(p^m, C)\big)$ (if any) and all the others. Since the number of the summands $M\big(X(p^m, C)\big)$ is p, the number of the summands $M_{m-1,C}$ of the first type is divisible by p.

The summands of the second type are summands of the summands $(i_1, \dots, i_p)$ with $\min_j v_p(i_j) = m - 1$. There is precisely one such summand $(i_1, \dots, i_p)$ with $i_1 = \cdots = i_p$, namely, the summand $(p^{m-1},\dots,p^{m-1})$. All the other such summands $(i_1,\dots,i_p)$ can be divided into disjoint groups which are orbits of the action of the cyclic group $\mathbb{Z}/p$ on the indices: the group containing a given $(i_1,\dots,i_p)$ consists of

$$(i_1,\dots,i_p),\ (i_2,\dots,i_p,i_1),\ \dots,\ (i_p,i_1,\dots,i_{p-1})$$

(note that these are p different summands because the integer p is prime). So, the number of the summands $M_{m-1,C}$ coming from the summands $(i_1,\dots,i_p)$ different from $(p^{m-1},\dots,p^{m-1})$ is divisible by p.

As to the remaining summand $(p^{m-1},\dots,p^{m-1}) = M\big(X(p^{m-1},C)^{\times p}\big)$, we can show (using the induction hypothesis) that the number of the summands $M_{m-1,C}$ in its complete motivic decomposition is also divisible by p. More generally, we can show that the number of the summands $M_{m-1,C}$ in the complete motivic decomposition of $X(p^{m-1},C)^{\times r}$ is divisible by p as far as $r \geq 2$. Indeed,

$$v_p\Big(\operatorname{rk} M\big(X(p^{m-1},C)^{\times r}\big)\Big) = r(n-m) > n-m$$

(we recall that $m<n$). The complete motivic decomposition of $X(p^{m-1},C)^{\times r}$ consists of the motives $M_{l,C}$ with $l \leq m-1$. Finally, $v_p(\operatorname{rk} M_{l,C}) > n-m$ for $l < m-1$ and $v_p(\operatorname{rk} M_{m-1,C}) = n-m$.

We have shown that the number of summands $M_{m-1,C}$ in the complete motivic decomposition of $X(p^m,D)_L$ is divisible by p. We finish by showing that the number of those summands $M_{m-1,C}$ which are not in the complete decomposition of M_L is also divisible by p.

If a summand $M_{m-1,C}$ is not in $M_L = (M_{m,D})_L$, then it is in $(M_{m-1,D})_L$. However the number of summands $M_{m-1,C}$ in $M_{m-1,D}$ is divisible by p because $v_p(\operatorname{rk} M_{m-1,C}) = n-m$ and $v_p(d) = n-m+1$, where d is the gcd of the degrees of the closed points on $X(p^{m-1},D)$. □

Theorem 3.6 together with Lemma 2.21 give the main theorem:

Theorem 3.7 (Main theorem)**.** *Let p be a positive prime integer. Let n be a non-negative integer. Let F be a field. Let D be a central division F-algebra of degree p^n. Let m be an integer in the interval $[0,\ n]$. Then the variety $X(p^m;D)$ is p-incompressible.* □

For an arbitrary central simple F-algebra A, canonical p-dimension of the varieties in $\mathfrak{X}_A$ is computed as follows:

Corollary 3.8. *Let X be a variety in $\mathfrak{X}_A$. Let $n = v_p(\operatorname{ind} A)$ and $m = v_p(\operatorname{ind} A_{F(X)})$. Then $\operatorname{cd}_p X = p^m(p^n - p^m)$.*

Proof. Let D be the p-primary part of a central division F-algebra Brauer-equivalent to A. Let L/F be a finite field extension of prime to p degree such that the L-algebra A_L is Brauer-equivalent to D_L. There exist rational maps $X_L \dashrightarrow X(p^m,D_L)$ and $X(p^m,D_L) \dashrightarrow X_L$. It follows that

$$\operatorname{cd}_p X = \operatorname{cd}_p X_L = \operatorname{cd}_p X(p^m,D_L) = \dim X(p^m,D) = p^m(\deg D - p^m) = p^m(p^n-p^m). \ \square$$

References

[1] Artin, M. Brauer-Severi varieties. In *Brauer groups in ring theory and algebraic geometry (Wilrijk, 1981)*, vol. 917 of *Lecture Notes in Math.* Springer, Berlin, 1982, pp. 194–210.

[2] Bass, H. *Algebraic K-theory.* W. A. Benjamin, Inc., New York-Amsterdam, 1968.

[3] Berhuy, G., and Favi, G. Essential dimension: a functorial point of view (after A. Merkurjev). *Doc. Math. 8* (2003), 279–330 (electronic).

[4] Berhuy, G., and Reichstein, Z. On the notion of canonical dimension for algebraic groups. *Adv. Math. 198*, 1 (2005), 128–171.

[5] Brosnan, P., Reichstein, Z., and Vistoli, A. Essential dimension, spinor groups, and quadratic forms. *Ann. of Math. (2),* to appear.

[6] Buhler, J., and Reichstein, Z. On the essential dimension of a finite group. *Compositio Math. 106*, 2 (1997), 159–179.

[7] Calmès, B., Petrov, V., Semenov, N., and Zainoulline, K. Chow motives of twisted flag varieties. *Compos. Math. 142*, 4 (2006), 1063–1080.

[8] Chernousov, V., Gille, S., and Merkurjev, A. Motivic decomposition of isotropic projective homogeneous varieties. *Duke Math. J. 126*, 1 (2005), 137–159.

[9] Chernousov, V., and Merkurjev, A. Motivic decomposition of projective homogeneous varieties and the Krull-Schmidt theorem. *Transform. Groups 11*, 3 (2006), 371–386.

[10] Elman, R., Karpenko, N., and Merkurjev, A. *The algebraic and geometric theory of quadratic forms*, vol. 56 of *American Mathematical Society Colloquium Publications.* American Mathematical Society, Providence, RI, 2008.

[11] Florence, M. On the essential dimension of cyclic p-groups. *Invent. Math. 171*, 1 (2008), 175–189.

[12] Karpenko, N. A. Hyperbolicity of hermitian forms over biquaternion algebras. Linear Algebraic Groups and Related Structures (preprint server).

[13] Karpenko, N. A. Hyperbolicity of orthogonal involutions. Preprint, available on the web page of the author.

[14] Karpenko, N. A. Grothendieck Chow motives of Severi-Brauer varieties. *Algebra i Analiz 7*, 4 (1995), 196–213.

[15] Karpenko, N. A. Cohomology of relative cellular spaces and of isotropic flag varieties. *Algebra i Analiz 12*, 1 (2000), 3–69.

[16] Karpenko, N. A. On anisotropy of orthogonal involutions. *J. Ramanujan Math. Soc. 15*, 1 (2000), 1–22.

[17] Karpenko, N. A., and Merkurjev, A. S. Canonical p-dimension of algebraic groups. *Adv. Math. 205*, 2 (2006), 410–433.

[18] Karpenko, N. A., and Merkurjev, A. S. Essential dimension of finite p-groups. *Invent. Math. 172*, 3 (2008), 491–508.

[19] Mathews, B. G. Canonical dimension of projective $\mathrm{PGL}_1(A)$-homogeneous varieties. Preprint, available on the web page of the author.

[20] Merkurjev, A. S. Essential dimension. *Contemp. Math.,* to appear.

[21] Merkurjev, A. S. Steenrod operations and degree formulas. *J. Reine Angew. Math. 565* (2003), 13–26.

[22] Merkurjev, A. S., Panin, I. A., and Wadsworth, A. R. Index reduction formulas for twisted flag varieties. I. *K-Theory 10*, 6 (1996), 517–596.

[23] Petrov, V., Semenov, N., and Zainoulline, K. J-invariant of linear algebraic groups. *Ann. Sci. École Norm. Sup. (4) 41* (2008), 1023–1053.

[24] Reichstein, Z., and Youssin, B. Essential dimensions of algebraic groups and a resolution theorem for G-varieties. *Canad. J. Math. 52*, 5 (2000), 1018–1056. With an appendix by János Kollár and Endre Szabó.

[25] Vishik, A. Integral motives of quadrics. Preprint of the Max Planck Institute at Bonn, (1998)-13, 82 pages.

[26] Vishik, A. Motives of quadrics with applications to the theory of quadratic forms. In *Geometric methods in the algebraic theory of quadratic forms*, vol. 1835 of *Lecture Notes in Math.* Springer, Berlin, 2004, pp. 25–101.

UPMC Univ Paris 06, Institut de Mathématiques de Jussieu, F-75252 Paris, FRANCE

Web page: www.math.jussieu.fr/~karpenko
E-mail address: karpenko *at* math.jussieu.fr