UPPER MOTIVES OF OUTER ALGEBRAIC GROUPS

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Abstract. Let G be a semisimple affine algebraic group over a field F. Let E/F be a minimal field extension such that the group G_E is of inner type. Assuming that the degree of E/F is a power of a prime p, we determine the structure of the Chow motives with coefficients in a finite field of characteristic p of the projective G-homogeneous varieties. More precisely, it is known that the motive of any such variety decomposes (in a unique way) into a sum of indecomposable motives, and we describe the indecomposable summands which appear in the decompositions. This description is already known for the groups G of inner type and is new for G of outer type.

1. Introduction

We fix an arbitrary base field F. Besides of that, we fix a finite field $\mathbb{F}$ and we consider the Grothendieck Chow motives over F with coefficients in $\mathbb{F}$. These are the objects of the category $\mathrm{CM}(F, \mathbb{F})$, defined as in [4].

Let G be a semisimple affine algebraic group over F. According to [3, Corollary 35(4)] (see also Corollary 2.2 here), the motive of any projective G-homogeneous variety decomposes (and in a unique way) into a finite direct sum of indecomposable motives. One would like to describe the indecomposable motives which appear this way. In this paper we do it under certain assumption on G formulated in terms of a minimal field extension E/F such that the group G_E is of inner type: the degree of E/F is assumed to be a power of p, where $p = \operatorname{char} \mathbb{F}$. (Note that in the case when $E = F$, that is, when G is of inner type, the problem has been solved in [5].)

Let us introduce a minimum of terminology and notation needed to formulate the answer. For any intermediate field L of the extension E/F and any projective G_L-homogeneous variety Y, we consider the upper (see [5, Definition 2.10]) indecomposable summand M_Y of the motive $M(Y) \in \mathrm{CM}(F, \mathbb{F})$ of Y (considered as an F-variety at this point). Note that L is the *constant field* (the algebraic closure of the base field in the function field) of the F-variety Y. The set of the isomorphism classes of the motives M_Y for all L and Y is called the set of *upper motives* of the algebraic group G.

We are going to claim that the complete motivic decomposition of any projective G-homogeneous variety X consists of shifts of upper motives of G. In fact, the information we have is a bit more precise:

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Theorem 1.1. *For F, G, E, and X as above, the complete motivic decomposition of X consists of shifts of upper motives of the algebraic group G. More precisely, any indecomposable summand of the motive of X is isomorphic the upper motive M_Y of a variety Y such that the Tits index of G over the function field of Y contains the Tits index of G over the function field of X_L, where L is the constant field of Y.*

The proof of Theorem 1.1 is given in §4. Before this, we get some preparation results which are also of independent interest. In §2, we prove the nilpotence principle for the quasi-homogeneous varieties. In §3, we establish some properties of a motivic corestriction functor.

By *sum* of motives we always mean the *direct* sum; a *summand* is a *direct* summand.

2. Nilpotence principle for quasi-homogeneous varieties

Let us consider the category $\mathrm{CM}(F, \Lambda)$ of Grothendieck Chow motives over a field F with coefficients in an *arbitrary* associative commutative unital ring Λ.

We say that a smooth complete F-variety X satisfies the nilpotence principle, if for any field extension K/F the kernel of the change of field homomorphism

$$\mathrm{End}\big(M(X)\big) \to \mathrm{End}\big(M(X_K)\big)$$

consists of nilpotents, where $M(X)$ stands for the motive of X in $\mathrm{CM}(F, \Lambda)$.

We say that an F-variety X is *quasi-homogeneous*, if each connected component X^0 of X has the following property: there exists a finite separable field extension L/F, a semisimple affine algebraic group G over L, and a projective G_L-homogeneous variety Y such that Y, considered as an F-variety via the composition $Y \to \mathrm{Spec}\, L \to \mathrm{Spec}\, F$, is isomorphic to X^0. (Note that the algebraic group G needs not to be defined over F in this definition.)

We note that any variety which is *projective quasi-homogeneous* in the sense of [1, §4] is also quasi-homogeneous in the above sense. The following statement generalizes [2, Theorem 8.2] (see also [1, Theorem 5.1]) and [3, Theorem 25]:

Theorem 2.1. *Any quasi-homogeneous variety satisfies the nilpotence principle.*

Proof. By [4, Theorem 92.4] it suffices to show that the quasi-homogeneous varieties form a *tractable class*. We first recall the definition of a tractable class $\mathcal{C}$ (over F). This is a disjoint union of classes $\mathcal{C}_K$ of smooth complete K-varieties, where K runs over *all* field extensions of F, having the following properties:

(1) if Y_1 and Y_2 are in $\mathcal{C}_K$ for some K, then the disjoint union $Y_1 \coprod Y_2$ is also in $\mathcal{C}_K$;
(2) if Y is in $\mathcal{C}_K$ for some K, then each component of Y is also in $\mathcal{C}_K$;
(3) if Y is in $\mathcal{C}_K$ for some K, then for any field extension K'/K the K'-variety $Y_{K'}$ is in $\mathcal{C}_{K'}$;
(4) if Y is in $\mathcal{C}_K$ for some K, Y is irreducible, $\dim Y > 0$, and $Y(K) \neq \emptyset$, then $\mathcal{C}_K$ contains a (not necessarily connected) variety Y_0 such that $\dim Y_0 < \dim Y$ and $M(Y) \simeq M(Y_0)$ in $\mathrm{CM}(K, \Lambda)$.

Let us define a class $\mathcal{C}$ as follows. For any field extension K/F, $\mathcal{C}_K$ is the class of all quasi-homogeneous K-varieties.

We claim that the class $\mathcal{C}$ is tractable. Indeed, the properties (1)–(3) are trivial and the property (4) is [2, Theorem 7.2]. □

We turn back to the case where the coefficient ring Λ is a finite field $\mathbb{F}$.

Corollary 2.2. *Let $M \in \mathrm{CM}(F, \mathbb{F})$ be a summand of the motive of a quasi-homogeneous variety. Then M decomposes in a finite direct sum of indecomposable motives; moreover, such a decomposition is unique (up to a permutation of the summands).*

Proof. Any quasi-homogeneous variety is *geometrically cellular*. In particular, it is *geometrically split* in the sense of [5, §2a]. Finally, by Theorem 2.1, it satisfies the nilpotence principle. The statement under proof follows now by [5, Corollary 2.6]. □

3. Corestriction of scalars for motives

As in the previous section, we fix an arbitrary coefficient ring Λ. We write Ch for the Chow group with coefficients in Λ. Let $\mathrm{C}(F, \Lambda)$ be the category whose objects are pairs (X, i), where X is a smooth complete equidimensional F-variety and i is an integer. A morphism $(X, i) \to (Y, j)$ in this category is an element of the Chow group $\mathrm{Ch}_{\dim X+i-j}(X \times Y)$ (and the composition is the usual composition of correspondences). The category $\mathrm{C}(F, \Lambda)$ is preadditive. Taking first the additive completion of it, and taking then the idempotent completion of the resulting category, one gets the category of motives $\mathrm{CM}(F, \Lambda)$, cf. [4, §63 and §64].

Let L/F be a finite separable field extension. We define a functor

$$\mathrm{cor}_{L/F} : \mathrm{C}(L, \Lambda) \to \mathrm{C}(F, \Lambda)$$

as follows: on the objects $\mathrm{cor}_{L/F}(X, i) = (X, i)$, where on the right-hand side X is considered as an F-variety via the composition $X \to \operatorname{Spec} L \to \operatorname{Spec} F$; on the morphisms, the map

$$\mathrm{Hom}_{\mathrm{C}(L,\Lambda)}\big((X, i), (Y, j)\big) \to \mathrm{Hom}_{\mathrm{C}(F,\Lambda)}\big((X, i), (Y, j)\big)$$

is the push-forward homomorphism $\mathrm{Ch}_{\dim X+i-j}(X \times_L Y) \to \mathrm{Ch}_{\dim X+i-j}(X \times_F Y)$ with respect to the closed imbedding $X \times_L Y \hookrightarrow X \times_F Y$. Passing to the additive completion and then to the idempotent completion, we get an additive and commuting with the Tate shift functor $\mathrm{CM}(L, \Lambda) \to \mathrm{CM}(F, \Lambda)$, which we also denote by $\mathrm{cor}_{L/F}$.

The functor $\mathrm{cor}_{L/F} : \mathrm{C}(L, \Lambda) \to \mathrm{C}(F, \Lambda)$ is left-adjoint and right-adjoint to the change of field functor $\mathrm{res}_{L/F} : \mathrm{C}(F, \Lambda) \to \mathrm{C}(L, \Lambda)$, associating to (X, i) the object (X_L, i). Therefore the functor $\mathrm{cor}_{L/F} : \mathrm{CM}(L, \Lambda) \to \mathrm{CM}(F, \Lambda)$ is also left-adjoint and right-adjoint to the change of field functor $\mathrm{res}_{L/F} : \mathrm{CM}(F, \Lambda) \to \mathrm{CM}(L, \Lambda)$. (This makes a funny difference with the category of varieties, where the functor $\mathrm{cor}_{L/F}$ is only left-adjoint to $\mathrm{res}_{L/F}$, while the right-adjoint to $\mathrm{res}_{L/F}$ functor is the Weil restriction.) It follows that for any $M \in \mathrm{CM}(L, \Lambda)$ and any $i \in \mathbb{Z}$, the Chow groups $\mathrm{Ch}^i(M)$ and $\mathrm{Ch}^i(\mathrm{cor}_{L/F} M)$ are canonically isomorphic as well as the Chow groups $\mathrm{Ch}_i(M)$ and $\mathrm{Ch}_i(\mathrm{cor}_{L/F} M)$ are. Indeed, since $\mathrm{res}_{L/F} \Lambda = \Lambda \in \mathrm{CM}(L, \Lambda)$, we have

$$\mathrm{Ch}^i(M) := \mathrm{Hom}\big(M, \Lambda(i)\big) = \mathrm{Hom}\big(\mathrm{cor}_{L/F} M, \Lambda(i)\big) =: \mathrm{Ch}^i(\mathrm{cor}_{L/F} M) \qquad \text{and}$$

$$\mathrm{Ch}_i(M) := \mathrm{Hom}\big(\Lambda(i), M\big) = \mathrm{Hom}\big(\Lambda(i), \mathrm{cor}_{L/F} M\big) =: \mathrm{Ch}_i(\mathrm{cor}_{L/F} M).$$

In particular, if the ring Λ is connected and $M \in \mathrm{CM}(L, \Lambda)$ is an *upper* (see [5, Definition 2.10]) motivic summand of an irreducible smooth complete L-variety X, then $\mathrm{cor}_{L/F} M$ is an upper motivic summand of the F-variety X.

Now we turn back to the situation where Λ is a finite field $\mathbb{F}$:

Proposition 3.1. *The following three conditions on a finite galois field extension E/F are equivalent:*

1. *for any intermediate field $F \subset K \subset E$, the K-motive of SpecE is indecomposable;*
2. *for any intermediate fields $F \subset K \subset L \subset E$ and any L-motive M, the K-motive $\mathrm{cor}_{L/K}(M)$ is indecomposable;*
3. *the degree of E/F is a power of p (where p is the characteristic of the coefficient field $\mathbb{F}$).*

Proof. We start by showing that (3) $\Rightarrow$ (2). So, we assume that $[E : F]$ is a power of p and we prove (2). The extension L/K decomposes in a finite chain of galois degree p extensions. Therefore we may assume that L/K itself is a galois degree p extension. Let $R = \mathrm{End}(M)$. This is an associative, unital, but not necessarily commutative $\mathbb{F}$-algebra. Moreover, since M is indecomposable, the ring R has no non-trivial idempotents. We have $\mathrm{End}\big(\mathrm{cor}_{L/K}(M)\big) = R \otimes_{\mathbb{F}} \mathrm{End}\big(M_K(\mathrm{Spec}\, L)\big)$. According to [3, §7], the ring $\mathrm{End}\big(M_K(\mathrm{Spec}\, L)\big)$ is isomorphic to the group ring $\mathbb{F}\Gamma$, where Γ is the Galois group of L/K. Since the group Γ is (cyclic) of order p, we have $\mathbb{F}\Gamma \simeq \mathbb{F}[t]/(t^p - 1)$. Since $p = \mathrm{char}\, \mathbb{F}$, $\mathbb{F}[t]/(t^p - 1) \simeq \mathbb{F}[t]/(t^p)$. It follows that the ring $\mathrm{End}\big(\mathrm{cor}_{L/K}(M)\big)$ is isomorphic to the ring $R[t]/(t^p)$. We prove (2) by showing that the latter ring does not contain non-trivial idempotents. An arbitrary element of $R[t]/(t^p)$ can be (and in a unique way) written in the form $a + b$, where $a \in R$ and b is divisible by t. Note that b is nilpotent. Let us take an idempotent of $R[t]/(t^p)$ and write it in the above form $a + b$. Then a is an idempotent of R. Therefore $a = 1$ or $a = 0$. If $a = 1$, then $a + b$ is invertible and therefore $a + b = 1$. If $a = 0$, then $a + b$ is nilpotent and therefore $a + b = 0$.

We have proved the implication (3) $\Rightarrow$ (2). The implication (2) $\Rightarrow$ (1) is trivial. We finish by proving that (1) $\Rightarrow$ (3).

We assume that $[E : F]$ is divisible by a different from p prime q and we show that (1) does not hold. Indeed, the galois group of E/F contains an element σ of order q. Let K be the subfield of E consisting of the elements of E fixed by σ. We have $F \subset K \subset E$ and E/K is galois of degree q. The endomorphisms ring of $M_K(\mathrm{Spec}\, E)$ is isomorphic to $\mathbb{F}[t]/(t^q - 1)$. Since $q \neq \mathrm{char}\, \mathbb{F}$, the factors of the decomposition $t^q - 1 = (t-1) \cdot (t^{q-1} + t^{q-2} + \cdots + 1) \in \mathbb{F}[t]$ are coprime. Therefore the ring $F[t]/(t^q - 1)$ is the direct product of the rings $\mathbb{F}[t]/(t - 1) = \mathbb{F}$ and $\mathbb{F}[t]/(t^{q-1} + \cdots + 1)$, and it follows that the motive $M_K(\mathrm{Spec}\, E)$ is not indecomposable. □

Corollary 3.2. *Let E/F be a finite p-primary galois field extension and let L be an intermediate field: $F \subset L \subset E$. Let $M \in \mathrm{CM}(L, \mathbb{F})$ be an upper indecomposable motivic summand of an irreducible smooth complete L-variety X. Then $\mathrm{cor}_{L/F} M$ is an upper indecomposable summand of the F-variety X.* □

4. Proof of Theorem 1.1

Before starting the proof of Theorem 1.1, let us recall some classical facts and introduce some notation.

We write $\mathcal{D}$ (or $\mathcal{D}_G$) for the set of vertices of the Dynkin diagram of G. The galois group Γ of the field extension E/F acts on $\mathcal{D}$.

Let L be an intermediate field of the extension E/F. Any subset τ in $\mathcal{D}$ which is stable under the action of the galois group of E/L, determines a projective G_L-homogeneous variety X_{τ,G_L} in the way described in [5, §3]. This is the variety corresponding to the set $\mathcal{D}\setminus\tau$ in the sense of [6]. For instance, $X_{\mathcal{D},G_L}$ is the variety of the Borel subgroups of G_L, and $X_{\emptyset,G_L} = \operatorname{Spec} L$. Any projective G_L-homogeneous variety is G_L-isomorphic to X_{τ,G_L} for some $\tau\subset\mathcal{D}$ stable under the action of the galois group of E/L.

For any intermediate field L of E/F and any $\operatorname{Gal}(E/L)$-stable subset $\tau\subset\mathcal{D}$, we write M_{τ,G_L} for the upper indecomposable motivic summand of the F-variety X_{τ,G_L}.

For any field extension L/F, we consider the Tits index (see [6]) of G over L and we write τ_L (or $\tau_{L,G}$) for the subset in $\mathcal{D}$ consisting of the circled vertices of the Dynkin diagram.

Proof of Theorem 1.1. This is a recast of [5, proof of Theorem 3.5].

We proof Theorem 1.1 simultaneously for all F, G, X using an induction on $n = \dim X$. The base of the induction is $n = 0$ where $X = \operatorname{Spec} F$ and the statement is trivial.

From now on we are assuming that $n \geq 1$ and that Theorem 1.1 is already proven for all varieties of dimension $< n$.

Let M be an indecomposable summand of $M(X)$. We have to show that M is isomorphic to a shift of M_{τ,G_L} for some intermediate field L of E/F and some $\operatorname{Gal}(E/L)$-stable subset $\tau\subset\mathcal{D}_G$ containing $\tau_{L(X)}$.

Let $G'/F(X)$ be the semisimple anisotropic kernel of the group $G_{F(X)}$. The set $\mathcal{D}_{G'}$ is identified with $\mathcal{D}_G\setminus\tau_{F(X),G}$.

We note that the group $G'_{E(X)}$ is of inner type. The field extension $E(X)/F(X)$ is galois with the galois group $\operatorname{Gal}(E/F)$. In particular, its degree is a power of p and any its intermediate field is of the form $L(X)$ for some intermediate field L of the extension E/F.

According to [1, Theorem 4.2], the motive of $X_{F(X)}$ decomposes into a sum of shifts of motives of projective $G'_{L(X)}$-homogeneous (where L runs over intermediate fields of the extension E/F) varieties Y, satisfying $\dim Y < \dim X = n$ (we are using the assumption that $n > 0$ here). It follows by the induction hypothesis and Corollary 3.2, that each summand of the complete motivic decomposition of $X_{F(X)}$ is a shift of $M_{\tau',G'_{L(X)}}$ for some L and some $\tau'\subset\mathcal{D}_{G'}$. The complete decomposition of $M_{F(X)}$ is a part of the above decomposition.

Let us choose a summand $M_{\tau',G'_{L(X)}}(i)$ with minimal i in the complete decomposition of $M_{F(X)}$. We set $\tau = \tau'\cup\tau_{L(X)}\subset\mathcal{D}_G$. We shall show that $M\simeq M_{\tau,G_L}(i)$ for these L, τ, and i.

We write Y for the F-variety X_{τ,G_L} and we write Y' for the $F(X)$-variety $X_{\tau',G'_{L(X)}}$. We write N for the F-motive M_{τ,G_L} and we write N' for the $F(X)$-motive $M_{\tau',G'_{L(X)}}$.

By [5, Lemma 2.14] and since M is indecomposable, it suffices to construct morphisms

$$\alpha : M(Y)(i) \to M \ \text{ and } \ \beta : M \to M(Y)(i)$$

satisfying $\operatorname{mult}(\beta \circ \alpha) = 1$.

We construct α first. Since $\tau' \subset \tau$, the $F(X)(Y)$-variety $Y' \times_{L(X)} \operatorname{Spec} F(X)(Y)$ has a rational point. Let $\alpha_1 \in \operatorname{Ch}_0\big(Y' \times_{L(X)} \operatorname{Spec} F(X)(Y)\big)$ be the class of a rational point. Let $\alpha_2 \in \operatorname{Ch}_i(X_{F(X)(Y)})$ be the image of α_1 under the composition

$$\operatorname{Ch}_0\big(Y' \times_{L(X)} \operatorname{Spec} F(X)(Y)\big) \to \operatorname{Ch}_0(Y'_{F(X)(Y)}) \to \operatorname{Ch}_0(N'_{F(X)(Y)}) \hookrightarrow \operatorname{Ch}_i(X_{F(X)(Y)}),$$

where the first map is the push-forward with respect to the closed imbedding

$$Y' \times_{L(X)} \operatorname{Spec} F(X)(Y) \hookrightarrow Y'_{F(X)(Y)} := Y' \times_{F(X)} \operatorname{Spec} F(X)(Y).$$

Since $\tau_{L(X)} \subset \tau$, the variety X has an $F(Y)$-point and therefore the field extension $F(X)(Y)/F(Y)$ is purely transcendental. Consequently, the element α_2 is $F(Y)$-rational and lifts to an element $\alpha_3 \in \operatorname{Ch}_{\dim Y+i}(Y \times X)$. We mean here a lifting with respect to the composition

$$\operatorname{Ch}_{\dim Y+i}(Y \times X) \twoheadrightarrow \operatorname{Ch}_i(X_{F(Y)}) \xrightarrow{\operatorname{res}_{F(X)(Y)/F(Y)}} \operatorname{Ch}_i(X_{F(X)(Y)})$$

where the first map is the epimorphism given by the pull-back with respect to the morphism $X_{F(Y)} \to Y \times X$ induced by the generic point of the variety Y.

We define the morphism α as the composition

$$M(Y)(i) \xrightarrow{\alpha_3} M(X) \longrightarrow M$$

where the second map is the projection of $M(X)$ onto its summand M.

We proceed by constructing β. Let $\beta_1 \in \operatorname{Ch}_{\dim Y'}(Y' \times_{F(X)} Y_{F(X)})$ be the class of the closure of the graph of a rational map of $L(X)$-varieties $Y' \dashrightarrow Y_{F(X)}$ (which exists because $\tau \subset \tau_{L(X)} \cup \tau'$). Note that this graph is a subset of $Y' \times_{L(X)} Y_{F(X)}$, which we consider as a subset of $Y' \times_{F(X)} Y_{F(X)}$ via the closed imbedding $Y' \times_{L(X)} Y_{F(X)} \hookrightarrow Y' \times_{F(X)} Y_{F(X)}$. Let β_2 be the image of β_1 under the composition

$$\begin{aligned}
&\operatorname{Ch}^{\dim Y}(Y' \times_{F(X)} Y_{F(X)}) = \\
&\qquad \operatorname{Ch}^{\dim Y}\big(M(Y') \otimes M(Y_{F(X)})\big) \to \operatorname{Ch}^{\dim Y}\big(N' \otimes M(Y_{F(X)})\big) \to \\
&\qquad\qquad \operatorname{Ch}^{\dim Y+i}\big(M(X_{F(X)}) \otimes M(Y_{F(X)})\big) = \operatorname{Ch}^{\dim Y+i}\big((X \times Y)_{F(X)}\big)
\end{aligned}$$

where the first arrow is induced by the projection $M(Y') \to N'$ and the second arrow is induced by the imbedding $N'(i) \to M(X_{F(X)})$. The element β_2 lifts to an element

$$\beta_3 \in \operatorname{Ch}^{\dim Y+i}(X \times X \times Y).$$

We mean here a lifting with respect to the epimorphism

$$\operatorname{Ch}^{\dim Y+i}(X \times X \times Y) \twoheadrightarrow \operatorname{Ch}^{\dim Y+i}\big((X \times Y)_{F(X)}\big)$$

given by the pull-back with respect to the morphism $X \times X \times Y \to (X \times Y)_{F(X)}$ induced by the generic point of the second factor in this triple direct product.

Let $\pi \in \operatorname{Ch}_{\dim X}(X \times X)$ be the projector defining the summand M of $M(X)$. Considering β_3 as a correspondence from X to $X \times Y$, we define

$$\beta_4 \in \operatorname{Ch}^{\dim Y+i}(X \times X \times Y)$$

as the composition $\beta_3 \circ \pi$. We get

$$\beta_5 \in \mathrm{Ch}^{\dim Y + i}(X \times Y) = \mathrm{Ch}_{\dim X - i}(X \times Y)$$

as the image of β_4 under the pull-back with respect to the diagonal of X. Finally, we define the morphism β as the composition

$$M \longrightarrow M(X) \xrightarrow{\beta_5} M(Y)(i).$$

The verification of the relation $\mathrm{mult}(\beta \circ \alpha) = 1$, finishing the proof, is similar to that of [5, proof of Theorem 3.5]. $\square$

Remark 4.1. Theorem 1.1 can be also proved using a weaker result in place of [1, Theorem 4.2], namely, [2, Theorem 7.5].

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