On Grothendieck—Serre's conjecture concerning principal G-bundles over reductive group schemes:I

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07.05.2009

Abstract

Let R be a semi-local regular domain containing an infinite perfect subfield and let K be its field of fractions. Let G be a reductive semi-simple simply connected R-group scheme such that each of its R-indecomposable factors is isotropic. We prove that in this case the kernel of the map

$$H^1_{\text{ét}}(R, G) \to H^1_{\text{ét}}(K, G)$$

induced by the inclusion of R into K is trivial. In other words, under the above assumptions every principal G-bundle P which has a K-rational point is itself trivial. This confirms a conjecture posed by Serre and Grothendieck. Our proof is based on a combination of methods of Raghunathan's paper [R1], Ojanguren—Panin's paper [OP1] and Panin's preprint [Pa1].

If R is the semi-local ring of several points on a k-smooth scheme, then it suffices to require that k is infinite and keep the same assumption concerning G.

1 Introduction

Recall that an R-group scheme G is called reductive (respectively, semi-simple or simple), if it is affine and smooth as an R-scheme and if, moreover, for each ring homomorphism $s : R \to \Omega(s)$ to an algebraically closed field $\Omega(s)$, its scalar extension $G_{\Omega(s)}$ is a reductive (respectively, semi-simple or simple) algebraic group over $\Omega(s)$. The class of reductive group schemes contains the class of semi-simple group schemes which in turn contains the class of simple group schemes. This notion of a simple R-group scheme coincides with the notion of a simple semi-simple R-group scheme from Demazure—Grothendieck [D-G, Exp. XIX, Defn. 2.7 and Exp. XXIV, 5.3]. *Throughout the paper R denotes an integral domain and G denotes a semi-simple R-group scheme, unless explicitly stated otherwise.*

*The author acknowledges support of the RFFI project 09-01-91333-NNIO-a

†The author acknowledges support of the RFFI project 09-01-00878

‡The author acknowledges support of the RFFI projects 08-01-00756, 09-01-00762, 09-01-00784, 09-01-00878 and 09-01-91333

Such an R-group scheme G is called *simply-connected* (respectively, *adjoint*), provided that for an inclusion $s : R \hookrightarrow \Omega(s)$ of R into an algebraically closed field $\Omega(s)$ the scalar extension $G_{\Omega(s)}$ is a simply-connected (respectively, adjoint) $\Omega(s)$-group scheme. This definition coincides with the one from [D-G, Exp. XXII. Defn.4.3.3].

A well-known conjecture due to J.-P. Serre and A. Grothendieck [Se, Remarque, p.31], [Gr1, Remarque 3, p.26-27], and [Gr2, Remarque 1.11.a] asserts that given a regular local ring R and its field of fractions K and given a reductive group scheme G over R the map

$$H^1_{\text{ét}}(R, G) \to H^1_{\text{ét}}(K, G),$$

induced by the inclusion of R into K, has trivial kernel. The following theorem, which is the main result of the present paper, asserts that for isotropic groups this is indeed the case (*recall that a simple R-group scheme is called isotropic if it contains a split tori* $\mathbb{G}_{m,R}$).

Theorem 1.1. *Let R be regular semi-local domain containing an infinite perfect field and let K be its field of fractions. Let G be an isotropic simple simply-connected group scheme over R. Then the map*

$$H^1_{\text{ét}}(R, G) \to H^1_{\text{ét}}(K, G),$$

induced by the inclusion R into K, has trivial kernel.

In other words, under the above assumptions on R and G each principal G-bundle P over R which has a K-rational point is itself trivial. More generally, it is natural to ask, whether two elements $\xi, \zeta \in H^1_{\text{ét}}(R, G)$ which are equal over K are already equal over R. In general, this does not follow from Theorem 1.1. However, this is indeed the case provided at least one of the group schemes $G(\xi)$ or $G(\zeta)$ is isotropic.

Theorem 1.1 can be extended to the case of semi-simple group schemes as well. However, in this generality its statement is a bit more technical and we postpone it till Section 12 (see Theorem 12.1). Let us list other known results in the same vein, corroborating Serre—Grothendieck's conjecture.

• For simple simply connected group schemes of *classical* series this result follows from more general results established by the first author, A. Suslin, M. Ojanguren and K. Zainoulline [PS], [OP1], [Z], [OPZ], In fact, unlike our Theorem 1.1, *no isotropy hypotheses* was imposed there. However, for the *exceptional* group schemes *our theorem is new*. Our proof is based on different ideas and treats classical and exceptional types in a unified way.

• The case of an arbitrary reductive group scheme over a discrete valuation ring is completely solved by Y. Nisnevich in [Ni].

• The case where G is an arbitrary torus over a regular local ring was settled by J.-L. Colliot-Thélène and J.-J. Sansuc in [C-T/S].

• In [Pa2] I.Panin extended Theorem 1.1 to the case of *an arbitrary reductive* group scheme satisfying a mild "isotropy" condition; **however his proof is heavily based on** Theorem 1.1 and on the main result of [C-T/S] concerning the case of tori.

• The case of *arbitrary simple adjoint* group schemes of type E_6 and E_7 is done by the first author, V.Petrov and A.Stavrova in [PPS]. *No isotropy condition is imposed there.*

• There exists a folklore result, concerning a simple R-group scheme of type G_2, where R is a regular semi-local ring containing an infinite perfect field or the semilocal ring of finitely many points on a k-smooth scheme with an infinite field k. That result gives affirmative answer in this case and also independent of isotropy hypotheses, see the paper by V. Chernousov and the first author [ChP].

• The case where the group scheme G comes from the ground field k is completely solved by J.-L. Colliot-Thélène, M. Ojanguren, M. S. Raghunatan and O. Gabber: in [C-T/O] when k is infinite; in [R1] when k is perfect; O. Gabber [Ga] announced a proof for an arbitrary ground field k.

• *The remaining unsolved open question* in case of simple simply-connected group scheme of *classical type* is this: prove the conjecture for the spinor group of an algebra with an orthogonal involution. The case of the spinor group of a quadratic space is done in [OPZ].

A geometric counterpart of Theorem 1.1 is the following result

Theorem 1.2. *Let k be an infinite field. Let $\mathcal{O}$ be the semi-local ring of finitely many points on a smooth irreducible k-variety X and let K be its field of fractions. Let G be an isotropic simple simply-connected group scheme over $\mathcal{O}$. Then the map*

$$H^1_{\text{ét}}(\mathcal{O}, G) \to H^1_{\text{ét}}(K, G),$$

induced by the inclusion $\mathcal{O}$ into K, has trivial kernel.

2 An outline of the proof

Since the proof of Theorem 1.1 is long and technical, in this section we briefly describe its general outline. The actual proofs are postponed till Section 11. Our proof consists of three parts.

(1) Reduction of the general case to Theorem 1.2.

Theorem 1.2 is usually called the *geometric case* of Theorem 1.1. The reduction itself is rather standard, the arguments are very similar to those in [OP2, Sect.7], and we mostly skip them.

In turn, analysis of the geometric case is subdivided into two parts:

(2) a *geometric part*, and

(3) a *group part*.

The geometric part of the proof starts with the following data. Fix a smooth affine k-scheme X, a finite family of points $x_1, x_2, \dots, x_n$ on X, and set $\mathcal{O} := \mathcal{O}_{X,\{x_1,x_2,\dots,x_n\}}$ and $U := \text{Spec}(\mathcal{O})$. Further, consider a simple simply-connected U-group scheme G and a principal G-bundle P over $\mathcal{O}$ which is trivial over the field of fractions K of $\mathcal{O}$. We may and will assume that for certain $\mathrm{f} \in \mathcal{O}$ the principal G-bundle P is trivial over $\mathcal{O}_{\mathrm{f}}$. Shrinking X if necessary, we may secure the following properties

(i) The points $x_1, x_2, \ldots, x_n$ are still in X.

(ii) The group scheme G is a simple group scheme defined over X. We often denote this X-group scheme by G_X and write G_U for the original G.

(iii) The principal G-bundle P is defined over X and $\mathrm{f} \in k[X]$.

(iv) The restriction P_{f} of the bundle P to a principal open sub-scheme X_{f} is trivial for a non-zero $\mathrm{f} \in k[X]$ and f vanishes at each x_i's.

In the sequel we may shrink X a little further, if necessary, always looking after verification of (i) to (iv). This way we construct in Section 5 a basic nice triple (9), see Definition 4.1. Beginning with that basic nice triple we construct in Section 6 the following data:

(a) a unitary polynomial $h(t)$ in $R[t]$,

(b) a principal G_U-bundle P_t over $\mathbf{A}^1_U$,

(c) a diagram of the form

$$\begin{tikzcd}
\mathbf{A}^1_U \arrow[dr, "\mathrm{pr}"'] & Y \arrow[l, "\sigma"'] \arrow[r, "q_X"] \arrow[d, "q_U"'] & X \\
& U \arrow[u, bend right, "\delta"'] \arrow[ur, "\mathrm{can}"'] &
\end{tikzcd} \tag{1}$$

(d) an isomorphism $\Phi : q_U^*(G_U) \to q_X^*(G)$ of Y-group schemes.

By Theorem 6.1 one may chose these objects to enjoy the following properties:

(1*) $q_U = \mathrm{pr} \circ \sigma$,

(2*) σ is étale,

(3*) $q_U \circ \delta = \mathrm{id}_U$,

(4*) $q_X \circ \delta = \mathrm{can}$,

(5*) the restriction of P_t to $(\mathbf{A}^1_U)_h$ is a trivial G_U-bundle,

(6*) $(\sigma)^*(P_t)$ and $q_X^*(P)$ are isomorphic as principal G_U-bundles. Here $q_X^*(P)$ is regarded as a principal G_U-bundle via the group scheme isomorphism Φ from Item (d).

This completes the geometric part. In the group part we prove the following result, see the end of Section 8.

Theorem 2.1. *Let $\mathcal{O}$ and G be the same as in Theorem 1.2. Let P be a principal G-bundle over the polynomial ring $\mathcal{O}[t]$ in one variable t such that for a monic polynomial $h(t) \in \mathcal{O}[t]$ the bundle $P_{h(t)}$ over the ring $\mathcal{O}[t]_{h(t)}$ is trivial. Then the G-bundle P is trivial.*

It remains to explain how Theorem 1.2 follows from Theorem 2.1. We have to check that the G-bundle P is trivial over U.

By assumption the group scheme G_U is isotropic. The restriction of the principal G_U-bundle P_t to $(\mathbf{A}^1_U)_h$ is trivial by Condition (5*). Then the principal G_U-bundle P_t is itself trivial by Theorem 2.1.

On the other hand, the G_U-bundle $(\sigma)^*(P_t)$ is isomorphic to $q_X^*(P)$ as principal G_U-bundle by (6*). Therefore, $q_X^*(G)$ is trivial as a $q_X^*(G)$-bundle. It follows that $\delta^*(q_X^*(P)) = \mathrm{can}^*(P)$ is trivial as a $\delta^*(q_X^*(G)) = \mathrm{can}^*(G)$-bundle. Thus, $\mathrm{can}^*(P) = P_U$ is trivial as a principal $\mathrm{can}^*(G) = G_U$-bundle.

3 Elementary fibrations

In this Section we extend a result of Artin from [A] concerning existence of nice neighborhoods. The following notion is due to Artin [A, Exp. XI, Déf. 3.1].

Definition 3.1. *An elementary fibration is a morphism of schemes $p : X \to S$ which can be included in a commutative diagram*

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "p"'] & \overline{X} \arrow[d, "\overline{p}"] & Y \arrow[l, "i"'] \arrow[dl, "q"] \\ & S & \end{tikzcd} \tag{2}$$

of morphisms satisfying the following conditions:

(i) *j is an open immersion dense at each fibre of $\overline{p}$, and $X = \overline{X} - Y$;*

(ii) *$\overline{p}$ is smooth projective all of whose fibres are geometrically irreducible of dimension one;*

(iii) *q is finite étale all of whose fibres are non-empty.*

The following Bertini type theorem is an extension of Artin's result [A, Exp.XI,Thm.2.1]

Theorem 3.2. *Let k be an infinite field, and let $V \subset \mathbf{P}^n_k$ be a locally closed sub-scheme of pure dimension r. Further, let $V' \subset V$ be an open sub-scheme consisting of all points $x \in V$ such that V is k-smooth at x. Finally, let $p_1, p_2, \ldots, p_m \in \mathbf{P}^n_k$ be a family of pair-wise distinct closed points. For a family $H_1(d), H_2(d), \ldots, H_s(d)$, with $s \leq r$, of hyper-planes of degree d containing all points p_i, $1 \leq i \leq m$, set*

$$Y = H_1(d) \cap H_2(d) \cap \cdots \cap H_s(d).$$

Then there exists an integer d depending on the family $p_1, p_2, \ldots, p_m$ such that if the family $H_1(d), H_2(d), \ldots, H_s(d)$ with $s \leq r$ is sufficiently general, then Y crosses V transversally at each point of $Y \cap V'$.

If, moreover, V is irreducible (respectively, geometrically irreducible) and $s < r$ then for the same integer d and for a sufficiently general family $H_1(d), H_2(d), \ldots, H_s(d)$ the intersection $Y \cap V$ is irreducible (respectively, geometrically irreducible).

Using this theorem, one can prove the following result, which is a slight extension of Artin's result [A, Exp.XI,Prop.3.3]

Proposition 3.3. *Let k be an infinite field, X/k be a smooth geometrically irreducible variety, $x_1, x_2, \dots, x_n \in X$ be closed points. Then there exists a Zariski open neighborhood X^0 of the family $\{x_1, x_2, \dots, x_n\}$ and an elementary fibration $p : X^0 \to S$, where S is an open sub-scheme of the projective space $\mathbf{P}^{\dim X - 1}$.*

If, moreover, Z is a closed co-dimension one subvariety in X, then one can choose X^0 and p in such a way that $p|_{Z \bigcap X^0} : Z \bigcap X^0 \to S$ is finite surjective.

We also omit for now the proof of the following proposition.

Proposition 3.4. *Let $p : X \to S$ be an elementary fibration. If S is a regular semi-local scheme, then there exists a commutative diagram of S-schemes*

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "\pi"'] & \overline{X} \arrow[d, "\overline{\pi}"] & Y \arrow[l, "i"'] \arrow[d] \\ \mathbf{A}^1 \times S \arrow[r, "\mathrm{in}"] & \mathbf{P}^1 \times S & \{\infty\} \times S \arrow[l, "i"'] \end{tikzcd} \qquad (3)$$

such that the left hand side square is Cartesian. Here, j and i are the same as in Definition 3.1, while $\mathrm{pr}_S \circ \pi = p$, where pr_S is the projection $\mathbf{A}^1 \times S \to S$.

In particular, $\pi : X \to \mathbf{A}^1 \times S$ is a finite surjective morphism of S-schemes, where X and $\mathbf{A}^1 \times S$ are regarded as S-schemes via the morphism p and the projection pr_S, respectively.

4 Nice triples

In the present section we introduce and study certain collections of geometric data and their morphisms. The concept of a *nice triple* is very similar to that of a *standard triple* introduced by Voevodsky [Vo, Defn.4.1], and was in fact inspired by that last notion. Let k be an infinite field, X/k be a smooth geometrically irreducible variety, and let $x_1, x_2, \dots, x_n \in X$ be its closed points. Further, let $\mathcal{O} = \mathcal{O}_{X,\{x_1,x_2,\dots,x_n\}}$ be the corresponding geometric semi-local ring.

Definition 4.1. *Let $U := \mathrm{Spec}(\mathcal{O}_{X,\{x_1,x_2,\dots,x_n\}})$. A* **nice triple** *over U consists of the following data:*

(i) *a smooth morphism $q_U : \mathcal{X} \to U$,*

(ii) *an element $f \in \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$,*

(iii) *a section Δ of the morphism q_U,*

subject to the following conditions:

(a) *each component of each fibre of the morphism q_U has dimension one,*

(b) *the module $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})/f \cdot \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is finite as a $\Gamma(U, \mathcal{O}_U) = \mathcal{O}$-module,*

(c) *there exists a finite surjective U-morphism* $\Pi : \mathcal{X} \to \mathbf{A}^1 \times U$,

(d) $\Delta^*(f) \neq 0 \in \Gamma(U, \mathcal{O}_U)$.

Definition 4.2. *A* **morphism** *of two nice triples* $(\mathcal{X}', f', \Delta') \to (\mathcal{X}, f, \Delta)$ *is an étale morphism of U-schemes* $\theta : \mathcal{X}' \to \mathcal{X}$ *such that*

(1) $q'_U = q_U \circ \theta$,

(2) $f' = \theta^*(f) \cdot g'$ *for an element* $g' \in \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$,

(3) $\Delta = \theta \circ \Delta'$.

Two observations are in order here.

• Item (2) implies in particular that $\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})/\theta^*(f) \cdot \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$ is a finite $\mathcal{O}$-module.

• It should be emphasized that no conditions are imposed on the inter-relation of Π' and Π.

Let us state two crucial results which will be used in our main construction. Their proofs are postponed till Sections 9 and 10.

Theorem 4.3. *Let U be as in Definition 4.1. Let $(\mathcal{X}, f, \Delta)$ be a nice triple over U. Let $G_{\mathcal{X}}$ be a simple simply-connected $\mathcal{X}$-group scheme, and let $G_U := \Delta^*(G_{\mathcal{X}})$. Finally, let G_{const} be the pull-back of G_U to $\mathcal{X}$. Then there exists a morphism $\theta : (\mathcal{X}', f', \Delta') \to (\mathcal{X}, f, \Delta)$ of nice triples and an isomorphism*

$$\Phi : \theta^*(G_{\mathrm{const}}) \to \theta^*(G_{\mathcal{X}})$$

of $\mathcal{X}'$-group schemes such that $(\Delta')^*(\Phi) = \mathrm{id}_{G_U}$.

The proof of this theorem is sketched in Section 9. Let U be as in Definition 4.1. Let $(\mathcal{X}, f, \Delta)$ be a nice triple over U. Then for each finite surjective U-morphism $\sigma : \mathcal{X} \to \mathbf{A}^1 \times U$ and the corresponding $\mathcal{O}$-algebra inclusion $\mathcal{O}[t] \hookrightarrow \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ the algebra $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is finitely generated as an $\mathcal{O}[t]$-module. Since both rings $\mathcal{O}[t]$ and $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ are regular, the algebra $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is finitely generated and projective as an $\mathcal{O}[t]$-module by theorem [E, Cor.18.17]. Take the characteristic polynomial $t^r - a_{n-1}t^{r-1} + \cdots \pm N(f)$ of the $\mathcal{O}[t]$-module endomorphism $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \xrightarrow{f} \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ and set

$$g_{f,\sigma} := f^{r-1} - a_{n-1}f^{r-2} + \cdots \pm a_1 \in \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}). \tag{4}$$

Lemma 4.4. $f \cdot g_{f,\sigma} = \pm N(f) \in \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$.

In fact, the characteristic polynomial of the operator $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \xrightarrow{f} \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ vanishes on f.

Theorem 4.5. *Let U be as in Definition 4.1. Let $(\mathcal{X}, f, \Delta)$ be a nice triple over U. There exists a distinguished finite surjective morphism*

$$\sigma : \mathcal{X} \to \mathbf{A}^1 \times U$$

of U-schemes which enjoys the following properties.

(1) *σ is étale along the closed subset $\{f=0\} \cup \Delta(U)$.*

(2) *For $g_{f,\sigma}$ and $N(f)$ defined by the distinguished σ, one has*

$$\sigma^{-1}\Big(\sigma\big(\{f=0\}\big)\Big) = \{N(f)=0\} = \{f=0\} \sqcup \{g_{f,\sigma}=0\}.$$

(3) *Denote by $\mathcal{X}^0 \hookrightarrow \mathcal{X}$ the largest open sub-scheme, where the morphism σ is étale. Write g for $g_{f,\sigma}$ in this item. Then the square*

$$\begin{tikzcd} \mathcal{X}^0_{N(f)} = \mathcal{X}^0_{fg} \arrow[r, "\mathrm{inc}"] \arrow[d, "\sigma^0_{fg}"'] & \mathcal{X}^0_g \arrow[d, "\sigma^0_g"] \\ (\mathbf{A}^1 \times U)_{N(f)} \arrow[r, "\mathrm{in}"] & \mathbf{A}^1 \times U \end{tikzcd} \qquad (5)$$

is an elementary Nisnevich square. More precisely, this square is Cartesian and the morphism of the reduced closed sub-schemes

$$\sigma^0_g|_{\{f=0\}_{\mathrm{red}}} : \{f=0\}_{\mathrm{red}} \to \{N(f)=0\}_{\mathrm{red}}$$

of the schemes $\mathcal{X}^0_g$ and $\mathbf{A}^1 \times U$ is an isomorphism.

(4) *One has $\Delta(U) \subset \mathcal{X}^0_g$.*

A sketch of the proof of this Theorem is given in Section 10.

Using Theorems 4.3 and 4.5 in the next Section we construct data (a) to (d) from the Introduction subject to Conditions (1*) to (6*).

5 A basic nice triple

With Propositions 3.3 and 3.4 at our disposal we may form *a basic nice triple*, namely the triple (9). This is the main aim of the present Section. Namely, fix a smooth irreducible affine k-scheme X, a finite family of points $x_1, x_2, \dots, x_n$ on X, and a non-zero function $\mathrm{f} \in k[X]$. We *always assume* that the set $\{x_1, x_2, \dots, x_n\}$ is contained in the vanishing locus of the function f.

Replacing k by its algebraic closure in $k[X]$, we may assume that X is a geometrically irreducible k-variety. By Proposition 3.3 there exist a Zariski open neighborhood X^0 of the family $\{x_1, x_2, \dots, x_n\}$ and an elementary fibration $p : X^0 \to S$, where S is an open sub-scheme of the projective space $\mathbf{P}^{\dim X - 1}$, such that

$$p|_{\{\mathrm{f}=0\}\cap X^0} : \{\mathrm{f}=0\} \cap X^0 \to S$$

is finite surjective. Let $s_i = p(x_i) \in S$, for each $1 \le i \le n$. Shrinking S, we may assume that S is *affine* and still contains the family $\{s_1, s_2, \dots, s_n\}$. Clearly, in this case $p^{-1}(S) \subseteq X^0$ contains the family $\{x_1, x_2, \dots, x_n\}$. We replace X by $p^{-1}(S)$ and f by its restriction to this new X.

In this way we get an elementary fibration $p : X \to S$ such that

$$\{x_1, \dots, x_n\} \subset \{\mathrm{f} = 0\} \subset X,$$

S is an open affine sub-scheme in the projective space $\mathbf{P}^{\dim X - 1}$, and the restriction of $p|_{\{\mathrm{f}=0\}} : \{\mathrm{f} = 0\} \to S$ to the vanishing locus of f is a finite surjective morphism. In other words, $k[X]/(\mathrm{f})$ is finite as a $k[S]$-module.

As an open affine sub-scheme of the projective space $\mathbf{P}^{\dim X - 1}$ the scheme S is regular. By Proposition 3.4 one can shrink S in such a way that S is still affine, contains the family $\{s_1, s_2, \dots, s_n\}$ and there exists a finite surjective morphism

$$\pi : X \to \mathbf{A}^1 \times S$$

such that $p = \mathrm{pr}_S \circ \pi$. Clearly, in this case $p^{-1}(S) \subseteq X$ contains the family $\{x_1, x_2, \dots, x_n\}$. We replace X by $p^{-1}(S)$ and f by its restriction to this new X.

In this way we get an elementary fibration $p : X \to S$ such that

$$\{x_1, \dots, x_n\} \subset \{\mathrm{f} = 0\} \subset X,$$

S is an open affine sub-scheme in the projective space $\mathbf{P}^{\dim X - 1}$, and the restriction of $p|_{\{\mathrm{f}=0\}} : \{\mathrm{f} = 0\} \to S$ to the vanishing locus of f is a finite surjective morphism. Eventually we conclude that there exists a finite surjective morphism

$$\pi : X \to \mathbf{A}^1 \times S$$

such that $p = \mathrm{pr}_S \circ \pi$.

Now, set $U := \mathrm{Spec}(\mathcal{O}_{X,\{x_1,x_2,\dots,x_n\}})$, denote by $\mathrm{can} : U \hookrightarrow X$ the canonical inclusion of schemes, and let $p_U = p \circ \mathrm{can} : U \to S$. Further, we consider the fibre product $\mathcal{X} := U \times_S X$. Then the canonical projections $q_U : \mathcal{X} \to U$ and $q_X : \mathcal{X} \to X$ and the diagonal morphism $\Delta : U \to \mathcal{X}$ can be included in the following diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r, "q_X"] \arrow[d, shift right, "q_U"'] & X \\ U \arrow[u, shift right, bend right, "\Delta"'] \arrow[ur, "\mathrm{can}"'] & \end{tikzcd} \tag{6}$$

where

$$q_X \circ \Delta = \mathrm{can} \tag{7}$$

and

$$q_U \circ \Delta = \mathrm{id}_U\,. \tag{8}$$

Note that q_U is *a smooth morphism with geometrically irreducible fibres of dimension one*. Indeed, observe that q_U is a base change via p_U of the morphism p which has the desired properties. Taking the base change via p_U of the finite surjective morphism $\pi : X \to \mathbf{A}^1 \times S$ we get *a finite surjective morphism*

$$\Pi : \mathcal{X} \to \mathbf{A}^1 \times U$$

such that $q_U = \mathrm{pr}_U \circ \Pi$. Set $f := q_X^*(\mathrm{f})$. The $\mathcal{O}_{X,\{x_1,x_2,\ldots,x_n\}}$-module $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})/f \cdot \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is finite, since the $k[S]$-module $k[X]/\mathrm{f} \cdot k[X]$ is finite.

Note that the data

$$(q_U : \mathcal{X} \to U, f, \Delta) \tag{9}$$

form an example of a *nice triple*, as defined in Definition 4.1.

Claim 5.1. *The schemes $\Delta(U)$ and $\{f = 0\}$ are both semi-local and the set of closed points of $\Delta(U)$ is contained in the set of closed points of $\{f = 0\}$.*

This holds since the set $\{x_1, x_2, \ldots, x_n\}$ is contained in the vanishing locus of the function f.

6 Main Construction

The main result of this Section is Theorem 6.1.

Fix a smooth affine k-scheme X, a finite family of points $x_1, x_2, \ldots, x_n$ on X, and set $\mathcal{O} := \mathcal{O}_{X,\{x_1,x_2,\ldots,x_n\}}$ and $U := \mathrm{Spec}(\mathcal{O})$. Further, consider a simple simply-connected U-group scheme G and a principal G-bundle P over $\mathcal{O}$ which is trivial over the field of fractions K of $\mathcal{O}$. We may and will assume that for certain $\mathrm{f} \in \mathcal{O}$ the principal G-bundle P is trivial over $\mathcal{O}_{\mathrm{f}}$. Shrinking X if necessary, we may secure the following properties

(i) The points $x_1, x_2, \ldots, x_n$ are still in X.

(ii) The group scheme G is defined over X and it is a simple group scheme. We will often denote this X-group scheme by G_X and write G_U for the original G.

(iii) The principal G-bundle P is defined over X and the function f belongs to $k[X]$.

(iv) The restriction P_{f} of the bundle P to a principal open sub-set X_{f} is trivial and f vanishes at each x_i's.

In particular, now we are given the smooth irreducible affine k-scheme X, the finite family of points $x_1, x_2, \ldots, x_n$ on X, and the non-zero function $\mathrm{f} \in k[X]$ vanishing at each point x_i. Recall, that starting from these data we constructed at the very end of Section 5 the nice triple (9). We did that shrinking X and secure properties (1) to (4) at the same time.

Let G be the simple simply-connected X-group scheme, P be the principal G-bundle over X. The restriction P_{f} of the bundle P to the principal open sub-scheme X_{f} is trivial by Item (iv) above. Set $G_{\mathcal{X}} := (q_X)^*(G)$. By Theorem 4.3 there exists a morphism of nice triples

$$\theta : (\mathcal{X}', f', \Delta') \to (\mathcal{X}, f, \Delta)$$

and an isomorphism

$$\Phi : \theta^*(G_{\mathrm{const}}) \to \theta^*(G_{\mathcal{X}}) =: G_{\mathcal{X}'} \tag{10}$$

of $\mathcal{X}'$-group schemes such that $(\Delta')^*(\Phi) = \mathrm{id}_{G_U}$.

Set

$$q'_X = q_X \circ \theta : \mathcal{X}' \to X. \tag{11}$$

Recall that

$$q'_U = q_U \circ \theta : \mathcal{X}' \to U, \tag{12}$$

since θ is a morphism of nice triples. Consider $(q'_X)^*(P)$ as a principal $(q'_U)^*(G_U) = G_{\mathrm{const}}$-bundle via the isomorphism Φ. Recall that P is trivial as a G-bundle over X_{f}. Therefore, $(q'_X)^*(P)$ is trivial as a principal $G_{\mathcal{X}'}$-bundle over $\mathcal{X}'_{\theta^*(f)}$. Since θ is a nice triple morphism one has $f' = \theta^*(f) \cdot g'$, and thus the principal $G_{\mathcal{X}'}$-bundle $(q'_X)^*(P) = (q_X \circ \theta)^*(G)$ is trivial over $\mathcal{X}'_{f'}$.

We can conclude that $(q'_X)^*(P)$ is trivial over $\mathcal{X}'_{f'}$, when regarded as a principal G_{const}-bundle via the isomorphism Φ.

By Theorem 4.5 there exists a finite surjective morphism $\sigma : \mathcal{X}' \to \mathbf{A}^1 \times U$ of U-schemes satisfying (1) to (3) from that Theorem. In particular, one has

$$\sigma^{-1}\Big(\sigma\big(\{f' = 0\}\big)\Big) = N(f') = \{f' = 0\} \sqcup \{g'_\sigma = 0\}$$

with $N(f')$ and g' defined in the item (2) of Theorem 4.5. Thus, replacing for brevity g'_σ by g', one gets the following elementary Nisnevich square

$$\begin{tikzcd}
(\mathcal{X}')^0_{N(f')} = (\mathcal{X}')^0_{f'g'} \arrow[r, "\mathrm{inc}"] \arrow[d, "\sigma^0_{f'g'}"'] & (\mathcal{X}')^0_{g'} \arrow[d, "\sigma^0_{g'}"] \\
(\mathbf{A}^1 \times U)_{N(f')} \arrow[r, "\mathrm{in}"] & \mathbf{A}^1 \times U
\end{tikzcd} \tag{13}$$

Regarded as a principal G_U-bundle via the isomorphism Φ, the bundle $(q'_X)^*(P)$ over $\mathcal{X}'$ becomes trivial over $\mathcal{X}'_{f'}$, and a fortiori over $(\mathcal{X}')^0_{f'g'}$. Now, taking the trivial G_U-bundle over $(\mathbf{A}^1 \times U)_{N(f')}$ and the isomorphism

$$\psi : (\mathcal{X}')^0_{N(f')} \times_U G_U \to (q'_X)^*(P)|_{(\mathcal{X}')^0_{N(f')}} \tag{14}$$

of principal G_U-bundles, we get a principal G_U-bundle P_t over $\mathbf{A}^1 \times U$ such that

(1) it is trivial over $(\mathbf{A}^1 \times U)_{N(f')}$,

(2) $(\sigma)^*(P_t)$ and $(q'_X)^*(P)$ are isomorphic as principal G_U-bundles. Here $(q'_X)^*(P)$ is regarded as a principal G_U-bundle via the $\mathcal{X}'$-group scheme isomorphism Φ from (10);

(3) over $(\mathcal{X}')^0_{N(f')}$ the two G_U-bundles are identified via the isomorphism ψ from (14).

Finally, form the following diagram

$$\begin{tikzcd}
\mathbf{A}^1_U \arrow[dr, "\mathrm{pr}"'] & (\mathcal{X}')^0_{g'} \arrow[l, "\sigma^0_{g'}"'] \arrow[r, "q'_X"] \arrow[d, "q'_U"'] & X \\
& U \arrow[u, bend right, "\Delta'"'] \arrow[ur, "\mathrm{can}"'] &
\end{tikzcd} \tag{15}$$

This diagram is well-defined since by Item (4) of Theorem 4.5 the image of the morphism Δ' lands in $(X')^0_{g'}$.

Theorem 6.1. *The unitary polynomial $N(f')$, the principal G_{const}-bundle P_t over $\mathbf{A}^1_U$, the diagram* (15) *and the isomorphism* (10) *constructed above satisfy Conditions* (1*) *to* (6*) *from the Introduction.*

Proof. By the very choice of σ it is an U-scheme morphism, which proves (1*). Since $(\mathcal{X}')^0 \hookrightarrow \mathcal{X}'$ is the largest open sub-scheme where the morphism σ is étale, one gets (2*). Property (3*) holds for Δ' since $(\mathcal{X}', \mathcal{Z}', \Delta')$ is a nice triple and, in particular, Δ' is a section of q'_U. Property (4*) can be established as follows:

$$q'_X \circ \Delta' = (q_X \circ \theta) \circ \Delta' = q_X \circ \Delta = \mathrm{can}.$$

The first equality here holds by the definition of q'_X, see (11); the second one holds, since θ is a morphism of nice triples; the third one follows from (7). Property (5*) is just Property (1) in the above construction of P_t. Property (6*) is precisely Property (2) in our construction of P_t. □

7 Group of points of isotropic simple groups

In this section we establish several results concerning groups of points of simple groups, in particular, Lemma 7.2, Proposition 7.7 and Lemma 7.8, which play crucial role in the rest of the paper.

Definition 7.1. *Let A be an arbitrary commutative ring, and let G be a reductive group scheme over A. Assume that G has a proper parabolic subgroup $P = P^+$ over A, and denote by U^+ its unipotent radical. By [D-G, Exp. XXVI Cor. 2.3, Th. 4.3.2] there exists a parabolic subgroup P^- of G opposite to P^+, and by [D-G, Exp. XXVI Cor. 1.8] any two such subgroups are conjugate by an element of $U^+(A)$. Let U^- be the unipotent radical of P^-. For any A-algebra B we define the P-elementary subgroup $E_P(B)$ of the group $G(B)$ as follows:*

$$E_P(B) = \langle U^+(B), U^-(B) \rangle.$$

Lemma 7.2. *Let $B \to \bar{B}$ be a surjective A-algebra homomorphism. Then the induced homomorphism of elementary groups $E_P(B) \to E_P(\bar{B})$ is also surjective.*

Proof. By [D-G, Exp.XXVI Cor. 2.5] the A-schemes U^+ and U^- are isomorphic to A-vector bundles of finite rank. Thus, the maps $U^\pm(B) \to U^\pm(\bar{B})$ are surjective. □

Let l be an infinite field and G_l be **an isotropic simple simply-connected** l-group scheme. Recall that an isotropic scheme contains an l-split rank one torus $\mathbb{G}_{m,l}$. Choose and fix two opposite parabolic subgroups $P_l = P_l^+$ and P_l^- of the l-group scheme G_l. Let U_l^+ and U_l^- be their unipotent radicals. We will be interested mostly in the group of points $G_l(l(t))$. The following definition originates from [T, MainTheorem].

Definition 7.3. *Define $G_l(l(t))^+$ as a subgroup of the group $G_l(l(t))$ generated by $l(t)$-points of unipotent radicals of all parabolic subgroups of G_l defined over the field l.*

Remark 7.4. *Clearly,* $l(t) = l(t^{-1})$. *Thus,*

$$G_l(l(t)) = G_l(l(t^{-1})) \qquad \textit{and} \qquad G_l(l(t))^+ = G_l(l(t^{-1}))^+.$$

By definition the group $G_l(l(t))^+$ is generated by unipotent radicals of *all* l-parabolic subgroups, and thus contains the elementary group $E_{P_l}(l(t))$, introduced in Definition 7.1. In fact they coincide.

Proposition 7.5. *The group* $G_l(l(t))^+$ *is generated by* $l(t)$*-points of unipotent radicals of any two opposite parabolic subgroups of the* l*-group scheme* G_l*. In particular, one has the equality*

$$G_l(l(t^{-1}))^+ = \Big\langle U_l^+(l(t^{-1})), U_l^-(l(t^{-1}))\Big\rangle = E_{P_l}(l(t^{-1})). \tag{16}$$

Proof. Set $G_{l(t)} = G_l \times_{\operatorname{Spec} l} \operatorname{Spec} l(t)$. The group $G_l(l(t))^+$ is contained in the subgroup of $G_l(l(t)) = G_{l(t)}(l(t))$ generated by $l(t)$-points of unipotent radicals of all parabolic subgroups of the group scheme $G_{l(t)}$ defined over the field $l(t)$. By [BT, Prop.6.2.(v)] the latter group is generated by $l(t)$-points of unipotent radicals of any two opposite parabolic subgroups of $G_{l(t)}$, in particular, by $l(t)$-points of $U^+_{l(t)}$ and $U^-_{l(t)}$. Since $U^{\pm}_{l(t)}(l(t)) = U^{\pm}_l(l(t))$, we have (16). □

Remark 7.6. *For any ring* A *and an* A*-algebra* B*, and any reductive* A*-group scheme* G *we can define the group* $G_A(B)^+$ *as in the Definition 7.3, that is, as the subgroup generated by* B*-points of unipotent radicals of all* A*-parabolic subgroups of* G*. The question, whether this subgroup coincides with* $E_P(B)$ *for an* A*-parabolic subgroup* P *of* G*, is in general rather subtle. See the paper* [PSt] *by V. Petrov and the second author for details.*

The following result is crucial for the sequel.

Proposition 7.7. *One has equality*

$$G_l(l(t^{-1})) = G_l(l(t^{-1}))^+ \cdot G_l(R) \tag{17}$$

where $G_l(l(t^{-1}))^+$ *is the group defined in Definition 7.3 (see also Remark 7.4) and* $R = l[t^{-1}]_{(t^{-1})}$ *is the localization of* $l[t^{-1}]$ *at the prime ideal* (t^{-1}).

Proof. Let S_l be a maximal l-split torus in G_l and let $\operatorname{Cent}_{G_l}(S_l)$ be its scheme theoretic centralizer in G_l. Then by [R1, Cor.1.7] one has

$$G_l((l(t^{-1}))) = G_l(l(t^{-1}))^+ \cdot S_l(l(t^{-1})) \cdot \operatorname{Cent}_{G_l}(S_l)(R). \tag{18}$$

We show that $S_l(l(t^{-1}))$ is contained in $G_l(l(t^{-1}))$. Choose a maximal torus T_l in G_l containing S_l. Let $\Psi := \Phi(S, G_l)$ be the *relative* root system of G_l with respect to S_l. Further, let l^{sep} be the separable closure of l. Denote by Φ the root system of $G_{l^{\mathrm{sep}}}$ with respect to $T_{l^{\mathrm{sep}}}$ — the *absolute* root system of G_l. Let

$$\Psi' = \{\alpha \in \Psi \mid 2\alpha \notin \Psi\}.$$

A semi-simple l-split l-subgroup scheme H_l of G_l is constructed in [BT, Thm.7.2]. That semi-simple l-split l-subgroup scheme H_l contains the torus S_l as its maximal l-split torus, the root system of H_l with respect to S_l coincides with Ψ' and for each $\alpha \in \Psi'$ the root subgroup $U_{\alpha,l}$ of the l-group scheme H_l is contained in the unipotent radical of an appropriate l-parabolic subgroup P_l of the l-group scheme G_l, and consequently, in $G_l(l(t^{-1}))^+$.

Further, by [BT, (4.6) (Corollary J.Humphreys)] the semi-simple l-split l-group scheme H_l is simply-connected provided that G_l is simply-connected. Since H_l is split, it follows that the group of points $H_l(l(t^{-1}))$ is generated by the root subgroups $U_{\alpha,l}(l(t^{-1}))$, $\alpha \in \Psi'$ (e.g. [D-G, Exp. XXII Cor. 5.7.6]). Since S_l is a maximal split torus of H_l, using the above inclusions $U_{\alpha,l}(l(t^{-1})) \leq G_l(l(t^{-1}))^+$, $\alpha \in \Psi'$, we conclude that $S_l(l(t^{-1})) \leq G_l(l(t^{-1}))^+$.

This completes the proof of the proposition, since the centralizer $\mathrm{Cent}_{G_l}(S_l)$ is an l-subgroup scheme of G_l.

$\square$

Let $f(t) \in l[t]$ be a polynomial of degree $n = \deg(f)$ in t such that $f(0) \neq 0$. We consider the reciprocal polynomial

$$f^*(t^{-1}) = f(t)/t^n \in l[t^{-1}],$$

clearly, $f^*(0) \neq 0$. Conversely, if $g(t^{-1}) \in l[t^{-1}]$ is a polynomial of degree $n = \deg(g)$ in t^{-1} such that $g(0) \neq 0$, the reciprocal polynomial

$$g^*(t) = g(t^{-1}) \cdot t^n \in l[t]$$

is defined by a similar formula. The above correspondences are mutually inverse. Further, when $f(t) \in l[t]$ runs over all polynomials in t with $f(0) \neq 0$, then the reciprocal polynomial $f^*(t^{-1})$ runs over all polynomials $g(t^{-1}) \in l[t^{-1}]$ with $g(0) \neq 0$.

Now let us return to the setting considered in (16) and (17) and Remark 7.6. Each non-constant $f(t) \in l[t]$ admits a unique factorisation of the form $f(t) = t^r \cdot g(t)$, where $g(t) \in l[t]$ and $g(0) \neq 0$. Clearly, for each $h(t) \in l[t]$ with $h(0) \neq 0$ one gets the following inclusions

$$G_l\big(l[t]_{f(t)}\big) \leq G_l\big(l[t^{-1}, t]_{g^*}\big) \leq G_l\big(l[t^{-1}, t]_{g^*h^*}\big).$$

This leads us to the following Lemma.

Lemma 7.8. *For each $\alpha \in G_l(l[t]_{f(t)})$ one can find a polynomial $h(t) \in l[t]$, $h(0) \neq 0$, and elements*

$$u \in E_{P_l}\big(l[t^{-1}, t]_{g^*h^*}\big), \qquad \beta \in G_l\big(l[t^{-1}]_{g^*h^*}\big)$$

such that

$$\alpha = u\beta \in G_l\big(l[t^{-1}, t]_{g^*h^*}\big). \tag{19}$$

The chain of l-algebra inclusions $l[t]_{fh} \subseteq l[t]_{tfh} = l[t, t^{-1}]_{gh} = l[t^{-1}, t]_{g^*h^*}$ shows that

$$u \in E_{P_l}\big(l[t]_{tfh}\big), \quad \textit{and} \quad \alpha \in G_l\big(l[t]_{tfh}\big).$$

Proof. As observed above, inclusions $\alpha \in G_l\big(l[t^{-1},t]_{g^*}\big) \leq G_l\big(l[t^{-1},t]_{g^*h^*}\big)$ are obvious. The equalities (17) and (16) imply that there exists a polynomial $h(t) \in l[t]$, $h(0) \neq 0$, and elements

$$u \in E_{P_l}\big([t^{-1},t]_{g^*h^*}\big), \qquad \beta \in G_l\big(l[t^{-1}]_{g^*h^*}\big),$$

such that $\alpha = u\beta$ in $G_l\big(l[t^{-1},t]_{g^*h^*}\big)$. The last claim follows from the obvious l-algebra inclusions

$$l[t]_{fh} \subseteq l[t]_{tfh} = l[t,t^{-1}]_{gh} = l[t^{-1},t]_{g^*h^*}.$$

$\square$

8 Principal G-bundles on a projective line

The main result of the present section is Corollary 8.7. Let B be a commutative ring, and let $\mathbf{A}^1_B$ and $\mathbf{P}^1_B$ be the affine line and the projective line over B, respectively. Usually we identify the affine line with a sub-scheme of the projective line as follows $\mathbf{A}^1_B = \mathbf{P}^1_B - (\{\infty\} \times \mathrm{Spec}(B))$, where $\infty = [0:1] \in \mathbf{P}^1$. Let G be a semi-simple B-group scheme, let P a principal G-bundle over $\mathbf{A}^1_B$, and let $p : P \to \mathbf{A}^1_B$ be the corresponding canonical projection.

For a monic polynomial

$$f = f(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_0 \in B[t]$$

we set $P_f = p^{-1}((\mathbf{A}^1_B)_f)$. Clearly, it is a principal G-bundle over $(\mathbf{A}^1_B)_f$. Further, we denote by

$$F(t_0,t_1) = t_1^n + a_{n-1}t_1^{n-1}t_0 + \cdots + a_0t_0^n$$

the corresponding homogeneous polynomial in two variables. Note that the intersection of the principal open set in $\mathbf{P}^1_B$ defined by the inequality $F \neq 0$ with the affine line $\mathbf{A}^1_B$ equals the principal open subset $(\mathbf{A}^1_B)_f$. As in the previous section in the case where $a_0 \neq 0$ we consider the reciprocal polynomial $f^*(t^{-1}) \in B[t^{-1}]$ equal to $f(t)/t^n$.

Definition 8.1. *Let $\varphi : G_{(\mathbf{A}^1_B)_f} \to P_f$ be a principal G-bundle isomorphism. We write $P(\varphi, f)$ for a principal G-bundle over the projective line $\mathbf{P}^1_B$ obtained by gluing P and $G_{(\mathbf{P}^1_B)_F}$ over $(\mathbf{A}^1_B)_f$ via the principal G-bundle isomorphism φ.*

Remark 8.2. *For any φ and f the principal G-bundles $P(\varphi,f)$ and $P(\varphi,fg)$ coincide for each monic polynomial $g \in B[t]$.*

For any φ and f, any monic polynomial $h(t) \in B[t]$ such that $h(0) \in B^*$ is invertible, and any $\beta \in G(B[t^{-1}]_{h^*})$ the principal G-bundles $P(\varphi,f)$ and $P(\varphi\circ\beta, tfh)$ are isomorphic. In fact, they differ by a co-boundary.

Now, let l be an infinite field and G_l be a semi-simple l-group scheme. Let $f \in l[t]$ be a polynomial. Let P be a principal G_l-bundle over $\mathbf{A}^1_l$ such that P_f is trivial over $\mathbf{A}^1_f$. Let $\varphi : G_{\mathbf{A}^1_f} \to P_f$ be a principal G_l-bundle isomorphism. Let $P(\varphi,f)$ be the corresponding principal G-bundle over $\mathbf{P}^1_l$.

Lemma 8.3. *In the above notation there exists an $\alpha \in G(l[t]_f)$ such that the principal G-bundle $P(\varphi \circ \alpha, f)$ is trivial over $\mathbf{P}^1_l$.*

Proof. By the main theorem of [RR] we may assume that there is an isomorphism $G_{\mathbf{A}^1_l} = P$ over $\mathbf{A}^1_l$. In this case the above isomorphism φ coincides with the right multiplication by an element $\beta \in G_l(l[t]_f)$. Clearly, $P(\beta \circ \beta^{-1}, f)$ is trivial over $\mathbf{P}^1_l$. Thus, $P(\beta \circ \alpha, f)$ is trivial for $\alpha = \beta^{-1}$. □

Corollary 8.4. *Let G_l be an isotropic simply-connected semi-simple l-group scheme and let P be a G_l-bundle over $\mathbf{A}^1_l$. Further, let $f(t) \in l[t]$ be a non-constant polynomial, $\varphi : G_{\mathbf{A}^1_f} \to P_{\mathbf{A}^1_f}$ be a principal G_l-bundle isomorphism and let $P(\varphi, f)$ be the corresponding principal G_l-bundle on the projective line $\mathbf{P}^1_l$. Then there exist $h(t) \in l[t]$ and $u \in G(l[t]_{tfh})^+$ such that the principal G_l-bundle $P(\varphi \circ u, tfh)$ is trivial over $\mathbf{P}^1_l$.*

Proof. By Lemma 8.3 there exists an $\alpha \in G_l(l[t]_f)$ such that the principal G_l-bundle $P(\varphi \circ \alpha, f)$ is trivial.

Let $f(t) = t^r g(t)$ be the unique factorisation such that $g(t) \in l[t]$ and $g(0) \neq 0$. By Lemma 7.8 for each $h(t) \in S$, $h(0) \neq 0$, one has the inclusion $\alpha \in G_l(l[t^{-1}, t]_{g^* h^*})$. Moreover, there exist an element $h(t) \in l[t]$ with $h(0) \neq 0$ and elements

$$u \in G_l(l[t]_{tfh})^+, \qquad \beta \in G_l(l[t^{-1}]_{g^* h^*})$$

such that

$$\alpha = u\beta \in G_l(l[t^{-1}, t]_{g^* h^*}). \tag{20}$$

The following chain of principal G_l-bundle isomorphisms completes the proof

$$G_l \times_{\mathrm{Spec}(l)} \mathbf{P}^1_l = P(\varphi \circ \alpha, f) = P(\varphi \circ \alpha,\ tfg) = P(\varphi \circ u \circ \beta, tfh) \cong P(\varphi \circ u,\ tfh).$$

All the equalities are obvious. The last isomorphism holds since $\beta \in G_l(l[t^{-1}]_{g^* h^*})$. □

Let k be an infinite field and let X be a k-smooth irreducible affine variety. Replacing k by its algebraic closure in $k[X]$ we may assume that X is smooth affine and geometrically irreducible over k. Let $x_1, x_2, \ldots, x_n$ be a finite family of points on X. Let $\mathcal{O} = \mathcal{O}_{X,\{x_1,x_2,\ldots,x_n\}}$ be the semi-local ring of the family $x_1, x_2, \ldots, x_n \in X$ and let U for $\mathrm{Spec}(\mathcal{O})$. Let G be a simple group scheme over $\mathcal{O}$. By $G(x_i)$, $1 \leq i \leq n$, we denote the fibre of G over the point x_i, in other words, $G(x_i) = G \times_X x_i$. Note that for each $i = 1, 2, \ldots, n$ the $k(x_i)$-group scheme $G(x_i)$ is an isotropic simple simply-connected $k(x_i)$-group scheme.

Let $\mathfrak{m}_i \subseteq k[X]$ be the maximal ideal corresponding to the point x_i. Let J be the intersection of all $\mathfrak{m}_i$, $1 \leq i \leq n$. Then $l = k[X]/J = l_1 \times l_2 \times \cdots \times l_n$, where $l_i = k(x_i)$. Let $G_l = G \times_X \mathrm{Spec}(l)$ be the fibre of G over $\mathrm{Spec}(l)$. In the sequel we write $\mathbf{P}^1$ and $\mathbf{A}^1$ for $\mathbf{P}^1_{\mathcal{O}}$ and $\mathbf{A}^1_{\mathcal{O}}$ respectively, whereas $\mathbf{P}^1_l$ and $\mathbf{A}^1_l$ denote the projective line and the affine line over l.

Let $f \in \mathcal{O}[t]$ be a monic polynomial and let P be a principal $G_{\mathcal{O}}$-bundle over $\mathbf{A}^1$ such that $P_{\mathbf{A}^1_f}$ is trivial. Let $\varphi : G_{\mathbf{A}^1_f} \to P_{\mathbf{A}^1_f}$ be a principal G-bundle isomorphism and let $P(\varphi, f)$ be the corresponding principal G-bundle on $\mathbf{P}^1_{\mathcal{O}}$ (see Definition 8.1).

Theorem 8.5. *Assume that the $\mathcal{O}$-group scheme G is isotropic simple and simply-connected. Then there exists a monic polynomial $h(t) \in \mathcal{O}[t]$ and an element $\alpha \in G(\mathcal{O}[t]_{tfh})$ such that the principal G-bundle $P(\varphi \circ \alpha, tfh)$ satisfies the condition*

(i) *$P(\varphi \circ \alpha, tfh)|_{\mathbf{P}^1_l}$ is a trivial principal G_l-bundle over the projective line $\mathbf{P}^1_l$.*

Proof. Denote by $\overline{P}(\overline{\varphi}, \overline{f})$ the restriction of $P(\varphi, f)$ to the projective line $\mathbf{P}^1_l$. By Corollary 8.4 there exists a monic polynomial $\overline{h}(t) \in l[t]$, such that $\overline{h}(0) \in l^\times$ and an element

$$u \in E_{P_l}\big(l[t]_{t\overline{f}\,\overline{h}}\big) \leq G_l\big(l[t]_{t\overline{f}\,\overline{h}}\big)$$

such that the principal G_l-bundle $\overline{P}(\overline{\varphi} \circ u, t\overline{f}\,\overline{h})$ is trivial over $\mathbf{P}^1_l$.

Choose a monic polynomial $h(t) \in \mathcal{O}[t]$ of degree equal to the degree of $\overline{h}(t)$ and such that $h(t)$ modulo J coincides with $\overline{h}(t)$. Clearly, the $\mathcal{O}$-algebras homomorphism $\mathcal{O}[t]_{tfh} \to l[t]_{t\overline{f}\,\overline{h}}$ is surjective. By Lemma 7.2 it induces a surjective group homomorphisms

$$E_{P_{\mathcal{O}}}\big(\mathcal{O}[t]_{tfh}\big) \to E_{P_{\mathcal{O}}}\big(l[t]_{t\overline{f}\,\overline{h}}\big) = E_{P_l}\big(l[t]_{t\overline{f}\,\overline{h}}\big).$$

Thus, there exists an $\alpha \in G(\mathcal{O}[t]_{tfh})$ such that $\overline{\alpha} = u$. In other words, α equals

$$u \in E_{P_l}(l[t]_{t\overline{f}\,\overline{h}}) \leq G_l(l[t]_{t\overline{f}\,\overline{h}})$$

modulo J.

Take the G-bundle $P(\varphi \circ \alpha, tfh)$. We claim that its restriction to the projective line $\mathbf{P}^1_l$ is trivial. Indeed, one has the following chain of equalities and isomorphisms of principal G_l-bundles over $\mathbf{P}^1_l$:

$$\overline{P}(\overline{\varphi \circ \alpha}, \overline{tfh}) = \overline{P}(\overline{\varphi} \circ \overline{\alpha}, t\overline{f}\,\overline{h}) = \overline{P}(\overline{\varphi} \circ u, t\overline{f}\,\overline{h}),$$

where the principal G_l-bundle $\overline{P}(\overline{\varphi} \circ u, t\overline{fh})$ is trivial over $\mathbf{P}^1_l$. □

We keep the same notation as in Theorem 8.5, see also the text immediately preceding its statement.

Theorem 8.6. *Let G be an simple simply-connected $\mathcal{O}$-group scheme. Let P be a principal G-bundle over $\mathbf{P}^1$ whose restriction to the closed fibre $P_{\mathbf{P}^1_l}$ is trivial. Then P is of the form: $P = \mathrm{pr}^*(P_0)$, where P_0 is a principal G-bundle over* $\mathrm{Spec}(\mathcal{O})$ *and* $\mathrm{pr} : \mathbf{P}^1 \to \mathrm{Spec}(\mathcal{O})$ *is the canonical projection.*

The proof of this Theorem is rather standard, and for the most part follows [R2]. However, our group scheme G does not come from the ground field k. Therefore, we have to somewhat modify Raghunathan's arguments. We skip details for now. Let us state an important corollary of the above theorems.

Corollary 8.7 (=Theorem 2.1). *Let G be an $\mathcal{O}$-group scheme satisfying the same assumptions as in Theorem 8.5. Further, let P be a principal G-bundle over $\mathbf{A}^1$. Assume, that there exists a monic polynomial $f \in \mathcal{O}[t]$ such that the principal G-bundle $P_{\mathbf{A}^1_f}$ is trivial. Then the principal G-bundle P is trivial. In other words, there exists a G-bundle isomorphism*

$$G \times_U \mathbf{A}^1 \cong P.$$

Proof of Corollary 8.7. Let $f \in \mathcal{O}[t]$ be a monic polynomial such that the principal G-bundle $P_{\mathbf{A}^1_f}$ is trivial. Choose a principal G-bundle isomorphism $\varphi : G_{\mathbf{A}^1_f} \to P_{\mathbf{A}^1_f}$. By Theorem 8.5 there exists a monic polynomial $h(t) \in \mathcal{O}[t]$ and an element $\alpha \in G(\mathcal{O}[t]_{tfh})$ such that the restriction $P(\varphi \circ \alpha, tfh)|_{\mathbf{P}^1_l}$ of the principal G-bundle $P(\varphi \circ \alpha, tfh)$ to the projective line $\mathbf{P}^1_l$ is a trivial principal G_l-bundle.

By Theorem 8.6 the principal G-bundle $P(\varphi \circ \alpha, tfh)$ is of the form: $P(\varphi \circ \alpha, tfh) = \mathrm{pr}^*(P_0)$, where P_0 is a principal G-bundle over $\mathrm{Spec}(\mathcal{O})$. Note that

$$G|_{\{\infty\} \times U} \cong P(\varphi \circ \alpha, tfh)|_{\{\infty\} \times U},$$

where $U = \mathrm{Spec}(\mathcal{O})$ (that is the restriction of $P(\varphi \circ \alpha, tfh)$ to $\{\infty\} \times U$ is trivial). Thus

$$G_{\mathbf{P}^1} \cong P(\varphi \circ \alpha, tfh).$$

Since the original principal G-bundle P over $\mathbf{A}^1$ is isomorphic to $P(\varphi \circ \alpha, tfh)|_{\mathbf{A}^1}$, it follows that P is trivial. This finishes the proof.

□

9 Equating Groups

The aim of this Section is to sketch a proof of Theorem 4.3. The following Proposition is a straightforward analogue of [**?**, Prop.7.1]

Proposition 9.1. *Let S be a regular semi-local irreducible scheme and let G_1, G_2 be two semi-simple simply-connected S-group schemes. Further, let $T \subset S$ be a closed sub-scheme of S and $\varphi : G_1|_T \to G_2|_T$ be an S-group scheme isomorphism. Then there exists a finite étale morphism $\widetilde{S} \xrightarrow{\pi} S$ together with its section $\delta : T \to \widetilde{S}$ over T and an $\widetilde{S}$-group scheme isomorphism $\Phi : \pi^* G_1 \to \pi^* G_2$ such that $\delta^*(\Phi) = \varphi$.*

Proof of Theorem 4.3. We can start by almost literally repeating arguments from the proof of [**?**, Lemma 8.1], which involve the following purely geometric lemma [**?**, Lemma 8.2].

For reader's convenience below we state that Lemma adapting notation to the ones of Section 4. Namely, let U be as in Definition 4.1 and let $(\mathcal{X}, f, \Delta)$ be a nice triple over U. Further, let $G_{\mathcal{X}}$ be a simple simply-connected $\mathcal{X}$-group scheme, $G_U := \Delta^*(G_{\mathcal{X}})$, and let G_{const} be the pull-back of G_U to $\mathcal{X}$. Finally, by the definition of a nice triple there exists a finite surjective morphism $\Pi : \mathcal{X} \to \mathbf{A}^1 \times U$ of U-schemes.

Lemma 9.2. *Let $\mathcal{Y}$ be a closed nonempty sub-scheme of $\mathcal{X}$, finite over U. Let $\mathcal{V}$ be an open subset of $\mathcal{X}$ containing $\Pi^{-1}(\Pi(\mathcal{Y}))$. There exists an open set $\mathcal{W} \subseteq \mathcal{V}$ still containing $q_U^{-1}(q_U(\mathcal{Y}))$ and endowed with a finite surjective morphism $\mathcal{W} \to \mathbf{A}^1 \times U$ (in general $\neq \Pi$).*

Let $\Pi : \mathcal{X} \to \mathbf{A}^1 \times U$ be the above finite surjective U-morphism. The following diagram summarises the situation:

$$\begin{tikzcd} & \mathcal{Z} \arrow[d] & \\ \mathcal{X} - \mathcal{Z} \arrow[r, hook] & \mathcal{X} \arrow[r, "\Pi"] \arrow[d, shift right, "q_U"'] & \mathbf{A}^1 \times U \\ & U \arrow[u, shift right, "\Delta"'] & \end{tikzcd}$$

Here $\mathcal{Z}$ is the closed sub-scheme defined by the equation $f = 0$. By assumption, $\mathcal{Z}$ is finite over U. Let $\mathcal{Y} = \Pi^{-1}(\Pi(\mathcal{Z} \cup \Delta(U)))$. Since $\mathcal{Z}$ and $\Delta(U)$ are both finite over U and since Π is a finite morphism of U-schemes, $\mathcal{Y}$ is also finite over U. Denote by $y_1, \ldots, y_m$ its closed points and let $S = \operatorname{Spec}(\mathcal{O}_{\mathcal{X}, y_1, \ldots, y_m})$. Set $T = \Delta(U) \subseteq S$. Further, let $G_U = \Delta^*(G_{\mathcal{X}})$ be as in the hypotheses of Theorem 4.3 and let G_{const} be the pull-back of G_U to $\mathcal{X}$. Finally, let $\varphi : G_{\mathrm{const}}|_T \to G_{\mathcal{X}}|_T$ be the canonical isomorphism. Recall that by assumption $\mathcal{X}$ is U-smooth, and thus S is regular.

By Proposition 9.1 there exists a finite étale covering $\theta_0 : \widetilde{S} \to S$, a section $\delta : T \to \widetilde{S}$ of θ_0 over T and an isomorphism

$$\Phi_0 : \theta_0^*(G_{\mathrm{const}, S}) \to \theta_0^*(G_{\mathcal{X}}|_S)$$

such that $\delta^* \Phi_0 = \varphi$. We can extend these data to a neighborhood $\mathcal{V}$ of $\{y_1, \ldots, y_n\}$ and get the diagram

$$\begin{tikzcd} & \widetilde{S} \arrow[r, hook] \arrow[d, "\theta_0"] & \widetilde{\mathcal{V}} \arrow[d, "\theta"'] & \\ T \arrow[r, hook] \arrow[ur, "\delta"] & S \arrow[r, hook] & \mathcal{V} \arrow[r, hook] & \mathcal{X} \end{tikzcd} \tag{21}$$

where $\pi : \widetilde{\mathcal{V}} \to \mathcal{V}$ finite étale, and an isomorphism $\Phi : \theta^*(G_{\mathrm{const}}) \to \theta^*(G_{\mathcal{X}})$.

Since T isomorphically projects onto U, it is still closed viewed as a sub-scheme of $\mathcal{V}$. Note that since $\mathcal{Y}$ is semi-local and $\mathcal{V}$ contains all of its closed points, $\mathcal{V}$ contains $\Pi^{-1}(\Pi(\mathcal{Y})) = \mathcal{Y}$. By Lemma 9.2 there exists an open subset $\mathcal{W} \subseteq \mathcal{V}$ containing $\mathcal{Y}$ and endowed with a finite surjective U-morphism $\Pi^* : \mathcal{W} \to \mathbf{A}^1 \times U$.

Let $\mathcal{X}' = \theta^{-1}(\mathcal{W})$, $f' = \theta^*(f)$, $q'_U = q_U \circ \theta$, and let $\Delta' : U \to \mathcal{X}'$ be the section of q'_U obtained as the composition of δ with Δ. We claim that the triple $(\mathcal{X}', f', \Delta')$ is a nice triple. Let us verify this. Firstly, the structure morphism $q'_U : \mathcal{X}' \to U$ coincides with the composition

$$\mathcal{X}' \xrightarrow{\theta} \mathcal{W} \hookrightarrow \mathcal{X} \xrightarrow{q_U} U.$$

Thus, it is smooth. The element f' belongs to the ring $\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$, the morphism Δ' is a section of q'_U. Each component of each fibre of the morphism q_U has dimension one, the morphism $\mathcal{X}' \xrightarrow{\theta} \mathcal{W} \hookrightarrow \mathcal{X}$ is étale. Thus, each component of each fibre of the morphism q'_U is also of dimension one. Since $\{f = 0\} \subset \mathcal{W}$ and $\theta : \mathcal{X}' \to \mathcal{W}$ is finite, $\{f' = 0\}$ is finite over $\{f = 0\}$ and hence also over U. In other words, the $\mathcal{O}$-module

$\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})/f' \cdot \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$ is finite. The morphism $\theta : \mathcal{X}' \to \mathcal{W}$ is finite and surjective. We have constructed above the finite surjective morphism $\Pi^* : \mathcal{W} \to \mathbf{A}^1 \times U$. It follows that $\Pi^* \circ \theta : \mathcal{X}' \to \mathbf{A}^1 \times U$ is finite and surjective.

Clearly, the étale morphism $\theta : \mathcal{X}' \to \mathcal{X}$ is a morphism of nice triples, with $g = 1$.

Denote the restriction of Φ to $\mathcal{X}'$ simply by Φ. The equality $(\Delta')^*\Phi = \mathrm{id}_{G_U}$ holds by the very construction of the isomorphism Φ. Theorem follows.

10 An elementary Nisnevich square

The aim of this Section is to sketch a proof of Theorem 4.5. We will use analogues of three lemmas from [**?**] making them characteristic free. Lemma 10.3 is a refinement of [**?**, Lemma 2].

Lemma 10.1. *Let k be an infinite field and let S be an k-smooth equidimensional k-algebra of dimension* 1. *Let $f \in S$ be a non-zero divisor*

Let $\mathfrak{m}_0$ be a maximal ideal with $S/\mathfrak{m}_0 = k$. Let $\mathfrak{m}_1, \mathfrak{m}_2, \dots, \mathfrak{m}_n$ be pair-wise distinct maximal ideals of S, (possibly $\mathfrak{m}_0 = \mathfrak{m}_i$ for some i). Then there exists a non-zero divisor $\bar{s} \in S$ such that S is finite over $k[\bar{s}]$ and

1. *the ideals $\mathfrak{n}_i := \mathfrak{m}_i \cap k[\bar{s}]$, $1 \leq i \leq n$, are pair-wise distinct. If $\mathfrak{m}_0$ is distinct from all $\mathfrak{m}_i$'s, then $\mathfrak{n}_i$ are all distinct from $\mathfrak{n}_0 := \mathfrak{m}_0 \cap k[\bar{s}]$;*
2. *the extension $S/k[\bar{s}]$ is étale at each $\mathfrak{m}_i$'s and at $\mathfrak{m}_0$;*
3. *$k[\bar{s}]/\mathfrak{n}_i = S/\mathfrak{m}_i$ for each $i = 1, 2, \dots, n$;*
4. *$\mathfrak{n}_0 = \bar{s}k[\bar{s}]$.*

Proof. Let x_i, $1 \leq i \leq n$, be the point on $\mathrm{Spec}(S)$ corresponding to the ideal $\mathfrak{m}_i$. Consider a closed embedding $\mathrm{Spec}(S) \hookrightarrow \mathbf{A}^n_k$ and find a generic linear projection $p : \mathbf{A}^n_k \to \mathbf{A}^1_k$ defined over k and such that for each i the following holds:

1. for all $i, j \geq 0$ one has $p(x_i) \neq p(x_j)$, provided that $x_i \neq x_j$;
2. for each index $i \geq 0$ the map $p|_{\mathrm{Spec}(S)} : \mathrm{Spec}(S) \to \mathbf{A}^1$ is étale at the point x_i;
3. the separable degree of the extension $k(x_i)/k(p(x_i))$ is one.

These items imply equalities $k(p(x_i)) = k(x_i)$, for all i. Indeed, the extension $k(x_i)/k(p(x_i))$ is separable by (2). By (3) we conclude that $k(p(x_i)) = k(x_i)$. Lemma follows.

□

Lemma 10.2. *Under the hypotheses of Lemma 10.1 let $f \in S$ be a non-zero divisor which does not belong to a maximal ideal distinct from $\mathfrak{m}_0, \mathfrak{m}_1, \dots, \mathfrak{m}_n$. Let $N(f) = N_{S/k[s]}(f)$ be the norm of f. Then for an element $\bar{s} \in S$ satisfying* (1) *to* (4) *of Lemma 10.1 one has*

(a) *$N(f) = fg$ for an element $g \in S$;*

(b) *$fS + gS = S$;*

(c) *the map $k[\bar{s}]/(N(f)) \to S/(f)$ is an isomorphism.*

Proof. Straightforward.

□

Lemma 10.3. *Let R be a semi-local essentially smooth k-algebra with maximal ideals $\mathfrak{p}_i$, $1 \le i \le r$. Let $A \supseteq R[t]$ be an $R[t]$-algebra smooth as an R-algebra and finite over $R[t]$. Assume that for each i the $R/\mathfrak{p}_i$-algebra $A_i = A/\mathfrak{p}_i A$ is equidimensional of dimension one. Let $\epsilon : A \to R$ be an R-augmentation and $I = \mathrm{Ker}(\epsilon)$. Given an $f \in A$ with*

$$0 \neq \epsilon(f) \in \bigcap_{i=1}^{r} \mathfrak{p}_i \subset R$$

and such that the R-module A/fA is finite, one can find an element $u \in A$ satisfying the following conditions

(1) *A is a finite projective module over $R[u]$;*

(2) *$A/uA = A/I \times A/J$ for some ideal J;*

(3) *$J + fA = A$;*

(4) *$(u-1)A + fA = A$;*

(5) *if $N(f) = N_{A/R[u]}(f)$, then $N(f) = fg \in A$ for some $g \in A$*

(6) *$fA + gA = A$;*

(7) *the composition map $\varphi : R[u]/(N(f)) \to A/(N(f)) \to A/(f)$ is an isomorphism.*

Proof. Replacing t by $t - \epsilon(t)$ we may assume that $\epsilon(t) = 0$. Since A is finite over $R[t]$ from a theorem of Grothendieck [E, Cor.17.18] it follows that it is a finite projective $R[t]$-module.

Since A is finite over $R[t]$ and A/fA is finite over R we conclude that $R[t]/(N(f))$ is finite over R, and thus $R/(tN(f))$ is finite over R. Setting $v = tN(f)$, we get an integral extension A over $R[v]$ and, moreover,

$$v = tN_{A/R[t]}(f) = (ft)g = tfg.$$

We claim that $A/R[v]$ is integral, $\epsilon(v) = 0$ and $v \in fA$. Indeed, $v = tN_{A/R[t]}(f) = t(fg)$ and thus $\epsilon(v) = \epsilon(t)\epsilon(fg) = 0$. Finally, by the very definition $v \in fA$.

Below, we use bar to denote the reduction modulo an ideal, at that subscript i indicates reduction modulo $\mathfrak{p}_i A$. Let $l_i = \bar{R}_i = R/\mathfrak{p}_i$. By the assumption of the lemma the l_i-algebra $\bar{A}_i$ is l_i-smooth equidimensional of dimension 1. Let $\mathfrak{m}_1^{(i)}, \mathfrak{m}_2^{(i)}, \dots, \mathfrak{m}_n^{(i)}$ be distinct maximal ideals of $\bar{A}_i$ dividing $\bar{f}_i$ and let $\mathfrak{m}_0^{(i)} = \mathrm{Ker}(\bar{\epsilon}_i)$. Let $\bar{s}_i \in \bar{A}_i$ be such that the extension $\bar{A}_i/l_i[\bar{s}_i]$ satisfies Conditions (1) to (4) of Lemma 10.1.

Let $s \in A$ be a common raising of $\bar{s}_i$'s, in other words, $\overline{s} = \bar{s}_i$ in $\bar{A}_i$, for all i. Replacing s by $s - \epsilon(s)$ we may assume that $\epsilon(s) = 0$ and, as above, $\overline{s} = \bar{s}_i$, for all i.

Let $s^n + p_1(v)s^{n-1} + \cdots + p_n(v) = 0$ be an integral dependence for s. Let N be an integer larger than $\max\{2, \deg(p_i(t))\}$, where $i = 1, 2, \ldots, n$. Then for any $r \in k^*$ the element $u = s - rv^N$ has the following property: v is integral over $R[u]$. Thus, for any $r \in k^*$ the ring R is integral over $A[u]$.

On the other hand, one has $\bar{v}_j \in \mathfrak{m}_i^j$ for all j and all $1 \leq i \leq n$. It follows, that the element $\bar{u}_j = \bar{s}_j - r\bar{v}_j{}^N$ still satisfies Conditions (1) to (4) of Lemma 10.1.

We claim that the element $u \in R$ has the required properties, for almost all $r \in k^*$.

In fact, for almost all $r \in k^*$ the element u satisfies Conditions (1) to (4) of Lemma [OP2, Lemma 5.2]. It remains to show that Conditions (5) to (7) hold for all $r \in k^*$.

Since A is finite over $R[u]$ a theorem of Grothendieck [E, Cor.17.18] implies that it is a finite projective $R[u]$-module. To prove (5), consider the characteristic polynomial of the operator $A \xrightarrow{f} A$ as an $R[u]$-module operator. This polynomial vanishes on f and its free term equals $\pm N(f)$, the norm of f. Thus, $f^n - a_1 f^{n-1} + \cdots \pm N(f) = 0$ and $N(f) = fg$ for some $g \in R$.

To prove (6), one has to verify that the above g is a unit modulo the ideal fA. It suffices to check that for each index i the element $\bar{g}_i \in \bar{A}_i$ is a unit modulo the ideal $\bar{f}_i\bar{A}_i$. With that end observe that the field $l_i = R/\mathfrak{p}_i$, the l_i-algebra $S_i = \bar{A}_i$, its maximal ideals $\mathfrak{m}_0^{(i)}, \mathfrak{m}_1^{(i)}, \ldots, \mathfrak{m}_n^{(i)}$ and the element $\bar{u}_i$ satisfy the hypotheses of Lemma 10.2, with u replaced by $\bar{u}_i$. Now, by Item (b) of Lemma 10.2 the reduction $\bar{g}_i$ is a unit modulo the ideal $\bar{f}_i\bar{R}_i$.

To prove (7), observe that $R[u]/(N_{A/k[X]}(f))$ and A/fA are finite A-modules. Thus, it remains to check that the map $\varphi : R[u]/(N_{A/k[X]}(f)) \to A/fA$ is an isomorphism modulo each maximal ideal $\mathfrak{m}_i^{(j)}$. With that end it suffices to verify that the map $\bar{\varphi}_i : l_i[\bar{u}_i]/(N(\bar{f}_i)) \to \bar{A}_i/\bar{f}_i\bar{A}_i$ is an isomorphism for each index i, where $N(\bar{f}_i) := N_{\bar{A}_i/l_i[\bar{u}]}(\bar{f}_i)$. Now, by Item (c) of Lemma 10.2 the map $\bar{\varphi}_j$ is an isomorphism. Lemma follows.

$\square$

Proof of Theorem 4.5. Let $U = \mathrm{Spec}(\mathcal{O}_{X,\{x_1,x_2,\ldots,x_r\}})$ be as in Definition 4.1. Write R for $\mathcal{O}_{X,\{x_1,x_2,\ldots,x_r\}}$. It is a semi-local essentially smooth k-algebra with maximal ideals $\mathfrak{p}_i$, $1 \leq i \leq r$. Let $(\mathcal{X}, f, \Delta)$ be a nice triple over U. We show that it gives rise to certain data subject to the hypotheses of Lemma 10.3.

Let $A = \Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X})$. It is an R-algebra via the ring homomorphism $q_U^* : R \to \Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X})$. Furthermore, it is smooth as an R-algebra. The triple $(\mathcal{X}, f, \Delta)$ is a nice triple. Thus, there exists a finite surjective U-morphism $\Pi : \mathcal{X} \to \mathbf{A}^1_U$. It induces an R-algebra inclusion $R[t] \hookrightarrow \Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X}) = A$ such that A is finitely generated as an $R[t]$-module. Since $(\mathcal{X}, f, \Delta)$ is a nice triple, $R/\mathfrak{p}_i$-algebra $A/\mathfrak{p}_iA$ is equidimensional of dimension one, for all i. Let $\epsilon = \Delta^* : A \to R$ be an R-algebra homomorphism induced by the section Δ of the morphism q_U. Clearly, this ϵ is an augmentation, let $I = \mathrm{Ker}(\epsilon)$. Further, since $(\mathcal{X}, f, \Delta)$ is a nice triple, $\epsilon(f) \neq 0 \in R$ and A/fA is finite as an R-module. Summarising the above, we can conclude that we find ourselves in the setting of Lemma 10.3, and may use the conclusion of that Lemma.

Thus, there exists an element $u \in A$ subject to Conditions (1) through (7) of Lemma 10.3. This u induces an R-algebra inclusion $R[u] \hookrightarrow A$ such that A is finite as an $R[u]$-module. Let

$$\sigma : \mathcal{X} \to \mathbf{A}^1 \times U$$

be the U-scheme morphism induced by the above inclusion $R[u] \hookrightarrow A$. Clearly, σ is finite and surjective. Let $N(f) \in R[u] \subseteq A$ and $g_{f,\sigma} \in A$ be elements defined just above Lemma 4.4. Write g for $g_{f,\sigma}$ in this proof.

We claim that this morphism σ and the chosen elements $N(f)$ and g satisfy conclusions (1) to (4) of Theorem 4.5. Let us verify this claim. Since A is finite as an $R[t]$-module and both rings $R[t]$ and A are regular, the $R[t]$-module A is finitely generated and projective, see [E, Corollary 18.17]). Thus, σ is étale at a point $x \in \mathcal{X}$ if and only if the $k(\sigma(x))$-algebra $k(\sigma(x)) \otimes_{R[t]} A$ is étale. If the point x belongs to the closed sub-scheme $\mathrm{Spec}(A/\mathfrak{p}_i A)$ for some maximal ideal $\mathfrak{p}_i$ of R, then

$$k(\sigma(x)) \otimes_{R[t]} A = k(\sigma(x)) \otimes_{(R/\mathfrak{p}_i)[t]} A/\mathfrak{p}_i A.$$

We can conclude that σ is étale at a specific point x if and only if the $(R/\mathfrak{p}_i)[t]$-algebra $A/\mathfrak{p}_i A$ is étale at the point x. It follows from the proof of Lemma 10.3 that the morphism σ induces a morphism $\mathrm{Spec}(A/\mathfrak{p}_i A) \xrightarrow{\sigma_i} \mathbf{A}^1_{l_i}$ on the closed fibre $\mathrm{Spec}(A/\mathfrak{p}_i A)$ for each i. This induced morphism is étale along the vanishing locus of the function $\bar{f}_i$ and along each point $\overline{\Delta}_i(\mathrm{Spec}\, l_i)$. In fact, for the vanishing locus of the function $\bar{f}_i$ this follows from items (6) and (7) of Lemma 10.3. It follows from the hypotheses of Lemma 10.3 that the function f vanishes at each maximal ideal containing I. Thus σ is étale along the closed sub-scheme $\mathcal{X}$ defined by the ideal I, that is along Δ.

Item (1) of Theorem 4.5 follows.

The first of the following equalities

$$\sigma^{-1}(\sigma(\{f=0\})) = \{N(f)=0\} = \{f=0\} \sqcup \{g_{f,\sigma}=0\}$$

is a commonplace. The second one follows from the equality $N(f) = \pm f \cdot g_{f,\sigma}$, proved in Lemma 4.4 and Item (6) of Lemma 10.3.

Write g for $g_{f,\sigma}$. Clearly, the square (5) is Cartesian and the morphism σ^0_g is étale. The scheme $\mathcal{X}^0_g$ contains a closed sub-scheme $\Delta(U)$, and hence is non-empty. Item (7) of Lemma 10.3 shows that the morphism of the reduced closed sub-schemes

$$\sigma^0_g|_{\{f=0\}_{\mathrm{red}}} : \{f=0\}_{\mathrm{red}} \to \{N(f)=0\}_{\mathrm{red}}$$

is an isomorphism. Thus, we have checked Item (3) of Theorem 4.5.

It remains only to check Item (4). We already know that $\{f=0\} \subset \mathcal{X}^0_g$. By Claim 5.1 both schemes $\Delta(U)$ and $\{f=0\}$ are semi-local and the set of closed points of $\Delta(U)$ is contained in the set of closed points of the closed set $\{f=0\}$. Thus, $\Delta(U) \subset \mathcal{X}^0_g$. This concludes the proof of Item (4) of Theorem 4.5 and thus of the theorem itself.

$\square$

11 Proof of Theorems 1.1 and 1.2

Proof of Theorem 1.2. We start with the following data. Fix a smooth affine k-scheme X, a finite family of points $x_1, x_2, \dots, x_n$ on X, and set $\mathcal{O} := \mathcal{O}_{X,\{x_1,x_2,\dots,x_n\}}$ and $U := \mathrm{Spec}(\mathcal{O})$. Further, consider a simple simply-connected U-group scheme G and a principal G-bundle P over $\mathcal{O}$ which is trivial over the field of fractions K of $\mathcal{O}$. *We have to check that the principal G-bundle P is trivial.* We may and will assume that for certain $\mathrm{f} \in \mathcal{O}$ the principal G-bundle P is trivial over $\mathcal{O}_{\mathrm{f}}$.

Shrinking X if necessary, we may secure the following properties

(i) The points $x_1, x_2, \dots, x_n$ are still in X;

(ii) The group scheme G is defined over X and it is a simple and simply-connected group scheme. We will often denote this X-group scheme by G_X and write G_U for the original G;

(iii) The principal G-bundle P is defined over X; $\mathrm{f} \in k[X]$ and f vanishes at each x_i's;

(iv) The restriction P_{f} of the bundle P to the principal open sub-scheme X_{f} is trivial.

We may shrink X a little further, if necessary, always looking after verification of (i) to (iv). This way we constructed in Section 5 a basic nice triple (9). Beginning with that basic nice triple we constructed in Section 6 the following data:

(a) a unitary polynomial $h(t)$ in $R[t]$,

(b) a principal G_U-bundle P_t over $\mathbf{A}^1_U$,

(c) a diagram of the form

$$\begin{tikzcd} \mathbf{A}^1_U \arrow[dr, "\mathrm{pr}"'] & Y \arrow[l, "\sigma"'] \arrow[r, "q_X"] \arrow[d, "q_U"', shift right] & X \\ & U \arrow[u, bend right, "\delta"'] \arrow[ur, "\mathrm{can}"'] & \end{tikzcd} \tag{22}$$

with $Y = \mathcal{X}'$,

(d) an isomorphism $\Phi : q_U^*(G_U) \to q_X^*(G)$ of Y-group schemes.

By Theorem 6.1 these data may be chosen to be subject to the following properties:

(1*) $q_U = \mathrm{pr} \circ \sigma$,

(2*) σ is étale,

(3*) $q_U \circ \delta = \mathrm{id}_U$,

(4*) $q_X \circ \delta = \mathrm{can}$,

(5*) the restriction of P_t to $(\mathbf{A}^1_U)_h$ is a trivial G_U-bundle,

(6*) $(\sigma)^*(P_t)$ and $q_X^*(P)$ are isomorphic as principal G_U-bundles. Here $q_X^*(P)$ is regarded as a principal G_U-bundle via the group scheme isomorphism Φ from the item (d).

Recall, that we have to check that the G-bundle P is trivial over U. By assumption the group scheme G_U is simple and simply-connected and isotropic. The restriction of the principal G_U-bundle P_t to $(\mathbf{A}^1_U)_h$ is trivial by Condition (5*). Then the principal G_U-bundle P_t is itself trivial by Theorem 2.1.

On the other hand, the G_U-bundle $(\sigma)^*(P_t)$ is isomorphic to $q_X^*(P)$ as principal G_U-bundle by (6*). Therefore, $q_X^*(G)$ is trivial as a $q_X^*(G)$-bundle. It follows that $\delta^*(q_X^*(P)) = \mathrm{can}^*(P)$ is trivial as a $\delta^*(q_X^*(G)) = \mathrm{can}^*(G)$-bundle. Thus, $\mathrm{can}^*(P) = P_U$ is trivial as a principal $\mathrm{can}^*(G) = G_U$-bundle.

$\square$

Proof of Theorem 1.1. Theorem follows from Theorem 1.2 using arguments similar to those in [OP2, Sect.7], and we skip them. Let us only indicate here that arguments in [OP2, Sect.7] are based on a Theorem due to D. Popescu [P] (see also [Sw] for a self-contained exposition of that theorem).

$\square$

12 Semi-simple case

In the present Section we extend Theorem 1.1 to the case of semi-simple simply-connected groups. By [D-G, Exp. XXIV 5.3, Prop. 5.10] the category of semi-simple simply-connected group schemes over a Noetherian domain R is semi-simple. In other words, each object has a unique decomposition into a product of indecomposable objects. Indecomposable objects can be described as follows. Take a domain R' such that $R \subseteq R'$ is a finite étale extension and a simple simply-connected group scheme G' over R'. Now, applying the Weil restriction functor $R_{R'/R}$ to the R-group scheme G' we get a simply-connected R-group scheme $R_{R'/R}(G')$, which is an indecomposable object in the above category. Conversely, each indecomposable object can be constructed in this way.

Theorem 12.1. *Let R be a regular semi-local domain containing an infinite perfect subfield and let K be the quotient field of R. Let G be a semi-simple simply-connected group scheme G all of whose indecomposable factors are isotropic. Then the map*

$$H^1_{\text{ét}}(R, G) \to H^1_{\text{ét}}(K, G)$$

induced by the inclusion of R into K has trivial kernel.

In other words, under the above assumptions on R and G each principal G-bundle P over R which has a K-rational point is itself trivial.

Proof. Take a decomposition of G into indecomposable factors $G = G_1 \times G_2 \times \cdots \times G_r$. Clearly, it suffices to check that for each index i the kernel

$$H^1_{\text{ét}}(R, G_i) \to H^1_{\text{ét}}(K, G_i)$$

is trivial. We know that there exists a finite étale extension R'_i/R such that R'_i is a domain and the Weil restriction $R_{R'_i/R}(G'_i)$ coincides with G_i.

The Faddeev—Shapiro Lemma [D-G, Exp. XXIV Prop. 8.4] states that there is a canonical isomorphism preserving the distinguished point

$$H^1_{\text{ét}}\big(R, R_{R'_i/R}(G'_i)\big) \cong H^1_{\text{ét}}\big(R', G_i\big).$$

To complete the proof it only remains to apply Theorem 1.1 to the semi-local regular ring R'_i, its quotient field K_i and the simple R'_i-group scheme G'_i. □

Theorem 12.2. *Let k be an infinite field. Let $\mathcal{O}$ be the semi-local ring of finitely many points on a smooth irreducible k-variety X and let K be its field of fractions. Let G be a semi-simple simply-connected group scheme G all of whose indecomposable factors are isotropic. Then the map*

$$H^1_{\text{ét}}(R, G) \to H^1_{\text{ét}}(K, G)$$

induced by the inclusion of R into K has trivial kernel.

Proof. Use Theorem 1.2 and argue literally as in the proof of Theorem 12.1

□

Acknowledgments The authors are very grateful to Konstantin Pimenov and Victor Petrov for useful discussions on the subject of the present article.

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