Motivic decomposability of generalized Severi-Brauer varieties

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Abstract

Let F be an arbitrary field. Let p be a positive prime number and D a central division F-algebra of degree p^n, with $n \geq 1$. We write $SB(p^m, D)$ for the generalized Severi-Brauer variety of right ideals in D of reduced dimension p^m for $m = 0, 1, \ldots, n-1$. We note by $M(SB(p^m, D))$ the Chow motive with coefficients in $\mathbb{F}_p$ of the variety $SB(p^m, D)$. It was proven by Nikita Karpenko that this motive is indecomposable for any prime p and $m = 0$ and for $p = 2, m = 1$ (cf. [7]). We prove decomposability of $M(SB(p^m, D))$ in all the other cases ($p = 2$, $1 < m < n$ and $p > 2$, $0 < m < n$).

Résumé

Soient F un corps arbitraire, p un nombre premier positif et D une F-algèbre de division de degré p^n. On écrit $SB(p^m, D)$ pour la variété de Severi-Brauer généralisée des idéaux à droite de dimension réduite p^m, $m = 0, 1, \ldots, n-1$. On note par $M(SB(p^m, D))$ le motif de Chow à coefficients dans $\mathbb{F}_p$ de la variété $SB(p^m, D)$. Il a été demontré par Nikita Karpenko que ce motif est indecomposable pour le nombre premier p arbitraire et $m = 0$ et pour $p = 2, m = 1$ (cf. [7]). Nous prouvons la décomposabilité de $M(SB(p^m, D))$ dans tous les autres cas ($p = 2$, $1 < m < n$ and $p > 2$, $0 < m < n$).

1 Chow motives with finite coefficients

Our basic reference for Chow groups and Chow motives (including notations) is [4]. We fix an associative unital commutative ring Λ (most frequently Λ will be the finite field $\mathbb{F}_p$ of p elements, where p is prime) and for a variety (i.e., a separated scheme of finite type over a field) X we write $\mathrm{Ch}(X)$ for its Chow group with coefficients in Λ (while we write $\mathrm{CH}(X)$ for its integral Chow group). Our category of motives is the category $\mathrm{CM}(F, \Lambda)$ of graded Chow motives with coefficients in Λ, [4, definition of § 64]. By a sum of motives we always mean the direct sum. We also write Λ for the motive $M(\mathrm{Spec} F) \in \mathrm{CM}(F, \Lambda)$. A Tate motive is the motive $\Lambda(i)$ with i an integer.

Let X be a smooth complete variety over F and let M be a motive. We call M split if it is a finite sum of Tate motives. We call X split, if its integral motive $M(X) \in \mathrm{CM}(F, \mathbb{Z})$ (and therefore the motive of X with an arbitrary coefficient ring Λ) is split. We call M or X geometrically split, if it splits over a field extension of F.

Let M be a geometrically split motive. Over an extension of F the motive M becomes isomorphic to a finite sum of Tate motives. We write $\operatorname{rk} M$ and $\operatorname{rk}_i M$ for respectively the number of all summands and the number of summands $\Lambda(i)$ in this decomposition, where i is an integer. Note that these two numbers do not depend on the choice of the splitting field extension: they are the ranks of the free Λ-modules $\operatorname{colim}_{L/F} \mathrm{Ch}_*(M_L)$ and $\operatorname{colim}_{L/F} \mathrm{Ch}_i(M_L)$ respectively, where the colimit is taken over all field extensions L/F.

We say that X satisfies the nilpotence principle, if for any field extension E/F and any coefficient ring Λ, the kernel of the change of field homomorphism $\operatorname{End}(M(X)) \to \operatorname{End}(M(X)_E)$ consists of nilpotents. Any projective homogeneous (under an action of a semisimple affine algebraic group) variety is geometrically split and satisfies the nilpotence principle, [4, Theorem 92.4 with Remark 92.3].

A complete decomposition of an object in an additive category is a finite direct sum decomposition with indecomposable summands. We say that the Krull-Schmidt principle holds for a given object of a given additive category, if every direct sum decomposition of the object can be refined to a complete one (in particular, a complete decomposition exists) and there is only one (up to a permutation of the summands) complete decomposition of the object. We have the following theorem:

Theorem 1.1. ([2, Theorem 3.6 of Chapter I]). *Assume that the coefficient ring Λ is finite. The Krull-Schmidt principle holds for any shift of any summand of the motive of any geometrically split F-variety satisfying the nilpotence principle.*

Lemma 1.2. *Assume that the coefficient ring Λ is finite. Let X be a variety satisfying the nilpotence principle. Let $f \in \operatorname{End}(M(X))$ and $1_E = f_E \in \operatorname{End}(M(X)_E)$ for some field extension E/F. Then $f^n = 1$ for some positive integer n.*

Proof. Since X satisfies the nilpotence principle, we have $f = 1 + \varepsilon$, where ε is nilpotent. Let n be a positive integer such that $\varepsilon^n = 0 = n\varepsilon$. Then $f^{n^n} = (1+\varepsilon)^{n^n} = 1$ because the binomial coefficients $\binom{n^n}{i}$ for $i < n$ are divisible by n. ☐

2 Motivic decomposability of generalized Severi-Brauer varieties

Let p be a positive prime integer. The coefficient ring Λ is $\mathbb{F}_p$ in this section. Let F be a field. Let D be a central division F-algebra of degree p^n. We write $SB(p^m, D)$ for the generalized Severi-Brauer variety of right ideals in D of reduced dimension p^m for $m = 0, 1, \ldots, n-1$. For the main Theorem 2.6 we will need the following definition.

Definition 2.1. Let $G_r(\mathbb{A}^n)$ be the Grassmann variety of r-planes in $\mathbb{A}^n$. Let $c_1 = c_1(Tav) \in \mathrm{Ch}^1(G_r(\mathbb{A}^n))$, where Tav is a tautological r-dimensional vector bundle on $G_r(\mathbb{A}^n)$. We define $t_p(r, n)$ as a maximal integer k, such that $c_1^k \neq 0$.

Remark 2.2. By [5, Example 14.6.6], the integer $t_p(r,n)$ does not depend on the base field and we have an inequality $\max\{r, n-r\} \leq t_p(r,n) \leq r(n-r) = \dim G_r(\mathbb{A}^n)$.

Lemma 2.3. *Let E/F be a splitting field extension for $X = SB(1,D)$. Then the subgroup of F-rational cycles in $\mathrm{Ch}_{\mathrm{dim}X}(X_E \times X_E)$ is generated by a diagonal class.*

Proof. We write $\bar{\mathrm{Ch}}(X)$ for the image of the homomorphism $\mathrm{Ch}(X) \to \mathrm{Ch}(X_E)$. By [8, Proposition 2.1.1], we have $\bar{\mathrm{Ch}}^i(X) = 0$ for $i > 0$. Since the (say, first) projection $X^2 \to X$ is a projective bundle, we have a (natural with respect to the base field change) isomorphism $\mathrm{Ch}_{\dim X}(X^2) \simeq \mathrm{Ch}(X)$. Passing to $\bar{\mathrm{Ch}}$, we get an isomorphism $\bar{\mathrm{Ch}}_{\dim X}(X^2) \simeq \bar{\mathrm{Ch}}(X) = \bar{\mathrm{Ch}}^0(X)$ showing that $\dim_{\mathbb{F}_p} \bar{\mathrm{Ch}}_{\dim X}(X^2) = 1$. Since the diagonal class in $\bar{\mathrm{Ch}}_{\dim X}(X^2)$ is nonzero, it generates all the group. □

Corollary 2.4. (cf. [8, Theorem 2.2.1]). *The motive with coefficients in $\mathbb{F}_p$ of the Severi-Brauer variety $X = SB(1,D)$ is indecomposable.*

Proof. To prove that our motive is indecomposable it is enough to show that $\mathrm{End}(M(X))= \mathrm{Ch}_{\mathrm{dimX}}(X \times X)$ does not contain nontrivial projectors. Let $\pi \in \mathrm{Ch}_{\mathrm{dim}X}(X \times X)$ be a projector. By Theorem 2.3, π_E is zero or equal to 1_E. Since X satisfies the nilpotence principle, π is nilpotent in the first case, but also idempotent, therefore π is zero. Lemma 1.2 gives us $\pi = 1$ in the second case. □

Nikita Karpenko has recently proved the motivic indecomposability of generalized Severi-Brauer varieties also in the case $p = 2$, $m = 1$.

Theorem 2.5. (cf. [7, Theorem 4.2]). *Let D be a central division F-algebra of degree 2^n with $n \geq 1$. Then the motive with coefficients in $\mathbb{F}_2$ of the variety $SB(2,D)$ is indecomposable.*

Taking into account the Corollary 2.4, Theorem 2.5 and the fact that any generalized Severi-Brauer variety $SB(p^m,D)$ is p-incompressible [7, Theorem 4.3] (this condition is weaker than motivic indecomposability), one can expect that the Chow motive with coefficients in $\mathbb{F}_p$ of any generalized Severi-Brauer variety $SB(p^m,D)$ is indecomposable. But the following theorem gives us the opposite answer.

Theorem 2.6. *Let D be a central division F-algebra of degree p^n with $n \geq 1$. Then the motive with coefficients in $\mathbb{F}_p$ of the variety $SB(p^m,D)$ is decomposable for $p = 2$, $1 < m < n$ and for $p > 2$, $0 < m < n$. In these cases $M(SB(1,D))(k)$ is a summand of $M(SB(p^m,D))$ for $2 \leq k \leq t_p(p^m,p^n)$.*

Proof. We use the notations: $X = SB(1,D)$, $Y = SB(p^m,D)$, $d = \dim(SB(1,D)) = p^n - 1$, $r = p^n - p^m$. Let $E = F(X)$, then E/F is a splitting field extension for the variety X (and also for Y). We will show that $M(X)(k)$ is a summand of $M(Y)$. By Lemma 1.2 it suffices to construct two F-rational morphisms

$$\alpha : M(X_E)(k) \to M(Y_E) \quad \text{and} \quad \beta : M(Y_E) \to M(X_E)(k)$$

satisfying $\beta \circ \alpha = 1 \in End(M(X_E)(k)) = \mathrm{Ch}_d(X_E \times X_E)$. By Theorem 2.3 we can replace condition $\beta \circ \alpha = 1$ by $\beta \circ \alpha \neq 0$.

Let Tav be the tautological vector bundle on X.The product $X \times Y$ considered over X (via the first projection) is isomorphic (as a scheme over X) to the Grassmann bundle $G_r(Tav)$ of r-dimensional subspaces in Tav (cf. [6, Proposition 4.3]). Let T be the tautological r-dimensional vector bundle on $G_r(Tav)$. Over the field E the algebra D becomes isomorphic to $\mathrm{End}_E(V)$ for some E-vector space V of dimension $d+1 = p^n$. We have $X_E \simeq \mathbb{P}^d(V)$ and $Y_E \simeq G_{p^m}(V)$. Let T_1 and T_{p^m} be the tautological bundles of rank 1 and p^m on X_E and Y_E respectively. Then we have an isomorphism (cf. [6, Proposition 4.3]): $T_E \simeq T_1 \boxtimes (-T_{p^m})^\vee$ (here we lift the bundles T_1 and T_{p^m} on $X_E \times Y_E$). Let $h = c_1(T_1) \in \mathrm{Ch}^1(X_E)$ (then $-h$ is a hyperplane class in $\mathrm{Ch}^1(X_E)$) and $c_i = c_i((-T_{p^m})^\vee) \in \mathrm{Ch}^i(Y_E)$. Then by [5, Remark 3.2.3(b)]

$$c_t(T_E) = c_t(T_1 \boxtimes (-T_{p^m})^\vee) = \sum_{i=0}^{r} (1 + (h \times 1)t)^{r-i}(1 \times c_i)t^i\,.$$

It follows from the conditions of the theorem that the binomial coefficients $\binom{p^n-p^m}{2}$, $\binom{p^n-p^m}{p^m-1}$ are divisible by p and $\binom{p^n-p^m-1}{p^m-2} \equiv (-1)^{p^m-2} \bmod p$. Therefore

$$c_1(T_E) = (p^n - p^m)h \times 1 + 1 \times c_1 = 1 \times c_1\,,$$

$$c_2(T_E) = \binom{p^n - p^m}{2} h^2 \times 1 + (p^n - p^m - 1)h \times c_1 + 1 \times c_2 = -h \times c_1 + 1 \times c_2\,,$$

$$\begin{aligned} c_{p^m-1}(T_E) = \binom{p^n - p^m}{p^m - 1} h^{p^m-1} \times 1 + \binom{p^n - p^m - 1}{p^m - 2} h^{p^m-2} \times c_1 + \ldots = \\ = (-1)^{p^m-2} h^{p^m-2} \times c_1 + \ldots\,, \end{aligned}$$

where "…" stands for a linear combination of only those terms whose second factor has codimension > 1. For the top Chern class we have:

$$c_r(T_E) = \sum_{i=0}^{r} h^{r-i} \times c_i\,.$$

Let $\beta_1 = c_r(T_E)c_{p^m-1}(T_E)c_2(T_E)c_1(T_E)^{k-2} = (-h)^d \times c_1^k + \ldots = x \times c_1^k + \ldots$, where "…" stands for a linear combination of only those terms whose second factor has codimension $> k$ and where x is the class of a rational point in $\mathrm{Ch}(X_E)$. We take $\beta = \beta_1^t$, where β_1^t is the transpose of β_1. Since the bundle T is defined over F, the morphism $\beta \in \mathrm{Ch}_{\dim Y - k}(Y_E \times X_E) = \mathrm{Hom}(M(Y_E), M(X_E)(k))$ is F-rational.

By Definition 2.1 the cycle c_1^k is non-zero. Let $a \in \mathrm{Ch}(Y_E)$ be the element dual to c_1^k with respect to the bilinear form $\mathrm{Ch}(Y_E) \times \mathrm{Ch}(Y_E) \to \mathbb{F}_p$, $(x_1, x_2) \mapsto \deg(x_1 \cdot x_2)$. The pull-back homomorphism $f : \mathrm{Ch}(X \times Y) \to \mathrm{Ch}(Y_{F(X)}) = \mathrm{Ch}(Y_E)$ with respect to the morphism $Y_{F(X)} = (\mathrm{Spec}\, F(X)) \times Y \to X \times Y$ given by the generic point of X is surjective by [4, Corollary 57.11]. Let $\alpha' \in \mathrm{Ch}(X \times Y)$ be a cycle whose image in $\mathrm{Ch}(Y_E)$ under the surjection f is a. We define α as α'_E and we have $\alpha = 1 \times a + \ldots$, where "…" stands for a linear combination of only those elements whose first factor is of positive codimension and where $1 = [X_E]$. Then $\beta \circ \alpha = 1 \times x + \dots$, where "$\dots$" stands for a linear combination of only those terms whose first factor is of positive codimension. It follows that $\beta \circ \alpha \neq 0$. $\square$

Remark 2.7. The Theorem 2.6 also gives us some information about the integral motive of the variety $SB(p^m, D)$. Indeed, according to [10, Corollary 2.7] the decomposition of $M(SB(p^m, D))$ with coefficients in $\mathbb{F}_p$ lifts (and in a unique way) to the coefficients $\mathbb{Z}/p^N\mathbb{Z}$ for any $N \geq 2$. Then by [10, Theorem 2.16] it lifts to $\mathbb{Z}$ (uniquely for $p = 2$ and $p = 3$ and non-uniquely for $p > 3$). See also Remark 2.10.

Remark 2.8. Let l be an integer such that $0 < l < p^n$ and $\gcd(l, p) = 1$. The complete decomposition of the motive $M(SB(l, D))$ with coefficients in $\mathbb{F}_p$ is described in [1, Proposition 2.4].

Example 2.9. As an application of Theorem 2.6 we describe the complete motivic decomposition of $SB(4, D)$ for a division algebra D of degree 8. Let E/F be a splitting field extension for the algebra D. We note $M = M(X)$. By Theorem 2.6 and Remark 2.2, the motives $M(2)$, $M(3)$, $M(4)$ and by duality $M(7)$, $M(6)$, $M(5)$ are direct summands of $M(SB(4, D))$. By [7, Theorem 4.1], we have an indecomposable direct summand $M_{2,D}$ of $M(SB(4, D))$ with a property: $\mathbb{F}_2(0)$ and $\mathbb{F}_2(16)$ are presented in the decomposition of $(M_{2,D})_E$. By [7, Theorem 3.8], and Theorems 2.4, 2.5 any other indecomposable summand of $M(SB(4, D))$ is some shift of either M or $M(SB(2, D))$. But the second case is impossible because $70 = \binom{8}{4} = \operatorname{rk} M(SB(4, D)) < 6 \operatorname{rk} M + \operatorname{rk} M(SB(2, D)) = 6 \cdot 8 + \binom{8}{2} = 76$.

We temporary note $X = SB(1, D), Y = SB(4, D)$. Let us assume that the motive $M(1)$ is a summand of $M(SB(4, D))$. Then there exist two correspondences $\alpha \in \operatorname{Ch}_{\dim X+1}(X \times Y)$ and $\beta \in \operatorname{Ch}_{\dim Y-1}(Y \times X)$, such that $\beta \circ \alpha = 1$, where 1 means the diagonal class in $\operatorname{Ch}_{\dim X}(X \times X)$. Let f be a projection $Y \times X \to Y$. Then the cycle $f_*(\beta_E) \in \operatorname{Ch}^1(X_E)$ is F-rational and non-zero. The contradiction follows from [3, Proposition 5.1] and [9, Corollary 2.7]. So $M(1)$ and by duality $M(8)$ could not be the summands of $M(Y)$.

Assume now that there are more than 6 motives M (with some shifts) in the decomposition of $M(SB(4, D))$. Then by duality there are at least 8 such summands. But the decomposition of any of these 8 summands M into the sum of Tate motives over the splitting field E contains $\mathbb{F}_2(7)$. We have a contradiction with $\operatorname{rk}_7 M(SB(4, D)) = 7$. Therefore

$$M(SB(4, D)) = M_{2,D} \oplus M(2) \oplus M(3) \oplus M(4) \oplus M(5) \oplus M(6) \oplus M(7)\,. \qquad (1)$$

We can write the decomposition of $M_{2,D}$ over the function field $L = F(SB(4, D))$:

$$(M_{2,D})_L = \mathbb{F}_2 \oplus M(SB(1, C))(1) \oplus M(SB(2, C))(4) \oplus M(SB(2, C))(8) \oplus M(SB(1, C))(12) \oplus \mathbb{F}_2(16)\,,$$

where C is a central division L-algebra (of degree 4) Brauer-equivalent to D_L.

Remark 2.10. We have the same decomposition as (1) for the integral motive of the variety $SB(4, D)$. To show this one can apply [10, Corollary 2.7] and then [10, Theorem 2.16].

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