ON THE ESSENTIAL DIMENSION OF INFINITESIMAL GROUP SCHEMES

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ABSTRACT. We discuss essential dimension of group schemes, with particular attention to infinitesimal group schemes. We prove that the essential dimension of a group scheme of finite type over a field k is at least equal to the difference between the dimension of its Lie algebra and its dimension. Furthermore, we show that the essential dimension of a trigonalizable group scheme of length p^n over a field of characteristic $p > 0$ is at most n. We give several examples.

1. INTRODUCTION

The notion of essential dimension of a finite group over a field k was introduced by Buhler and Reichstein ([BR97]). It was later extended to various contexts. First Reichstein generalized it to linear algebraic groups ([Rei00]) in characteristic zero; afterwards Merkurjev gave a general definition for covariant functors from the category of extension fields of the base field k to the category of sets ([BF03]). Brosnan, Reichstein and Vistoli ([BRV09a]) studied the essential dimension of an algebraic stack, a general class which includes almost all the examples of interest.

Important results on the essential dimension of finite groups in characteristic 0 have been proved by Florence ([Flo07]) and Karpenko and Merkurjev ([KM08]). The essential dimension of finite groups in positive characteristic has been studied by Ledet [Led04]. As to higher dimensional groups, there are works of Reichstein and Youssin ([RY00]), Chernousov and Serre ([CS06]), Gille and Reichstein [GR09], Brosnan, Reichstein and Vistoli [BRV09b] on algebraic groups, Brosnan [Bro07] on abelian varieties over $\mathbb{C}$, and Brosnan and Shreekantan [BS08] on abelian varieties over number fields.

The case of non-smooth group schemes (necessarily in positive characteristic) has not been investigated. It is certainly known to experts that the essential dimensions of $\boldsymbol{\alpha}_{p^m}$ and $\boldsymbol{\mu}_{p^m}$ over a field of characteristic $p > 0$ are 1, and that the essential dimension of $\boldsymbol{\mu}_{p^m}^n$ is n. To the authors' knowledge nothing else was known. The purpose of this paper is to throw some light on this subject. In particular we will focus on the essential dimension of infinitesimal (i.e. connected and finite) group schemes. This could give, for instance, some information about the essential dimension of supersingular Abelian varieties in characteristic $p > 0$ (an Abelian varieity is *supersingular* if its p-torsion subgroup scheme is infinitesimal). If an Abelian variety A is not supersingular, its essential dimension is conjecturally infinite, since for any $n > 0$ the group scheme A contains, over an algebraic closure of k, the group scheme $\mathbb{Z}/p^n\mathbb{Z}$, whose essential dimension is conjecturally equal to n.

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If G is a group scheme of finite type over a field k, its essential dimension $\operatorname{ed}_k G$ is the essential dimension of the stack $\mathcal{B}_k G$. Let us recall the definition. Let k be a field, and G be a group scheme of finite type over k. If X is a k-scheme, a G-torsor is a k-scheme P with a right action of G, with a G-invariant morphism $P \to X$, such that fppf locally on X the scheme P is G-equivariantly isomorphic to $X \times_{\operatorname{Spec} k} G$. Isomorphism classes of G-torsors on X form a pointed set $\mathrm{H}^1(X, G)$; if G is commutative, then $\mathrm{H}^1(X, G)$ is a group, and coincides with the cohomology group of G in the fppf topology.

Definition 1.1. Let G be a group scheme of finite type over a field k. Let $k \subseteq K$ be an extension field and $[\xi] \in \mathrm{H}^1(X, \operatorname{Spec}(K))$ the class of a G-torsor ξ. Then the essential dimension of ξ over k, which we denote by $\operatorname{ed}_k(\xi)$, is the smallest nonnegative integer n such that there exists a subfield K' of K containing k, with $\operatorname{tr.d.}(K'/k) \leq n$ such that $[\xi]$ is in the image of the morphism $\mathrm{H}^1(X, \operatorname{Spec}(K')) \to \mathrm{H}^1(X, \operatorname{Spec}(K))$.

The essential dimension of G over k, which we denote by $\operatorname{ed}_k(G)$, is the supremum of $\operatorname{ed}_k(\xi)$, where K/k ranges through all the extension of K, and ξ ranges through all the G-torsors over $\operatorname{Spec}(K)$.

If G is smooth, then G-torsors are locally trivial in the étale topology, and our definition coincides with that of Berhuy and Favi ([BF03]); in the general case, to get meaningful results one needs to use the fppf topology. For example, if G is an infinitesimal group scheme, the G-torsors over a reduced scheme that are locally trivial in the étale topology are in fact trivial.

Our first result is the determination of a general lower bound for the essential dimension.

Theorem 1.2. *Let G be a group scheme of finite type over a field k. Then*

$$\operatorname{ed}_k G \geq \dim_k \operatorname{Lie} G - \dim G\,.$$

Considering how hard it is in general to prove lower bounds for the essential dimension, it is a little surprising that the proof of Theorem 1.2 is little more than an observation.

We also have a general upper bound. We firstly recall the definition of a trigonalizable group scheme.

Definition 1.3. Let G be an affine group scheme of finite type over a field k. G is called trigonalizable if it has a normal unipotent subgroup U scheme such that G/U is diagonalizable.

The name is justified by the fact that, if G is trigonalizable and of finite type over k, then there exists a monomorphism of G in some trigonal group. Any affine commutative group scheme over an algebraically closed field is trigonalizable (see [DG70, IV, §3, 1.1]).

Theorem 1.4. *Let G be a finite trigonalizable group scheme, over a field of characteristic $p > 0$, of order p^n. Then $\operatorname{ed}_k G \leq n$.*

For constant p-group schemes (which are unipotent) the above result has already been proved by Ledet ([Led04]).

The second and the third sections are devoted to the proofs of these two theorems. In the last section, combining the lower and upper bounds above, we calculate the essential dimension of some classes of infinitesimal group schemes. In particular we prove that the essential dimension of a trigonalizable group scheme of height ≤ 1, i.e. such that the Frobenius F is trivial on it, is equal to the dimension of its Lie algebra (Corollary 4.3). We do not have examples of infinitesimal group schemes with essential dimension strictly bigger than the dimension of its Lie algebra. For not trigonalizable group schemes such an example should probably exist: for example, there should be twisted forms of $\boldsymbol{\mu}_p$ whose essential dimension is larger than 1. Over an algebraically closed field, the issue is less clear. In Example 4.8 we propose a class of a commutative unipotent group scheme whose essential dimensions we are unable to determine, which should be an important test case to determine whether it is reasonable to conjecture that equality hold for trigonalizable group schemes.

2. The proof of Theorem 1.2

We first state a well known lemma.

Lemma 2.1. *If G is a subgroup scheme of a group scheme H, then* $\operatorname{ed}_k G + \dim G \leq \operatorname{ed}_k H + \dim H$.

Proof. This follows from [BRV09a, 3.2] applied to the stack morphism $\mathcal{B}_k G \to \mathcal{B}_k H$. ♠

We now prove the Theorem. If the characteristic of k is 0, then G is smooth and there is nothing to prove. Suppose that the characteristic of k is $p > 0$. Since the essential dimension does not increase after a base change ([BF03, 1.5]), we may assume that k is algebraically closed.

Let G_1 be the kernel of the Frobenius map $G \to G$ defined at the level of algebras by $a \mapsto a^p$; then G_1 is an infinitesimal group scheme, and $\operatorname{Lie} G_1 = \operatorname{Lie} G$. Since by 2.1 we have $\operatorname{ed}_k G_1 \leq \operatorname{ed}_k G - \dim G$, it is sufficient to show that $\operatorname{ed}_k G_1 \geq \dim_k \operatorname{Lie} G_1$; in other words, we may assume that $G = G_1$, i.e., by definition, G has height at most 1. This implies that $k[G]$ is isomorphic to $k[x_1, \dots, x_n]/(x_1^p, \dots, x_n^p)$ as a k-algebra ([DG70, II §7, 4.2]).

Let G act freely on an open subscheme X of a representation of G. If K is the function field of the quotient X/G and E is the function field of X, then we have a generic G-torsor $\operatorname{Spec} E \to \operatorname{Spec} K$. Set $n \stackrel{\text{def}}{=} \dim_k \operatorname{Lie} G$. Suppose that the G-torsor is defined over an extension L of k contained in K; we need to show that the transcendence degree of L over k is at least n. Let $\operatorname{Spec} R \to \operatorname{Spec} L$ the G-torsor yielding $\operatorname{Spec} E \to \operatorname{Spec} K$ by base change. Clearly R is a field. We claim that $R \subseteq L^{1/p}$; in other words, the p^{th}-power homomorphism $R \to R$ has its image contained in $L \subseteq R$.

In order to prove this we may base change to an algebraic closure $\overline{K}$ of K; then $\overline{K} \otimes_K E$ is isomorphic as a $\overline{K}$-algebra to the Hopf algebra $\overline{K}[G]$. Since $\overline{K}[G]$ is isomorphic to $\overline{K}[x_1, \dots, x_n]/(x_1^p, \dots, x_n^p)$ as a $\overline{K}$-algebra, it is clearly true that the Frobenius map carries $\overline{K} \otimes_K E$ into $\overline{K}$; hence $R \subseteq L^{1/p}$, as claimed. Since the degree $[R : L]$ is p^n, the result follows from the following well known fact.

Lemma 2.2. *Let L be a finitely generated extension of transcendence degree d of a perfect field k of characteristic $p > 0$. Then $[L^{1/p} : L] = p^d$.*

3. The proof of Theorem 1.4

Let us start with stating a few Lemmas that we will use in the proof. The first two are well known.

Lemma 3.1. [DG70, IV §2, 2.5] *Let G be a commutative unipotent group scheme over a field K. Then there exists a central decomposition series*

$$1 = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_r = G$$

of G, such that each successive quotient G_i/G_{i-1} is a subgroup scheme of $\mathbb{G}_{\mathrm{a}}$.

Lemma 3.2. [DG70, IV §2, 2.6] *If G is a group scheme over a field K and E is an extension of K, then $\operatorname{Spec} E \times_{\operatorname{Spec} K} G \to \operatorname{Spec} E$ is unipotent if and only if G is unipotent. In particular, any twisted form of a unipotent group scheme is unipotent.*

Lemma 3.3. *Let G be a commutative unipotent group scheme over a field K. Then $\mathrm{H}^i(K, G) = 0$, for $i \geq 2$.*

Proof. By Lemma 3.1, we may assume that G is a subgroup of $\mathbb{G}_{\mathrm{a}}$. The quotient $\mathbb{G}_{\mathrm{a}}/G$ is isomorphic to $\mathbb{G}_{\mathrm{a}}$ ([DG70, IV §2, 1.1]); then the result follows from the fact that $\mathrm{H}^i(K, \mathbb{G}_{\mathrm{a}}) = 0$ for $i \geq 1$. ♠

We here prove the key Lemma.

Lemma 3.4. *Suppose that we have an extension*

$$1 \longrightarrow G_1 \longrightarrow G \longrightarrow G_2 \longrightarrow 1$$

of group schemes over k, where G_1 is a commutative unipotent normal group subscheme of a group scheme G. Let $P \to \operatorname{Spec} K$ be a G-torsor over an extension K of k. Then there exists an intermediate extension $k \subseteq E \subseteq K$ and a twisted form $\widetilde{G_1} \to \operatorname{Spec} E$ of G_1 over E, such that P is defined over an intermediate extension of transcendence degree at most $\operatorname{ed}_k G_2 + \operatorname{ed}_E \widetilde{G_1}$ over k.

Furthermore, if G_1 is central in G, then $\widetilde{G_1} = \operatorname{Spec} E \times_{\operatorname{Spec} k} G_1$.

Proof. Consider the induced G_2-torsor $Q \overset{\text{def}}{=} P/G_1 \to \operatorname{Spec} K$. There exists an intermediate extension $k \subseteq E \subseteq K$ with $\operatorname{tr\,deg}_k E \leq \operatorname{ed} G_2$, such that Q comes by base change from a G_2-torsor $Q_E \to \operatorname{Spec} E$. I claim that Q_E lifts to a G-torsor $P_E \to \operatorname{Spec} E$.

To see this, consider the fppf gerbe of liftings $\mathcal{L} \to (\mathrm{Sch}/E)$. It is a fibered category over the category (Sch/E) of E-schemes, whose objects over an E-scheme $T \to \operatorname{Spec} E$ are G-torsors $P_T \to T$, together with isomorphisms of G_2-torsors $P_T/G_1 \simeq T \times_{\operatorname{Spec} E} Q_E$, or, equivalently, G-equivariant morphisms of T-schemes $P_T \to T \times_{\operatorname{Spec} E} Q_E$. The arrows from $P_T \to T \times_{\operatorname{Spec} E} Q_E$ to $P'_{T'} \to T' \times_{\operatorname{Spec} E} Q_E$ are defined in the obvious way, as diagrams

$$\begin{array}{ccccc} P_T & \longrightarrow & T \times_{\operatorname{Spec} E} Q_E & \xrightarrow{\mathrm{pr}_1} & T \\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle f \times \mathrm{id}} & & \downarrow{\scriptstyle f} \\ P_{T'} & \longrightarrow & T' \times_{\operatorname{Spec} E} Q_E & \xrightarrow{\mathrm{pr}_1} & T' \end{array}$$

in which F is G-equivariant. We need to show that $\mathcal{L}$ has a global section over $\operatorname{Spec} E$.

The action of G on G_1 by conjugation descends to an action of G_2 on G_1, since G_1 is commutative. Denote by $\widetilde{G_1}$ the twisted form of the group scheme $\operatorname{Spec} E \times_{\operatorname{Spec} k} G_1$ coming from the G_2-torsor $Q_E \to \operatorname{Spec} E$; in other words, $\widetilde{G_1}$ is the quotient $(Q_E \times_{\operatorname{Spec} k} G_1)/G_2$, where G_2 acts on the right on Q_E, and on G_1 by right conjugation. We claim that the gerbe $\mathcal{L}$ is banded by $\widetilde{G_1}$; that is, if $P_T \to T \times_{\operatorname{Spec} E} Q_E$ is an object of $\mathcal{L}(T)$, the automorphism group $\operatorname{Aut}(P_T)$ is isomorphic to $\widetilde{G_1}(T)$, and this isomorphism is functorial in T. In fact, the twisted form $\widetilde{G}$ of G obtained as the quotient $P \times_{\operatorname{Spec} k} G$ by the action of G on conjugation on G is the automorphism group scheme of the G-torsor $P_T \to T$, and it contains $\widetilde{G_1}$ as the subgroup scheme of automorphisms inducing the identity on $T \times_{\operatorname{Spec} E} Q_E$. Hence $\widetilde{G_1}$ is the automorphism group scheme of the object $P_T \to T \times_{\operatorname{Spec} E} Q_E$, and this proves the claim.

By [Gir71, IV 3.4], the equivalence classes of gerbes banded by $\widetilde{G_1}$ are parametrized by the group $\mathrm{H}^2(K, \widetilde{G_1})$; a gerbe corresponds to 0, i.e., it is equivalent to the classifying stack $\mathcal{B}_E\widetilde{G_1}$, if and only if it has a section. Now, by Lemma 3.2, $\widetilde{G_1}$ is unipotent; hence by Lemma 3.3 $\mathrm{H}^2(E, \widetilde{G_1}) = 0$, so $\mathcal{L}$ has a section, and the G_2-torsor $Q_E \to \operatorname{Spec} E$ lifts to a G-torsor $P_E \to Q_E \to \operatorname{Spec} E$.

There is no reason why $\operatorname{Spec} K \times_{\operatorname{Spec} E} P_E$ should be isomorphic to $P \to \operatorname{Spec} K$ as a G-torsor. However, by construction, we have $P/G_1 \simeq \operatorname{Spec} K \times_{\operatorname{Spec} E} Q_E$. Since as we just saw $\mathcal{L}$ is a trivial gerbe banded by $\widetilde{G_1}$, we have that $\mathcal{L}$ is equivalent to the classifying stack $\mathcal{B}_E\widetilde{G_1}$, in such a way that the lifting $P_E \to Q_E$ corresponds to the trivial torsor $\widetilde{G_1} \to \operatorname{Spec} E$. Then $P \to Q$ gives an object of $\mathcal{L}(\operatorname{Spec} K)$; this will be defined over a intermediate extension $E \subseteq F \subseteq K$ of transcendence degree at most $\operatorname{ed}_E \widetilde{G_1}$ over E. So over F there will exist an object $P_F \to \operatorname{Spec} L \times_{\operatorname{Spec} E} Q_E$ which is isomorphic to $P \to Q$ when pulled back to K. Hence P is defined over F, and, since the transcendence degree of F is at most equal to $\operatorname{ed}_k G_2 + \operatorname{ed}_E \widetilde{G_1}$, the result follows. ♠

Now we are ready to prove the theorem.

First of all, suppose that G is a diagonalizable group of order p^n; then G is a product $\boldsymbol{\mu}_{p^{d_1}} \times \cdots \times \boldsymbol{\mu}_{p^{d_r}}$ for certain positive integers $d_1, \ldots, d_r$ with $d_1 + \cdots + d_r = n$. Then G is a subgroup scheme of $\mathbb{G}_{\mathrm{m}}^r$, hence by the Lemma 2.1 we have $\operatorname{ed} G \leq r \leq n$.

Now, assume that G is commutative unipotent of order p^n, with $n > 0$. Suppose that G is a subgroup of $\mathbb{G}_{\mathrm{a}}$; then again by Lemma 2.1 we have $\operatorname{ed}_k G \leq \dim \mathbb{G}_{\mathrm{a}} = 1 \leq n$, and we are done.

If it is not a subgroup scheme of $\mathbb{G}_{\mathrm{a}}$, we proceed by induction on n. Assume that the result holds for all commutative unipotent subgroup schemes of order p^m with $m < n$. Let G_1 be a nontrivial subgroup scheme that is a subgroup scheme of $\mathbb{G}_{\mathrm{a}}$ and call p^m its order. The group scheme G_1 exists by 3.1. Then by Lemma 3.4 we have

$$\operatorname{ed}_k G \leq \operatorname{ed}_k(G/G_1) + \operatorname{ed}_E G_{1E} \leq (n - m) + m = n;$$

so the result holds for G.

Let G be a trigonalizable infinitesimal group scheme of order p^n. Once again, we proceed by induction on n. Let us suppose that the result is true for all trigonalizable groups of order p^m with $m < n$. By definition, G is an extension

$$1 \longrightarrow G_{\mathrm{u}} \longrightarrow G \longrightarrow G_{\mathrm{d}} \longrightarrow 1\,,$$

where G_{u} is unipotent and G_{d} is diagonalizable. We may assume that G_{u} is nontrivial, otherwise G is diagonalizable and we are done. If G_1 denotes the center of G_{u}, then G_1 is a nontrivial commutative unipotent normal subgroup of G; set $G_2 \overset{\mathrm{def}}{=} G/G_1$. Call p^m the order of G_2; by induction hypothesis we have $\mathrm{ed}_k\, G_2 \leq m$. Once again using Lemma 3.4, we have that $\mathrm{ed}_k\, G \leq \mathrm{ed}_k\, G_2 + \mathrm{ed}_E\, \widetilde{G_1}$ for some twisted form of G_1; but by Lemma 3.2 the group scheme $\widetilde{G_1}$ is still commutative unipotent, hence by the previous case $\mathrm{ed}_E\, \widetilde{G_1} \leq n - m$, and we are done.

4. The essential dimension of some group schemes

The two Theorems above give rise to two natural questions.

- For which group schemes G of finite type over a field k does the equality

$$\mathrm{ed}_k(G) = \dim_k(\mathrm{Lie}(G)) - \dim G \tag{4.1}$$

  hold?
- Is it true that for a finite group scheme of order p^n over a field of characteristic $p > 0$ we have $\mathrm{ed}_k(G) \leq n$?

It is clear that if G is a finite group scheme of order p^n over a field of characteristic $p > 0$ that satisfies (4.1), then $\mathrm{ed}_k\, G \leq n$.

The class of smooth group schemes satisfying the first equality is well known, it is that of the so-called *special groups*. This were introduced by J.P. Serre in [Ser58] and studied by Grothendieck in [Gro58], and by many other authors since then.

For singular groups the question becomes substantially harder. The observation above about smooth group schemes motivates the following definition.

Definition 4.1. Let G be a group scheme of finite type over a field k. If

$$\mathrm{ed}_k(G) = \dim_k(\mathrm{Lie}(G)) - \dim G$$

then G is called *almost special*.

In the following we give several examples of almost special group schemes. Most of them are connected and trigonalizable. We do not know whether all connected trigonalizable group scheme are almost special. For diagonalizable group scheme this is true (see Example 4.2). In the unipotent case this is open: in 4.9 we discuss what we consider to be a key example to analyze to clarify this question.

It is easy to give examples of infinitesimal almost special group schemes that are not trigonalizable: for example, if $\mathcal{G}$ is a non-trigonalizable smooth group scheme (e.g. $\mathcal{G} = \mathrm{Gl}_n$), the kernel of its Frobenius ${}_{\mathrm{F}}\mathcal{G}$ is almost special, as it follows from Theorem 1.2 and Lemma 2.1.

Finally we remark that if G and H are almost special then the product $G \times H$ is also almost special, and $\mathrm{ed}_k(G \times H) = \mathrm{ed}_k(G) + \mathrm{ed}_k(H)$.

About the second question above, our guess is that the upper bound does not hold in general. An example could be given by some twisted form of $\boldsymbol{\mu}_p$; however, we are unable to prove this, as, unlike what happens for smooth group schemes, a group scheme of essential dimension 1 is not necessarily contained in PGL_2.

The following simple strategy lets us to find some almost special group schemes. If an infinitesimal group scheme G can be embedded in a special smooth algebraic group scheme of dimension $n \overset{\mathrm{def}}{=} \dim_k \mathrm{Lie}(G)$ then, using Theorem 1.2 and Lemma 2.1, we can conclude that the essential dimension is exactly n, i.e. G is almost special. Here is an example when this happens. Later in 4.8 we will see another example in which this argument can not be applied.

Example 4.2. If n is a positive integer, we denote by W_n the group scheme of truncated Witt vectors of length n (see [Ser79, Chapter 2, § 6], [Haz78, Chapter 3, § 1] and [DG70, Chapter 5, § 1]). Let G be the group scheme

$$\prod_{j=1}^{t} {}_{\mathrm{F}^{m_j}}W_{n_j} \times \prod_{i=1}^{s} \boldsymbol{\mu}_{p^{l_i}},$$

where ${}_{\mathrm{F}^m}W_n$ is the kernel of the iterated Frobenius $\mathrm{F}^m : W_n \to W_n$. Then G is almost special, i.e.

$$\mathrm{ed}_k G = s + \sum_{j=1}^{t} n_j = \dim_k \mathrm{Lie}\, G. \tag{4.2}$$

Indeed, by the remark above, it is enough to remark that G is a closed subgroup of $\prod_{j=1}^{t} W_{n_j} \times \mathbb{G}_m^s$, which is special.

Another example in which the equality (4.1) holds is given by the following Corollary.

Corollary 4.3. *Any trigonalizable group scheme G of height ≤ 1 and of finite type over k is almost special.*

Proof. If G has order p^n then, since G has height ≤ 1, the dimension of its Lie algebra is n. Therefore the result follows from Theorems 1.2 and 1.4. ♠

As a consequence of Theorem 1.4 and Lemma 3.4 we prove the following result.

Corollary 4.4. *Let*

$$1 \longrightarrow G_1 \longrightarrow G \longrightarrow G_2 \longrightarrow 1$$

be an extension of group schemes over a field k of characteristic $p > 0$, with G_1 unipotent commutative of order p^n, then

$$\mathrm{ed}_k(G_1) \leq n + \mathrm{ed}_k(G_2).$$

Remark 4.5. In particular one can apply the corollary under the further hypothesis $\mathrm{ed}_k(G_1) = n$. In this situation we have the interesting upper bound

$$\mathrm{ed}_k(G) \leq \mathrm{ed}_k(G_1) + \mathrm{ed}_k(G_2).$$

Later we will see some cases in which equality holds.

By Corollary 4.3, the hypothesis $\mathrm{ed}_k(G_1) = n$ is satisfied if G_1 is commutative unipotent of height at most 1. If k is a perfect field, any unipotent commutative k-group scheme of height at most 1 is isomorphic to a finite direct product of ${}_{\mathrm{F}}W_n$, $n \in \mathbb{N}$, where ${}_{\mathrm{F}}W_n$ is the kernel of the Frobenius $\mathrm{F} : W_n \to W_n$ (see [DG70, IV, §2, 2.14]).

This hypothesis is also satisfied when G_1 is a twisted form of $\mathbb{Z}/p^n\mathbb{Z}$ with $n \leq 2$ and conjecturally for any n (see Example 4.9).

Proof. Let $P \to \mathrm{Spec}\, K$ be a G-torsor over an extension K of k. Then, from Lemma 3.4, there exists an intermediate extension $k \subseteq E \subseteq K$ and a twisted form $\widetilde{G_1} \to \mathrm{Spec}\, E$ of G_1 over E, such that

$$\mathrm{ed}_k P \leq \mathrm{ed}_E(\widetilde{G_1}) + \mathrm{ed}_k(G_2).$$

By Lemma 3.2, $\widetilde{G_1}$ is unipotent. Hence by Theorem 1.4 we have $\mathrm{ed}_E(\widetilde{G_1}) \leq n$, and the result follows. ♠

Example 4.6. In addition to the hypotheses of the Corollary 4.4, let us suppose that G_1 and G_2 are almost special and

$$\dim_k \mathrm{Lie}(G) = \dim_k \mathrm{Lie}(G_1) + \dim_k \mathrm{Lie}(G_2).$$

In particular G_1 must be of height ≤ 1. From Corollary 4.4 and Theorem 1.2 it follows that

$$\mathrm{ed}_k(G) = \mathrm{ed}_k(G_1) + \mathrm{ed}_k(G_2).$$

The hypothesis on the dimension of the Lie algebra of G is satisfied, for instance, if the extension is split.

So, for example, if $G_1$is a commutative unipotent group scheme of height ≤ 1 (hence almost special) and G_2 is almost special then the group scheme $G = G_1 \rtimes G_2$ is almost special and its essential dimension is $\mathrm{ed}_k(G_1) + \mathrm{ed}_k(G_2)$. This could be applied for example when G_1 is a commutative unipotent group scheme of height ≤ 1 and G_2 is one of the finite group schemes of the Example 4.2.

Example 4.7. The above example lets us to compute, in particular, the essential dimension of $\boldsymbol{\alpha}_p \rtimes \boldsymbol{\mu}_{p^n}$, which is 2. But the argument does not apply to $\boldsymbol{\alpha}_{p^m} \rtimes \boldsymbol{\mu}_{p^n}$, for $m > 1$. However in this case the essential dimension is also 2. From the Theorem 1.2 it is at least 2 and we now prove it is exactly 2. One observes (quite easily) that for any field K of positive characteristic the automorphism group scheme of $\boldsymbol{\alpha}_{p^n}$ is smooth and trigonalizable. In particular this is true for the field $\mathbb{F}_p$. Hence we have that $\mathrm{Aut}_{\mathbb{F}_p-gr}(\boldsymbol{\alpha}_{p^n})$ is special. Indeed it is an extension of a smooth unipotent group scheme by a smooth diagonalizable group scheme and both are special. For a smooth diagonalizable group scheme it is immediate to see that it is special while for an unipotent smooth group scheme it is enough to observe that, since $\mathbb{F}_p$ is perfect, it has a central decomposition series with quotients isomorphic to $\mathbb{G}_{\mathrm{a}}$ (see [DG70, IV, §2 3.9]). A fortiori $\mathrm{Aut}_{K-gr}(\boldsymbol{\alpha}_{p^n})$ is almost special for any field K of characteristic $p > 0$, since the essential dimension does not increase by base change (see ([BF03, 1.5])). This implies that there are no non trivial twisted forms of $\boldsymbol{\alpha}_{p^n}$. The claim now follows from the Lemma 3.4, since $\mathrm{ed}_k(\boldsymbol{\alpha}_{p^n}) = 1$. With the same argument one obtains, if G is an almost special group scheme G acting on $\boldsymbol{\alpha}_{p^r}$, that $\boldsymbol{\alpha}_{p^r} \rtimes G$ is almost special, i.e.

$$\mathrm{ed}_k(\boldsymbol{\alpha}_{p^r} \rtimes G) = \mathrm{ed}_k G + 1.$$

Example 4.8. We do not have examples of infinitesimal group schemes that are not almost special. Some such group schemes could be maybe found among the twisted forms of diagonalizable group schemes, i.e. group schemes of multiplicative type. If such an example exists among trigonalizable group schemes, a candidate should be the following.

Let us consider the kernel $G_{r,m,n}$ of the morphism $\mathrm{F}^m + V^r : \mathrm{W}_n \to \mathrm{W}_n$, where V is the Verschiebung (called "shift" in [Ser79] and "decalage" in [DG70]), $m > 0$ and $0 < r < n$. This is a group scheme of order p^{mn}. The dimension of its Lie algebra is r, and it it is embedded in a special group scheme of dimension n, therefore $r \leq \mathrm{ed}_k(G_{r,m,n}) \leq n$.

Now consider the case $r = 1$. Then we will prove that G can not be embedded in any special algebraic group scheme $\mathcal{G}$ of dimension 1, so we can not use an argument as in the Example 4.2 to conclude that $\mathrm{ed}_k(G_{1,m,n}) = 1$. To prove this fact we can clearly suppose that k is algebraically closed. Now let us suppose that such a group $\mathcal{G}$ exists. First of all we can suppose it is connected. Secondly we observe that it should be unipotent. Indeed, since it is special, it should be affine ([Ser58, Théorème 1]) and smooth (see Theorem 1.2); therefore, if it was not unipotent, from [DG70, IV §3.11] we conclude that it contains a subgroup scheme isomorphic to $\mathbb{G}_m$. This implies that the kernel of Frobenius, ${}_{\mathrm{F}}\mathcal{G}$, is not unipotent, since it contains a subgroup scheme isomorphic to $\boldsymbol{\mu}_p$. But since the Lie algebra of $G_{1,m,n}$ is equal to the Lie algebra of $\mathcal{G}$ we have that ${}_{\mathrm{F}}\mathcal{G} = {}_{\mathrm{F}}G_{1,m,n}$, which is a contradiction since $G_{1,m,n}$ is unipotent. But any unipotent smooth and connected group scheme of dimension 1 over a perfect field is isomorphic to $\mathbb{G}_a$ ([DG70, IV, §2, 2.10]). And $G_{1,m,n}$ is not a subgroup scheme of $\mathbb{G}_a$.

This example is also interesting since, if k is perfect, the group scheme $G_{1,1,2}$ is the p-torsion group scheme of a supersingular elliptic curve. The p^n-torsion group scheme of the same curve has a decomposition series with quotients isomorphic to $G_{1,1,2}$

Example 4.9. It is conjectured that the essential dimension of $\mathbb{Z}/p^n\mathbb{Z}$ over a field k of characteristic p is n. The conjecture has been proved (easily) for $n \leq 2$. Since $\mathbb{Z}/p^n\mathbb{Z}$ is contained in the group scheme of Witt vectors of dimension n, W_n, and this group is special then $\mathrm{ed}_k(\mathbb{Z}/p^n\mathbb{Z}) \leq n$. The same result also follows from Theorem 1.4, which for constant p-group schemes was already known. The open problem (for $n > 2$) is to prove the opposite inequality. We remark that if this conjecture is true over an algebraic closure $\bar{k}$ of k then $\mathrm{ed}_k(G)$ is equal to n for any twisted form G of $\mathbb{Z}/p^n\mathbb{Z}$. Indeed such a group scheme would be unipotent (see Lemma 3.1) so, by the Theorem 1.4, its essential dimension is smaller or equal to n. On the other hand it is at least n, since the essential dimension does not increase by base change (see [BF03, 1.5]).

Even if the essential dimension of a cyclic constant p-group is not known it is possible to determine the essential dimension of group schemes with $\mathbb{Z}/p^n\mathbb{Z}$ as direct summand. Let G be an infinitesimal group scheme with essential dimension n and contained in W_n. Then

$$\mathrm{ed}_k(G \times \mathbb{Z}/p^m\mathbb{Z}) = n$$

if $m \leq n$. First $\mathbb{Z}/p^m\mathbb{Z}$ is a subgroup of $\mathbb{Z}/p^n\mathbb{Z}$ and this one is the kernel of $\mathrm{F} - 1 : \mathrm{W}_n \to \mathrm{W}_n$. Moreover since G is infinitesimal and $\mathbb{Z}/p^n\mathbb{Z}$ is étale the morphism, induced by multiplication inside W_n,

$$G \times \mathbb{Z}/p^m\mathbb{Z} \longrightarrow \mathrm{W}_n$$

is injective. Therefore

$$n = \mathrm{ed}_k(G) \leq \mathrm{ed}_k(G \times \mathbb{Z}/p^m\mathbb{Z}) \leq n$$

For instance, if $m, n, r \in \mathbb{N}$ and $m \leq n$ then we have

$$\mathrm{ed}_k({}_{\mathrm{F}^r}\mathrm{W}_n \times \mathbb{Z}/p^m\mathbb{Z}) = n.$$

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