ON THE MAKAR-LIMANOV, DERKSEN INVARIANTS, AND FINITE AUTOMORPHISM GROUPS OF ALGEBRAIC VARIETIES

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To Peter Russell on the occasion of his 70th birthday

Abstract. A simple method of constructing a big stock of algebraic varieties with trivial Makar-Limanov invariant is described, the Derksen invariant of some varieties is computed, the generalizations of the Makar-Limanov and Derksen invariants are introduced and discussed, and some results on the Jordan property of automorphism groups of algebraic varieties are obtained.

Introduction

The subject matter of this note are automorphism groups of algebraic varieties.

In Section 1 I discuss the Makar-Limanov and Derksen invariants. As is known, they have been first introduced as the means for distinguishing the Koras-Russell threefolds from affine spaces. Since then studying varieties with certain properties of these invariants (for instance, with trivial Makar-Limanov invariant) became an independent line of research, see, e.g., [Dai], [Dub], [FZ], and references therein. At the conference *Affine Algebraic Geometry*, June 1–5, 2009, Montreal, I was surprised to find that a simple general method of constructing a big stock of such varieties remained unnoticed by the experts. In Section 1 I expand my comment on this point made after one of the talks and give the related proofs and some illustrating examples. Then I consider the Derksen invariant and show that in many cases in presence of an algebraic group action it coincides with the coordinate algebra. At the end of this section I introduce and discuss the natural generalizations of the Makar-Limanov and Derksen invariants. In Section 2 some results on the Jordan property of automorphism groups of algebraic varieties are obtained.

Conventions and notation.

Below variety means algebraic variety. All varieties are taken over an algebraically closed field k of characteristic zero. I use the standard conventions of [Bo] and [Sp] and the following notation.

A^* is the group of units of the commutative ring A with identity.

$\mathbf{M}_{n\times m}$ is the affine space of all $n \times m$-matrices with entries in k.

$\mathbf{A}^1_*$ is the punctured affine line $\mathbf{A}^1 \setminus \{0\}$.

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$\mathbf{Z}_{>0}$ is the set of positive integers.
$|M|$ is the number of elements of the set M.
$\operatorname{Rad} G$ is the radical of the linear algebraic group G.
$\operatorname{Rad}_u G$ is the unipotent radical of the linear algebraic group G.
(G, G) is the commutator subgroup of the group G.
$k[X]$ is the k-algebra of regular function on the variety X.
$k(X)$ is the field of rational function on the irreducible variety X.
$\mathrm{T}_{x,X}$ is the tangent space to the variety X at the point $x \in X$.
$\operatorname{Aut}(X)$ is the automorphism group of the variety X.
$\operatorname{Bir}(X)$ is the group of birational automorphisms of the irreducible variety X.

Given the varieties X and Y (not necessarily affine), $k[X]$ and $k[Y]$ are naturally identified with the k-subalgebras of $k[X \times Y]$. Recall that then $k[X \times Y]$ is generated by $k[X]$ and $k[Y]$ and, moreover, $k[X \times Y] = k[X] \otimes_k k[Y]$, see [SW]. If A and B are the k-subalgebras of resp. $k[X]$ and $k[Y]$, then the subalgebra of $k[X \times Y]$ generated by A and B is $A \otimes_k B$.

Below action of an algebraic group on an algebraic variety means algebraic action. Homomorphism of algebraic groups means algebraic homomorphism.

Let X be a variety endowed with an action of an algebraic group G. Then the natural homomorphism $\varphi\colon G \to \operatorname{Aut}(X)$ defined by this action is called *algebraic* and $\varphi(G)$ is called the *algebraic subgroup* of $\operatorname{Aut}(X)$. If φ is injective, $\varphi(G)$ is identified with G by means of φ.

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1. The Makar–Limanov and Derksen invariants

1.1. **The Makar–Limanov invariant.**

Recall that the *Makar-Limanov invariant* of a variety X is the following k-subalgebra of $k[X]$:

$$\mathrm{ML}(X) := \bigcap_H k[X]^H \tag{1}$$

where H in (1) runs over the images of all homomorphisms $\mathbf{G}_a \to \operatorname{Aut}(X)$.

Below is described a simple method of constructing varieties whose Makar-Limanov invariant is trivial (i.e., equal to k). The starting point is

Lemma 1.1. *For every connected linear algebraic group G, the following are equivalent*:

(i) *G has no nontrivial characters*;
(ii) *G is generated by one-dimensional unipotent subgroups*;
(iii) *G is generated by unipotent elements*;
(iv) $\operatorname{Rad} G = \operatorname{Rad}_u G$.

Proof. Let G_0 be the subgroup of G generated by all one-dimensional unipotent subgroups of G; it is normal and, by [Sp, 2.2.7], closed. Since $\operatorname{char} k = 0$, for every nonidentity unipotent element $u \in G$, the closure of $\{u^n \mid n \in \mathbb{Z}\}$ is a one-dimensional unipotent subgroup of G (cf., e.g., [OV, Chap. 3, §2, no. 2, Theorem 1]). Hence G_0 coincides with the subgroup generated by all unipotent element of G. This yields (ii)$\Leftrightarrow$(iii).

Since homomorphisms of algebraic groups preserve Jordan decompositions, G_0 is contained in the kernel of every character of G and every element of the G/G_0 is semisimple. The latter yields that G/G_0 is a torus (cf., e.g., [Bo, I.4.6]). Hence G has no nontrivial characters if and only if $G = G_0$. This proves (i)$\Leftrightarrow$(ii).

Since $\operatorname{char} k = 0$, there is a reductive subgroup L in G such that G is the semidirect product of $\mathrm{Rad}_u G$ and L (cf., e.g. [OV, Chap. 6, Sect. 4]). Let Z and Z^0 be resp. the center of L and the identity component of Z. Put $H := (L, L)\mathrm{Rad}_u G$. Then Z^0 is a torus, $F := (L, L) \cap Z^0$ is finite, $L = Z^0(L, L)$, and H is connected and normal. Being connected semisimple, (L, L) has no nontrivial characters. Hence H is generated by unipotent elements. This yields $H \subseteq G_0$. As G/H is isomorphic to Z^0/F and the latter is a torus, all elements of G/H are semisimple. Hence $H = G_0$. Thus, (i) holds if and only if Z^0 is the identity. Since $\mathrm{Rad}\, G = Z^0 \mathrm{Rad}_u G$, this proves (i)$\Leftrightarrow$(iv). □

Corollary 1.2. $\mathrm{ML}(X) = \bigcap_{H \subseteq \mathrm{Aut}(X)} k[X]^H$, *where H runs over all connected linear algebraic subgroups of* $\mathrm{Aut}(X)$ *that have no nontrivial characters.*

Theorem 1.3. *Let X be a variety and let G be a connected linear algebraic subgroup of* $\mathrm{Aut}(X)$ *that has no nontrivial characters. Then*

$$\mathrm{ML}(X) \subseteq k[X]^G. \tag{2}$$

Proof. From Lemma 1.1 we infer that $k[X]^G = \bigcap_H k[X]^H$ where H runs over all one-parameter unipotent subgroups of G. This and (1) imply (2). □

Corollary 1.4. *Maintain the notation of Theorem* 1.3. *If G has no nontrivial characters and $k[X]^G = k$, then* $\mathrm{ML}(X) = k$.

Since there are no nonconstant invariant functions on orbit closures, this yields the following.

Corollary 1.5. *Maintain the notation of Theorem* 1.3. *If G has no nontrivial characters and X is the closure of a G-orbit, then* $\mathrm{ML}(X) = k$.

Corollary 1.6. *Let G be a connected algebraic group that has no nontrivial characters. Let H be a reductive subgroup of G. Then G/H is an irreducible affine variety with trivial Makar-Limanov invariant.*

Proof. As G acts on G/H transitively, Corollary 1.5 yields $\mathrm{ML}(G/H) = k$. By [Bo, Theorem 6.8] and [PV_2, Theorem 4.9] reductivity of H implies that G/H is affine. □

The following generalizes Corollary 1.5.

Theorem 1.7. *Let X be a variety endowed with an action of a connected linear algebraic group G. Let d be the dimension of the center of $G/\mathrm{Rad}_u G$. If X contains a dense G-orbit, then*

$$\operatorname{tr\,deg}_k \mathrm{ML}(X) \leqslant d. \tag{3}$$

Proof. Let X be the closure of the G-orbit of a point $x \in X$. The morphism $G \to X$, $g \mapsto g \cdot x$, is G-equivariant with respect to the action of G on itself by left translations. Since its image is dense in X, the corresponding comorphism is a G-equivariant embedding of the k-algebras

$$k[X] \hookrightarrow k[G]. \tag{4}$$

Let L, Z, Z^0, F, and H be as in the proof of Lemma 1.1. From (4) and Theorem 1.3 we then infer that

$$\mathrm{ML}(X) \subseteq k[X]^H \hookrightarrow k[G]^H. \tag{5}$$

Since G/H is isomorphic to Z^0/F and $\dim Z^0/F = \dim Z^0 = \dim Z = d$, we have $\dim G/H = d$. As $k[G]^H$ is isomorphic to $k[G/H]$, this and (5) imply the claim. □

Corollary 1.8. *Let X be the closure in $\mathbf{P}^n$ of an orbit of a connected algebraic subgroup G in $\mathrm{Aut}(\mathbf{P}^n)$. Let $\widehat{X} \subseteq k^{n+1}$ be the affine cone over X. Then $\operatorname{tr\,deg}_k \mathrm{ML}(\widehat{X}) \leqslant d+1$, where d is the dimension of the center of $G/\mathrm{Rad}_u G$.*

Proof. Let $\widehat{G}$ be the pullback of G with respect to the natural projection $\mathrm{Aut}(k^{n+1}) \to \mathrm{Aut}(\mathbf{P}^n)$. Then $\widehat{X}$ is the closure of a $\widehat{G}$-orbit in k^{n+1} and the dimension of the center of $\widehat{G}/\mathrm{Rad}_u\widehat{G}$ is $d+1$; whence the claim by Theorem 1.7. □

Lemma 1.9. *For any varieties X_1 and X_2,*

$$\mathrm{ML}(X_1 \times X_2) \subseteq \mathrm{ML}(X_1) \otimes_k \mathrm{ML}(X_2). \tag{6}$$

Proof. Take an element $f \in \mathrm{ML}(X_1 \times X_2)$. Since $k[X_1 \times X_2]$ is generated by $k[X_1]$ and $k[X_2]$, there is a decomposition

$$f = \sum_{i=1}^{n} s_i t_i, \quad s_1, \ldots, s_n \in k[X_1],\ t_1, \ldots, t_n \in k[X_2]. \tag{7}$$

We may (and shall) assume that $t_1, \ldots, t_n$ in (7) are linearly independent over k. As $k[X_1 \times X_2] = k[X_1] \otimes_k k[X_2]$, then they are also linearly independent over $k[X_1]$.

Consider an action α of $\mathbf{G}_a$ on X_1. Then $k[X_1]$ is stable and $k[X_2]$ is pointwise fixed with respect to the diagonal action of $\mathbf{G}_a$ on $X_1 \times X_2$ determined by α and trivial action on X_2. For every element $g \in \mathbf{G}_a$ and this diagonal action, (1) and (7) imply that

$$\sum_{i=1}^{n} s_i t_i = f = g \cdot f = \sum_{i=1}^{n} (g \cdot s_i) t_i. \tag{8}$$

Since $t_1, \ldots, t_n$ are linearly independent over $k[X_1]$, we infer from (8) that every s_i is invariant with respect to α. As α is arbitrary, (1) implies that $s_1, \ldots, s_n \in \mathrm{ML}(X_1)$.

Hence f is decomposed as

$$f=\sum_{i=1}^{m} s_i' t_i', \quad s_1',\dots,s_m'\in \mathrm{ML}(X_1),\ t_1',\dots,t_m'\in k[X_2], \tag{9}$$

where $s_1',\dots,s_m'$ are linearly independent over k. The same argument as above then yields $t_1',\dots,t_m'\in \mathrm{ML}(X_2)$. Now (6) follows from (9). ☐

Corollary 1.10. *For any varieties X_1 and X_2, the following are equivalent:*

(i) $\mathrm{ML}(X_1)$ *and* $\mathrm{ML}(X_2)$ *lie in* $\mathrm{ML}(X_1\times X_2)$*;*
(ii) $\mathrm{ML}(X_1\times X_2)=\mathrm{ML}(X_1)\otimes_k \mathrm{ML}(X_2)$.

Corollary 1.11. *If* $\mathrm{ML}(X_1)=k$ *and* $\mathrm{ML}(X_2)=k$*, then* $\mathrm{ML}(X_1\times X_2)=k$*.*

Corollary 1.12. *Let X_1 and X_2 be the varieties such that* $\mathrm{ML}(X_1)$ *and* $\mathrm{ML}(X_2)$ *are generated by units. Then* $\mathrm{ML}(X_1\times X_2)=\mathrm{ML}(X_1)\otimes_k \mathrm{ML}(X_2)$.

Proof. This follows from Corollary 1.10 since $k[X_1]^*$ and $k[X_2]^*$ lie in $k[X_1\times X_2]^*$ and $k[X_1\times X_2]^*\subset \mathrm{ML}(X_1\times X_2)$, cf. [F, 1.4]. ☐

Definition 1.13. A variety is called *toral* if it is isomorphic to a closed subvariety of a linear algebraic torus.

Note that closed subvarieties and products of toral varieties are toral.

Lemma 1.14. *Let X be an affine variety.*

(a) *The following are equivalent:*
 (a_1) *X is toral;*
 (a_2) *$k[X]$ is generated by $k[X]^*$.*
(b) *For every finite subgroup G of* $\mathrm{Aut}(X)$*, there is a covering of X by G-stable open toral sets.*
(c) *If X is toral, then*
 (c_1) *for every unipotent linear algebraic group H, every algebraic homomorphism $\varphi\colon H\to \mathrm{Aut}(X)$ is trivial*;
 (c_2) $\mathrm{ML}(X)=k[X]$.

Proof. (a) Every character of a linear algebraic torus T is an element of $k[T]^*$ and $k[T]^*$ is the k-linear span of the set of all characters [Bo, Sect. 8.2]; this and Definition 1.13 imply (a_1)⇒(a_2).

Conversely, if (a_2) holds, let $k[X]=k[f_1,\dots,f_n]$ for some $f_i\in k[X]^*$. Then $\iota\colon X\to \mathbf{A}^n$, $x\mapsto (f_1(x),\dots,f_n(x))$, is a closed embedding since X is affine. The standard coordinate functions on $\mathbf{A}^n$ do not vanish on $\iota(X)$ since f_i does not vanish on X. Hence $\iota(X)\subset (\mathbf{G}_m)^n$. This proves ($\mathrm{a}_2$)⇒($\mathrm{a}_1$) and completes the proof of (a).

(b) Let x be a point of X. We have to show that x is contained in a G-stable open toral subset of X. Let $k[X]=k[h_1,\dots,h_s]$. Replacing h_i by $h_i+\alpha_i$ for an appropriate $\alpha_i\in k$, we may)and shall) assume that every h_i vanishes nowhere on the G-orbit $G\cdot x$ of x. Enlarging the set $\{h_1,\dots,h_s\}$ by including in it $g\cdot h_i$ for every i and $g\in G$, we may (and shall) assume that $\{h_1,\dots,h_s\}$ is G-stable. Then $h:=h_1\cdots h_s\in k[X]^G$.

Hence the affine open set $X_h := \{z \in X \mid h(z) \neq 0\}$ is G-stable and contains $G \cdot x$. Since $k[X_h] = k[h_1, \ldots, h_s, 1/h]$ we have $h_i \in k[X_h]^*$ for every i. Hence, X_h is toral by (a). This proves (b).

(c) Consider the action of H on X determined by φ. Let $H \cdot x$ be the H-orbit of a point $x \in X$. Since $\operatorname{char} k = 0$, $H \cdot x$ is isomorphic to $\mathbf{A}^d$ for some d, see [P$_1$, Cor. of Theorem 2]. Since H is unipotent and X is affine, $H \cdot x$ is closed in X, cf. [Bo, 4.10]. Hence $H \cdot x$ is toral. Since $k[\mathbf{A}^d]^* = k^*$, from (a) we then infer that $d = 0$, i.e., x is a fixed point. This proves (c$_1$). In turn, (c$_1$) implies (c$_2$) by (1). □

Corollary 1.15. *If* $\mathrm{ML}(X_1) = k$ *and* X_2 *is toral, then* $\mathrm{ML}(X_1 \times X_2) = k[X_2]$.

Utilizing the above statements one gets many interesting varieties with trivial Makar-Limanov invariant. The following construction is typical (but not the only possible, see Example 1.21).

Let G be a connected semisimple algebraic group acting on an affine variety X. By Hilbert's theorem, $k[X]^G$ is a finitely generated k-algebra. Let $k[X]^G = k[f_1, \ldots, f_n]$. For every $\alpha_1, \ldots, \alpha_n \in k$, denote by $X(\alpha_1, \ldots, \alpha_n)$ the closed subvariety of X whose underlying topological space is $\{x \in X \mid f_1(x) = \alpha_1, \ldots, f_n(x) = \alpha_n\}$ (warning: in general, the ideal $(f_1 - \alpha_1, \ldots, f_n - \alpha_n)$ of $k[X]$ is not radical). Let Y be a G-stable closed suvariety of X. It is well-known that $k[X]^G \to k[Y]^G$, $f \mapsto f|_Y$, is an epimorphism [PV$_2$, 3.4]. Hence $k[Y]^G = k$ if and only if Y is contained in some $X(\alpha_1, \ldots, \alpha_n)$. From Theorem 1.3 we then infer that the Makar-Limanov invariant of every G-stable closed subvariety of $X(\alpha_1, \ldots, \alpha_n)$ is trivial.

There are many instances where $f_1, \ldots, f_n$ can be explicitly described. E.g., classical invariant theory yields such a description for a number of finite-dimensional modules X of classical linear groups G; for some of them, it is proved that $(f_1 - \alpha_1, \ldots, f_n - \alpha_n)$ is radical. If the latter happens, one obtains the instances of affine algebras with trivial Makar-Limanov invariant that are explicitly described by equations.

Below are several illustrating examples.

Example 1.16 (Closures of adjoint orbits)**.** Let f_s be the sum of all principal $s \times s$-minors of the $n \times n$-matrix (x_{ij}) where $x_{11}, \ldots, x_{nn}$ are variables considered as the standard coordinate functions on $\mathbf{M}_{n\times n}$. For $\alpha_1, \ldots, \alpha_n \in k$,

$$\mathbf{M}_{n\times n}(\alpha_1, \ldots, \alpha_n) := \{a \in \mathbf{M}_{n\times n} \mid f_1(a) = \alpha_1, \ldots, f_n(a) = \alpha_n\}$$

is the set of all matrices whose characteristic polynomial is $t^n + \sum_{i=1}^n (-1)^i \alpha_i t^{n-i}$.

Consider the action of $\mathbf{SL}_n$ on $\mathbf{M}_{n\times n}$ by conjugation. Then $k[\mathbf{M}_{n\times n}]^{\mathbf{SL}_n}$ is freely generated by $f_1, \ldots, f_n$ (cf., e.g., [PV$_2$, 0.6]). Moreover, $\mathbf{M}_{n\times n}(\alpha_1, \ldots, \alpha_n)$ is irreducible and the ideal $(f_1 - \alpha_1, \ldots, f_n - \alpha_n)$ of $k[\mathbf{M}_{n\times n}]$ is radical (see the next paragraph). Hence, $\mathbf{M}_{n\times n}(\alpha_1, \ldots, \alpha_n)$ is a closed subvariety $\mathbf{M}_{n\times n}$ of such that

$$\mathrm{ML}(\mathbf{M}_{n\times n}(\alpha_1, \ldots, \alpha_n)) = k$$

and $k[\ldots, x_{ij}, \ldots]/(f_1 - \alpha_1, \ldots, f_n - \alpha_n)$ is the k-domain with trivial Makar-Limanov invariant.

This admits the following generalization. Let G be a connected reductive algebraic group and let $\mathrm{Lie}(G)$ be the Lie algebra of G endowed with the adjoint action of G. By [K] the graded k-algebra $k[\mathrm{Lie}(G)]^G$ is free and, for every minimal system of its homogeneous generators $f_1, \dots, f_r$ and constants $\alpha_1, \dots, \alpha_r \in k$,

(i) $\mathrm{Lie}(G)(\alpha_1, \dots, \alpha_r) := \{a \in \mathrm{Lie}(G) \mid f_1(a) = \alpha_1, \dots, f_r(a) = \alpha_r\}$ is the closure of a G-orbit;

(ii) the ideal $(f_1 - \alpha_1, \dots, f_r - \alpha_r)$ of $k[\mathrm{Lie}(G)]$ is radical.

Since the center Z of G acts trivially on $\mathrm{Lie}(G)$ and G/Z is semisimple, this yields

$$\mathrm{ML}(\mathrm{Lie}(G)(\alpha_1, \dots, \alpha_r)) = k$$

and $k[\mathrm{Lie}(G)]/(f_1-\alpha_1, \dots, f_r-\alpha_r)$ is the k-domain with trivial Makar-Limanov invariant.

For $G = \mathbf{GL}_n$, we have $\mathrm{Lie}(G)(\alpha_1, \dots, \alpha_r) = \mathbf{M}_{n\times n}(\alpha_1, \dots, \alpha_n)$.

Example 1.17 (Determinantal varieties)**.** Given positive integers $n \geqslant m > r$, let $\{x_{ij} \mid i = 1, \dots, n,\ j = 1, \dots, m\}$ be the set of variables considered as the standard coordinates functions on $\mathbf{M}_{n\times m}$. Let $I_{n,m,r}$ be the ideal of $k[\mathbf{M}_{n\times m}] = k[\dots, x_{ij}, \dots]$ generated by all $(r + 1) \times (r + 1)$-minors of the matrix (x_{ij}). Then $I_{n,m,r}$ is radical, cf., e.g., [Pr]. The (affine) determinantal variety $D_{n,m,r}$ is the subvariety of $\mathbf{M}_{n\times m}$ defined by $I_{n,m,r}$. Its underlying set is that of $n \times m$-matrices of rank $\leqslant r$. It is stable with respect to the action of $\mathbf{SL}_n \times \mathbf{SL}_m$ on $\mathbf{M}_{n\times m}$ by $(g, h)\cdot a := gah^{-1}$ and contains a dense orbit. Whence

$$\mathrm{ML}(D_{n,m,r}) = k$$

and $k[\mathbf{M}_{n\times m}]/I_{n,m,r}$ is the k-domain with trivial Makar-Limanov invariant.

Example 1.18 (S-varieties in the sense of [PV_1])**.** Denote by $\mathsf{S}^d k^n$ the dth symmetric power of the coordinate vector space (of columns) k^n. The natural $\mathbf{SL}_n$-action on k^n induces that on $\mathsf{S}^d k^n$. The (affine) Veronese morphism

$$\nu_n^d \colon k^n \to \mathsf{S}^d k^n, \qquad v \mapsto v^d,$$

is $\mathbf{SL}_n$-equivariant. Its image $\nu_n^d(k^n)$ is closed and contains a dense $\mathbf{SL}_n$-orbit. The ideal of $\nu_n^d(k^n)$ is generated by all 2×2-minors of a certain symmetric matrix whose entries are the coordinates on $\mathsf{S}^d k^n$, cf. [Ha]. Thus,

$$\mathrm{ML}\big(\nu_n^d(k^n)\big) = k$$

and the coordinate algebra of $\nu_n^d(k^n)$ is the explicitly described k-domain with trivial Makar-Limanov invariant.

More generally, the following combination of the Veronese and Segre morphisms

$$\nu_{n_1,\dots,n_s}^{d_1,\dots,d_s} \colon k^{n_1} \times \cdots \times k^{n_s} \to \mathsf{S}^{d_1} k^{n_1} \otimes \cdots \otimes \mathsf{S}^{d_s} k^{n_s},$$
$$(v_1, \dots, v_s) \mapsto v^{d_1} \otimes \cdots \otimes v^{d_s},$$

is equivariant with respect to the natural $\mathbf{SL}_{n_1} \times \cdots \times \mathbf{SL}_{n_s}$-actions, its image is closed and contains a dense orbit. Whence,

$$\mathrm{ML}\big(\nu_{n_1,\dots,n_s}^{d_1,\dots,d_s}(k^{n_1} \times \cdots \times k^{n_s})\big) = k.$$

In turn, this construction admits a further generalization. Namely, any matrix

$$A = \begin{pmatrix} a_{11} \dots a_{1s} \\ \dots\dots\dots \\ a_{r1} \dots a_{rs} \end{pmatrix}$$

with the entries in $\mathbf{Z}_{>0}$ defines the diagonal morphism

$$\nu^A_{n_1,\dots,n_s} := \nu^{a_{11},\dots,a_{1s}}_{n_1,\dots,n_s} \times \dots \times \nu^{a_{r1},\dots,a_{rs}}_{n_1,\dots,n_s}.$$

This morphism is $\mathbf{SL}_{n_1} \times \dots \times \mathbf{SL}_{n_s}$-equivariant and its image

$$\mathbf{H}^A_{n_1,\dots,n_s} := \nu^A_{n_1,\dots,n_s}(k^{n_1} \times \dots \times k^{n_s})$$

is closed and contains a dense orbit. Thus,

$$\mathrm{ML}(\mathbf{H}^A_{n_1,\dots,n_s}) = k.$$

In fact, $\mathbf{H}^A_{n_1,\dots,n_s}$'s are special examples of varieties with trivial Makar-Limanov invariant obtained by the following general construction [PV_1].

Let G be a connected semisimple algebraic group and let $E(\lambda)$ be a simple G-module with the highest weight λ (with respect to a fixed Borel subgroup and its maximal torus). Let v_{λ_i} be a highest vector of $E(\lambda_i)$. For $x = v_{\lambda_1} + \dots + v_{\lambda_s} \in E(\lambda_1) \oplus \dots \oplus E(\lambda_s)$, let $X(\lambda_1, \dots, \lambda_s)$ be the closure of the G-orbit of x. Up to G-isomorphism, $X(\lambda_1, \dots, \lambda_s)$ depends only on $\lambda_1, \dots, \lambda_s$. By Corollary 1.5

$$\mathrm{ML}(X(\lambda_1, \dots, \lambda_s)) = k.$$

The ideal $I(\lambda)$ of $X(\lambda)$ in $k[E(\lambda)]$ is generated by quadratic forms that can be explicitly described. Namely, $k[E(\lambda)]$ is the symmetric algebra of the dual G-module $E(\lambda)^* = E(\lambda^*)$. The submodule $\mathsf{S}^2E(\lambda^*)$ of the G-module $k[E(\lambda)]$ contains a unique simple submodule with the highest weight $2\lambda^*$, the Cartan component of $\mathsf{S}^2E(\lambda^*)$. Hence $\mathsf{S}^2E(\lambda^*)$ contains a unique submodule L complement to the Cartan component. This L generates $I(\lambda)$, see [Br, Theorem 4.1].

For $G = \mathbf{SL}_{n_1} \times \dots \times \mathbf{SL}_{n_s}$ and $\lambda_i = a_{i1}\varpi^{(1)}_1 + \dots + a_{is}\varpi^{(s)}_1$ where $\varpi^{(j)}_1$ is the highest weight of the natural $\mathbf{SL}_{n_j}$-module k^{n_j}, we have

$$X(\lambda_1, \dots, \lambda_s) = \mathbf{H}^A_{n_1,\dots,n_s}.$$

Example 1.19 (Irreducible affine surfaces quasihomogeneous with respect to an algebraic group in the sense of [G])**.** By [P_1], up to isomorphism, such surfaces are exhausted by the following list (we maintain the notation of Example 1.18):

(i) smooth surfaces:

$$\mathbf{A}^2,\ \mathbf{A}^1 \times \mathbf{A}^1_*,\ \mathbf{A}^1_* \times \mathbf{A}^1_*,\ (\mathbf{P}^1 \times \mathbf{P}^1) \setminus \Delta,\ \mathbf{P}^2 \setminus C, \tag{10}$$

where Δ is the diagonal in $\mathbf{P}^1 \times \mathbf{P}^1$, and C is a nondegenerate conic in $\mathbf{P}^2$;

(ii) singular surfaces:

$$\mathbf{V}(n_1, \dots, n_r) := \mathbf{H}^A_2 \qquad \text{for } A = (n_1, \dots, n_r)^{\mathsf{T}}, \quad n_1, \dots, n_r \geqslant 2.$$

Each of these surfaces but $\mathbf{A}^1\times\mathbf{A}^1_*$ and $\mathbf{A}^1_*\times\mathbf{A}^1_*$ admits an $\mathbf{SL}_2$-action with a dense orbit. Namely, $(\mathbf{P}^1\times\mathbf{P}^1)\backslash\Delta=\mathbf{SL}_2/T$ and $\mathbf{P}^2\backslash C=\mathbf{SL}_2/N(T)$, where T is a maximal torus of $\mathbf{SL}_2$ and $N(T)$ its normalizer, see [P_1, Lemma 2]. The surface $\mathbf{V}(n_1,\ldots,n_r)$ is the closure of the $\mathbf{SL}_2$-orbit of $v_1+\cdots+v_r\in\mathbf{R}_{n_1}\oplus\cdots\oplus\mathbf{R}_{n_r}$, where v_i is a highest vector of the simple $\mathbf{SL}_2$-module $\mathbf{R}_{n_i}$ of dimension n_i+1 (such a module is unique up to isomorphism),

see [P_1, §2]. By Corollary 1.5 this yields

$$\mathrm{ML}((\mathbf{P}^1\times\mathbf{P}^1)\setminus\Delta)=\mathrm{ML}(\mathbf{P}^2\setminus C)=\mathrm{ML}(\mathbf{V}(n_1,\ldots,n_r))=k, \tag{11}$$

As $\mathbf{A}^m_*:=(\mathbf{A}^1_*)^m$ is toral and $\mathrm{ML}(\mathbf{A}^n)=k$, Corollary 1.15 implies that

$$\mathrm{ML}(\mathbf{A}^n\times\mathbf{A}^m_*)=k[\mathbf{A}^m_*]. \tag{12}$$

From (12) we get the Makar-Limanov invariants of the remaining three surfaces in (10).

Example 1.20 (Irreducible affine threefolds quasihomogeneous with respect to an algebraic group in the sense of [G])**.** We maintain the notation of Examples 1.18 and 1.19. Identify $\mathrm{Pic}((\mathbf{P}^1\times\mathbf{P}^1)\setminus\Delta)$ with $\mathbf{Z}$ by a fixed isomorphism φ. Let $\mathbf{X}_n$ be the total space of the one-dimensional vector bundle over $(\mathbf{P}^1\times\mathbf{P}^1)\setminus\Delta$ corresponding to $n\in\mathbf{Z}$, and let $\mathbf{X}^*_n$ be the complement of the zero section in $\mathbf{X}_n$. In fact, $\mathbf{X}_n$ is isomorphic to $\mathbf{X}_{-n}$ and $\mathbf{X}^*_n$ to $\mathbf{X}^*_{-n}$, so $\mathbf{X}_n$ and $\mathbf{X}^*_n$ do not depend on the choice of φ, see [P_2].

The group $\mathrm{Pic}\big(\mathbf{P}^2\setminus C\big)$ has order 2. Let $\mathbf{Y}_0$ and $\mathbf{Y}_1$ be the total spaces of, resp., trivial and nontrivial one-dimensional vector bundles over $\mathbf{P}^2\setminus C$. Let $\mathbf{Y}^*_n$ be the complement of the zero section in $\mathbf{Y}_n$.

Let $\widetilde{\mathrm{T}}$, $\widetilde{\mathrm{O}}$, $\widetilde{\mathrm{I}}$, and $\widetilde{\mathrm{D}}_n$ be, resp., the binary tetrahedral, octahedral, icosahedral, and dihedral subgroup of order $4n$ in $\mathbf{SL}_2$. Put $\mathbf{S}_3=\mathbf{SL}_2/\widetilde{\mathrm{T}}$, $\mathbf{S}_4=\mathbf{SL}_2/\widetilde{\mathrm{O}}$, $\mathbf{S}_5=\mathbf{SL}_2/\widetilde{\mathrm{I}}$, and $\mathbf{W}_n=\mathbf{SL}_2/\widetilde{\mathrm{D}}_n$.

By [P_1] up to isomorphism irreducible affine threefolds quasihomogeneous with respect to an algebraic group in the sense of [G] are exhausted by the following list:

(i) smooth threefolds:

$$\begin{gathered}\mathbf{X}_n,\ \mathbf{X}^*_n,\ \mathbf{W}_n,\ \mathbf{Y}_0,\ \mathbf{Y}^*_0,\ \mathbf{Y}^*_1,\ \mathbf{S}_3,\ \mathbf{S}_4,\ \mathbf{S}_5,\\ \mathbf{A}^3,\ \mathbf{A}^2\times\mathbf{A}^1_*,\ \mathbf{A}^1\times\mathbf{A}^2_*,\ \mathbf{A}^3_*;\end{gathered} \tag{13}$$

(ii) singular threefolds:

$$\begin{gathered}\mathbf{P}(A):=\mathbf{H}_3(A)\quad\text{where all entries of } A \text{ are }\geqslant 1,\\ \mathbf{Q}(B):=\mathbf{H}_{2,2}(B)\quad\text{where }\ \mathrm{rk}\,B=1.\end{gathered}$$

By construction, $\mathbf{S}_3$, $\mathbf{S}_4$, $\mathbf{S}_5$, $\mathbf{W}_n$ are homogeneous with respect to $\mathbf{SL}_2$ while $\mathbf{P}(A)$ and $\mathbf{Q}(B)$ admit an action of resp. $\mathbf{SL}_2\times\mathbf{SL}_2$ and $\mathbf{SL}_3$ with a dense orbit. In fact, $\mathbf{X}^*_n$ for $n\neq 0$ is homogeneous with respect to $\mathbf{SL}_2$ as well (it is the quotient of $\mathbf{SL}_2$ modulo a cyclic subgroup of order $|n|$). By [P_2, Theorem 9] every $\mathbf{X}_n$ is homogeneous with respect to the nonreductive linear algebraic group $\mathbf{SL}_{2,|n|}:=\mathbf{SL}_2\ltimes\mathbf{R}_{|n|}$ (see Example

1.19); the radical of $\mathbf{SL}_{2,|n|}$ is unipotent. By [P$_2$, Prop. 18], $\mathbf{Y}_0$ is homogeneous with respect to $\mathbf{SL}_{2,d}$ for every even $d>0$. From Theorem 1.7 we then deduce that

$$\begin{gathered}\mathrm{ML}(\mathbf{S}_3)=\mathrm{ML}(\mathbf{S}_4)=\mathrm{ML}(\mathbf{S}_5)=\mathrm{ML}(\mathbf{Y}_0)=k,\\ \mathrm{ML}(\mathbf{X}_n)=\mathrm{ML}(\mathbf{W}_n)=\mathrm{ML}(\mathbf{P}(A))=\mathrm{ML}(\mathbf{Q}(B))=k,\\ \mathrm{ML}(\mathbf{X}_n^*)=k \quad \text{for } n\neq 0.\end{gathered}$$

As $\mathbf{X}_0^*=((\mathbf{P}^1\times\mathbf{P}^1)\setminus\Delta)\times\mathbf{A}^1_*$ and $\mathbf{Y}_0^*=(\mathbf{P}^2\setminus C)\times\mathbf{A}^1_*$, we deduce from (11) and Corollary 1.15 that

$$\mathrm{ML}(\mathbf{X}_0^*)=\mathrm{ML}(\mathbf{Y}_0^*)=k[\mathbf{A}^1_*].$$

By [P$_2$, Prop. 16], $\mathbf{Y}_1^*$ is homogeneous with respect to $\mathbf{SL}_2\times\mathbf{G}_m$. Since the latter is a reductive group with one-dimensional center, Theorem 1.7 implies that $\operatorname{tr}\deg_k\mathrm{ML}(\mathbf{Y}_1^*)\leqslant 1$. On the other hand, by [P$_2$, Prop. 19], $k[\mathbf{Y}_1^*]/k^*$ is a free abelian group of rank 1. Since $k[X]^*\subseteq \mathrm{ML}(X)$ for every X, this yields

$$\operatorname{tr}\deg_k\mathrm{ML}(\mathbf{Y}_1^*)=1.$$

Finally, (12) yields the Makar-Limanov invariants of the last four threefolds in (13).

Example 1.21 (Schubert varieties)**.** Let G be a connected semisimple algebraic group and let $\mathbf{P}E$ be the projective space of 1-dimensional linear subspaces in a nonzero simple G-module E. There is a unique closed G-orbit $\mathcal{O}$ in $\mathbf{P}E$. Let U be a maximal unipotent subgroup of G. There are only finitely many U-orbits in $\mathcal{O}$; their closures are called Schubert varieties, cf., e.g., [Sp, 8.3–8.5]. Let $X\subseteq\mathcal{O}$ be a Schubert variety and let $\widehat{X}$ be the affine cone over X in E. As U is unipotent, Corollary 1.8 yields

$$\operatorname{tr}\deg_k\mathrm{ML}(\widehat{X})\leqslant 1.$$

The ideal of $\widehat{X}$ in $k[E]$ is generated by certain forms of degree $\leqslant 2$ that admit an explicit description, see, e.g., [BL, 2.10].

1.2. **The Derksen invariant.**

Let X be a variety. Recall that the *Derksen invariant* $\mathrm{D}(X)$ of X is the k-subalgebra of $k[X]$ generated by all $k[X]^H$'s where H runs over all subgroups of $\mathrm{Aut}(X)$ isomorphic to $\mathbf{G}_a$. If there are no such subgroups, we put $\mathrm{D}(X)=\varnothing$.

Example 1.22. If X is toral, then $\mathrm{D}(X)=\varnothing$ by Lemma 1.14(c$_1$).

In this section we deduce some information on $\mathrm{D}(X)$ in case when $\mathrm{Aut}(X)$ contains a connected noncommutative reductive algebraic subgroup.

Recall that if an algebraic group G acts linearly on a (not necessarily finite-dimensional) k-vector space V, then the G-module V is called *algebraic* if every element of V is contained in an algebraic finite-dimensional G-submodule of V, cf., e.g., [PV$_2$, 3.4].

The starting point is

Lemma 1.23. *Let G be a connected noncommutative reductive algebraic group. Then every algebraic G-module V is a k-linear span of the set*

$$\bigcup_{H\subset G} V^H, \tag{14}$$

where H in (14) *runs over all one-parameter unipotent subgroups of G.*

Proof. The assumptions that G is reductive, $\operatorname{char} k = 0$, and V is algebraic imply that V is a sum of simple G-submodules. Hence we may (and shall) assume that V is a nonzero simple G-module. Since G is a connected noncommutative reductive algebraic group, it contains a one-dimensional unipotent subgroup U (indeed, since (G,G) is a nontrivial semisimple group, a root subgroup of (G,G) with respect to a maximal torus may be taken as U). By the Lie–Kolchin theorem $V^U \neq \{0\}$. Let v be a nonzero vector of V^U. As $g\cdot v \in V^{gUg^{-1}}$ for every element $g\in G$, the G-orbit $G\cdot v$ of v is contained in set (14). Hence the k-linear span of $G\cdot v$ is contained in the k-linear span of this set. But the k-linear span of $G\cdot v$ is G-stable and therefore coincides with V since V is simple. This completes the proof. □

Theorem 1.24. *Let X be a variety. If* $\operatorname{Aut}(X)$ *contains a connected noncommutative reductive algebraic subgroup, then*

$$\mathrm{D}(X) = k[X]. \tag{15}$$

Proof. Let G be a connected noncommutative reductive algebraic subgroup of $\operatorname{Aut}(X)$. Since the G-module $k[X]$ is algebraic (see [PV_2, Lemma 1.4]), the claim follows from Lemma 1.23 and the definition on $\mathrm{D}(X)$. □

Remark 1.25. The following are equivalent:

(i) $\operatorname{Aut}(X)$ contains a connected noncommutative reductive algebraic subgroup;
(ii) $\operatorname{Aut}(X)$ contains SL_2 or PSL_2.

Indeed, SL_2 and PSL_2 are connected noncommutative reductive algebraic groups and every connected noncommutative reductive algebraic group contains SL_2 or PSL_2, cf. [Bo, Theorem 13.18(4)], [Sp, 7.2.4].

The following example shows that the assumption of noncommutativity in Theorem 1.24 cannot be dropped.

Example 1.26. By [Der], for the Koras–Russell cubic threefold X, the following inequality distinguishing X from $\mathbf{A}^3$ holds:

$$\mathrm{D}(X) \neq k[X]. \tag{16}$$

On the other hand, since X is defined in $\mathbf{A}^4$ by $x_1+x_1^2x_2+x_3^2+x_4^3=0$ where $x_1,\ldots,x_4$ are the standard coordinate functions on $\mathbf{A}^4$, it is stable with respect to the action of $\mathbf{G}_m$ on $\mathbf{A}^4$ defined by $t\cdot(a_1,a_2,a_3,a_4) = (t^6a_1, t^{-6}a_2, t^3a_3, t^2a_4)$. Hence $\operatorname{Aut}(X)$ contains a one-dimensional connected commutative reductive subgroup, cf. [DM-JP, Sect. 3].

One can apply Theorem 1.24 to proving that, for some varieties X, there are no connected noncommutative reductive algebraic subgroups in $\mathrm{Aut}(X)$.

Example 1.27. For the Koras–Russell cubic threefold X, Theorem 1.24 and (16) imply that $\mathrm{Aut}(X)$ contains no connected noncommutative reductive algebraic subgroups.

Since $\mathbf{A}^n$ for $n \geqslant 2$ contains a connected noncommutative reductive algebraic subgroup (for instance, $\mathbf{GL}_n$), the next corollary generalizes the well-known fact that $\mathrm{D}(X \times \mathbf{A}^n) = k[X \times \mathbf{A}^n]$ for $n \geqslant 2$ (see, e.g., [CM]).

Corollary 1.28. *Let Z be a variety such that $\mathrm{Aut}(Z)$ contains a connected noncommutative reductive algebraic subgroup. Then, for every variety X,*

$$\mathrm{D}(X \times Z) = k[X \times Z]. \tag{17}$$

Proof. Consider the natural action of $\mathrm{Aut}(Z)$ on Z and its trivial action on X. Then the diagonal action of $\mathrm{Aut}(Z)$ on $X \times Z$ identifies $\mathrm{Aut}(Z)$ with a subgroup of $\mathrm{Aut}(X \times Z)$. Whence the claim by Theorem 1.24. □

The following example shows that the assumption of noncommutativity in Corollary 1.28 cannot be dropped.

Example 1.29. Let x_1, x_2 be the standard coordinate functions on $\mathbf{A}^2$. The principal open set Y in $\mathbf{A}^2$ defined by $x_1 \neq 0$ is isomorphic to $\mathbf{A}^1_* \times \mathbf{A}^1$ and

$$k[Y] = k[t, t^{-1}, s], \qquad \text{where} \quad t := x_1|_Y,\ s := x_2|_Y. \tag{18}$$

Since t is the unit of $k[Y]$, for every action of $\mathbf{G}_a$ on Y we have

$$t, t^{-1} \in k[Y]^{\mathbf{G}_a}. \tag{19}$$

As, clearly, $\mathrm{Aut}(Y)$ contains a one-dimensional unipotent subgroup, (19) and the definition of $\mathrm{D}[Y]$ yield $k[t, t^{-1}] \subseteq \mathrm{D}[Y]$. We also deduce from (19) that, for every point $y \in Y$, the $\mathbf{G}_a$-orbit of y lies in the line defined by the equation $t = t(y)$. But this orbit is closed in Y since Y is affine and $\mathbf{G}_a$ is unipotent, cf. [Bo, 4.10]. Hence, if y is not a fixed point, this orbit coincides with the aforementioned line. Therefore, if $\mathbf{G}_a$ acts on Y nontrivially, t separates orbits in general position. Since $\mathrm{char}\, k = 0$, by [PV$_2$, Lemma 2.1] this means that $k(Y)^{\mathbf{G}_a} = k(t)$. Whence by (18) we have $k[Y]^{\mathbf{G}_a} = k[t, t^{-1}]$. From this, (12) and (18) we then infer that

$$k[t, t^{-1}] = \mathrm{ML}(\mathbf{A}^1_* \times \mathbf{A}^1) = \mathrm{D}(\mathbf{A}^1_* \times \mathbf{A}^1) \subsetneqq k[\mathbf{A}^1_* \times \mathbf{A}^1] = k[t, t^{-1}, s].$$

Thus, (17) does not hold for $X = \mathbf{A}^1_*$, $Z = \mathbf{A}^1$ while both $\mathrm{Aut}(\mathbf{A}^1_*)$ and $\mathrm{Aut}(\mathbf{A}^1)$ contain a one-dimensional connected commutative reductive algebraic subgroup.

Theorem 1.30. *If X_i is a variety such that $\mathrm{ML}(X_i) \neq k[X_i]$, $i = 1, 2$, then*

$$\mathrm{D}(X_1 \times X_2) = k[X_1 \times X_2].$$

Proof. As $\mathrm{ML}(X_1) \neq k[X_1]$, there is a nontrivial $\mathbf{G}_a$-action α on X_1. The diagonal $\mathbf{G}_a$-action on $X_1 \times X_2$ determined by α and trivial action on X_2 is a nontrivial $\mathbf{G}_a$-action for which $k[X_2]$ lies in the algebra of invariants. Hence, $k[X_2] \subseteq \mathrm{D}(X_1 \times X_2)$. Similarly, $k[X_2] \subseteq \mathrm{D}(X_1 \times X_2)$. As $k[X_1 \times X_2]$ is generated by $k[X_1]$ and $k[X_2]$, the claim follows. □

Example 1.31. If X is the Koras–Russell cubic threefold X, then $\mathrm{D}(X) \neq k[X]$ by [Der]. But for the square of X we have $\mathrm{D}(X \times X) = k[X \times X]$— since $\mathrm{ML}(X) = k[x_1|_X] \neq k[X]$ (cf., e.g., [F, Chap. 9]), this follows from Theorem 1.30.

1.3. Generalizations.

The Makar-Limanov and Derksen invariants can be naturally generalized.

Namely, let X be a variety and let F be an algebraic group.

Definition 1.32. The F-*kernel* of X is the following k-subalgebra of $k[X]$:

$$\mathrm{Ker}_F(X) := \bigcap_H k[X]^H, \tag{20}$$

where H in (20) runs over the images of all algebraic homomorphisms $F \to \mathrm{Aut}(X)$.

Definition 1.33. The F-*envelope* of X is the k-subalgebra

$$\mathrm{Env}_F(X)$$

of $k[X]$ generated by all $k[X]^H$'s where H runs over all subgroups of $\mathrm{Aut}(X)$ isomorphic to H. If there are no such subgroups, we put $\mathrm{Env}_F(X) = \varnothing$.

Example 1.34. The definitions imply that

$$\mathrm{Ker}_{\mathbf{G}_a}(X) = \mathrm{ML}(X), \qquad \mathrm{Env}_{\mathbf{G}_a}(X) = \mathrm{D}(X).$$

Definition 1.35. We say that an algebraic group G is F-*generated* if it is generated by the images of all homomorphisms $F \to G$.

Examples 1.36. (1) By Lemma 1.1 a connected linear algebraic group G is $\mathbf{G}_a$-generated if and only if G has no nontrivial characters that, in turn, is equivalent to the condition $\mathrm{Rad}\, G \neq \mathrm{Rad}_u G$.

(2) Every connected reductive algebraic group G is $\mathbf{G}_m$-generated. This is clear if G is a torus. In the general case this follows from the fact that the subgroup generated by algebraic subgroups is closed (see, e.g., [Sp, 2.2.7]) and the union of maximal tori of G contains a dense open subset ([Sp, 6.4.5(iii), 7.6.4(ii)].

(3) Clearly, the subgroup generated by the images of all homomorphisms $F \to G$ is normal. Hence, if G is simple as abstract group and there exists a nontrivial homomorphism $F \to G$, then G is F-generated.

The following are the generalizations of the above statements on $\mathrm{ML}(X)$ and $\mathrm{D}(X)$.

Theorem 1.37. *If a variety X is endowed with an action of an F-generated algebraic group G, then* $\mathrm{Ker}_F(X) \subseteq k[X]^G$.

Proof. This follows from Definitions 1.32 and 1.35. □

Corollary 1.38. *If a variety X is endowed with an action of an F-generated algebraic group G and X contains a dense G-orbit, then* $\mathrm{Ker}_F(X) = k$.

Corollary 1.39. *If H is a reductive subgroup of an F-generated linear algebraic group G, then G/H is an affine variety with* $\mathrm{Ker}_F(G/H) = k$.

Corollary 1.40. *Let X be an irreducible variety. If there is an action of $\mathbf{G}_m$ on X with a fixed point and without other closed orbits, then*

$$\mathrm{Ker}_{\mathbf{G}_m}(X) = k. \tag{21}$$

Proof. The assumptions imply that the fixed point is unique and lies in the closure of every $\mathbf{G}_m$-orbit; whence $k[X]^{\mathbf{G}_m} = k$. In turn, this and (20) yield (21). Note that, in fact, X is affine [P4]. □

Corollary 1.41. *Let X be a closed subset of $\mathbf{P}^n$ and let $\widehat{X} \subseteq k^{n+1}$ be the affine cone over X. Then* $\mathrm{Ker}_{\mathbf{G}_m}(\widehat{X}) = k$.

Example 1.42. Consider the case $F = \mathbf{G}_m$. If G is a connected reductive subgroup of $\mathrm{Aut}(X)$ and X contains a dense G-orbit, then Corollary 1.38 and Example 1.36(2) imply that (21) holds. In particular, this is so for every toric variety X; for instance,

$$\mathrm{Ker}_{\mathbf{G}_m}(\mathbf{A}^n \times \mathbf{A}^m_*) = k.$$

(compare with (12)). Applying this to the varieties considered in Examples 1.16–1.20, we see that (21) holds for every X from the following list:

$$\mathrm{Lie}(G)(\alpha_1, \ldots, \alpha_n) \quad \text{(see Example 1.16)};$$
$$D_{n,m,r} \quad \text{(see Example 1.17)};$$
$$X(\lambda_1, \ldots, \lambda_s) \quad \text{(see Example 1.18)};$$
$$(\mathbf{P}^1 \times \mathbf{P}^1) \setminus \Delta,\ \ \mathbf{P}^2 \setminus C,\ \ \mathbf{V}(n_1, \ldots, n_r) \ \text{ where } n_1, \ldots, n_r \geqslant 2 \ \text{ (see Example 1.19)};$$
$$\mathbf{S}_3,\ \ \mathbf{S}_4,\ \ \mathbf{S}_5,\ \ \mathbf{W}_n,\ \ \mathbf{P}(A),\ \ \mathbf{Q}(B),\ \ \mathbf{X}^*_n \ \text{ where } n \neq 0,\ \ \mathbf{Y}^*_1 \ \text{ (see Example 1.20)}.$$

The threefold $\mathbf{X}_n$ from Example 1.20 is homogeneous with respect to $\mathbf{SL}_{2,|n|}$. One can prove that $\mathbf{SL}_{2,|n|}$ is $\mathbf{G}_m$-generated; whence $\mathrm{Ker}(\mathbf{X}_n) = k$.

The remaining threefolds $\mathbf{X}^*_0$, $\mathbf{Y}_0$, and $\mathbf{Y}^*_0$ from Example 1.20 are considered in Example 1.45 below.

The same proof as that of Lemma 1.9 yields

Lemma 1.43. *For any varieties X_1 and X_2,*

$$\mathrm{Ker}_F(X_1 \times X_2) \subseteq \mathrm{Ker}_F(X_1) \otimes_k \mathrm{Ker}_F(X_2).$$

Corollary 1.44. *For any varieties X_1 and X_2, the following are equivalent:*

(i) $\mathrm{Ker}_F(X_1)$ *and* $\mathrm{Ker}_F(X_2)$ *lie in* $\mathrm{Ker}_F(X_1 \times X_2)$*;*
(ii) $\mathrm{Ker}_F(X_1 \times X_2) = \mathrm{Ker}_F(X_1) \otimes_k \mathrm{Ker}_F(X_2)$.

Example 1.45. Since $\mathbf{X}^*_0 = ((\mathbf{P}^1 \times \mathbf{P}^1) \setminus \Delta) \times \mathbf{A}^1_*$, $\mathbf{Y}_0 = (\mathbf{P}^2 \setminus C) \times \mathbf{A}^1$, and $\mathbf{Y}^*_0 = (\mathbf{P}^2 \setminus C) \times \mathbf{A}^1_*$ (see Example 1.20), we deduce from Lemma 1.43 and Example 1.42 that $\mathrm{Ker}_{\mathbf{G}_m}(\mathbf{X}^*_0) = \mathrm{Ker}_{\mathbf{G}_m}(\mathbf{Y}_0) = \mathrm{Ker}_{\mathbf{G}_m}(\mathbf{Y}^*_0) = k$.

Lemma 1.46. *For any connected algebraic group F that has no nontrivial characters,*

$$k[X]^* \subseteq \operatorname{Ker}_F(X). \tag{22}$$

Proof. Let H be the image of an algebraic homomorphism $F \to \operatorname{Aut}(X)$. We claim that $k[X]^* \subseteq k[X]^H$; by virtue of Definition 1.32 this inclusion implies (22). Since H is connected, every irreducible component of X is H-stable, so proving the claim we may (and shall) assume that X is irreducible. In this case every element of $k[X]^*$ is H-semiinvariant by [PV$_2$, Theorem 3.1], hence lies in $k[X]^H$ since H has no nontrivial characters. This completes the proof. □

Corollary 1.47. *Let F be a connected algebraic group that has no nontrivial characters. Let X_1 and X_2 be varieties such that* $\operatorname{Ker}_F(X_1)$ *and* $\operatorname{Ker}_F(X_2)$ *are generated by units. Then* $\operatorname{Ker}_F(X_1 \times X_2) = \operatorname{Ker}_F(X_1) \otimes_k \operatorname{Ker}_F(X_2)$.

Lemma 1.48. *Let G be a connected reductive algebraic group of rank $\geqslant 2$. Then every algebraic G-module V is a k-linear span of the set*

$$\bigcup_{H \subseteq G} V^H, \tag{23}$$

where H in (23) *runs over all one-dimensional tori of G.*

Proof. Like in the proof of Lemma 1.23 we may (and shall) assume that V is a nonzero simple G-module. Let T be a maximal torus of G and let $v \in V$, $v \neq 0$, be a weight vector of T. Since $\dim T \geqslant 2$, the T-stabilizer T_v of v is a diagonalizable group of dimension $\geqslant 1$. Hence T_v contains a one-dimensional torus S. Thus, $v \in V^S$. Like in the proof of Lemma 1.23 we then conclude that the orbit $G \cdot v$ is contained in set (23). Since V is simple, the k-linear span of $G \cdot v$ coincides with V; whence the claim. □

Theorem 1.49. *Let X be a variety such that* $\operatorname{Aut}(X)$ *contains a connected reductive algebraic group G of rank $\geqslant 2$. Then*

$$\operatorname{Env}_{\mathbf{G}_m}(X) = k[X].$$

Proof. Since the G-module $k[X]$ is algebraic, the claim follows from Lemma 1.48 and Definition 1.33. □

Remark 1.50. Clearly, $\operatorname{Env}_{\mathbf{G}_m}(\mathbf{A}^1) = k$. This shows that in Lemma 1.48 and Theorem 1.49 the condition "$\geqslant 2$" can not be replaced by "$\geqslant 1$".

2. Finite automorphism groups of algebraic varieties

2.1. **Jordan groups.**

The following definition is inspired by the classical Jordan theorem (Theorem 2.2).

Definition 2.1. A group G is called a *Jordan group* if there exists a positive integer J_G, depending only on G, such that every finite subgroup K of G contains a normal abelian subgroup whose index in K is at most J_G.

Jordan's theorem (see [CR, Theorem 36.13]) can be then reformulated as follows:

Theorem 2.2. *Every* $\mathbf{GL}_n(k)$ *is Jordan.*

Remark 2.3. For $G = \mathbf{GL}_n(k)$, the explicit upper bounds J_G are known, see [CR, §36].

Since subgroups of Jordan groups are Jordan and every linear algebraic group is isomorphic to a subgroup of some $\mathbf{GL}_n(k)$ (see [Sp, 2.3.7]), Theorem 2.2 yields the following more general

Theorem 2.4. *Every linear algebraic group is Jordan.*

Lemma 2.5. *Let H be a finite normal subgroup of a group G. If G is Jordan, then G/H is Jordan.*

Proof. Let $\pi\colon G \to G/H$ be the natural projection and let K be a finite subgroup of G/H. Since H is finite, $\pi^{-1}(K)$ is a finite subgroup of G. As G is Jordan, $\pi^{-1}(K)$ contains a normal abelian subgroup A whose index is at most J_G. Hence $\pi(A)$ is a normal abelian subgroup of K whose index is at most J_G. □

Lemma 2.6. *If the groups G_1 and G_2 are Jordan, then $G_1 \times G_2$ is Jordan.*

Proof. Let $\pi_i\colon G := G_1 \times G_2 \to G_i$ be the natural projection and let K be a finite subgroup of G. Since G_i is Jordan, $K_i := \pi_i(K)$ contains an abelian normal subgroup A_i such that

$$[K_i : A_i] \leqslant J_{K_i}. \tag{24}$$

The subgroup $\widetilde{A}_i := \pi_i^{-1}(A_i) \cap K$ is normal in K and $K/\widetilde{A}_i$ is isomorphic to K_i/A_i. From (24) we then conclude that

$$[K : \widetilde{A}_i] \leqslant J_{K_i}. \tag{25}$$

Since $A := \widetilde{A}_1 \cap \widetilde{A}_2$ is the kernel of the diagonal homomorphism

$$K \longrightarrow \prod_{i=1}^{2} K/\widetilde{A}_i$$

determined by the canonical projections $K \to K/\widetilde{A}_i$, we infer from (25) that

$$[K : A] = |K/A| \leqslant |\prod_{i=1}^{2} K/\widetilde{A}_i| \leqslant J_{K_1} J_{K_2} \tag{26}$$

By construction, $A \subseteq A_1 \times A_2$. Since A_i is abelian, this implies that A is abelian. As A is normal in K, this and (26) complete the proof. □

The following definition distinguishes a special class of Jordan groups.

Definition 2.7. A group G is called *bounded* if there is a positive integer b_G, depending only on G, such that the order of every finite subgroup of G is at most b_G.

Examples 2.8. (1) Finite groups and torsion free groups are bounded.

(2) Every finite subgroup of $\mathbf{GL}_n(\mathbf{Q})$ is conjugate to a subgroup of $\mathbf{GL}_n(\mathbf{Z})$ (see, e.g., [CR, Theorem 73.5]). On the other hand, by Minkowski's theorem (see, e.g., [Hu, Theorem 39.4]) $\mathbf{GL}_n(\mathbf{Z})$ is bounded. Hence $\mathbf{GL}_n(\mathbf{Q})$ is bounded. Note that H. Minkowski and I. Schur obtained explicit upper bounds of the orders of finite subgroups in $\mathbf{GL}_n(\mathbf{Z})$, see [Hu, §39].

(3) It is immediate from the definition that every extension of a bounded group by bounded is bounded as well.

Lemma 2.9. *Let H be a normal subgroup of a group G such that G/H is bounded. Then G is Jordan if and only if H is Jordan.*

Proof. A proof is needed only for the sufficiency. So assume that H is Jordan; we have to prove that G is Jordan. Let K be a finite subgroup of G. By Definition 2.1

$$L := K \cap H \tag{27}$$

contains an abelian normal subgroup A such that

$$[L : A] \leqslant J_H. \tag{28}$$

Let g be an element of K. Since L is a normal subgroup of K, we infer that gAg^{-1} is a normal abelian subgroup of L and

$$[L : A] = [L : gAg^{-1}]. \tag{29}$$

Consider now the group

$$M := \bigcap_{g \in K} gAg^{-1}. \tag{30}$$

It is a normal abelian subgroup of K. We claim that $[K : M]$ is upper bounded by a constant not depending on K. To prove this, fix the representatives $g_1, \ldots, g_{|K/L|}$ of all cosets of L in K. Then (30) and normality of A in L imply that

$$M = \bigcap_{i=1}^{|K/L|} g_i A g_i^{-1}. \tag{31}$$

From (31) we deduce that M is the kernel of the diagonal homomorphism

$$L \longrightarrow \prod_{i=1}^{|K/L|} L/g_i A g_i^{-1}$$

determined by the canonical projections $L \to L/g_i A g_i^{-1}$. This, (29), and (28) yield

$$[L : M] \leqslant [L : A]^{|K/L|} \leqslant J_H^{|K/L|}. \tag{32}$$

Let $\pi \colon G \to G/H$ be the canonical projection. By (27) the finite subgroup $\pi(K)$ of G/H is isomorphic to K/L. Since G/H is bounded, this yields $|K/L| \leqslant b_{G/H}$. We then deduce from (32) and $[K : M] = [K : L][L : M]$ that

$$[K : M] \leqslant b_{G/H} J_H^{b_{G/H}}.$$

This completes the proof. $\square$

Corollary 2.10. *Let H be a finite normal subgroup of a group G such that the center of H is trivial. If G/H is Jordan, then G is Jordan.*

Proof. The conjugating action of G on H determines a homomorphism $\varphi\colon G \to \mathrm{Aut}(H)$. The definition of φ and triviality of the center of H implies that

$$H \cap \ker \varphi = \{1\}. \tag{33}$$

In turn, (33) yields that the restriction of the natural projection $G \to G/H$ to $\ker \varphi$ is an embedding $\ker \varphi \hookrightarrow G/H$. Hence $\ker \varphi$ is Jordan since G/H is Jordan. But $G/\ker \varphi$ is finite since it is isomorphic to a subgroup of $\mathrm{Aut}(H)$ for the finite group H. Whence G is Jordan by Lemma 2.9. This completes the proof. □

Example 2.12, Theorems 2.13, 2.16 and their corollaries below give, for some varieties X, the affirmative answer to the following

Question 2.11. Let X be an irreducible affine variety. Is it true that $\mathrm{Aut}(X)$ is Jordan?

Example 2.12. $\mathrm{Aut}(\mathbf{A}^n)$ is Jordan for $n \leqslant 2$. For $n = 1$ this clear, for $n = 2$ follows from Theorem 2.2 and the well-known fact that every finite subgroup of $\mathrm{Aut}(\mathbf{A}^2)$ is linearizable, i.e., conjugate to a subgroup of $\mathbf{GL}_2(k)$ (see also Subsection 2.2 below).

Theorem 2.13. *The automorphism group of every irreducible toral variety* (*see Definition* 1.13) *is Jordan.*

Proof. By [R], for any irreducible variety X, the abelian group

$$\Gamma := k[X]^*/k^*$$

is free and of finite rank. Let X be toral and let H be the kernel of the natural action of $\mathrm{Aut}(X)$ on Γ. We claim that H is abelian. Indeed, for every element $f \in k[X]^*$, the line spanned by f in $k[X]$ is H-stable. Since $\mathbf{GL}_1$ is abelian, this yields that

$$h_1 h_2 \cdot f = h_2 h_1 \cdot f \qquad \text{for any elements} \quad h_1, h_2 \in H. \tag{34}$$

As X is toral, $k[X]^*$ generates the k-algebra $k[X]$ by Lemma 1.14. Hence (34) holds for every $f \in k[X]$. Since X is affine, the automorphisms of X coincide if and only if they induce the same automorphisms of $k[X]$. Whence H is abelian, as claimed.

Let n be the rank of Γ. Then $\mathrm{Aut}(\Gamma)$ is isomorphic to $\mathbf{GL}_n(\mathbf{Z})$. By the definition of H, the natural action of $\mathrm{Aut}(X)$ on Γ induces an embedding of $\mathrm{Aut}(X)/H$ in $\mathrm{Aut}(\Gamma)$. Hence $\mathrm{Aut}(X)/H$ is isomorphic to a subgroup of $\mathbf{GL}_n(\mathbf{Z})$. Example 2.8(2) then implies that $\mathrm{Aut}(X)/H$ is bounded. Thus, $\mathrm{Aut}(X)$ is an extension of a bounded group by an abelian group, hence Jordan by Lemma 2.9. This completes the proof. □

Remark 2.14. Maintain the notation of the proof of Theorem 2.13. Let $f_1, \ldots f_n$ be a basis of Γ. There are the homomorphisms $\lambda_i\colon H \to k^*$, $i = 1, \ldots, n$, such that $g \cdot f_i = \lambda(g) f_i$ for every $g \in H$ and i. Since $k[X]^*$ generates $k[X]$, the diagonal map $H \to (k^*)^n$, $h \mapsto (\lambda_1(g), \ldots, \lambda_n(g))$, is injective. This and the proof of Theorem 2.13

show that the automorphism group of X is an extension of a subgroup of $\mathbf{GL}_n(\mathbf{Z})$ by a subgroup of the torus $(k^*)^n$.

The following lemma is well-known (see, e.g., [FZ, Lemma 2.7(b)]).

Lemma 2.15. *Let X be a variety and let G be a reductive algebraic subgroup of $\mathrm{Aut}(X)$. Let $x \in X$ be a fixed point of G. Then the kernel of the induced action of G on $\mathrm{T}_{x,X}$ is trivial.*

Theorem 2.16. *Let $\sim$ be the equivalence relation on the set of points of a variety X defined by*

$$x \sim y \iff \text{the local rings of } X \text{ at } x \text{ and } y \text{ are } k\text{-isomorphic.}$$

If there is a finite equivalence class of $\sim$, then $\mathrm{Aut}(X)$ is Jordan.

Proof. Every equivalence class of $\sim$ is $\mathrm{Aut}(X)$-stable. Let C be a finite equivalence class of $\sim$ and let G be the kernel of the action of $\mathrm{Aut}(X)$ on C. Then G is a normal subgroup of finite index in $\mathrm{Aut}(X)$. By Lemma 2.9 it suffices to prove that G is Jordan.

Let K be a finite subgroup of G and let x be a point of C. As x is fixed by K, the action of K on X induces an action of K on $\mathrm{T}_{x,X}$. The latter is linear and hence determined by a homomorphism $\tau\colon K \to \mathbf{GL}(\mathrm{T}_{x,X})$. Being finite, K is reductive. Hence τ is injective by Lemma 2.15. Theorem 2.2 then yields that K contains an abelian normal subgroup A such that $[K : A] \leqslant J_{\mathbf{GL}_n(k)}$, $n := \dim \mathrm{T}_{x,X}$. This completes the proof. $\square$

Given a variety X, we say that its point x is a *vertex* of X if

$$\dim \mathrm{T}_{x,X} \geqslant \dim \mathrm{T}_{y,X} \ \text{ for every point } \ y \in X.$$

Clearly, an irreducible X is smooth if and only if every its point is a vertex.

Corollary 2.17. *The automorphism group of every variety with only finitely many vertices is Jordan.*

Corollary 2.18. *Let $\approx$ be the equivalence relation on the set of points of a variety X defined by*

$$x \approx y \iff \text{the tangent cones of } X \text{ at } x \text{ and } y \text{ are isomorphic.}$$

If there is a finite equivalence class of $\approx$, then $\mathrm{Aut}(X)$ is Jordan.

Corollary 2.19. *The automorphism group of every nonsmooth variety with only finitely many singular points is Jordan.*

Corollary 2.20. *Let $\widehat{X} \subset k^{n+1}$ be the affine cone of a smooth closed proper subvariety X in $\mathbf{P}^n = \mathbf{P}(k^{n+1})$ that does not lie in any hyperplane. Then $\mathrm{Aut}(\widehat{X})$ is Jordan.*

Proof. The assumptions imply that the singular locus of $\widehat{X}$ consists of a single point, the origin; whence the claim by Corollary 2.19. $\square$

Remark 2.21. Smoothness in Corollary 2.20 may be replaced by the assumption that X is not a cone. Indeed, in this case the origin constitutes a single equivalence class of $\approx$ for points of $\widehat{X}$; whence the claim by Corollary 2.18.

Theorem 2.22. *For every variety X, every finite subgroup G of* $\mathrm{Aut}(X)$ *such that $X^G \neq \varnothing$ contains an abelian normal subgroup whose index in G is at most $J_{\mathbf{GL}_d(k)}$ where $d = \max\limits_x \dim \mathrm{T}_{x,X}$.*

Proof. Like in the above proof of Theorem 2.16, this follows from Lemma 2.15 and Theorem 2.2. □

Corollary 2.23. *Let p be a prime number. Then every finite p-subgroup G of* $\mathrm{Aut}(\mathbf{A}^n)$ *contains an abelian normal subgroup whose index in G is at most $J_{\mathbf{GL}_n(k)}$.*

Proof. This follows from Theorem 2.22 since in this case $(\mathbf{A}^n)^G \neq \varnothing$, see [Se$_3$, Theorem 1.2]. □

Remark 2.24. To date, it is not known whether or not $(\mathbf{A}^n)^G \neq \varnothing$ for every finite subgroup G of $\mathrm{Aut}(\mathbf{A}^n)$. By Theorem 2.22 the affirmative answer would imply that $\mathrm{Aut}(\mathbf{A}^n)$ is Jordan.

Remark 2.25. The statement of Corollary 2.23 remains true if $\mathbf{A}^n$ is replaced by any p-acyclic variety X and n in $J_{\mathbf{GL}_n(k)}$ by $\max\limits_x \dim \mathrm{T}_{x,X}$. This is because in this case $X^G \neq \varnothing$ for every finite p-subgroup G of $\mathrm{Aut}(X)$, see [Se$_3$, Sect. 7–8].

Theorem 2.26. *For every variety X, there is an integer m_X such that any finite subgroup G of any connected linear algebraic subgroup L of* $\mathrm{Aut}(X)$ *contains an abelian normal subgroup whose index in G is at most m_X.*

Proof. Being reductive, G is contained in a maximal reductive subgroup R of L. Then R is a Levi subgroup, i.e., L is a semidirect product of R and $\mathrm{Rad}_u L$, cf., e.g., [OV, Chap. 6]. As L is connected, R is connected as well. Since the kernel of the action of R on X is trivial, $\mathrm{rk}\, R \leqslant \dim X$, see [P$_2$, §3]. The claim then follows from Theorem 2.4 as there are only finitely many connected reductive groups of rank at most $\dim X$. □

2.2. **Generalizations.** One may ask whether "affine" in Question 2.11 can be dropped:

Question 2.27. Is there an irreducible variety X such that $\mathrm{Aut}(X)$ is not Jordan?

The negative answer to Question 2.27 would follow from that to

Question 2.28. Is there an irreducible variety X such that $\mathrm{Bir}(X)$ is not Jordan?

If X is a curve, then the answer to Question 2.28 is negative.

Indeed, we may assume that X is smooth and projective. Then $\mathrm{Bir}(X) = \mathrm{Aut}(X)$.

If $g(X)$, the genus of X, is 0, then $X = \mathbf{P}^1$, hence $\mathrm{Bir}(X) = \mathbf{PGL}_2(k)$, so $\mathrm{Bir}(X)$ is Jordan by Theorem 2.4.

If $g(X) = 1$, then X is an elliptic curve; whence $\mathrm{Bir}(X)$ is the extension of a finite group by the abelian algebraic group X, hence Jordan by Lemma 2.9.

If $g(X) \geqslant 2$, then $\mathrm{Bir}(X)$ is finite, hence Jordan.

Note that all curves (not necessarily smooth and projective) with infinite automorphism group are classified in [P_3].

Answering Question 2.28 for surfaces X, one may assume that X is a smooth projective minimal model.

If X is of general type, then by Matsumura's theorem $\mathrm{Bir}(X)$ is finite, hence Jordan.

If X is rational, then $\mathrm{Bir}(X)$ is the Cremona group of rank 2 over k, hence Jordan by [Se_1, Theorem 5.3], [Se_2, Théorème 3.1].

If X is a nonrational ruled surface, it is birationally isomorphic to $\mathbf{P}^1 \times B$ where B is a smooth projective curve such that $g(B) > 0$; we may then take $X = \mathbf{P}^1 \times B$. As $g(B) > 0$, there are no dominant rational maps $\mathbf{P}^1 \to B$, hence the elements of $\mathrm{Bir}(X)$ permute the fibers of the natural projection $\mathbf{P}^1 \times B \to B$. The set of elements inducing trivial permutation is a normal subgroup $\mathrm{Bir}_B(X)$ of $\mathrm{Bir}(X)$. The definition implies that $\mathrm{Bir}_B(X) = \mathbf{PGL}_2(k(B))$, hence Jordan by Theorem 2.4. Naturally identifying $\mathrm{Aut}(B)$ with the subgroup of $\mathrm{Bir}(X)$, we get the decomposition $\mathrm{Bir}(X) = \mathrm{Bir}_B(X) \rtimes \mathrm{Aut}(B)$. Note that $\mathrm{Aut}(X) = \mathbf{PGL}_2(k) \times \mathrm{Aut}(B) \neq \mathrm{Bir}(X)$ (see [M, pp. 98–99]), so $\mathrm{Aut}(X)$ is Jordan by Lemma 2.6. Let $g(B) \geqslant 2$. Then $\mathrm{Aut}(B)$ is finite, hence $[\mathrm{Bir}(X) : \mathrm{Bir}_B(X)] < \infty$. Lemma 2.9 then implies that $\mathrm{Bir}(X)$ is Jordan. For $g(B) = 1$, this argument does not work as B is an elliptic curve, so $\mathrm{Aut}(B)$ is infinite.

The canonical class of all other surfaces X is numerically effective, so, for them, $\mathrm{Bir}(X) = \mathrm{Aut}(X)$ (cf. [IS, Sect. 7.3, Theorem 2].

If X is an abelian surface, then $\mathrm{Bir}(X)$ is an extension of a subgroup of $\mathbf{GL}_4(\mathbf{Z})$ by the abelian algebraic group X, hence Jordan by Lemma 2.9 and Example 2.8(2).

If X is a bielliptic surface, then X is the quotient of an abelian surface $\widetilde{X}$ by a finite automorphism group F and $\mathrm{Aut}(X)$ is isomorphic to $\mathrm{Aut}(\widetilde{X})/F$. Since $\mathrm{Aut}(\widetilde{X})$ is Jordan, Lemma 2.5 then yields that $\mathrm{Aut}(X)$ is Jordan.

In the other cases let K be the kernel of the natural action of $\mathrm{Bir}(X)$ on $H^2(X, \mathbf{Q})$ (we may assume that $k = \mathbf{C}$) and let D be the image of $\mathrm{Bir}(X)$ in $\mathbf{GL}(H^2(X, \mathbf{Q}))$ defined by this action. Since $\mathrm{Bir}(X)/K$ is isomorphic to D and D is bounded (see Example 2.8(2)), Lemma 2.9 implies that $\mathrm{Bir}(X)$ is Jordan if K is Jordan (and $\mathrm{Bir}(X)$ is bounded if K is bounded, see Example 2.8(3)). In many cases this yields that $\mathrm{Bir}(X)$ is Jordan. Namely:

- If X is a $K3$-surface, then K is trivial (cf. [IS, Sect. 12.4]); whence $\mathrm{Bir}(X)$ is bounded.
- If X is an Enriques surface, then K is finite by [MM]; whence $\mathrm{Bir}(X)$ is bounded.
- For elliptic surfaces X of Kodaira dimension one, there is an extensive information on the cases when K is finite (hence $\mathrm{Bir}(X)$ is bounded), see [C] and references therein. For instance, by [L] if $H^0(X, \mathcal{T}_X) = 0$ (i.e., X admits no nonzero regular vector fields), then K is finite.

So to answer Question 2.28 for surfaces it only remains to consider the following two types of elliptic surfaces that are not covered by the above considerations:

(a) $X = \mathbf{P}^1 \times B$ where B is an elliptic curve;

(b) elliptic surfaces X of Kodaira dimension one for which the kernel of the natural action of $\mathrm{Bir}(X)$ on $H^2(X, \mathbf{Q})$ is infinite (note that every such X admits a nonzero regular vector field).

It looks plausible that $\mathrm{Bir}(X)$ is Jordan for both types (a) and (b). If this is indeed so, this would complete the proof of the following

Conjecture 2.29. $\mathrm{Bir}(X)$ *is Jordan for every surface* X.

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