LEVELS AND SUBLEVELS OF QUATERNION ALGEBRAS

DETLEV W. HOFFMANN

Dedicated to Professor David W. Lewis on the occasion of his 65th birthday

Abstract. The level s (resp. sublevel $\underline{s}$) of a ring R with $1 \neq 0$ is the smallest positive integer such that -1 (resp. 0) can be written as a sum of s (resp. $\underline{s}+1$) nonzero squares in R, provided -1 (resp. 0) is a sum of nonzero squares at all. D.W. Lewis showed that any value of type 2^n or 2^n+1 can be realized as level of a quaternion division algebra, and in all these examples, the sublevel was 2^n, which prompted the question whether or not the level and sublevel of a quaternion division algebra will always differ at most by one. In this note, we give a positive answer to that question.

1. Introduction

Let D be a division ring. The *level* $s(D)$ and the *sublevel* $\underline{s}(D)$ of D are defined as follows:

(1) If -1 is a sum of squares in D, then

$$s(D) = \min\{n \mid \exists x_1, \ldots, x_n \in D : -1 = x_1^2 + \ldots + x_n^2\} .$$

Otherwise, $s(D) = \infty$.

(2) If 0 is a sum of nonzero squares in D, then

$$\underline{s}(D) = \min\{n \mid \exists x_1, \ldots, x_{n+1} \in D^* = D \setminus \{0\} : 0 = x_1^2 + \ldots + x_{n+1}^2\} .$$

Otherwise, $\underline{s}(D) = \infty$.

It is clear from the definition that $\underline{s}(D) \leq s(D)$, and one readily sees that if D is a (commutative) field, the $s(D) = \underline{s}(D)$.

The study of level and sublevel of rings has a history dating back at least to the early 20th century. A famous result by Pfister [9] states that the level of a field, if finite, is always a 2-power, and that each 2-power can be realized as level of a field. This answered a question posed by Van der Waerden in the 1930s.

The study of levels and sublevels in the above sense for noncommutative division rings started in the mid-1980s. In [5], [6], David Lewis showed that for every $k \in \mathbb{N}$, there exist quaternion division algebras with $s = \underline{s} = 2^k$ and with $s = \underline{s}+1 = 2^k+1$, and that for any quaternion division algebra D with $s(D) = 2^k$ one also has $\underline{s}(D) = 2^k$. Leep [4] gave slight improvements on some of Lewis's results, and he asked the following questions (already implicit in [5], [6] and reiterated in [7]):

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Question. (1) Can the level (resp. sublevel) of a quaternion division algebra D take values that are not of the form 2^k, 2^k+1 (resp. 2^k)?
(2) Does one always have $s(D) \leq \underline{s}(D)+1$?

As for the first question, quaternion division algebras of sublevel 3 were constructed by Krüskemper and Wadsworth [2]. It was shown in [1] that for each $k \geq 2$, there exist quaternion division algebras D with $2^k+2 \leq s(D) \leq 2^{k+1}-1$ (although the method used there to construct such D by employing function fields of quadrics does not allow to give the exact value for $s(D)$). O'Shea [8] observed that this function field method also allows to construct quaternion division algebras D of sublevel not of the form 2^k and >3. It is still not fully known what exact values can be realized as (sub)levels of quaternion division algebras.

In this note, we give a positive answer to the second question:

Theorem. *Let D be a quaternion division algebra. Then $\underline{s}(D) \leq s(D) \leq \underline{s}(D)+1$.*

2. Proof of the Theorem

We first recall a few simple facts about quaternion algebras. We refer to [3, chapter III] for any facts we use without further reference.

Let F be a field of characteristic different from 2 and let $D=(a,b)_F$ $(a,b \in F^*)$ be the quaternion algebra with F-basis $\{1,i,j,k\}$ subject to the relations $i^2=a$, $j^2=b$, $ij=-ji=k$. We assume D to be a division algebra, which is equivalent to saying that its norm form $\langle 1,-a,-b,ab\rangle$ is anisotropic.

For $\zeta = x+yi+zj+wk \in D$ $(x,y,z,w\in F)$, we call x the scalar part of ζ, and $\zeta' = yi+zj+wk$ its pure part. We put $D' = Fi+Fj+Fk$, the subspace of pure quaternions. We have $\zeta^2 = x^2+2x\zeta'+\zeta'^2$ with $\zeta'^2 = ay^2+bz^2-abw^2 \in F$. The quadratic form $\langle a,b,-ab\rangle$ will be denoted by T_P. We immediately get the following well known lemma:

Lemma. *$c\in F$ is a sum of m squares of pure quaternions in D (not all squares equal to 0 if $c=0$) if and only if the quadratic form*

$$m\times T_P = \underbrace{T_P \perp \ldots \perp T_P}_{m}$$

represents c (nontrivially if $c=0$, i.e. $m\times T_P$ is isotropic in that case).

Proof of the Theorem. Let D be a quaternion division algebra as above and assume that $\underline{s}(D)=m$. We only have to show that $s(D)\leq m+1$. Let $\zeta_\ell \in D^*$, $1\leq \ell \leq m+1$ be such that

$$0 = \zeta_1^2 + \ldots + \zeta_{m+1}^2 \ .$$

Write $\zeta_\ell = x_\ell + \zeta'_\ell$ with $x_\ell \in F$ and $\zeta'_\ell \in D'$. We get

$$0 = \sum_{\ell=1}^{m+1} x_\ell^2 + 2x_\ell\zeta'_\ell + \zeta'^2_\ell$$

and thus

$$\sum_{\ell=1}^{m+1} x_\ell^2 + \zeta'^2_\ell \ = \ 0 \ = \ \sum_{\ell=1}^{m+1} x_\ell \zeta'_\ell \ .$$

1. case: All $x_\ell = 0$, $1\leq \ell\leq m+1$.
In this case, 0 is a nontrivial sum of squares of $m+1$ pure quaternions, so $(m+1)\times T_P$

is isotropic by the Lemma. But then $(m+1)\times T_P$ contains a hyperbolic plane $\langle 1,-1\rangle$ as subform, in particular, $(m+1)\times T_P$ represents -1. Again by the Lemma, we have that -1 is a sum of squares of $m+1$ pure quaternions, hence $s(D)\leq m+1$.

2. case: $\sum_{\ell=1}^{m+1} x_\ell^2 = 0$ *but not all* $x_\ell = 0$.
In this case, 0 is a nontrivial sum of $m+1$ squares already in F, and thus $s(D)\leq s(F)=\underline{s}(F)\leq m$.

3. case: $\sum_{\ell=1}^{m+1} x_\ell^2 \neq 0$.
Let

$$c_\ell = \frac{x_\ell}{x_1^2+\cdots+x_{m+1}^2}\ .$$

We then get

$$\sum_{\ell=1}^{m+1} c_\ell\zeta_\ell \;=\; \frac{1}{x_1^2+\ldots+x_{m+1}^2}\left(\sum_{\ell=1}^{m+1} x_\ell^2 \;+\; \underbrace{\sum_{\ell=1}^{m+1} x_\ell\zeta_\ell'}_{=0}\right) \;=\; 1\ .$$

Put $c=c_1^2+\ldots+c_{m+1}^2=(x_1^2+\ldots+x_{m+1}^2)^{-1}$. This yields

$$\begin{aligned}\sum_{\ell=1}^{m+1}\left[\left(\frac{c+1}{2}\right)\zeta_\ell - c_\ell\right]^2 \;&=\; \left(\frac{c+1}{2}\right)^2 \underbrace{\sum_{\ell=1}^{m+1}\zeta_\ell^2}_{=0} \;-\; (c+1)\underbrace{\sum_{\ell=1}^{m+1} c_\ell\zeta_\ell}_{=1} \;+\; \underbrace{\sum_{\ell=1}^{m+1} c_\ell^2}_{=c} \\ &=\; -1\ ,\end{aligned}$$

which shows that $s(D)\leq m+1$. □

Remark. The above proof can be used more or less *verbatim* in the case of octonion division algebras (with the appropriate notions of pure octonion and of the form T_P corresponding to squares of pure octonions). So if $\mathcal{O}$ is an octonion division algebra, one also gets that $\underline{s}(\mathcal{O})\leq s(\mathcal{O})\leq \underline{s}(\mathcal{O})+1$.

References

[1] D.W. Hoffmann, Levels of quaternion algebras, *Arch. Math. (Basel)* **90** (2008), 401–411.

[2] M. Krüskemper and A.R. Wadsworth, A quaternion algebra of sublevel 3, *Bull. Soc. Math. Belg. Sér. B* **43** (1991), 181–185.

[3] T.Y. Lam, *Introduction to quadratic forms over fields*, Graduate Studies in Mathematics 67, American Mathematical Society, Providence, RI, 2005.

[4] D.B. Leep, Levels of division algebras, *Glasgow Math. J.* **32** (1990), 365–370.

[5] D.W. Lewis, Levels and sublevels of division algebras, *Proc. Roy. Irish Acad. Sect. A* **87** (1987), 103–106.

[6] D.W. Lewis, Levels of quaternion algebras, *Quadratic forms and real algebraic geometry (Corvallis, OR, 1986). Rocky Mountain J. Math.* **19** (1989), 787–792.

[7] D.W. Lewis, Levels of fields and quaternion algebras—a survey, *Théorie des nombres, Années 1996/97–1997/98, Publ. Math. UFR Sci. Tech. Besançon,* Univ. Franche-Comté, Besançon, 1999, 9 pp.

[8] J. O'Shea, Bounds on the levels of composition algebras, *Math. Proc. R. Ir. Acad.* (to appear).

[9] A. Pfister, Zur Darstellung von -1 als Summe von Quadraten in einem Körper, *J. London Math. Soc.* **40** (1965), 159–165.

School of Mathematical Sciences, University of Nottingham, University Park, Nottingham NG7 2RD, UK

E-mail address: `detlev.hoffmann@nottingham.ac.uk`