FIELD PATCHING, FACTORIZATION, AND LOCAL-GLOBAL PRINCIPLES

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1. Introduction

The goal of this paper is twofold - first to give an introduction to the method of field patching as first presented in [HH], and later used in [HHKa], paying special attention to the relationship between factorization and local-global principles and second, to extend the basic factorization result in [HHKa] to the case of retract rational groups, thereby answering a question posed to the author by J. L. Colliot-Thélène.

Throughout, we fix a complete discrete valuation ring T with field of fractions K and residue field k. Let $t \in T$ be a uniformizer. Let X/K be a smooth projective curve and F its function field.

Broadly speaking, the method of field patching is a procedure for constructing new fields F_ξ which will be in certain ways simpler than F, and to reduce problems concerning F to problems about the various F_ξ. Overall, there are two ways in which this is done. Let us suppose that we are interested in studying a particular type of arithmetic object, such as a quadratic form, a central simple algebra, etc.

Constructive Strategy (Patching) : This consists in showing that under suitable hypotheses, algebraic objects defined over the fields F_ξ which are "compatible," exactly correspond to objects defined over F (see Theorem 3.2.3). One may then use this idea to construct new examples and counterexamples of such objects by building them "locally."

Deconstructive Strategy (Local-global principle) : We say that a particular type of algebraic object satisfies a local global principle if whenever an object defined over F becomes "trivial" when scalars are extended to each F_ξ, it must in fact have been trivial to begin with (see Section 2.2).

In this paper, we will not focus on these applications, which are discussed for example in [HH, HHKa, HHKb, CTPS09]. Instead, we focus on elucidating and extending the underlying methods used.

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2. Patches and local-global principles

2.1. **Fields associated to patches.** The fields F_ξ are not canonically defined - they depend on a number of choices, beginning with the choice of a model for X over T.

Definition 2.1.1 ***Models***
*A **model** for the scheme X/K is defined to be a connected normal projective $\mathbb{P}^1_T$-scheme $\widehat{X}$ such that*

(1) *the structure morphism $f : \widehat{X} \to \mathbb{P}^1_T$ is finite,*
(2) *considered as a T-scheme, the generic fiber $\widehat{X}_K$ is isomorphic to X,*
(3) *The reduced closed fiber $\widehat{X}_k^{red}$ is a normal crossings divisor in $\widehat{X}$,*
(4) *$f^{-1}(\infty)$ contains all the singular points of the reduced closed fiber $\widehat{X}_k^{red}$.*

Given a model $\widehat{X}$ (we will generally suppress the morphism $f : \widehat{X} \to \mathbb{P}^1_T$ from the notation), we let $S(\widehat{X})$ denote the set of closed points in $f^{-1}(\infty)$ and $\mathcal{U}(\widehat{X})$ denote the set of connected (or equivalently, irreducible) components of $\widehat{X}_k^{\text{red}} \setminus S(\widehat{X})$. These sets play a critical role in what follows.

Warning 2.1.2
In other sources such as [HH, HHKa, HHKb]*, $\widehat{X}$ is not given the structure of a $\mathbb{P}^1_T$-scheme, but rather the structure of a T scheme together with a distinguished set of closed points S. In this context, one is allowed more general sets S. The reader must keep in mind that a model $\widehat{X}$ comes with the extra structure of a morphism to $\mathbb{P}^1$ throughout!*

It is perhaps a bit odd to include the finite morphism to $\mathbb{P}^1_T$ as part of the definition of a model — by comparison, in [HH], it is only assumed that one should start with a projective T-curve with a set S of closed points such that there exists a finite T-morphism to a curve with smooth reduced closed fiber and such that the set S is the inverse image of a set of closed points under this morphism. We include the morphism to $\mathbb{P}^1_T$ as part of our definition simply as a matter of convenience of exposition. The following lemma shows that it is not much of an extra assumption, however:

Lemma 2.1.3 [HH]***, Proposition 6.6***
Suppose $\widehat{X}$ is a projective T-curve and $S \subset \widehat{X}$ a finite set of closed points. Then there exists a finite morphism $f : \widehat{X} \to \mathbb{P}^1$ such that $S \subset f^{-1}(\infty)$.

For the remainder of the section, we will suppose that we are given such a model $\widehat{X}$, and we let $F = F(\widehat{X})$ be its function field. Given any nonempty subset of points $Z \subset \widehat{X}$, we define

$$R_Z = \{f \in F \mid \forall P \in Z, f \in \mathcal{O}_{\widehat{X},P}\}$$

We will define fields associated to two particular types of subsets Z:

Definition 2.1.4 ***Fields associated to closed points***
Let $P \in \widehat{X}$ be a closed point. We define $R_P = R_{\{P\}}$, $\widehat{R}_P$ its completion with respect to its maximal ideal, and F_P the field of fractions of $\widehat{R}_P$.

Definition 2.1.5 ***Fields associated to open subsets of $\widehat{X}_k^{red}$***
Let $U \subset \widehat{X}_k^{red}$ be a nonempty irreducible Zariski open affine subset of the reduced closed fiber which is disjoint from the singular locus of $\widehat{X}_k^{red}$. We let $\widehat{R}_U$ be the completion of R_U with respect to the t-adic valuation, and F_U the field of fractions of $\widehat{R}_U$.

Note that there are natural maps $F \subset F_U, F_P$ for any such P and U, as well as inclusions $F_U \to F_V$ and $F_U \to F_P$, whenever $V \subset U$ or $P \in U$ respectively.

2.2. **Some local-global principles.** We may now give some examples of local-global principles. For these, we assume that X/K is a smooth projective curve where K a complete discretely valued field with valuation ring T, and that we are given a model $\widehat{X} \to \mathbb{P}^1$. We let F be the function field of X.

Theorem 2.2.1 ***Local-global principle for the Brauer group (see*** [HH]***, Theorem 4.10)***
The natural homomorphism

$$\mathrm{Br}(F) \to \left(\prod_{P \in S(\widehat{X})} \mathrm{Br}(F_P)\right) \times \left(\prod_{U \in \mathcal{U}(\widehat{X})} \mathrm{Br}(F_U)\right)$$

is injective.

We give a proof of this result on page 5. In fact, we will see later, using patching, that this may be extended to a three term exact sequence by adding a term on the right (see Theorem 3.3.1).

Theorem 2.2.2 ***Local-global principle for isotropy (see*** [HHKa]***, Theorem 4.2)***
Suppose q is a regular quadratic form of dimension at least 3, and $\mathrm{char}(F) \neq 2$. *If q_{F_P} and q_{F_U} are isotropic for every $P \in S(\widehat{X})$ and $U \in \mathcal{U}(\widehat{X})$ then q is also isotropic.*

The proof of this is given on page 5. See also Theorem 3.3.2 for a related result.

Both of these principles in fact, may be regarded as special cases of either of the following results, the main new results of this paper:

Definition 2.2.3
Suppose H is a variety over F and G is an algebraic group which acts on H. We say that G ***acts transitively*** *on H if for every field extension L/F, the group $G(L)$ acts transitively on the set $H(L)$.*

The following result generalizes [HHKa], Theorem 3.7 by weakening the hypothesis of rationality to allow for retract rational groups as well:

Theorem 2.2.4 ***Local-global principle for varieties with transitive actions***
Suppose G is a connected retract rational algebraic group defined over F, and H is a variety on which G acts transitively. Then $H(F) \neq \emptyset$ if and only if $H(F_P), H(F_U) \neq \emptyset$ for all $P \in S(\widehat{X})$ and $U \in \mathcal{U}(\widehat{X})$.

This theorem follows quickly from Theorem 5.1.1, and its proof may be found just after the statement of this theorem on page 14. The proof of this in the case of retract rationality will occupy a good portion of this paper. Along the way, we will explore the connections between this local global principles and the notion of "factorization" for the group G. The following corollary is particularly useful.

Corollary 2.2.5
Suppose G is a retract rational reductive group over F and H a homogeneous variety for G. Then $H(F) \neq \emptyset$ if and only if $H(F_P), H(F_U) \neq \emptyset$ for all $P \in S(\widehat{X})$ and $U \in \mathcal{U}(\widehat{X})$.

From these theorems (or even the versions assuming only rationality of G from [HHKa]), we may prove the above local global results concerning the Brauer group and quadratic forms.

proof of Theorem 2.2.1. Let $\alpha \in \mathrm{Br}(F)$ and suppose $\alpha_{F_P} = 0 = \alpha_{F_U}$ for every $P \in S(\widehat{X})$, $U \in \mathcal{U}(\widehat{X})$. We need to show that $\alpha = 0$.

Let A be a central simple F algebra in the class α and let H be the Severi-Brauer variety for A. Note that this is a homogeneous variety for the group $GL(A)$ which is rational, connected and reductive. Recall that for a field extension L/F, $H(L)$ is nonempty exactly when $A \otimes_F L$ is a split algebra — that is to say, $\alpha_L = 0$. But since $\alpha_{F_P}, \alpha_{F_U} = 0$, we have $H(F_P), H(F_U) \neq \emptyset$ for every U, P. Consequently, by Corollary 2.2.5, it follows that $H(F) \neq \emptyset$ and so $\alpha = 0$ as desired. □

proof of Theorem 2.2.2. Let q be a quadratic form over F satisfying the hypotheses of the Theorem. We wish to show that q is isotropic. Let H be the quadratic hypersurface of projective space defined by the equation $q = 0$. Recall that this is a homogeneous variety for the group $SO(q)$ which under the hypotheses is a rational, connected, reductive group (see [KMRT98], page 209, excercise 9). As above, we immediately see that since $H(F_P), H(F_U)$ are nonempty for each $P \in S(\widehat{X})$ and $U \in \mathcal{U}(\widehat{X})$, we have by Corollary 2.2.5, $H(F) \neq \emptyset$ as desired. □

3. Patching

The fundamental idea of patching is that defining an algebraic object over the field F is equivalent to defining objects over each of the fields F_P for $P \in S(\widehat{X})$ and F_U for $U \in \mathcal{U}(\widehat{X})$, together with the data of how these objects agree on overlaps. This will be stated in this section in terms of an equivalence of categories. We will simply cite the results of [HH] section 6 and 7 for the most part, but we focus more on the equivalence of tensor categories, and explore how to produce other examples of algebraic patching.

Suppose we are given a model $\widehat{X}$ for a curve X/K. Given a point $P \in S(\widehat{X})$, the height 1 primes of R_P which contain t correspond to the components of $\widehat{X}_k^{\mathrm{red}}$ incident to P. Each such component is the closure of a uniquely determined element $U \in \mathcal{U}(\widehat{X})$.

Definition 3.0.6 ***Branches, and their fields***
Given such a height 1 prime $\mathcal{P}$ of R_P, corresponding to an element $U \in \mathcal{U}(\widehat{X})$, a ***branch*** *along U at P is an irreducible component of the scheme $\widehat{R}_P/\mathcal{P}\widehat{R}_P$. Alternately, these are in correspondence with the*

height one primes of $\widehat{R}_P$ containing $\mathcal{P}\widehat{R}_P$. Given such a height 1 prime $\wp$, we let $\widehat{R}_\wp$ be the t-adic completion of the localization of $\widehat{R}_P$ at $\wp$, and $F_\wp$ its field of fractions. We let $\mathcal{B}(\widehat{X})$ denote the set of all branches at all points in $S(\widehat{X})$.

The fields F_P and F_U come equipt with natural inclusions into $F_\wp$ which we now describe. We note that the natural inclusion $\widehat{R}_P \to \widehat{R}_\wp$ induces an inclusion of fields $F_P \to F_\wp$. Further, we note that $\widehat{R}_\wp$ is a 1 dimensional regular local ring, and hence a DVR, whose valuation is determined by considering order of vanishing along the branch corresponding to $\wp$. In particular, considering the inclusion $F \subset F_P \subset F_\wp$, we find that all the elements of R_U, having no poles along U, can also not have poles along any branch lying along U, and in particular, we see we have an inclusion $R_U \subset \widehat{R}_\wp$. Since the t-adic topology on $\widehat{R}_\wp$ is the same as the $\wp$-adic topology, we further find that $\widehat{R}_\wp$ is t-adically complete, and we therefore have an induced inclusion $F_U \to F_\wp$.

3.1. **Patching finite dimensional vector spaces.**

Definition 3.1.1 ***Patching problems***
*A **patching problem** is a collection V_ξ for $\xi \in S(\widehat{X}) \cup \mathcal{U}(\widehat{X})$, where V_ξ is a finite dimensional F_ξ vector space together with a collection of isomorphisms $\phi_\wp : V_P \otimes_{F_P} F_\wp \to V_U \otimes_{F_U} F_\wp$ of $F_\wp$ vector spaces for every branch $\wp$ at P on U. We denote this problem by (V, ϕ).*
*We define a **morphism of patching problems** $f : (V, \phi) \to (W, \psi)$ to be a collection of homomorphisms $f_\xi : V_\xi \to W_\xi$ such that whenever $\wp$ is a branch at P lying on U, the following diagram commutes:*

$$\begin{tikzcd}
V_P \otimes_{F_P} F_\wp \arrow[r, "f_P \otimes F_\wp"] \arrow[d, "\phi_\wp"'] & W_P \otimes_{F_P} F_\wp \arrow[d, "\psi_\wp"] \\
V_U \otimes_{F_U} F_\wp \arrow[r, "f_U \otimes F_\wp"'] & V_P \otimes_{F_P} F_\wp
\end{tikzcd}$$

We see then that patching problems naturally form a category, which we denote by $\mathcal{PP}(\widehat{X}, S)$. In fact, this category has a $\otimes$-structure as well defined by $(V, \phi) \otimes (W, \psi) = (V \otimes W, \phi \otimes \psi)$ where $(V \otimes W)_\xi = V_\xi \otimes_{F_\xi} W_\xi$ and

$$(\phi \otimes \psi)_\wp : (V \otimes W)_P \to (V \otimes W)_U$$

is given by $\phi_\wp \otimes_{F_\wp} \psi_\wp$ via the above identification. One may also verify that this monoidal structure is symmetric and closed (see [ML98], VII.7).

Definition 3.1.2
If V is a vector space over F, we let $(\widetilde{V}, \mathbb{I})$ denote the patching problem defined by $\widetilde{V}_P = V_{F_P}$ and $\widetilde{V}_U = V_{F_U}$ and where $\mathbb{I}_\wp$ is induced by the natural identifications

$$(V \otimes_F F_P) \otimes_{F_P} F_\wp = V_\wp = (V \otimes_F F_U) \otimes_{F_U} F_\wp$$

Theorem 3.1.3
[[HH]*, Theorem 6.4] Consider the functor*

$$\Omega : \mathcal{V}ect_{f.d.}(F) \to \mathcal{PP}(\widehat{X}, S)$$

from the category of finite dimensional F-vector spaces to the category of patching problems defined by sending a finite dimensional vector space V to the patching problem $(\widetilde{V}, \mathbb{I})$. Then Ω is an equivalence of categories.

3.2. **Patching algebraic objects.**

Definition 3.2.1
*A **type of algebraic object** (generally abbreviated to simply a "type") is a symmetric closed monoidal category $\mathscr{T}$. If $\mathscr{T}$ is a type and L a field, then an algebraic object of type $\mathscr{T}$ over L is a strict symmetric closed monoidal functor (see* [ML98]*, § VII.1, § VII.7 and* [Hov99] *§4.1 for definitions) from the category $\mathscr{T}$ to the category of finite dimensional vector spaces over L (with its natural symmetric closed monoidal structure). Morphisms between algebraic objects of type $\mathscr{T}$ are defined simply to be natural transformations between functors. We let $\mathscr{T}(L)$ denote the category of such objects.*

Note that $\mathscr{T}$ in fact defines a (pseudo-)functor from the category of fields to the 2-category of categories (see [Gra74] for definitions).

Despite the formality of this definition, one may observe that one may interpret an algebraic object of a given type $\mathscr{T}$ to be given by a vector space, or a collection of vector spaces, together with extra structure encoded by perhaps a collection of morphisms between various tensor powers of the vector spaces satisfying certain axioms, and where morphisms between these objects are given by collections of linear maps satisfying certain compatibilies with the extra structures given. For example, we might consider:

- Lie algebras,
- Alternative (or Jordan) algebras,
- Operads,
- Central simple algebras
- Quadratic forms, where morphisms are isometries,
- Quadratic forms, where morphisms are similarities,
- Separable commutative or noncommutative algebras,
- G-Galois extensions of rings in the sense of [DI71]
- and so on...

In these cases, the category $\mathscr{T}$ in question is simply given as the symmetric closed monoidal category generated by some set of objects (corresponding to the underlying vector spaces of the structure) and some morphisms (defining the structure of the algebra or form), such that certain diagrams commute which define the structure in question. For example, a central simple algebra is a vector space A together with a bilinear product $A \otimes A \to A$, and F-algebra structure $F \to A$ such that the canonical "sandwich map" of algebras

$$A \otimes A^{\mathrm{op}} \to \mathrm{Hom}(A, A)$$

is an isomorphism (see [DI71], Chapter 2, Theorem 3.4(iii)). In this case, the category $\mathscr{T}$ is generated by a single element a, a morphism $a \otimes a \to a$ and $\mathbb{1} \to a$ (where $\mathbb{1}$ is the unit for the monoidal structure), and such that the natural map $a \otimes a \to \mathrm{Hom}(a, a)$ (where the Hom is defined by the closed structure) has an inverse.

To see quadratic forms and isometries in this way, one may simply let the category $\mathscr{T}$ be generated by a single element v a morphism $v \otimes v \to \mathbb{1}$, assumed to commute with the morphism switching the order of the v's. In the case of similarities instead of isometries, one may add a new object ℓ, and replace $v \otimes v \to \mathbb{1}$ with a morphism $v \otimes v \to \ell$. To force ℓ to correspond to a 1-dimensional vector space, one may then add to this category an inverse to the natural morphism $\mathbb{1} \to \ell \otimes \ell^* \cong \mathrm{Hom}(\ell, \ell)$.

Definition 3.2.2 ***Patching problems***
Let $\mathscr{T}$ be a type of algebraic object. A ***patching problem of objects of type*** *$\mathscr{T}$ is a collection A_ξ for $\xi \in S(\widehat{X}) \cup \mathfrak{U}(\widehat{X})$, where A_ξ is an object of type $\mathscr{T}$ over F_ξ, together with a collection of isomorphisms $\phi_\wp : A_P \otimes_{F_P} F_\wp \to A_U \otimes_{F_U} F_\wp$ in $\mathscr{T}(F_\wp)$. We denote this problem by (A, ϕ).*

Just as with vector spaces, we may define morphisms of patching problems of objects of type $\mathscr{T}$, and again find that these form a tensor category, which we denote $\mathcal{PP}_{\mathscr{T}}(\widehat{X})$. Again as before, if A is an algebraic object of type $\mathscr{T}$ over F, we may form a natural patching problem $(\widetilde{A}, \mathbb{I})$, and obtain a functor from $\mathscr{T}(F)$ to $\mathcal{PP}_{\mathscr{T}}(\widehat{X})$.

Theorem 3.2.3
Consider the functor

$$\Omega_{\mathscr{T}} : \mathscr{T}(F) \to \mathcal{PP}_{\mathscr{T}}(\widehat{X})$$

defined by sending an algebraic object A to the patching problem $(A, \mathbb{I})$. Then $\Omega_{\mathscr{T}}$ is an equivalence of categories.

Proof. Since we have an equivalence of categories $\mathcal{V}ect_{\text{f.d.}}(F) \cong \mathcal{PP}(\widehat{X})$ by Theorem 3.1.3, it is immediate that this equivalence also induces an equivalence of functor categories

$$\mathscr{T}(F) = Fun(\mathscr{T}, \mathcal{V}ect_{\text{f.d.}}(F) \cong Fun(\mathscr{T}, \mathcal{PP}(\widehat{X})) \cong \mathcal{PP}_{\mathscr{T}}(\widehat{X}).$$

One may now check that this gives the desired equivalence. □

Remark 3.2.4
It would be interesting to know if one could extend this to equivalences of other kinds of objects. In particular, infinite dimensional vector spaces, finitely generated commutative algebras, or perhaps even to (some suitably restricted) categories of schemes. None of these fall under the definition of an algebraic object given above, and it is therefore not at all clear if the conclusions of Theorem 3.2.3 will still hold.

3.3. **Central simple algebras and quadratic forms.** For the following results, we suppose we are given $\widehat{X}$ a normal, connected, projective, finite $\mathbb{P}^1_T$-scheme. The machinery of patching gives the exactness of various exact sequences relating to field invariants derived from algebraic objects.

Theorem 3.3.1 ***(see*** [HH]***, Theorem 7.2)***
We have an exact sequence:

$$0 \to \mathrm{Br}(F) \to \left(\prod_{P \in S(\widehat{X})} \mathrm{Br}(F_P) \right) \times \left(\prod_{U \in \mathcal{U}(\widehat{X})} \mathrm{Br}(F_U) \right) \to \prod_{\wp \in \mathcal{B}(\widehat{X})} \mathrm{Br}(F_\wp)$$

Proof. Exactness on the left was noted in Theorem 2.2.1. To see exactness in the middle, suppose we have classes α_P, α_U such that $(\alpha_U)_{F_\wp} \cong (\alpha_P)_{F_\wp}$ whenever $\wp$ is a branch at P on U. Since there are only a finite number of points and components, we may choose an integer n such that each of the Brauer classes α_U, α_P may be represented by central simple algebras A_U, A_P of degree n. Now, by hypothesis, for each branch $\wp$ as above, we may find an isomorphism of central simple algebras $\phi_\wp : (A_P)_{F_\wp} \to (A_U)_{F_\wp}$. But this gives the data of a patching problem for central simple algebras, and therefore we may find a central simple F-algebra A such that $A_{F_P} \cong A_P$ and $A_{F_U} \cong A_U$ as desired. □

Theorem 3.3.2
We have an exact sequence:

$$\mathrm{W}(F) \to \left(\prod_{P \in S(\widehat{X})} \mathrm{W}(F_P) \right) \times \left(\prod_{U \in \mathcal{U}(\widehat{X})} \mathrm{W}(F_U) \right) \to \prod_{\wp \in \mathcal{B}(\widehat{X})} \mathrm{W}(F_\wp)$$

Proof. The proof is very similar to the last one. Suppose we have Witt classes α_P, α_U such that $(\alpha_U)_{F_\wp} \cong (\alpha_P)_{F_\wp}$ whenever $\wp$ is a branch at P on U. Since there are only a finite number of points and components, we may choose an integer n such that each of the Witt classes α_U, α_P may be represented by quadratic forms q_U, q_P of the same dimension n. Now, by hypothesis, for each branch $\wp$ as above, we may find an isometry $\phi_\wp : (q_P)_{F_\wp} \to (q_U)_{F_{wp}}$. But this gives the data of a patching problem for quadratic forms, and therefore we may find a form q over F such that $q_{F_P} \cong q_P$ and $q_{F_U} \cong q_U$ as desired. □

We note that exactness on the left is discussed in Theorem 2.2.2.

3.4. **Properties of $\widehat{R}_P, \widehat{R}_U, F_P, F_U$.** Let us now gather together some fundamental facts which we will need in the sequel.

Lemma 3.4.1 [HH]***, Lemma 6.2***
Suppose $\widehat{Y} \to \widehat{X}$ is a finite morphisms of projective, normal, finite $\mathbb{P}^1_T$-schemes. Then the natural inclusions of fields yield isomorphisms:

$$F_P \otimes_{F(\widehat{X})} F(\widehat{Y}) \cong \prod F_{P'} \quad F_U \otimes_{F(\widehat{X})} F(\widehat{Y}) \cong \prod F_{U'} \quad F_\wp \otimes_{F(\widehat{X})} F(\widehat{Y}) \cong \prod F_{\wp'}$$

where P' (resp. U', $\wp'$) range over all the points (resp. components, branches) lying over P (resp. U, $\wp$).

Lemma 3.4.2 [HH]***, Lemma 6.3***
Let $\widehat{X}$ be a projective, normal, finite $\mathbb{P}^1_T$-scheme. Then the natural

inclusions of fields yield an exact sequence of $F = F(\widehat{X})$-vector spaces:

$$0 \to F \to \left(\prod_{P \in S(\widehat{X})} F_P \right) \times \left(\prod_{U \in \mathcal{U}(\widehat{X})} F_U \right) \to \prod_{\wp \in \mathcal{B}(\widehat{X})} F_\wp$$

Lemma 3.4.3
Suppose $\mathfrak{V}, \mathfrak{W} \subset \mathfrak{U}$ are $\widehat{R}_0$ modules, where $t \in \widehat{R}_0$ and $\mathfrak{V}, \mathfrak{W}, \mathfrak{U}$ are all t-adically complete. Suppose further that $\mathfrak{V}/t\mathfrak{V} + \mathfrak{W}/t\mathfrak{W} = \mathfrak{U}/t\mathfrak{U}$. Then $\mathfrak{V} + \mathfrak{W} = \mathfrak{U}$.

Proof. Suppose $u \in \mathfrak{U}$. Let $v_0 = w_0 = 0$. We will inductively construct a sequence of elements $v_i \in \mathfrak{V}$, $w_i \in \mathfrak{W}$ such that $v_i - v_{i+1} \in t^i\mathfrak{V}, w_i - w_{i+1} \in t^i\mathfrak{W}, v_i + w_i - u \in t^i\mathfrak{U}$. By completeness, these will converge to elements $v \in \mathfrak{V}, w \in \mathfrak{W}$ such that $v + w = u$.

Suppose we have constructed v_i, w_i satisfying the above hypotheses. Since $u - v_i - w_i \in t^i\mathfrak{U}$, we may write $u - v_i - w_i = t^i r$. By hypothesis, we may write $r = v' + w' + tr'$ for some $v' \in \mathfrak{V}, w' \in \mathfrak{W}, r' \in \mathfrak{U}$. Setting $v_{i+1} = v_i + t^i v', w_{i+1} = w_i + t^i w'$, completes the inductive step. □

Lemma 3.4.4
Considering $\mathbb{P}^1_T$, we have $\widehat{R}_{\mathbb{A}^1} + \widehat{R}_\infty = \widehat{R}_\wp$, where $\wp$ is the unique branch at ∞.

Proof. Using Lemma 3.4.3, we need only check that $\overline{R}_{\mathbb{A}^1} + \overline{R}_\infty = \overline{R}_\wp$, where

$$\overline{R}_{\mathbb{A}^1} \cong \widehat{R}_{\mathbb{A}^1}/t\widehat{R}_{\mathbb{A}^1}, \quad \overline{R}_\infty \cong \widehat{R}_\infty/t\widehat{R}_\infty, \quad \overline{R}_\wp \cong \widehat{R}_\wp/t\widehat{R}_\wp$$

But, we may compute $\overline{R}_{\mathbb{A}^1} = k[\mathbb{A}^1_k]$, $\overline{R}_\infty = \widehat{\mathcal{O}_{\mathbb{P}^1_{k,\infty}}}$, $\overline{R}_\wp = \mathrm{frac}(\widehat{\mathcal{O}_{\mathbb{P}^1_{k,\infty}}})$. Writing x for the coordinate function on the affine part of the k-line, we may explicitly identify

$$\overline{R}_{\mathbb{A}^1} = k[x], \quad \overline{R}_\infty = k[[x^{-1}]], \quad \overline{R}_\wp = k((x^{-1})),$$

and the result follows. □

4. Local-global principles, factorization and patching

Let $\widehat{X} \to \mathbb{P}^1$ be a model for X/K, and let G be an algebraic group defined over F.

4.1. **Local-global principles for rational points.**

Definition 4.1.1
*We say that **factorization holds for** G, with respect to $\widehat{X}$, if for every tuple $(g_\wp)_{\wp \in \mathcal{B}(\widehat{X})}$, there exist collections of elements g_P for each*

$P \in S(\widehat{X})$ and g_U for each $U \in \mathfrak{U}(\widehat{X})$ such that whenever $\wp$ is a branch at P on U we have

$$g_\wp = g_P g_U$$

with respect to the natural embeddings $F_P, F_U \to F_\wp$.

Definition 4.1.2
*We say that **the local-global principle holds** for an F scheme V, with respect to a model $\widehat{X}$ if $X(F) \neq \emptyset$ holds if and only if $X(F_P), X(F_U) \neq \emptyset$ for every $P \in S(\widehat{X})$ and $U \in \mathfrak{U}(\widehat{X})$.*

Definition 4.1.3
*Let G be an algebraic group over F and H a scheme over F. We say that H is a **transitive** G-scheme if G acts transitively on H (see Definition 2.2.3).*

Proposition 4.1.4
If factorization holds for a group G, then the local-global principle holds for all transitive schemes over G.

Proof. See the proof of Theorem 3.7 in [HHKa]. □

4.2. **Local-global principles for algebraic objects and torsors.**

Definition 4.2.1
*We say that **the local-global principle holds for an algebraic object** A (of some given type) if for any algebraic object B (of the same type), we have $A \cong B$ if and only if $A_{F_P} \cong B_{F_P}$ and $A_{F_U} \cong B_{F_U}$ for all P, U. We say that the local-global principle holds for a type of algebraic object if it holds for every algebraic object of the type.*

Proposition 4.2.2
The local-global principle holds for an algebraic object A' if and only for every patching problem (A, ϕ) of algebraic objects such that $A_P \cong (A')_{F_P}, A_U \cong (A')_{F_U}$ for all P, U, the isomorphism class of (A, ϕ) is independent of ϕ.

Proof. Suppose that the local-global principle holds for A', and let $(A, \phi), (A, \psi)$ be two patching problems, such that $A_P \cong (A')_{F_P}$ and $A_U \cong (A')_{F_U}$ for each P, U. Since we may patch algebraic objects, we may find algebraic objects B_1, B_2 over F whose patching problems are equivalent to $(A, \phi), (A, \psi)$ respectively. Since $(B_1)_{F_U} \cong A_{F_U} \cong (B_2)_{F_U}$ and similarly for F_P, we find that by the local-global principle, $B_1 \cong B_2$. But therefore their associated patching problems are isomorphic, implying $(A, \phi) \cong (A, \psi)$ as desired.

Conversely, suppose that (A, ϕ)'s isomorphism class is independent of ϕ for every patching problem. Suppose we are given A', B' be algebraic objects over F with associated patching problems (A, ϕ) and (B, ψ) respectively. Suppose further that $(A')_{F_U} \cong (B')_{F_U}$ and similarly for F_P. Since $A_U \cong (A')_{F_U} \cong (B')_{F_U} \cong B_U$ and $A_P \cong (A')_{F_P} \cong (B')_{F_P} \cong B_P$ for all U, P by definition, we may change ψ via these isomorphisms to find $(B, \psi) \cong (A, \psi')$ for some ψ'. But therefore by hypothesis, $(A, \phi) \cong (A, \psi') \cong (B, \psi)$. Since patching gives an equivalence of categories, we further conclude $A' \cong B'$, completing the proof. □

Remark 4.2.3
Let $\mathscr{T}$ be a type of algebraic object, and A is a particular object of type $\mathscr{T}$. Let $\mathscr{T}_A$ denote the subclass of objects which are isomorphic to A (more precisely, $\mathscr{T}_A$ is the sub-pseudofunctor of $\mathscr{T}$ which associates to every field extension L/F the category of algebraic objects of type $\mathscr{T}$ over L which are isomorphic to the object A_L). Then $\mathscr{T}_A$ satisfies the hypotheses of patching — i.e. we have an equivalence of categories between the category $\mathcal{PP}_{\mathscr{T}_A}(\widehat{X})$ and $\mathscr{T}_A(F)$ — if and only if the local-global principle holds for A. Note that in general $\mathscr{T}_A$ is not a "type of algebraic object," described by some monoidal category in the sense described above.

Definition 4.2.4
*Let G be an algebraic group over F. We say that **the local-global principle holds for** G if for $\alpha \in H^1(F, G)$, with $\alpha_{F_P}, \alpha_{F_U}$ trivial for each P, U we have α trivial.*

Note that since elements of $H^1(F, G)$ correspond to torsors for G, we see immediately that the local-global principle will hold for G if and only if the local-global principle holds for all G-torsors, in the sense of Definition 4.1.2. From this observation and from Proposition 4.1.4, we immediately obtain:

Proposition 4.2.5
Suppose G is a linear algebraic group defined over F, and suppose that factorization holds for G with respect to $\widehat{X}$. Then the local global principle holds for G as well.

Proposition 4.2.6
Suppose A is an algebraic object of some type $\mathscr{T}$, whose automorphism group is the linear algebraic group G. Then the following are equivalent:

(1) *the local-global principle holds for A,*
(2) *the local-global principle holds for G.*

(3) *factorization holds for G,*

Proof. Since G is the automorphism group of A, by descent (see [Ser79], X.§2, Proposition 4), we may identify $H^1(L, G_L) = \mathrm{Forms}(A_L)$, the pointed set of twisted forms of A_L. In particular, it is immediate from the definition that the local global principle for A is equivalent to the local global principle for G.

Suppose we have a local-global principle for A, and consider a collection of elements $g_\wp \in G(F_\wp)$. Consider the patching problem (B, ϕ) where $B_P = A_{F_P}, B_U = A_{F_U}$, and $\phi_\wp = g_\wp$. By the local-global principle, this is isomorphic to the patching problem $(\widetilde{A}, \mathbb{I})$. By definition, we may find an isomorphism $h : (B, \mathbb{I}) \to (B, \phi)$. Let $g_P = h_P^{-1}$ and $g_U = h_U$. By definition of a morphism of patching problems, we find that $g_\wp = g_P g_U$, and that $g_P \in Aut(B_P) = G(F_P)$, $g_U \in Aut(B_U) = G(F_U)$ as desired.

Conversely, suppose we have factorization for G. In this case it is immediate from Proposition 4.2.5 that the local global principle must hold for G, completing the proof. □

Remark 4.2.7
Theorem 4.2.6 raises the question of whether it would be possible to show the equivalence of the local global principle for a group G and factorization for this group without the presence of an algebraic object with G as its automorphism group. This would give a converse to Proposition 4.2.5. In turn since G-torsors are, in particular, transitive G-schemes, one would then also obtain a converse to Proposition 4.1.4.

5. Factorization for retract rational groups

5.1. **Overview and preliminaries.** The goal of this section will be to prove the following theorem:

Theorem 5.1.1
Suppose $\widehat{X}$ is a connected normal finite $\mathbb{P}^1_T$-scheme, with function field F and let G be a connected retract rational algebraic group over F. Then factorization holds for G with respect to $\widehat{X}$.

Using this theorem, we may easily proceed to the proof of the local global principle for schemes with transitive action stated earlier in Theorem 2.2.4: If G is a connected retract rational group over F, then by the theorem, factorization holds for G with respect to $\widehat{X}$. But then by Proposition 4.1.4, the local global principle must hold for transitive G schemes, as desired.

The proof of this theorem will occupy the remainder of the section. Our strategy will be to reduce this to a more abstract factorization problem, arising from the case when $\widehat{X} = \mathbb{P}^1_T$. Overall, the proof stategy is roughly parallel to that followed in [HHKa], where retractions of open subsets of affine space take the place of open subsets of affine space.

Definition 5.1.2
*Suppose we have commutative rings $F \subset F_1, F_2 \subset F_0$, and an algebraic group G over F. We will say that **factorization holds** with respect to G, F, F_1, F_2, F_0 if for every $g_0 \in G(F_0)$ there exist $g_1 \in G(F_1)$ and $g_2 \in G(F_2)$ such that $g_0 = g_1 g_2$.*

Note that here we are omitting from the notation the homomorphism $G(F_i) \to G(F_0)$ for $i = 1, 2$. We are particularly interested in this in the following case. Suppose $\widehat{X}$ is a connected, normal, finite $\mathbb{P}^1_T$-scheme. In this case, we set $F = F(\widehat{X})$, and we let

$$F_1 = \prod_{P \in S(\widehat{X})} F_P, \quad F_2 = \prod_{U \in \mathcal{U}(\widehat{X})} F_U, \quad F_0 = \prod_{\wp \in \mathcal{B}(\widehat{X})} F_\wp$$

Remark 5.1.3
It follows immediately from the definitions that factorization holds for the group G with respect to $\widehat{X}$ in the sense of Definition 4.1.1 if and only if factorization holds for G, F, F_1, F_2, F_0 in the sense of Definition 5.1.2 where F, F_1, F_2, F_0 are as above.

Back to the somewhat more abstract setting, suppose that F is some field, and let L/F be a finite algebra. Recall that if G is a linear algebraic group scheme, we may define its Weil restriction, also referred to as its corestriction or transfer, as the linear algebraic group with the functor of points defined by:

$$\mathrm{R}_{L/F}\, G(R) = G(R \otimes_F L)$$

where R ranges through all F-algebras ([Gro62], Exp. 195, p. 13 for the definition and Exp. 221, p. 19 for proof of existence). We note that the corestriction in fact comes from a Weil restriction functor from the category of quasi-projective L-schemes to the category of quasi-projective F-schemes, and that this functor takes open inclusions to open inclusions, and takes affine space to affine space (of a different dimension). In particular, it follows that the corestriction of a rational (or retract rational) variety is itself rational (resp. retract rational).

We note the following Lemma, which is an consequence of the definition of the corestriction in terms of the functor of points given above.

Lemma 5.1.4
Let F be a field, and suppose we are given rings $F \subset F_1, F_2 \subset F_0$, and an finite algebra L/F. Let G be a linear algebraic group over L. Then factorization holds for $G, L, L \otimes_F F_1, L \otimes_F F_2, L \otimes F_0$ if and only if it holds for $\mathrm{R}_{L/F}\, G, F, F_1, F_2, F_0$.

Lemma 5.1.5
Suppose that we are given a morphism of connected projective normal finite $\mathbb{P}^1_T$-schemes $f : \widehat{Y} \to \widehat{X}$. Let L be the function field of $\widehat{Y}$ and F the function field of $\widehat{X}$. Then factorization holds for $G, \widehat{Y}$ if and only if it holds for $\mathrm{R}_{L/F}\, G, \widehat{X}$.

Proof. This follows immediately from the universal property of the Weil restriction, together with Lemma 3.4.1. □

Lemma 5.1.6
Let F be the function field of $\mathbb{P}^1_T$, which we consider as a $\mathbb{P}^1_T$-scheme via the identity map. Suppose that for every connected retract rational group G over F, factorization holds for G with respect to $\mathbb{P}^1_T$ (as in Definition 4.1.1). Then for every normal finite $\mathbb{P}^1_T$-scheme $\widehat{X}$ with function field L, and every connected retract rational group H over L, factorization holds for H with respect to $\widehat{X}$.

Proof. This follows immediately from Lemma 5.1.5. □

As a consequence of this, in order to prove Theorem 5.1.1, we may restrict to the setting where F is the function field of $\mathbb{P}^1_T$, and where $F_1 = F_\infty$, $F_2 = F_{\mathbb{A}^1_k}$, and where $F_0 = F_\wp$ is the field associated to the unique branch $\wp$ along $\mathbb{A}^1_k$ at ∞. We let $\widehat{R}_0 = \widehat{R}_\wp, \mathfrak{V} = \widehat{R}_\infty, \mathfrak{W} = \widehat{R}_{\mathbb{A}^1_k}$. For convenience, in the sequel we will often refer to the following hypothesis for factorization.

Hypothesis 5.1.7 *see* [HHKa], *Hypothesis 2.4*
We assume that the complete discrete valuation ring $\widehat{R}_0$ contains a subring T which is also a complete discrete valuation ring having uniformizer t, and that F_1, F_2 are subfields of F_0 containing T. We further assume that $\mathfrak{V} \subset F_1 \cap \widehat{R}_0$, $\mathfrak{W} \subset F_2 \cap \widehat{R}_0$ are t-adically complete T-submodules satisfying $\mathfrak{V} + \mathfrak{W} = \widehat{R}_0$.

Lemma 5.1.8
With respect to the scheme $\mathbb{P}^1_T$ consider $F = F(\mathbb{P}^1_T)$, $F_0 = F_\wp$, $F_1 = F_\infty$,

$F_2 = F_{\mathbb{A}^1_k}$, $\widehat{R}_0 = \widehat{R}_\wp$, $\mathfrak{V} = \widehat{R}_\infty, \mathfrak{W} = \widehat{R}_{\mathbb{A}^1_k}$. Then these rings and modules satisfy the Hypothesis 5.1.7.

Proof. The completeness of $\mathfrak{V}, \mathfrak{W}$ is satisfied by definition. The fact that $\mathfrak{V} + \mathfrak{W} = \widehat{R}_0$ follows from Lemma 3.4.4. □

5.2. **Retractions – basic definitions and properties.** Before attacking the problem of factorization directly, it is necessary to collect some facts concerning retractions and retract rational varieties.

Definition 5.2.1
*We say that a variety Y is a **retraction** of a variety U if there exist morphisms $i : Y \to U$ and $p : U \to Y$ such that $pi = id_Y$. We say that it is a **closed retraction** if i is a closed embedding.*

Remark 5.2.2
In the case of a closed retraction, we will occasionally abuse notation by simply regarding i as an inclusion.

Lemma 5.2.3 ***Retractions are open***
Suppose we are given topological spaces Y, U and continuous maps $i : Y \to U$ and $p : U \to Y$ such that $pi = id_Y$. Then p is an open map (i.e. for $V \subset U$ open, $p(U)$ is open).

Proof. This follows immediately from the fact that $p(V) = i^{-1}(V)$ and the continuity of i. □

Definition 5.2.4
*We say that Y is a **rational retraction** of U if there are rational maps $i : Y \dashrightarrow U$ and $p : U \dashrightarrow Y$ such that $pi = id_Y$ on some open set on which pi is defined.*

Definition 5.2.5
*We say a variety Y is **retract rational** if it is a rational retraction of $\mathbb{A}^n$ for some n.*

Lemma 5.2.6 ***Rational retractions shrink to retractions***
Suppose Y is a rational retraction of U via rational maps i, p. Then we may find dense open subsets $Y_0 \subset Y$ and $U_0 \subset U$ such that i, p make Y_0 a retraction of U_0.

Proof. We may find open subsets $\widetilde{Y} \subset Y$ and $\widetilde{U} \subset U$ such that i, p restrict to morphisms on these sets, i.e. we have:

$$\begin{tikzcd} Y \arrow[r, dashed] & \mathbb{A}^n \arrow[r, dashed] & Y \\ \widetilde{Y} \arrow[u] \arrow[ur, "i"] & \widetilde{U} \arrow[u] \arrow[ur, "p"] & \widetilde{Y} \arrow[u] \end{tikzcd}$$

We choose $Y' = i^{-1}p^{-1}\widetilde{Y} \subset \widetilde{Y}$. We note that pi is defined on Y' and so by definition, we may find $Y_0 \subset Y'$ such that $pi|_{Y_0} = id_{Y_0}$. Let $U = p^{-1}(Y_0)$. Then we have $pi(Y_0) \subset Y_0$ and so $i(Y_0) \subset p^{-1}(Y_0) = U$. Since $p(U) \subset Y_0$ by definition, we have constructed the desired morphisms. $\square$

Lemma 5.2.7 ***Retractions shrink to closed retractions***
Suppose Y is a retraction of U via morphisms i, p. Then we may find dense open subvarieties $Y_0 \subset Y$ and $U_0 \subset U$ such that Y_0 is a closed retraction of U_0 via the restrictions of i, p.

Proof. Since we may identify Y with the image of p it follows that Y is locally constructible in U [EGA 4-1, p. 239 (Chevalley's thm)]. By [EGA 3-1, p. 12], it follows that Y is the intersection of a closed and an open set in U. By setting U_0 to be this open set, and $Y_0 = Y \cap U_0$ it follows that Y_0 is closed in U_0. Now it is easy to see that the restrictions of i, p exhibit Y_0 as a retraction of U_0. $\square$

Corollary 5.2.8 ***Rational retractions shrink to closed retractions***
Suppose Y is a rational retraction of U via rational maps i, p. Then we may find dense open subsets $Y_0 \subset Y$ and $U_0 \subset U$ such that i, p make Y_0 a closed retraction of U_0.

Proof. This follows immediately from Lemmas 5.2.6 and 5.2.7. $\square$

The following lemma gives us a first hint that retractions inherit some of the geometry of the larger spaces.

Lemma 5.2.9 ***Retractions of smooth schemes are smooth***
Suppose Y is a retraction of a smooth scheme U. Then Y is smooth.

Proof. This follows from the formal criterion for smoothness (see for example [Gro67] §17 or [Ill96] §2). From this formulation, in the language of [Ill96], we must show that if $S_0 \to S$ is a thickening, and $f : S_0 \to Y$ is a morphism, then we must be able to find a cover $\{V_i\}$ of S and morphisms $g_i : V_i \to Y$ extending $f|_{S_0 \cap V_i}$. To see this, we first use the smoothness of U to find $\widetilde{g}_i : V_i \to U$ extending $i \circ f|_{S_0 \cap V_i}$. Now we set $g_i = p \circ \widetilde{g}_i$. We then have

$$\begin{aligned} g_i|_{S_0 \cap V_i} &= p \circ \widetilde{g}_i|_{S_0 \cap V_i} \\ &= p \circ i \circ f|_{S_0 \cap V_i} \\ &= f|_{S_0 \cap V_i} \end{aligned}$$

as desired. □

Lemma 5.2.10 ***Standard position for retractions***
Suppose Y is a smooth d-dimensional variety which is a closed retraction of an open subscheme $U \subset \mathbb{A}^n$. We also suppose that with respect to the inclusion of Y in $\mathbb{A}^n$, that $0 \in Y$. Then we may shrink U and choose coordinates on U so that Y is the zero locus of polynomials $f_1, \ldots, f_{n-d}$ with

$$f_i = x_i + P_i$$

where the x_i's are the coordinate functions on $\mathbb{A}^n$ and P_i is a polynomial in the x_j's, each of whose terms is of degree at least 2.

Further, we may alter i and p defining the retraction so that the morphism $ip : U \to Y \to U$ is given by

$$(x_1, \ldots, x_n) \mapsto (M_1 + Q_1, \ldots, M_n + Q_n)$$

where

$$M_i = \begin{cases} 0 & \text{if } 1 \le i \le n-d \\ x_i & \text{if } n-d < i \le n \end{cases}$$

and Q_i is a rational function in the variables x_i, regular on U, such that $\frac{\partial}{\partial x_j} Q_i \Big|_0 = 0$ for all i, j.

Proof. For purposes of skimmability, we have placed this proof at the end of the section. □

5.3. **Adic convergence of Taylor series.** The basic strategy for factorization will be to produce closer and closer approximations to a particular factorization. In order to carry this out, it is necessary to discuss notions of convergence and approximations in the adic setting, paralleling the discussion of [HHKa], Section 2.

Suppose F_0 is a field complete with respect to a discrete valuation v with uniformizer t, and let $|a| = e^{-v(a)}$ be a corresponding norm. Let $A = F_0[x_1, \ldots, x_N]$, $\mathfrak{m}$ the maximal ideal at 0, $A_\mathfrak{m}$ the local ring at 0 and $\widehat{A} = F_0[[x_1, \ldots, x_N]]$ the complete local ring at 0. For $I =$

$(i_1, \ldots, i_N) \in \mathbb{N}^N$, we let $|I| = \sum_j i_j$. Define for $r \in \mathbb{R}$, $r > 0$

$$\widehat{A}_r = \left\{ \sum_I a_I x^I \,\middle|\, \lim_{|I| \to \infty} |a_I| r^{|I|} = 0 \right\}$$

and for $f = \sum a_I x^I \in \widehat{A}_r$, we set

$$|f|_r = \sup_I |a_I| r^{|I|}.$$

We give $\mathbb{A}^n(F_0)$ a norm via the supremum of the coordinates

$$|(a_1, \ldots, a_N)| = \max_i \{|a_i|\}$$

and we let $D(a, r)$ be the closed disk of radius r about $a \in \mathbb{A}^n(F_0)$ with respect to the induced metric. We note that since the values of the metric are discrete, this disk is in fact both open and closed in the t-adic topology.

We note the following elementary lemma:

Lemma 5.3.1
Suppose $a \in D(0, r)$, and $f, g \in \widehat{A}_r$. Then

1. $f + g, fg \in \widehat{A}_r$,
2. $|f + g|_r \leq \max\{|f|_r, |g|_r\}$,
3. *the group*
$$\{a \in \widehat{A}_r \mid |a|_r < M\} \subset \widehat{A}_r$$
is complete with respect to the filtered collection of subgroups $\mathfrak{m}^i \cap \widehat{A}_r$,
4. $|f|_r$ *is finite,*
5. $|fg|_r \leq |f|_r |g|_r$,
6. *if* $r' < r$, *then* $|f|_{r'} \leq \max\{|f(0)|, \frac{r'}{r}|f|_r\}$,
7. $f(a)$ *is well defined (i.e. is a convergent series), and*
8. $|f(a)| \leq |f|_r$, *and if* $f(0) = 0$ *then* $|f(a)| \leq |f|_r |a| r^{-1}$.

Lemma 5.3.2
Suppose $f \in A_{\mathfrak{m}}$. Then for all $\varepsilon \geq |f(0)|$ with $\varepsilon > 0$, there exists $r > 0$ such that $f \in \widehat{A}_r$ and $|f|_r < \varepsilon$. Further, for any $\delta > 0$ we may choose $r < \delta$.

Proof. Write $f = g/h$, $g, h \in A$ with $h \notin \mathfrak{m}$. Since $A/\mathfrak{m}$ is a field, we may find $h' \in A$ with $hh' - 1 = -b \in \mathfrak{m}$. Therefore, in $\widehat{A}$, we have $f = gh'(\sum b^i)$. Since g, h', b are polynomials, they are in A_r for any r. Further, by Lemma 5.3.1(6), we may reduce r so that $|gh'|_r \leq |f(0)|$, and since $b(0) = 0$, we may also ensure $|b|_r < 1$. In doing this, note that we may also ensure that $r < \delta$. We note that by Lemma 5.3.1(5), $|b^i|_r < 1$. Now, by Lemma 5.3.1(3), it follows that $|\sum b^i|_r < 1$. But therefore $|f|_r = |gh' \sum b^i|_r < \varepsilon$ as desired. □

Lemma 5.3.3
The t-adic topology on $\mathbb{A}^N(F_0)$ is finer than the Zariski topology.

Proof. It suffices to show that if $p \in \mathbb{A}^N(F_0)$ and f is a polynomial not vanishing on p, we may find a disk about p on which f is nonvanishing. Without loss of generality, we may apply a translation and assume that $p = 0$. Let $g = f - f(0)$. By Lemma 5.3.2, since $g(0) = 0$, we may find an $r > 0$ such that $f \in \widehat{A}_r$ and such that $|g|_r < |f(0)|$ (using $\varepsilon = |f(0)|$). In particular, if $a \in \mathbb{A}^N(F_0)$ with $|a| < r$, we have $|g(a)| \leq |g|_r < |f(0)|$ by Lemma 5.3.1(8). Therefore, for such an a, $f(a) = g(a) + f(0) \neq 0$. Therefore f does not vanish on a disk of radius r about the origin as desired. □

Proposition 5.3.4 ***Linear approximations and error term***
Suppose $f \in \widehat{A}_r$ for $r \leq 1$. Write

$$f = c_0 + L + P \text{ where } P(\vec{x}) = \sum_{|\nu| \geq 2} c_\nu x^\nu,$$

where L is a linear form with coefficients in F_0 and all $c_\nu \in F_0$. Suppose $|L + P|_r \leq 1$. Let $0 < \varepsilon \leq |t| r^2$, and suppose $a, h \in \mathbb{A}^N(F_0)$ with $|h|, |a| \leq \varepsilon$. Then

$$|f(a+h) - f(a) - L(h)| \leq |t||h|$$

.

Proof. This proof is a very slight modification of Lemma 2.2 in [HHKa]. Choose a real number s so that we may write $|h| = \varepsilon |t|^s$. We may rearrange the quantity of interest as:

$$f(a+h) - f(a) - L(h) = \sum_{|\nu| \geq 2} c_\nu \left((a+h)^\nu - a^\nu \right).$$

Since the absolute value is nonarchimedean, it suffices to show that for every term $m = c_\nu x^\nu$ with $|\nu| \geq 2$ we have

$$|m(a+h) - m(a)| \leq \varepsilon |t|^{s+1}.$$

For a given ν with $|\nu| \geq 2$, consider the expression $(x + x')^\nu - x^\nu$, regarded as a homogeneous element of degree $j = |\nu|$ in the polynomial ring $F_0[x_1, \ldots, x_N, x'_1, \ldots, x'_N]$. Since the terms of degree j in $x_1, \ldots, x_N$ cancel, the result is a sum of terms of the form $\lambda \ell$ where λ

is an integer and ℓ is a monomial in the variables x, x' with total degree d in $x_1, \ldots, x_N$ and total degree d' in $x'_1, \ldots, x'_N$, such that $d + d' = j$ and $d < j$. Hence $d' \geq 1$. Consequently, for each term of this form,

$$|\lambda\ell(a, h)| \leq |\ell(a, h)| \leq \varepsilon^d(\varepsilon|t|^s)^{d'} = \varepsilon^j|t|^{sd'} \leq \varepsilon^j|t|^s.$$

Since $(a + h)^\nu - a^\nu$ is a sum of such terms, and the norm is nonarchimedean, we conclude $|(a + h)^\nu - a^\nu| \leq \varepsilon^j|t|^s$.

Since $m = c_\nu x^\nu$, it follows that

$$|m(a + h) - m(a)| \leq |c_\nu|\varepsilon^j|t|^s \leq r^{-j}\varepsilon^j|t|^s.$$

Now $\varepsilon \leq |t|r^2$, so $\varepsilon^{j-1} \leq |t|^{j-1}r^{2j-2}$. Since $|t| < 1$, $r \leq 1$, and $j \geq 2$, we have

$$\varepsilon^{j-1} \leq |t|^{j-1}r^{j+j-2} \leq |t|r^j.$$

Rearranging this gives the inequality $(\varepsilon/r)^j \leq \varepsilon|t|$ and so $(\varepsilon/r)^j|t|^s \leq \varepsilon|t|^{s+1}$. Therefore

$$|m(a + h) - m(a)| \leq r^{-j}\varepsilon^j|t|^s \leq \varepsilon|t|^{s+1} = |t||h|,$$

as desired.

□

Lemma 5.3.5 ***Local bijectivity***
Suppose $f : U \to V$ is a morphism between Zariski open subschemes of $\mathbb{A}^d_{F_0}$ containing the origins and such that $f(0) = 0$. Suppose further, that after writing the coordinates of f as elements of $\widehat{A}$, we have $f = (f_1, \ldots, f_d)$ with $f_i = x_i + Q_i$ and Q_i consisting of terms of degree higher than 2. Then we may find t-adic neighborhoods $U' \subset U(F_0)$ and $V' \subset V(F_0)$ of 0 such that f maps U' bijectively onto V'. Further, we may assume that U' and V' are disks about the origin of equal radii.

Proof. By Lemma 5.3.2, since $f(0) = 0$, we may find $0 < r \leq 1$ such that $f \in \widehat{A}_r$ and $|f|_r \leq 1$. Choose $\varepsilon \leq |t|r^2$ as in the statement of Proposition 5.3.4 and such that $D_0(\varepsilon) \subset V(F_0)$ and $D_0(\varepsilon) \subset U(F_0)$. Let $V' = D_0(\varepsilon) \subset V(F_0)$ and $U' = D_0(\varepsilon) \subset V(F_0)$. We claim that for $b \in U'$, we have $|f(b)| \leq \varepsilon$ and so $f(b) \in V'$. To see this, we note that $Q_i \in \widehat{A}_r$ and $|Q_i|_r \leq 1$, and hence we may apply Proposition 5.3.4 (with 0 linear and constant term) to see that $|Q_i(b)| \leq |t||b| < |b| = \max\{|b_i|\}$. By the nonarchimedean property, this gives

$$\begin{aligned}|f(b)| = \max\{|f_i(b)|\} = \max\{|b_i + Q_i(b)|\} &\leq \max\{|b_i|, |Q_i(b)|\}\\ &= \max\{|b_j|\} = |b|\end{aligned}$$

We consider first surjectivity. Note that both U' and V' are both closed and open. Since they are closed in a complete metric space, they contain all limits of their Cauchy sequences. Let $a \in V'$, and let $b_0 = 0$. We will inductively construct elements $b_i \in U'$ such that $|f(b_i) - a| \leq \varepsilon |t|^i$. In particular, since $|a| \leq \varepsilon$, we have $|f(b_0) - a| = |a| \leq \varepsilon$. Assuming we have constructed b_{i-1}, we let $h = a - f(b_{i-1})$, and note $|h| \leq \varepsilon |t|^{i-1}$ by hypothesis, and $|b_{i-1}| \leq \varepsilon$ since $b_{i-1} \in U'$. Therefore, by Proposition 5.3.4, we have

$$|f(b_{i-1} + h) - f(b_{i-1}) - h| \leq |t||h| \leq \varepsilon |t|^i$$

and so setting $b_i = b_{i-1} + h$, we find that since $f(b_{i-1}) + h = a$, we have

$$\begin{aligned} |f(b_i) - a| &= |f(b_{i-1} + h) - a| \\ &= |f(b_{i-1} + h) - f(b_{i-1}) - h + f(b_{i-1}) + h - a| \\ &= \left| \big(f(b_{i-1} + h) - f(b_{i-1}) - h\big) + f(b_{i-1}) - a + a - f(b_{i-1}) \right| \\ &= |f(b_{i-1} + h) - f(b_{i-1}) - h| \leq |t||h| \leq \varepsilon |t|^i \end{aligned}$$

as desired. Since b_i is a Cauchy sequence, using the completeness of U', we may set $b = \lim b_i \in U'$ and we find by continuity that $f(b) = a$ as desired.

Next, we consider injectivity. Suppose $a, b \in U'$, let $h = b - a$ and suppose $h \neq 0$. We need to show that $f(a) \neq f(b)$. Since the valuation is nonarchimedean we have $a, h \leq \varepsilon$. Let $E = f(a+h) - f(a) - h$. Then we find $|E| \leq |h||t|$ by Proposition 5.3.4. But this means in particular that $|E + h| = |h|$ by the nonarchimedean triangle inequality. Therefore

$$|f(b) - f(a)| = |f(a + h) - f(a)| = |E + h| = |h| \neq 0$$

so $f(b) \neq f(a)$ as desired. $\square$

Lemma 5.3.6
Suppose that Y is a closed retraction of an open subscheme $U \subset \mathbb{A}^n$ in the standard form of Lemma 5.2.10 with respect to morphisms i, p. Then we may find a t-adic neighborhood V' of $0 \in Y$ (regarding Y as a subscheme of U via i) such that the composition $V' \to \mathbb{A}^n(F_0) \to \mathbb{A}^d(F_0)$ is bijective onto a t-adic disk, where the last map is given by projection onto the last d coordinates.

Proof. As in Lemma 5.2.10, we suppose that Y is the zero locus of polynomials $f_1, \ldots, f_{n-d}$ with

$$f_i = x_i + P_i$$

where P_i is a polynomial in the x_j's of degree at least 2. Using Lemma 5.3.2, we may choose $0 < r \leq 1$ such that $f_i \in \widehat{A}_r$ and $|f_i|_r \leq 1$ for each of the finitely many functions f_i. Choose $\varepsilon \leq |t| r^2$ as in Proposition 5.3.4. Let $U' \in \mathbb{A}^n(F_0)$ be the disk about the origin of radius ε, and let V' be the intersection of the inverse image of U' in $Y(F_0)$ with the disk of radius ε in $\mathbb{A}^n(F_0)$. Let $g : V' \to U'$ be the composition map in the statement.

First, suppose $a, b \in V'$ with $a \neq b$, and let $h = b - a$. If $g(a) = g(b)$, then by definition of g, the last d coordinates of a and b must match. Since $a \neq b$, we therefore know that $x_i(h) \neq 0$ for some $i = 1, \ldots, n-d$ where x_i is the i'th coordinate function on $\mathbb{A}^d$. We may therefore choose i such that $|x_i(h)|$ has the largest possible value, and in particular, we then would have $|x_i(h)| = |h|$. But, estimating $|f_i(b) - f_i(a)| = |f_i(a+h) - f_i(a)|$ using Proposition 5.3.4, we find

$$|f_i(a+h) - f_i(a) - x_i(h)| \leq |t||h|.$$

We claim that $|f_i(a+h) - f_i(a)| \geq |h|$ and in particular that $f_i(a) \neq f_i(b)$. To see this must hold, assume by contradiction that $|f_i(a+h) - f_i(a)| < |h|$. In this case, we have

$$|f_i(a+h) - f_i(a) - x_i(h)| = |x_i(h)|$$

since $|x_i(h)| = |h|$. But therefore we have $|h| \leq |t||h|$ which is a contradiction since $|t| < 1$. Therefore, $f_i(b) \neq f_i(a)$ as claimed, forcing g to be injective.

For surjectivity, consider $a \in U'$, and consider its image $b \in \mathbb{A}^d$. Using the form for the retraction in Lemma 5.2.10, we may apply Lemma 5.3.5 to the composition (shrinking ε if necessary)

$$\begin{tikzcd} \mathbb{A}^d \cap U \arrow[r] & U \arrow[r, "p"] & Y \arrow[r] \arrow[rr, bend left, "g"] & U \arrow[r] & \mathbb{A}^d \end{tikzcd}$$

By Lemma 5.3.5, we may find an inverse image b' of b in $\mathbb{A}^d$ of norm less than ε. Consequently, by definition, the image of b' in Y must actually live in V', and this is an inverse image for a as desired. $\square$

Corollary 5.3.7
In the notation of the previous lemma, we may choose t-adic neighborhoods of the origin $U' \in \mathbb{A}^d$ and $V' \in Y$ such that the composition $U' \to U \to Y$ takes U' bijectively to V' and the composition $V' \to Y \to U \to \mathbb{A}^n \to \mathbb{A}^d$ takes V' bijectively to U'.

Proof. By Proposition 5.2.10 and Lemma 5.3.5, we may find $U' \subset \mathbb{A}^d$, $V' \subset Y$ so that the composition $U' \to V' \to U'$ is bijective. By Lemma 5.3.6, we may find $V'' \subset V'$ such that $V'' \to U'$ in bijective onto a t-adic disk $U'' \subset U'$. But now again the composition $U'' \to U''$ is bijective, and since $V'' \to U''$ is also bijective, we find $U'' \to V''$ is bijective as well. $\square$

5.4. **Factorization.**

Theorem 5.4.1
Under Hypothesis 5.1.7, let $f : \mathbb{A}^d_{F_0} \times \mathbb{A}^d_{F_0} \dashrightarrow \mathbb{A}^d_{F_0}$ *be an* F_0*-rational map that is defined on a Zariski open set* $U \subseteq \mathbb{A}^d_{F_0} \times \mathbb{A}^d_{F_0}$ *containing the origin* $(0,0)$*. Suppose further that we may write:*

$$f = (f_1, \dots, f_d), \quad f_i \in \widehat{k}[x_1, y_1 \dots, x_d, y_d]_{\mathfrak{m}}$$
$$\text{where } f_i = x_i + y_i + \sum_{|(\nu,\rho)|\geq 2} c_{\nu,\rho,i} x^\nu y^\rho.$$

Then there is a real number $\varepsilon > 0$ *such that for all* $a \in \mathbb{A}^d(F_0)$ *with* $|a| \leq \varepsilon$*, there exist* $v \in \mathfrak{V}^d$ *and* $w \in \mathfrak{W}^d$ *such that* $(v, w) \in U(F_0)$ *and* $f(v, w) = a$*.*

Proof. The proof of this theorem is exactly as in [HHKa], Theorem 2.5, wherein in the first paragraph, the problem is reduced to exactly the hypotheses which we assume. □

Theorem 5.4.2
Under Hypothesis 5.1.7, let $m : Y \times Y \to Y$ *be a rational* F*-morphism defined at* $(0,0)$*, and suppose that* $m(y, 0) = y = m(0, y)$ *where it is defined. Suppose that* Y *is a closed retraction of an open subscheme of* $\mathbb{A}^n$*. Then there exists* $\varepsilon > 0$ *such that for* $y \in Y(F_0) \subset \mathbb{A}^n(F_0)$*,* $|y| \leq \varepsilon$*, there exist* $y_i \in Y(F_i)$*,* $i = 1, 2$ *such that* $y = m(y_1, y_2)$*.*

Proof. We consider as in Corollary 5.3.7, t-adic neighborhoods of 0 $U' \subset \mathbb{A}^d(F_0)$ and $V' \subset Y(F_0)$ such that we have bijections $U' \to V'$ and $V' \to U'$ defined by algebraic rational morphisms $p' : \mathbb{A}^d \dashrightarrow Y$ and $i' : Y \dashrightarrow \mathbb{A}^d$. We consider

$$\begin{tikzcd} V' \times V' \arrow[r, "m|_{V' \times V'}"] & V' \arrow[d, "i'"] \\ U' \times U' \arrow[u, "p'"] \arrow[r] & U' \end{tikzcd}$$

By hypothesis, the composition in the bottom $U' \times U' \to U'$ is given as the restriction of an algebraic rational morphism $\mu : \mathbb{A}^d \times \mathbb{A}^d \to \mathbb{A}^d$. By Corollary 5.3.7, it is sufficient to show that μ is surjective when restricted to a sufficiently small t-adic neighborhood.

We first shrink V', U' if necessary to make them contained in Zariski neighborhoods V, U as in Lemma 5.2.10. Now, we note that the rational map $\mu|_{\mathbb{A}^d \times \{0\}}$ is just ip, since $m|_{Y \times \{0\}} = id_Y$. By Lemma 5.2.10, we find

$$m|_{\mathbb{A}^d \times \{0\}}(x_1, \dots, x_d) = (x_1 + Q_1, \dots, x_d + Q_d)$$

where Q_i is a rational function in the variables x_i, regular on U, such that $\frac{\partial}{\partial x_j} Q_i\Big|_0 = 0$ for all i, j. But now we are done, using Theorem 5.4.1. □

Theorem 5.4.3 ***Factorization for retract rational groups***
Under Hypothesis 5.1.7, assume that $F = F(\mathbb{P}^1_T)$, $F_1 = F_{\mathbb{A}^1_k}$, $F_2 = F_\infty$, and $F_0 = F_\wp$, where $\wp$ is the unique branch at ∞. Let G be a retract rational connected linear algebraic group defined over F. Then for any $g_0 \in G(F_0)$ there exist $g_i \in G(F_i)$, $i = 1, 2$, such that $g_1 g_2 = g_0$ — that is to say, factorization holds for G with respect to $\mathbb{P}^1_T$ (see Definition 4.1.1).

Proof. Using Lemma 5.2.7, we may find an open subscheme $Y \subset G$ which is a retraction of an open subscheme U of affine space. In particular, Y must contain an F-rational point $y \in Y(F)$, and after replacing Y by $y^{-1}Y$ if necessary, we may assume Y contains the identity element of G. Using 5.4.2, where m is the multiplication map, we find that there exists $\varepsilon > 0$ such that factorization holds for $g_0 \in G(F_0)$ provided that $|g_0| < \varepsilon$. Fix such an epsilon, and suppose $g_0 \in G(F_0)$ is an arbitrary element. Since G is retract rational, it follows that $G(F)$ is Zariski dense in $G(F_0)$. Therefore, we have the existence of an element $g' \in G(F)$ such that ${g'}^{-1} g_0 \in Y$. Since Y is a retraction of affine space, it follows that $Y(F_2)$ is t-adically dense in $Y(F_0)$. Therefore we may find $g'' \in Y(F_2)$ such that $|{g'}^{-1} g_0 {g''}^{-1}| < \varepsilon$. Writing ${g'}^{-1} g_0 {g''}^{-1} = g_1 g_2$ where $g_i \in G(F_i)$, we conclude that $g_0 = (g' g_1)(g_2 g'')$. Since $g' g_1 \in G(F_1)$ and $g_2 g'' \in G(F_2)$, we are done. □

By Lemma 5.1.5 and the comments just following, we conclude that Theorem 5.1.1 holds.

5.5. **Proof of Lemma 5.2.10.**

Lemma 5.5.1
Suppose $f = g/h$ for $g, h \in k[x_1, \ldots, x_n]$ with $h(0) \neq 0, g(0) = 0$ and $(\partial f/\partial x_i)|_0 = 0$ for all i. Then if R is a k-algebra with $h(0) \in R^*$ and containing and element $\epsilon \in R$, $\epsilon^2 = 0$ then $f(\epsilon v) = 0$ for $v \in k^n$.

Proof. Since $g(0) = 0$, we may write $g = L + Q$ where L is a linear polynomial, and Q is a sum of homogeneous terms of degree at least 2. Now we simply note that

$$\frac{\partial f}{\partial x_i} = \frac{h(\partial L/\partial x_i + \partial Q/\partial x_i) - (L + Q)(\partial h/\partial x_i)}{h^2}$$

and in particular since $h(0) \neq 0$, we find $(\partial f/\partial x_i)|_0 = 0$ implies that $h(0)(\partial L/\partial x_i(0)) = 0$, which implies that all the coefficients of the linear form L are 0 and so $L = 0$. Since $h(0) \neq 0$, it follows that $h(\epsilon v)$ is a unit, and we therefore may note that $f(\epsilon v) = Q(\epsilon v)/h(\epsilon v)$ is well defined and $\epsilon^2 = 0$ implies $Q(\epsilon v) = 0$, showing that $f(\epsilon v) = 0$ as desired. □

We now proceed with the proof of Lemma 5.2.10. To begin, choose $f_1, \ldots, f_r$ which are regular on a neighborhood of $0 \in U$ and which cut out Y. Writing $f_i = g_i/h_i$, for g_i and h_i with no common factors, we see that since the h_i don't vanish at 0, after shrinking U so that the h_i don't vanish on U, we may ensure that the h_i are units, and hence Y is cut out by the g_i. Therefore we may assume (after replacing f_i by g_i and shrinking U), that the f_i are polynomials. Next, we write

$$f_i = L_i + P_i$$

where L_i is a linear polynomial and P_i has degree at least 2. Note that f_i has no constant term since it must vanish at 0. Since Y is smooth of dimension d, by the Jacobian criterion, the L_i's (which we may identify with the gradient of f_i at 0), span a $n - d$ dimensional space. After relabelling, we may assume that $L_1, \ldots, L_{n-d}$ give a basis for this space. Let $\widetilde{Y}$ be the zero locus of $f_1, \ldots, f_{n-d}$. Since $Y \subset \widetilde{Y}$ we have the codimension of $\widetilde{Y}$ at 0, $codim_0(\widetilde{Y}) \leq codim(Y) = n - d$. By construction, the Jacobian matrix of the defining equations for $\widetilde{Y}$ at 0 has rank $n - d$, and so by [Eis95], page 402, $n - d \leq codim(\widetilde{Y})$. But then

$$n - d \neq codim_0(\widetilde{Y}) \leq codim(Y) = n - d$$

so $codim_0(\widetilde{Y}) = n - d$ and also by the Jacobian criterion, we conclude that $\widetilde{Y}$ is smooth at 0. We may therefore, after shrinking U assume that $\widetilde{Y}$ is smooth, irreducible, and of the same dimension as Y. But $Y \subset \widetilde{Y}$ therefore implies $Y = \widetilde{Y}$, and in particular we may assume $r = n - d$, and the L_i are independent.

After choosing a new basis for $\mathbb{A}^n$, it is clear that we may assume $L_i = x_i$ while preserving our hypotheses.

Finally, consider the morphism $\gamma = ip : U \to U$ (where i and p are as in the definition of the retraction), and write $\gamma(\vec{x}) = (\gamma_1(\vec{x}), \ldots, \gamma_n(\vec{x}))$, where each γ_i is a regular function on U. Since $\gamma_i(0) = 0$, we may write $\gamma_i = M_i + Q_i$ for the linear function

$$M_i = \sum \frac{\partial}{\partial x_i} \gamma_i \bigg|_{\vec{x}=0} x_i$$

and have all the partial derivatives of the Q_i vanishing. Let $\mathbb{T} = \operatorname{Spec} k[\epsilon]/(\epsilon^2)$, and consider a $\mathbb{T}$-valued point $\tau : \mathbb{T} \to U$ given by $\vec{a}\epsilon = (a_1\epsilon, \ldots, a_n\epsilon) \in \mathbb{A}^n(k[\epsilon]/(\epsilon^2))$. We note that τ maps $\mathbb{T}$ into Y if and only if $f_i(\vec{a}\epsilon) = 0$ for each i. But we have

$$f_i(\vec{a}\epsilon) = L_i(\vec{a}\epsilon) = \epsilon L_i(\vec{a}).$$

In particular, this occurs exactly when $a_i = 0$ for $1 \leq i \leq n-d$. Since $\gamma(\vec{a}\epsilon) \in Y$, we therefore have $M_i = 0$ for $1 \leq i \leq n-d$. Since $\gamma|_Y = \mathrm{id}_Y$, looking on $\mathbb{T}$-valued points of Y under γ, we find $M_i = M_i' + x_i$ for $n-d < i \leq n$ where M_i' is a linear function of $x_1, \ldots, x_{n-d}$. Consider the linear function $\mathbb{A}^n \to \mathbb{A}^n$ given by

$$\phi : (x_1, \ldots x_n) \mapsto (y_1, \ldots, y_n)$$

where

$$y_i = \begin{cases} x_i & \text{if } 1 \leq i \leq n-d \\ x_i - M_i & n-d < i \leq n \end{cases}$$

Define rational maps $i' = i \circ \phi : Y \dashrightarrow \mathbb{A}^n$ and $p' = \phi^{-1} \circ p : \mathbb{A}^n \dashrightarrow Y$. We then have $p' \circ i' = \phi^{-1} p i \phi = id_Y$ as rational maps, and therefore define a rational retraction. By Lemma 5.2.7 we may shrink U and Y to make this a closed retraction. Note also that $i'p' = ip = \gamma$.

As before, let $\tau : \mathbb{T} \to U$ be given by $\vec{a}\epsilon = (a_1\epsilon, \ldots, a_n\epsilon) \in \mathbb{A}^n(k[\epsilon]/(\epsilon^2))$, where $\vec{a} \in \mathbb{A}^n(k)$. We consider the morphism $i'p' : U \to Y \to U$, which we write as

$$(x_1, \ldots, x_n) \mapsto (M_1' + Q_1', \ldots, M_n' + Q_n')$$

with M_i' linear and the first derivatives of the Q_i' vanishing at the origin. Computing using Lemma 5.5.1 applied to functions Q_i', we then find

$$i'p'(\tau) = \epsilon(M_1'(\vec{a}), \ldots, M_n'(\vec{a}))$$

and also, using the linearity of ϕ and the fact that $i'p' = ip = \gamma = M + Q$,

$$\begin{aligned} i'p'(\tau) &= \epsilon\phi^{-1}(M_1(\phi(\vec{a}), \ldots, M_n(\phi(\vec{a}))) \\ &= \epsilon(0, \ldots, 0, a_{n-d+1}, \ldots, a_n), \end{aligned}$$

and so

$$M_i' = \begin{cases} 0 & \text{if } 1 \leq i \leq n-d \\ x_i & \text{if } n-d < i \leq n \end{cases}$$

Therefore, upon replacing p, i by p', i', we obtain the desired conclusion.

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