SPRINGER'S THEOREM FOR TAME QUADRATIC FORMS OVER HENSELIAN FIELDS

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Abstract. A quadratic form over a Henselian-valued field of arbitrary residue characteristic is tame if it becomes hyperbolic over a tamely ramified extension. The Witt group of tame quadratic forms is shown to be canonically isomorphic to the Witt group of graded quadratic forms over the graded ring associated to the filtration defined by the valuation, hence also isomorphic to a direct sum of copies of the Witt group of the residue field indexed by the value group modulo 2.

A celebrated theorem of Springer [8] establishes an isomorphism between the Witt group of a complete discretely valued field and the direct sum of two copies of the Witt group of the residue field, provided the residue characteristic is different from 2. Springer also considered the case where the residue characteristic is 2, and he pointed out the extra complications that arise from the fact that the residue forms are not necessarily nonsingular, even when the residue field is perfect. The exhaustive analysis by Aravire–Jacob [1] shows that the description of the Witt group of a dyadic Henselian field is extremely delicate, even when the field is maximally complete of characteristic 2.

Our purpose is to show that, by contrast, Springer's original techniques—as revisited in [7]—yield a very general characteristic-free version of Springer's theorem for the tame part of the Witt group. Our main result is the following: let F be a field with a Henselian valuation v, with value group Γ_F and residue field $\overline{F}$ of arbitrary characteristic; there is a group isomorphism

$$\partial\colon I_{qt}(F) \xrightarrow{\sim} \bigoplus_{\Gamma_F/2\Gamma_F} I_q(\overline{F})$$

where $I_q(\overline{F})$ is the quadratic Witt group of $\overline{F}$, consisting of Witt-equivalence classes of even-dimensional nonsingular quadratic forms over $\overline{F}$, and $I_{qt}(F) \subseteq I_q(F)$ is the subgroup of Witt classes that are split by a tamely ramified extension of F. The isomorphism ∂ is defined by residue forms, which depend on the choice of "uniformizing parameters" $\pi_\delta \in F^\times$ for representatives δ of the various cosets of Γ_F modulo $2\Gamma_F$. It is actually obtained as the composition of a *canonical* isomorphism

$$I_{qt}(F) \xrightarrow{\sim} I_q\big(\mathsf{gr}(F)\big), \tag{1}$$

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where $\mathsf{gr}(F)$ is the graded ring associated with the filtration of F defined by the valuation v, and an isomorphism

$$I_q\big(\mathsf{gr}(F)\big) \xrightarrow{\sim} \bigoplus_{\Gamma_F/2\Gamma_F} I_q(\overline{F}) \tag{2}$$

depending on the choice of uniformizing parameters. If $\operatorname{char}\overline{F} \neq 2$, the isomorphism ∂ extends to the whole Witt group $W(F)$, as is well-known (see [10, Satz 3.1] of [7], for instance). The case where $\operatorname{char}\overline{F} = 2$ was also considered by Tietze in [10][1], who also obtained an isomorphism from a subgroup $U(F) \subseteq I_q(F)$ onto $\bigoplus_{\Gamma_F/2\Gamma_F} I_q(\overline{F})$. Tietze's method is quite different: it uses a presentation of Witt groups by generators and relations, and $U(F)$ is defined in [10] as the subgroup generated by the Witt classes of quadratic forms $ax^2+bxy+cy^2$ with $v(a) = v(b) = v(c)$. Comparing Tietze's results with ours, it appears that actually $U(F) = I_{qt}(F)$, see Corollary 17.

Sections 1 and 2 introduce the formalism of norms on vector spaces and graded rings. The Witt group of graded fields is discussed in §1, where we prove the isomorphism (2): see Theorem 2. The isomorphism (1) is obtained in §2 (see Corollary 11), where $I_{qt}(F)$ is defined as the group of Witt classes of quadratic spaces that admit a certain type of norms, which we call *tame norms*. In §3, we establish the alternative characterization of $I_{qt}(F)$ in terms of tame splitting fields.

The following notation will be used consistently throughout: F is a field with a (Krull) valuation $v\colon F \to \Gamma \cup \{\infty\}$, where Γ is a totally ordered group. Without loss of generality, we assume Γ is divisible, since we may substitute for Γ its divisible hull. We let Γ_F denote the value group of F, i.e., $\Gamma_F = v(F^\times) \subseteq \Gamma$, and let $\overline{F}$ denote the residue field of F.

For quadratic forms, we generally follow the terminology of [3]: if $q\colon V \to F$ is a quadratic form on an F-vector space V, we let $b\colon V \times V \to F$ denote the polar form

$$b(x,y) = q(x+y) - q(x) - q(y) \qquad \text{for } x,\, y \in V.$$

The form b is *nondegenerate* if $x = 0$ is the only vector such that $b(x,y) = 0$ for all $y \in V$. We call the quadratic form q *regular* if $x = 0$ is the only vector in V such that $q(x) = 0$ and $b(x,y) = 0$ for all $y \in V$, and *nondegenerate* if it remains regular after scalar extension to an algebraic closure of F. We say q is *nonsingular* if its polar form is nondegenerate; thus, if $\operatorname{char} F \neq 2$ or if $\dim q$ is even, q is nonsingular if and only if it is nondegenerate. If $\operatorname{char} F = 2$, all nonsingular forms are even-dimensional since their polar form is nondegenerate and alternating, but $x_1x_2 + x_3^2$ is an odd-dimensional nondegenerate form.

1. Quadratic Witt group of graded fields

Let Γ be a divisible torsion-free abelian group, which will contain the degrees of all the graded structures we shall consider. Graded commutative rings in which every nonzero homogeneous element is invertible are called *graded fields*, and graded modules over graded fields are called *graded vector spaces*. Since Γ is torsion-free, graded fields are domains and graded vector spaces are free modules, see [5, §1].

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[1] We are grateful to S. Garibaldi for calling our attention on this reference.

The rank of a graded vector space is called its *dimension*. Let

$$\mathsf{F} = \bigoplus_{\gamma\in\Gamma} \mathsf{F}_\gamma$$

be a graded field and

$$\mathsf{V} = \bigoplus_{\gamma\in\Gamma} \mathsf{V}_\gamma$$

be a graded F-vector space. We let Γ_{F}, Γ_{V} denote the sets of degrees of F and V, i.e.,

$$\Gamma_{\mathsf{F}} = \big\{\gamma\in\Gamma \mid \mathsf{F}_\gamma \neq \{0\}\big\}, \qquad \Gamma_{\mathsf{V}} = \big\{\gamma\in\Gamma \mid \mathsf{V}_\gamma \neq \{0\}\big\}.$$

The set Γ_{F} is a subgroup of Γ, and Γ_{V} is a union of cosets of Γ_{F}. For each coset $\Lambda\in\Gamma/\Gamma_F$, let

$$\mathsf{V}_\Lambda = \bigoplus_{\lambda\in\Lambda} \mathsf{V}_\lambda$$

(so $\mathsf{V}_\Lambda = \{0\}$ if $\Lambda\not\subset\Gamma_V$). If Γ_{V} is the disjoint union of cosets $\Lambda_1, \ldots, \Lambda_n \in \Gamma/\Gamma_F$, i.e., $\Gamma_{\mathsf{V}} = \Lambda_1 \sqcup \ldots \sqcup \Lambda_n$, we have a canonical decomposition of V into graded sub-vector spaces

$$\mathsf{V} = \bigoplus_{i=1}^{n} \mathsf{V}_{\Lambda_i}. \tag{3}$$

Note that the homogeneous component F_0 of F is a field, and each V_γ for $\gamma\in\Gamma$ is an F_0-vector space. Picking an element $\lambda_i\in\Lambda_i$ for each $i = 1, \ldots, n$, we have

$$\dim_{\mathsf{F}} \mathsf{V} = \sum_{i=1}^{n} \dim_{\mathsf{F}} \mathsf{V}_{\Lambda_i} = \sum_{i=1}^{n} \dim_{\mathsf{F}_0} \mathsf{V}_{\lambda_i}.$$

Let V be a finite-dimensional graded vector space over a graded field F. A *graded quadratic form* on V is a map

$$q\colon \mathsf{V}\to\mathsf{F}$$

satisfying the following conditions involving q and its polar form $b\colon \mathsf{V}\times\mathsf{V}\to\mathsf{F}$ defined by

$$b(x,y) = q(x+y) - q(x) - q(y):$$

(i) $q(xa) = q(x)a^2$ for all $x\in\mathsf{V}$, $a\in\mathsf{F}$;
(ii) b is an F-bilinear form on V;
(iii) $q(\mathsf{V}_\gamma)\subseteq\mathsf{F}_{2\gamma}$ for all $\gamma\in\Gamma$;
(iv) $b(\mathsf{V}_\gamma,\mathsf{V}_\delta)\subseteq\mathsf{F}_{\gamma+\delta}$ for all γ, $\delta\in\Gamma$.

These conditions imply in particular that

$$q(\mathsf{V}_\gamma) = \{0\} \text{ if } \gamma\notin\tfrac{1}{2}\Gamma_{\mathsf{F}} \quad\text{and}\quad b(\mathsf{V}_\gamma,\mathsf{V}_\delta) = \{0\} \text{ if } \gamma+\delta\notin\Gamma_{\mathsf{F}}. \tag{4}$$

The graded quadratic form q is called *nonsingular* if its polar form b is nondegenerate, i.e., $x = 0$ is the only vector such that $b(x,y) = 0$ for all $y\in V$. It is called *hyperbolic* if it is nonsingular and $\mathsf{V} = \mathsf{U}\oplus\mathsf{W}$ for some graded subspaces U, W such that $q(\mathsf{U}) = q(\mathsf{W}) = \{0\}$.

Proposition 1. *Let q be a nonsingular graded quadratic form on V. The canonical decomposition* (3) *yields an orthogonal decomposition*

$$\mathsf{V} = \Big(\overset{\perp}{\bigoplus_{\Lambda\in\frac{1}{2}\Gamma_{\mathsf{F}}/\Gamma_{\mathsf{F}}}} \mathsf{V}_\Lambda\Big) \overset{\perp}{\oplus} \mathsf{W} \qquad \textit{where} \quad \mathsf{W} = \bigoplus_{\Lambda\notin\frac{1}{2}\Gamma_{\mathsf{F}}/\Gamma_{\mathsf{F}}} \mathsf{V}_\Lambda.$$

For each $\Lambda \in \frac{1}{2}\Gamma_{\mathsf{F}}/\Gamma_{\mathsf{F}}$, the restriction of q to V_Λ is nonsingular. Moreover, the restriction of q to W is hyperbolic.

Proof. All assertions except the last one really concern the polar form b; their easy proof (based on the observation (4)) can be found in [7, Proposition 1.1]. It is also proven there that the restriction of b to W is hyperbolic. Since moreover $q(\mathsf{V}_\Lambda) = \{0\}$ for all cosets $\Lambda \not\subset \frac{1}{2}\Gamma_{\mathsf{F}}$ by (4), the proposition follows. □

For each coset $\Delta \in \Gamma_{\mathsf{F}}/2\Gamma_{\mathsf{F}}$, fix an element $\delta \in \Delta$ and a nonzero homogeneous element $\pi_\delta \in \mathsf{F}_\delta$ ($\delta = 0$ and $\pi_0 = 1$ for $\Delta = 2\Gamma_{\mathsf{F}}$) and let

$$q_\Delta\colon \mathsf{V}_{\frac{1}{2}\delta} \to \mathsf{F}_0, \qquad x \mapsto \pi_\delta^{-1}q(x). \tag{5}$$

Since the restriction of q to $\mathsf{V}_{\frac{1}{2}\Delta}$ is nonsingular, it follows that q_Δ is a nonsingular quadratic form on the F_0-vector space $\mathsf{V}_{\frac{1}{2}\delta}$. Of course, this quadratic form depends on the choice of π_δ, except for $\Delta = 2\Gamma_{\mathsf{F}}$.

Mimicking the usual construction, we may define a Witt equivalence of nonsingular graded quadratic forms over a given graded field F and endow the set of Witt-equivalence classes with a group structure using the orthogonal sum. We let $I_q(\mathsf{F})$ denote the quadratic Witt group of F, consisting of Witt-equivalence classes of even-dimensional nonsingular graded quadratic forms over F.

Theorem 2. *The map that carries each nonsingular graded quadratic form q to the collection of nonsingular F_0-quadratic forms $(q_\Delta)_{\Delta\in\Gamma_{\mathsf{F}}/2\Gamma_{\mathsf{F}}}$ defines a group isomorphism:*

$$I_q(\mathsf{F}) \xrightarrow{\sim} \bigoplus_{\Gamma_{\mathsf{F}}/2\Gamma_{\mathsf{F}}} I_q(\mathsf{F}_0).$$

This isomorphism depends on the choice of the homogeneous elements $\pi_\delta \in \mathsf{F}_\delta$.

The proof is routine. See [7, Proposition 1.5(iv)] for the (more complicated) case of even hermitian forms over graded division algebras with involution.

2. Norms and residues

Let (F, v) be an arbitrary valued field and V a finite-dimensional F-vector space. We recall from [7] and [11] that a *v-value function* on V is a map

$$\alpha\colon V \to \Gamma \cup \{\infty\}$$

satisfying the following properties, for x, $y \in V$ and $\lambda \in F$:

(i) $\alpha(x) = \infty$ if and only if $x = 0$;
(ii) $\alpha(x\lambda) = \alpha(x) + v(\lambda)$;
(iii) $\alpha(x + y) \geq \min\bigl(\alpha(x), \alpha(y)\bigr)$.

The v-value function α is called a *v-norm* if there is a base $(e_i)_{i=1}^n$ of V that splits α in the following sense:

$$\alpha\Bigl(\sum_{i=1}^n e_i\lambda_i\Bigr) = \min\bigl(\alpha(e_i\lambda_i) \mid i = 1, \dots, n\bigr) \qquad \text{for } \lambda_1, \dots, \lambda_n \in F.$$

Any v-value function α on V defines a filtration of V by modules over the valuation ring of F: for any $\gamma \in \Gamma$ we let

$$V^{\geq\gamma} = \{x \in V \mid \alpha(x) \geq \gamma\}, \quad V^{>\gamma} = \{x \in V \mid \alpha(x) > \gamma\}, \quad V_\gamma = V^{\geq\gamma}/V^{>\gamma},$$

and we define

$$\mathsf{gr}_\alpha(V) = \bigoplus_{\gamma\in\Gamma} V_\gamma.$$

Similarly, let $\mathsf{gr}(F)$ be the graded ring associated to the filtration of F defined by the valuation. Since every nonzero homogeneous element of $\mathsf{gr}(F)$ is invertible, this ring is a graded field. The F-vector space structure on V induces on $\mathsf{gr}_\alpha(V)$ a structure of graded $\mathsf{gr}(F)$-module, so $\mathsf{gr}_\alpha(V)$ is a graded $\mathsf{gr}(F)$-vector space. For any nonzero $x \in V$ we let

$$\widetilde{x} = x + V^{>\alpha(x)} \in V_{\alpha(x)} \subseteq \mathsf{gr}_\alpha(V).$$

We also let $\widetilde{0} = 0$ and use a similar notation for elements in $\mathsf{gr}(F)$. It is shown in [7, Corollary 2.3] that a base $(e_i)_{i=1}^n$ of V splits α if and only if $(\widetilde{e}_i)_{i=1}^n$ is a $\mathsf{gr}(F)$-base of $\mathsf{gr}_\alpha(V)$, and that α is a norm if and only if

$$\dim_{\mathsf{gr}(F)} \mathsf{gr}_\alpha(V) = \dim_F V.$$

Now, let $q\colon V \to F$ be an arbitrary quadratic form, with polar form $b\colon V\times V \to F$. We say that a v-value function α is *bounded by* q, and write $\alpha \prec q$, if the following two conditions hold:

(a) $v\big(b(x,y)\big) \geq \alpha(x) + \alpha(y)$ for all x, $y \in V$, and
(b) $v\big(q(x)\big) \geq 2\alpha(x)$ for all $x \in V$.

Of course, letting $x = y$ in (a) yields (b) if $\operatorname{char} F \neq 2$. On the other hand, (b) does not imply (a): see Example 3 below. Note that Springer in [9] requires only (b), whereas Goldman–Iwahori in [4] require only (a). Both conditions are required by Bruhat–Tits in [2, Définition 2.1].

For each $\gamma \in \Gamma$, let $p_\gamma\colon F^{\geq\gamma} \to F_\gamma$ be the canonical map. When $\alpha \prec q$ we define maps

$$\widetilde{q}_\alpha\colon V_\gamma \to F_{2\gamma} \quad\text{and}\quad \widetilde{b}_\alpha\colon V_\gamma\times V_\delta \to F_{\gamma+\delta} \quad \text{for } \gamma,\ \delta\in\Gamma$$

by

$$\widetilde{q}_\alpha(\widetilde{x}) = p_{2\gamma}\big(q(x)\big) \quad\text{and}\quad \widetilde{b}_\alpha(\widetilde{x},\widetilde{y}) = p_{\gamma+\delta}\big(b(x,y)\big)$$

for x, $y \in V$ with $\alpha(x) = \gamma$ and $\alpha(y) = \delta$. We extend $\widetilde{b}_\alpha$ to $\mathsf{gr}_\alpha(V)$ by bilinearity and define

$$\widetilde{q}_\alpha\colon\ \mathsf{gr}_\alpha(V) \to \mathsf{gr}(F)$$

as follows: for $\widetilde{x}_\gamma \in V_\gamma$,

$$\widetilde{q}_\alpha\Big(\sum_{\gamma\in\Gamma}\widetilde{x}_\gamma\Big) = \sum_\gamma \widetilde{q}_\alpha(\widetilde{x}_\gamma) + \sum_{\gamma<\delta}\widetilde{b}_\alpha(\widetilde{x}_\gamma,\widetilde{x}_\delta).$$

The map $\widetilde{q}_\alpha$ is a quadratic form on $\mathsf{gr}_\alpha(V)$ with polar form $\widetilde{b}_\alpha$. It is a graded quadratic form since $\widetilde{q}_\alpha(V_\gamma) \subseteq F_{2\gamma}$ and $\widetilde{b}_\alpha(V_\gamma, V_\delta) \subseteq F_{\gamma+\delta}$ for γ, $\delta \in \Gamma$. The straightforward verifications are omitted.

Example 3. Suppose v is a discrete valuation on F with $\Gamma_F = \mathbb{Z}$ and let $V = F^{\oplus 2}$ with the hyperbolic quadratic form $q(x_1,x_2) = x_1x_2$. Define a norm $\alpha\colon V \to \frac{1}{2}\mathbb{Z}\cup\{\infty\}$ by

$$\alpha(x_1,x_2) = \min(v(x_1), v(x_2)+\tfrac{1}{2}) \qquad \text{for } x_1,\ x_2\in F.$$

Clearly, we have $v\big(q(x)\big) \geq 2\alpha(x)$ for all $x \in V$; but $b\big((1,0),(0,1)\big) = 1$, so

$$v\big(b\big((1,0),(0,1)\big)\big) = 0 < \alpha(1,0) + \alpha(0,1) = \tfrac{1}{2}.$$

Thus, condition (b) holds but not (a). Also, note that for $x = (1,0)$ and $x' = (1,1)$ we have $\widetilde{x} = \widetilde{x}'$ in $\mathsf{gr}_\alpha(V)$, but $q(x) = 0$ and $q(x') = 1$, so $\widetilde{q}_\alpha(\widetilde{x})$ is not well-defined.

By contrast, the map $\beta\colon V \to \mathbb{Z} \cup \{\infty\}$ defined by

$$\beta(x_1, x_2) = \min\bigl(v(x_1), v(x_2)\bigr) \qquad \text{for } x_1,\, x_2 \in F$$

is bounded by q. The induced quadratic form $\widetilde{q}_\beta$ is hyperbolic.

Example 4. Let (F, v) be an arbitrary valued field and let K/F be an arbitrary quadratic extension. Let $N\colon K \to F$ be the norm form, which is a quadratic form. We consider two cases:

Case 1: The valuation v extends to two different valuations v_1, v_2 on K.

Then K/F is not purely inseparable, hence it is a Galois extension. Let $\sigma\colon K \to K$ be the nontrivial automorphism of K/F, so $v_2 = v_1 \circ \sigma$. We have $v\bigl(N(x)\bigr) = v_1(x) + v_2(x)$ for all $x \in V$. The map $\alpha\colon K \to \Gamma \cup \{\infty\}$ defined by

$$\alpha(x) = \min\bigl(v_1(x), v_2(x)\bigr) \qquad \text{for } x \in K$$

is a v-norm on K by [11, Corollary 1.7], and it is readily checked that $\alpha \prec N$. Let $u \in K$ be such that $v_1(u) \neq v_2(u)$. Then $\sigma(u) \neq -u$, hence after scaling by a suitable factor in $F^\times$ we may assume $u + \sigma(u) = 1$. Since $v_1\bigl(\sigma(u)\bigr) = v_2(u) \neq v_1(u)$, this equation yields

$$0 = \min\bigl(v_1(u), v_1\bigl(\sigma(u)\bigr)\bigr) = \alpha(u).$$

It follows that $\widetilde{b}_\alpha(\widetilde{1}, \widetilde{u}) = \widetilde{1}$, so the form $\widetilde{N}_\alpha$ is nonsingular. Moreover, $\widetilde{N}_\alpha(\widetilde{u}) = 0$, so $\widetilde{N}_\alpha$ is hyperbolic.

Case 2: The valuation v has a unique extension to K.

We again write v for the valuation on K extending v. This valuation is a v-value function on K and it is readily checked that $v \prec N$; in fact $v\bigl(N(x)\bigr) = 2v(x)$ for all $x \in K$. If K/F is immediate (i.e., $\overline{K} = \overline{F}$ and $\Gamma_K = \Gamma_F$), then $\mathsf{gr}_v(K) = \mathsf{gr}(F)$, so v is not a norm. Otherwise, $\mathsf{gr}_v(K)$ is a quadratic extension of $\mathsf{gr}(F)$, and $\widetilde{N}_\alpha$ is the norm form of that extension. It is nonsingular if and only if K/F is tame, i.e., either K/F is totally ramified and $\operatorname{char} \overline{F} \neq 2$, or $\overline{K}/\overline{F}$ is a separable quadratic extension (i.e., K/F is inertial).

This last example suggests the following definition:

Definition 5. Let (F, v) be an arbitrary valued field and (V, q) be a quadratic space over F. A v-norm α on V is called a *tame norm compatible with* q if $\alpha \prec q$ and the induced quadratic form $\widetilde{q}_\alpha$ on $\mathsf{gr}_\alpha(V)$ is nonsingular.

For any quadratic space (V, q) and any v-value function α on V such that $\alpha \prec q$, it is clear that $\widetilde{b}_\alpha(\widetilde{x}, \eta) = 0$ for all $\eta \in \mathsf{gr}_\alpha(V)$ if $b(x, y) = 0$ for all $y \in V$. Therefore, tame compatible norms exist only for nonsingular forms. If $\operatorname{char} \overline{F} \neq 2$, the tame norms compatible with a quadratic form are exactly the norms that are compatible with its polar form, in the terminology of [7]; see [7, Remark 3.2]. In that case, for every nonsingular quadratic form there is a tame compatible norm, see [7, Corollary 3.6]. If $\operatorname{char} \overline{F} = 2$, the tame norms compatible with a quadratic form q are those that are compatible with its polar form and that moreover satisfy condition (b): $v\bigl(q(x)\bigr) \geq 2\alpha(x)$ for all $x \in V$. There are nonsingular forms for which there is no tame compatible norm, for instance the norm forms of totally ramified quadratic extensions of Henselian dyadic fields: see Theorem 18.

Lemma 6. *Let (V_1, q_1) and (V_2, q_2) be quadratic spaces over F, and let α_1, α_2 be tame v-norms on V_1, V_2 that are compatible with q_1 and q_2 respectively. Define $\alpha_1 \oplus \alpha_2 \colon V_1 \oplus V_2 \to \Gamma \cup \{\infty\}$ by*

$$(\alpha_1 \oplus \alpha_2)(x_1, x_2) = \min\bigl(\alpha_1(x_1), \alpha_2(x_2)\bigr) \qquad \text{for } x_1 \in V_1 \text{ and } x_2 \in V_2.$$

Then $\alpha_1 \oplus \alpha_2$ is a tame v-norm on $V_1 \oplus V_2$ compatible with $q_1 \perp q_2$, and there is a canonical identification of graded quadratic spaces

$$(\mathsf{gr}_{\alpha_1 \oplus \alpha_2}(V_1 \oplus V_2), \widetilde{(q_1 \perp q_2)}_{\alpha_1 \oplus \alpha_2}) = (\mathsf{gr}_{\alpha_1}(V_1), \widetilde{q_1}_{\alpha_1}) \perp (\mathsf{gr}_{\alpha_2}(V_2), \widetilde{q_2}_{\alpha_2}).$$

The easy proof is omitted; see [7, Example 3.7(iii)].

Lemma 7. *Let (V, q) be a quadratic space over F and let α be a tame v-norm on V compatible with q. The Witt indices of q and $\widetilde{q}_\alpha$ are related by*

$$\mathfrak{i}_0(q) \le \mathfrak{i}_0(\widetilde{q}_\alpha).$$

In particular, if q is hyperbolic, then $\widetilde{q}_\alpha$ is hyperbolic.

Proof. If $U \subseteq V$ is a totally q-isotropic subspace, then $\mathsf{gr}_\alpha(U) \subseteq \mathsf{gr}_\alpha(V)$ is a totally $\widetilde{q}_\alpha$-isotropic subspace with $\dim \mathsf{gr}_\alpha(U) = \dim U$. □

Let $I_q(F)$ denote the quadratic Witt group of F, consisting of Witt-equivalence classes of even-dimensional nonsingular quadratic forms; see [3, §8.B]. From Lemma 6, it follows that the Witt-equivalence classes of even-dimensional nonsingular quadratic forms that admit a tame compatible norm form a subgroup of $I_q(F)$. We let $I_{qt}(F)$ denote this subgroup, which we call the *tame quadratic Witt group of F*.

Proposition 8. *There is a well-defined group epimorphism*

$$\partial \colon I_{qt}(F) \to I_q\bigl(\mathsf{gr}(F)\bigr)$$

that carries the Witt class of any even-dimensional nonsingular quadratic form q with a tame compatible norm α to the Witt class of $\widetilde{q}_\alpha$.

Proof. If $\operatorname{char} \overline{F} \neq 2$, this is shown in [7, Theorem 3.11] (and holds for odd-dimensional nondegenerate quadratic forms as well). The arguments also hold if $\operatorname{char} \overline{F} = 2$: to see that the Witt equivalence class of $\widetilde{q}_\alpha$ does not depend on the choice of the tame compatible norm α, suppose β is another tame compatible norm. Then $\alpha \oplus \beta$ is a tame norm compatible with the hyperbolic form $q \perp -q$, hence Lemmas 6 and 7 show that $\widetilde{q}_\alpha \perp -\widetilde{q}_\beta$ is hyperbolic.

To prove surjectivity of ∂ when $\operatorname{char} \overline{F} = 2$, it suffices to show that the Witt class of every nonsingular binary form over $\mathsf{gr}(F)$ is in the image. Let V be a 2-dimensional graded vector space over $\mathsf{gr}(F)$, and let $\varphi \colon \mathsf{V} \to \mathsf{gr}(F)$ be a nonsingular quadratic form. Since the corresponding polar form b_φ is nonsingular, we may find homogeneous vectors ε_1, $\varepsilon_2 \in \mathsf{V}$ such that $b_\varphi(\varepsilon_1, \varepsilon_2) = 1$. These vectors form a base of V. If $\varphi(\varepsilon_1) = 0$ or $\varphi(\varepsilon_2) = 0$, then φ is hyperbolic hence its Witt class is zero, which lies in the image of ∂. If $\varphi(\varepsilon_1)$, $\varphi(\varepsilon_2)$ are nonzero, then we may find a_1, $a_2 \in F^\times$ such that $\varphi(\varepsilon_1) = \widetilde{a}_1$ and $\varphi(\varepsilon_2) = \widetilde{a}_2$. Note that the condition $b_\varphi(\varepsilon_1, \varepsilon_2) = 1$ implies that $\deg \varepsilon_1 + \deg \varepsilon_2 = 0$, hence $v(a_1) + v(a_2) = 0$. Consider a 2-dimensional F-vector space U with base e_1, e_2 and quadratic form

$$q(e_1 x_1 + e_2 x_2) = a_1 x_1^2 + x_1 x_2 + a_2 x_2^2 \qquad \text{for } x_1,\, x_2 \in F.$$

Straightforward computations show that q is a nonsingular quadratic form and that the map $\alpha\colon U \to \Gamma\cup\{\infty\}$ defined by

$$\alpha(e_1x_1+e_2x_2)=\min\bigl(\tfrac12 v(a_1)+v(x_1),\tfrac12 v(a_2)+v(x_2)\bigr)\qquad\text{for } x_1,\, x_2\in F$$

is a tame norm compatible with q, such that $(\mathsf{gr}_\alpha(U),\widetilde{q}_\alpha)\simeq(\mathsf{V},\varphi)$ under the map $e_1x_1+e_2x_2\mapsto\varepsilon_1\widetilde{x}_1+\varepsilon_2\widetilde{x}_2$. Thus, the Witt class of φ is in the image of ∂. □

Our next goal is to show that ∂ is an isomorphism when the valuation v on F is Henselian.

Lemma 9. *Let (V,q) be a quadratic space over F. Suppose v is Henselian and x, $y\in V$ are such that*

$$2v\bigl(b(x,y)\bigr)<v\bigl(q(x)\bigr)+v\bigl(q(y)\bigr).$$

Then there is a q-isotropic vector in the span of x and y.

Proof. This was observed by several authors, in particular Springer [8, Proposition 1] (see also [10, Lemma (2.2)]). We give a proof for completeness. First, note that if z_1, $z_2\in V$ are multiple of each other, then

$$2v\bigl(b(z_1,z_2)\bigr)=v\bigl(q(z_1)\bigr)+v\bigl(q(z_2)\bigr)+2v(2)\geq v\bigl(q(z_1)\bigr)+v\bigl(q(z_2)\bigr).\tag{6}$$

In particular, under the hypotheses of the lemma, x and y are not multiple of each other. Set $z=yq(x)b(x,y)^{-1}$, so $b(x,z)=q(x)$. For all $\lambda\in F$ we have

$$q(x\lambda+z)=q(x)(\lambda^2+\lambda+q(x)q(y)b(x,y)^{-2}).\tag{7}$$

The second factor on the right side is a polynomial $P(\lambda)$ with coefficients in the valuation ring of F. Its image in $\overline{F}[\lambda]$ is $\lambda(\lambda+1)$. By Hensel's Lemma, $P(\lambda)$ has a root $\lambda_0\in F$. The vector $x\lambda_0+z$ is nonzero since x and y are not multiple of each other, and it is an isotropic vector of q. □

Theorem 10. *Let (V,q) be a quadratic space over F. Assume the valuation v on F is Henselian. Suppose $\alpha\colon V\to\Gamma\cup\{\infty\}$ is a tame norm compatible with q, and consider the induced quadratic form $\widetilde{q}_\alpha$ on $\mathsf{gr}_\alpha(V)$. The forms q and $\widetilde{q}_\alpha$ have the same Witt index:*

$$\mathfrak{i}_0(q)=\mathfrak{i}_0(\widetilde{q}_\alpha).$$

Proof. Lemma 7 already yields $\mathfrak{i}_0(q)\leq\mathfrak{i}_0(\widetilde{q}_\alpha)$. To prove the reverse inequality, we argue by induction on $\dim V$, and copy the proof of [7, Proposition 4.3]. If $\widetilde{q}_\alpha$ is isotropic, we may find a homogeneous isotropic vector $\widetilde{x}$ by taking the homogeneous component of smallest degree of an arbitrary isotropic vector. Since α is a tame norm compatible with q, the polar form $\widetilde{b}_\alpha$ is nondegenerate. Therefore, we may find a homogeneous vector $\widetilde{y}$ such that $\widetilde{b}_\alpha(\widetilde{x},\widetilde{y})\neq 0$. Let $W\subseteq V$ be the subspace spanned by x and y, so $\mathsf{gr}_\alpha(W)\subseteq\mathsf{gr}_\alpha(V)$ is the graded subspace spanned by $\widetilde{x}$ and $\widetilde{y}$. We claim that W is a hyperbolic plane. To see this, observe that the equation $\widetilde{q}_\alpha(\widetilde{x})=0$ yields $v\bigl(q(x)\bigr)>2\alpha(x)$, while $\widetilde{b}_\alpha(\widetilde{x},\widetilde{y})\neq 0$ shows that $v\bigl(b(x,y)\bigr)=\alpha(x)+\alpha(y)$. Since $v\bigl(q(y)\bigr)\geq 2\alpha(y)$, it follows that $2v\bigl(b(x,y)\bigr)<v\bigl(q(x)\bigr)+v\bigl(q(y)\bigr)$. Therefore, Lemma 9 shows that W contains a q-isotropic vector. The restriction of q to W is nonsingular since $b(x,y)\neq 0$, hence W is a hyperbolic plane.

Let q' denote the restriction of q to $W^\perp$, so $\mathfrak{i}_0(q)=1+\mathfrak{i}_0(q')$. By the choice of x and y, the bilinear form $\widetilde{b}_\alpha$ is nondegenerate on $\mathsf{gr}_\alpha(W)$. Therefore, the norm $\alpha|_W$ is compatible with the restriction of the polar form $b|_W$, and [7, Proposition 3.8] shows that $W^\perp$ is a splitting complement of W with respect to α, i.e., $\alpha = \alpha|_W \oplus \alpha|_{W^\perp}$. Thus,

$$\mathsf{gr}_\alpha(V) = \mathsf{gr}_\alpha(W) \perp \mathsf{gr}_\alpha(W^\perp),$$

hence $\mathfrak{i}_0(\widetilde{q}_\alpha) = 1 + \mathfrak{i}_0(\widetilde{q}'_\alpha)$ since $\mathsf{gr}_\alpha(W)$ is a hyperbolic plane. The induction hypothesis yields $\mathfrak{i}_0(q') \geq \mathfrak{i}_0(\widetilde{q}'_\alpha)$, hence $\mathfrak{i}_0(q) \geq \mathfrak{i}_0(\widetilde{q}_\alpha)$ and the proof is complete. □

Corollary 11. *The homomorphism ∂ of Proposition 8 is an isomorphism if F is Henselian.*

Proof. Surjectivity of ∂ was shown in Proposition 8, and injectivity follows from Theorem 10. □

3. Tame quadratic forms over Henselian fields

In this section, we show that the tame quadratic Witt group of a Henselian field is the Witt kernel of the scalar extension map to the maximal tame extension.

Let F be a field with a valuation $v\colon F \to \Gamma \cup \{\infty\}$. Throughout this section, we assume v is Henselian. Let (V, q) be a quadratic space over F with polar form b. If q is anisotropic, we define a map $\widehat{\alpha}\colon V \to \Gamma \cup \{\infty\}$ by

$$\widehat{\alpha}(x) = \tfrac{1}{2}v\big(q(x)\big) \qquad \text{for } x \in V. \tag{8}$$

Proposition 12. *The map $\widehat{\alpha}$ is a v-value function and $\widehat{\alpha} \prec q$.*

Proof. This was proved by Springer [8, Proposition 1] (and attributed to M. Eichler) in the case where the valuation is discrete and F is complete. Springer's arguments hold without change under the more general hypotheses of this section. We include the proof for the reader's convenience.

By definition of $\widehat{\alpha}$, we clearly have $\widehat{\alpha}(x\lambda) = \widehat{\alpha}(x) + v(\lambda)$ for $x \in V$ and $\lambda \in F$, and $\widehat{\alpha}(x) = \infty$ if and only if $x = 0$ because q is anisotropic. It is also clear from the definition that $v\big(q(x)\big) \geq 2\widehat{\alpha}(x)$ for all $x \in V$. Thus, it only remains to show

$$v\big(b(x, y)\big) \geq \widehat{\alpha}(x) + \widehat{\alpha}(y) \qquad \text{for } x,\, y \in V \tag{9}$$

and

$$\widehat{\alpha}(x + y) \geq \min\big(\widehat{\alpha}(x), \widehat{\alpha}(y)\big) \qquad \text{for } x,\, y \in V. \tag{10}$$

The inequality (9) readily follows from Lemma 9 since q is anisotropic. Property (10) follows, since (9) implies $v\big(b(x,y)\big) \geq \min\big(v\big(q(x)\big), v\big(q(y)\big)\big)$ for x, $y \in V$, hence

$$v\big(q(x+y)\big) = v\big(q(x) + b(x,y) + q(y)\big) \geq \min\big(v\big(q(x)\big), v\big(q(y)\big)\big).$$

□

Now, consider the following property for an anisotropic quadratic form q over F:

(S) $v\big(b(x, y)\big) > \widehat{\alpha}(x) + \widehat{\alpha}(y)$ for all nonzero x, $y \in V$ such that $\widehat{\alpha}(x) = \widehat{\alpha}(y)$.

Note that this property cannot hold if $\operatorname{char} \overline{F} \neq 2$ since for $x = y$ it implies $v(2) > 0$ (see (6)). By (9), we have $v\big(b(x,y)\big) > 2\min\big(\widehat{\alpha}(x), \widehat{\alpha}(y)\big)$ if $\widehat{\alpha}(x) \neq \widehat{\alpha}(y)$, hence Property (S) has the following equivalent formulation: for all nonzero x, $y \in V$,

$$v\big(b(x,y)\big) > 2\min\big(\widehat{\alpha}(x), \widehat{\alpha}(y)\big) = \min\big(v\big(q(x)\big), v\big(q(y)\big)\big). \tag{S'}$$

Our goal is to prove that Property (S) characterizes the quadratic forms that remain anisotropic over any inertial extension. Thus, *until the end of this section, except for the last theorem, we assume* $\operatorname{char} \overline{F} = 2$.

We first consider inertial *quadratic* extensions K/F. The residue extension $\overline{K}/\overline{F}$ is obtained by adjoining to $\overline{F}$ a root of an irreducible polynomial of the form $X^2+X+\overline{u}$ for some $\overline{u}\in\overline{F}$, hence $K=F(\mu)$ where $\mu^2+\mu+u=0$ for some $u\in F$ with $v(u)=0$. The Henselian valuation on F has a unique extension to K, for which we also use the notation v. Since 1, $\overline{\mu}$ are linearly independent over $\overline{F}$, we have

$$v(a+b\mu)=\min\bigl(v(a),v(b)\bigr) \qquad \text{for } a,\, b\in F. \tag{11}$$

Lemma 13. *An anisotropic quadratic form satisfies (S) if and only if it remains anisotropic over every inertial quadratic extension.*

Proof. If (S) does not hold, then we can find nonzero vectors x, $y\in V$ such that $\widehat{\alpha}(x)=\widehat{\alpha}(y)$ and $v\bigl(b(x,y)\bigr)=\widehat{\alpha}(x)+\widehat{\alpha}(y)$. In view of (6), the vectors x and y are not multiple of each other. Letting $z=yq(x)b(x,y)^{-1}$ as in the proof of Lemma 9, we see from (7) that q becomes isotropic over $F(\lambda_0)$, where λ_0 is a root of the polynomial $\lambda^2+\lambda+q(x)q(y)b(x,y)^{-2}$. Since $v(q(x)q(y)b(x,y)^{-2})=0$, the field $F(\lambda_0)$ is an inertial extension of F.

Conversely, suppose q becomes isotropic over an inertial quadratic extension K of F. We have $K=F(\mu)$ where $\mu^2+\mu+u=0$ for some $u\in F$ with $v(u)=0$. Let $x_0\otimes 1+x_1\otimes\mu\in V\otimes_F K$ be an isotropic vector of q. We have

$$q(x_0\otimes 1+x_1\otimes\mu)=\bigl(q(x_0)-uq(x_1)\bigr)+\bigl(b(x_0,x_1)-q(x_1)\bigr)\mu, \tag{12}$$

hence

$$q(x_0)=uq(x_1) \qquad\text{and}\qquad b(x_0,x_1)=q(x_1).$$

Since $v(u)=0$, it follows that $v\bigl(q(x_0)\bigr)=v\bigl(q(x_1)\bigr)=v\bigl(b(x_0,x_1)\bigr)$, hence (S) does not hold. □

Lemma 14. *If an anisotropic quadratic form satisfies (S), then its scalar extension to any inertial quadratic extension also satisfies (S).*

Proof. Suppose q is an anisotropic quadratic form over F satisfying (S), and K is an inertial quadratic extension of F. By Lemma 13, we know q remains anisotropic over K. We may therefore extend to $V\otimes_F K$ the definition of the v-value function $\widehat{\alpha}$ of (8). Let $K=F(\mu)$ where $\mu^2+\mu+u=0$ for some $u\in F$ with $v(u)=0$. For $x=x_0\otimes 1+x_1\otimes\mu\in V\otimes_F K$ we have by (12) and (11)

$$\widehat{\alpha}(x)=\tfrac{1}{2}\min\Bigl(v\bigl(q(x_0)-uq(x_1)\bigr),v\bigl(b(x_0,x_1)-q(x_1)\bigr)\Bigr).$$

We claim that

$$\widehat{\alpha}(x)=\min\bigl(\widehat{\alpha}(x_0),\widehat{\alpha}(x_1)\bigr).$$

We check this formula case-by-case: if $v\bigl(q(x_0)\bigr)=v\bigl(q(x_1)\bigr)$, then $v\bigl(q(x_0)-uq(x_1)\bigr)\geq v\bigl(q(x_1)\bigr)$ and, by (S), $v\bigl(b(x_0,x_1)-q(x_1)\bigr)=v\bigl(q(x_1)\bigr)$; thus, $\widehat{\alpha}(x)=\widehat{\alpha}(x_0)=\widehat{\alpha}(x_1)$.

If $v\bigl(q(x_0)\bigr)>v\bigl(q(x_1)\bigr)$, then $v\bigl(q(x_0)-uq(x_1)\bigr)=v\bigl(q(x_1)\bigr)$ and $v\bigl(b(x_0,x_1)\bigr)>v\bigl(q(x_1)\bigr)$ by (S'), hence $\widehat{\alpha}(x)=\widehat{\alpha}(x_1)$.

Likewise, if $v\bigl(q(x_0)\bigr)<v\bigl(q(x_1)\bigr)$, then $v\bigl(b(x_0,x_1)\bigr)>v\bigl(q(x_0)\bigr)$, hence $\widehat{\alpha}(x)=\widehat{\alpha}(x_0)$. Thus, the claim is proved.

Now, suppose $x=x_0\otimes 1+x_1\otimes\mu$ and $y=y_0\otimes 1+y_1\otimes\mu$ are nonzero vectors in $V\otimes_F K$ such that $\widehat{\alpha}(x)=\widehat{\alpha}(y)$, and let $\gamma=\widehat{\alpha}(x)$, hence

$$\gamma=\min\bigl(\widehat{\alpha}(x_0),\widehat{\alpha}(x_1)\bigr)=\min\bigl(\widehat{\alpha}(y_0),\widehat{\alpha}(y_1)\bigr).$$

We have

$$b(x,y) = \big(b(x_0,y_0) - ub(x_1,y_1)\big) + \big(b(x_0,y_1) + b(x_1,y_0) - b(x_1,y_1)\big)\mu,$$

hence by (11)

$$v\big(b(x,y)\big) = \min\Big(v\big(b(x_0,y_0) - ub(x_1,y_1)\big), v\big(b(x_0,y_1) + b(x_1,y_0) - b(x_1,y_1)\big)\Big).$$

By Property (S') we have

$$v\big(b(x_i,y_j)\big) > \min\big(\widehat{\alpha}(x_i), \widehat{\alpha}(y_j)\big) \geq 2\gamma \qquad \text{for } i,\, j = 0,\, 1,$$

hence

$$v\big(b(x_0,y_0) - ub(x_1,y_1)\big) > 2\gamma \quad \text{and} \quad v\big(b(x_0,y_1) + b(x_1,y_0) - b(x_1,y_1)\big) > 2\gamma.$$

Therefore, $v\big(b(x,y)\big) > 2\gamma$, and it follows that Property (S) holds for the extension q_K of q to K. □

We now turn to odd-degree extensions. Let L/F be an inertial extension of odd degree d. The Henselian valuation v has a unique extension to L, for which we also use the notation v, and we have $L = F(\lambda)$ for some λ with $v(\lambda) = 0$ such that $\overline{L} = \overline{F}(\overline{\lambda})$ and the minimal polynomial of $\overline{\lambda}$ over $\overline{F}$ has degree d. Every anisotropic quadratic form $q\colon V \to F$ remains anisotropic over L by a theorem of Springer (see [3, Corollary 18.5]), hence we may extend the value function $\widehat{\alpha}$ of (8) to $V \otimes_F L$.

Lemma 15. *For $x = x_0 \otimes 1 + x_1 \otimes \lambda + \cdots + x_{d-1} \otimes \lambda^{d-1} \in V \otimes_F L$, we have*

$$\widehat{\alpha}(x) = \min\big(\widehat{\alpha}(x_0), \ldots, \widehat{\alpha}(x_{d-1})\big).$$

If q satisfies (S), then its extension q_L to L also satisfies (S).

Proof. If $x = 0$, the formula trivially holds. We may therefore assume $x_0, \ldots, x_{d-1}$ are not all zero and set

$$\gamma = \min\big(\widehat{\alpha}(x_0), \ldots, \widehat{\alpha}(x_{d-1})\big).$$

We have

$$q_L(x) = \sum_{0 \leq i \leq d-1} q(x_i)\lambda^{2i} + \sum_{0 \leq i < j \leq d-1} b(x_i, x_j)\lambda^{i+j}.$$

Since $v\big(q(x_i)\big) \geq 2\gamma$ by definition of γ and

$$v\big(b(x_i,x_j)\big) \geq \widehat{\alpha}(x_i) + \widehat{\alpha}(x_j) \geq 2\gamma$$

by (9), it follows that $v\big(q_L(x)\big) \geq 2\gamma$, hence $\widehat{\alpha}(x) \geq \gamma$. To prove that this inequality is an equality, consider the homogeneous component V_γ of the graded vector space $\mathsf{gr}_{\widehat{\alpha}}(V)$ and the quadratic map $\widetilde{q}_{\widehat{\alpha}}\colon V_\gamma \to F_{2\gamma}$ induced by q, as in §2. This map is anisotropic since $v\big(q(z)\big) = 2\widehat{\alpha}(z)$ for all $z \in V$, by definition of $\widehat{\alpha}$. Since $\overline{L}/\overline{F}$ is an odd-degree extension, this map remains anisotropic after scalar extension to $\overline{L}$, by a theorem of Springer (see [3, Corollary 18.5]). Therefore, letting $x_i' = x_i + V^{>\gamma} \in V_\gamma$ for $i = 0, \ldots, d-1$, so that $x_i' = \widetilde{x}_i \neq 0$ if $\widehat{\alpha}(x_i) = \gamma$ and $x_i' = 0$ if $\widehat{\alpha}(x_i) > \gamma$, we have

$$(\widetilde{q}_{\widehat{\alpha}})_{\overline{L}}(x_0' \otimes 1 + x_1' \otimes \overline{\lambda} + \cdots + x_{d-1}' \otimes \overline{\lambda}^{d-1}) \neq 0.$$

The left side is the image of $q_L(x)$ under the canonical map $L^{\geq 2\gamma} \to L_{2\gamma}$, hence $v\big(q_L(x)\big) = 2\gamma$, and $\widehat{\alpha}(x) = \gamma$. The formula for $\widehat{\alpha}(x)$ is thus established.

Now, let $x,\, y \in V \otimes_F L$ be nonzero vectors with $\widehat{\alpha}(x) = \widehat{\alpha}(y)$. Let $\gamma = \widehat{\alpha}(x) = \widehat{\alpha}(y)$ and

$$x = x_0 \otimes 1 + \cdots + x_{d-1} \otimes \lambda^{d-1}, \qquad y = y_0 \otimes 1 + \cdots + y_{d-1} \otimes \lambda^{d-1}$$

with $x_0, \dots, y_{d-1} \in V$ and

$$\gamma = \min\big(\widehat{\alpha}(x_0), \dots, \widehat{\alpha}(x_{d-1})\big) = \min\big(\widehat{\alpha}(y_0), \dots, \widehat{\alpha}(y_{d-1})\big).$$

We have

$$b(x,y) = \sum_{i,j=0}^{d-1} b(x_i, y_j)\lambda^{i+j}.$$

If Property (S) (hence also (S')) holds for q, we have

$$v\big(b(x_i, y_j)\big) > 2\min\big(\widehat{\alpha}(x_i), \widehat{\alpha}(y_j)\big) \geq 2\gamma \qquad \text{for all } i,\ j,$$

hence also $v\big(b(x,y)\big) > 2\gamma$. Thus, Property (S) holds for q_L. □

Theorem 16. *Suppose F is a field with a Henselian dyadic valuation. An anisotropic quadratic form over F satisfies (S) if and only if it remains anisotropic over every inertial extension of F.*

Proof. The "if" part follows from Lemma 13. For the converse, suppose q is an anisotropic form over F satisfying (S), and let M be an inertial extension of F. We have to show that q remains anisotropic over M. Substituting for M its Galois closure, we may assume M is Galois over F. Let $L \subseteq M$ be the subfield fixed under some 2-Sylow subgroup of the Galois group. Then L/F is an odd-degree extension and M/L is a Galois 2-extension, hence there is a sequence of field extensions

$$L = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_r = M$$

with $[L_i : L_{i-1}] = 2$ for $i = 1, \dots, r$. By a theorem of Springer (see [3, Corollary 18.5]), q remains anisotropic over L, and its extension q_L satisfies (S) by Lemma 15. Therefore, q_{L_1} is anisotropic by Lemma 13, and it satisfies (S) by Lemma 14. Arguing iteratively, we see that q_M is anisotropic (and satisfies (S)). □

Theorem 16 also yields precise information on the quadratic forms that are split by an inertial extension of the base field.

Corollary 17. *Let q be an anisotropic quadratic form over the Henselian dyadic field F. If q becomes hyperbolic over some inertial extension of F, then there exist inertial quadratic extensions $K_1, \dots, K_n$ of F and $a_1, \dots, a_n \in F^\times$ such that*

$$q \simeq \langle a_1\rangle N_1 \perp \ldots \perp \langle a_n\rangle N_n$$

where N_i is the (quadratic) norm form of K_i/F.

Proof. Since q becomes hyperbolic over some inertial extension of F, Theorem 16 shows that it does not satisfy Property (S). Therefore, by Lemma 13, q becomes isotropic over some inertial quadratic extension K_1/F. By [3, Proposition 34.8], we may then find $a_1 \in F^\times$ and a quadratic form q_1 such that

$$q \simeq \langle a_1\rangle N_1 \perp q_1.$$

Since q becomes hyperbolic over some inertial extension of F and N_1 becomes hyperbolic over the inertial extension K_1, the form q_1 also becomes hyperbolic over some inertial extension of F. The corollary follows by induction on the dimension. □

If we include the split quadratic F-algebra $F \times F$, with hyperbolic norm form, among the inertial quadratic extensions of F, Witt's decomposition theorem shows that Corollary 17 also holds for isotropic quadratic forms that become hyperbolic over some inertial extension. It also holds in the nondyadic case, because then Springer's theorem can be used to show that every anisotropic quadratic form split by an inertial extension becomes isotropic over some inertial quadratic extension. Thus, the same argument as in the proof of Corollary 17 applies.

Our last theorem holds without the hypothesis that $\operatorname{char}\overline{F} = 2$.

Theorem 18. *Let F be a field with a Henselian valuation v (with arbitrary residue characteristic), and let (V, q) be an even-dimensional nondegenerate quadratic space over F. The following conditions are equivalent:*

(a) *q becomes hyperbolic over some tame extension of F;*
(b) *there exists a tame norm α on V compatible with q.*

Proof. If $\operatorname{char}\overline{F} \neq 2$, both conditions hold for all even-dimensional nondegenerate quadratic spaces: (a) because the quadratic closure of F is tame and (b) by [7, Corollary 3.6]. For the rest of the proof, we may thus assume $\operatorname{char}\overline{F} = 2$. We may also assume q is anisotropic because if (b) holds for an anisotropic form it also holds for every Witt-equivalent form by Lemma 6 since hyperbolic forms admit tame compatible norms (see Example 3).

If (a) holds, then by Corollary 17 we may find a decomposition

$$q \simeq \langle a_1 \rangle N_1 \perp \ldots \perp \langle a_n \rangle N_n$$

for some $a_1, \ldots, a_n \in F^\times$ and for $N_1, \ldots, N_n$ the norm forms of some inertial quadratic extensions $K_1, \ldots, K_n$ of F. Example 4 shows that for each $i = 1, \ldots, n$ the unique valuation on K_i extending v is a tame norm compatible with N_i, hence also with $\langle a_i \rangle N_i$. The direct sum of these norms is a tame norm compatible with q by Lemma 6, hence condition (b) holds.

Conversely, suppose there is a tame norm α on V compatible with q. Then we may find a separable extension L' of $\overline{F}$ such that $\widetilde{q}_\alpha$ splits after scalar extension to $\mathsf{gr}(F) \otimes_{\overline{F}} L'$. Let L be an inertial lift of L', i.e., L/F is an inertial extension such that $\overline{L} = L'$. Write again v for the unique extension of v to L. Then $\alpha \otimes v$ is a tame norm on $V \otimes_F L$ compatible with q_L, and $(\widetilde{q_L})_{\alpha \otimes v} = (\widetilde{q}_\alpha)_{\mathsf{gr}(L)}$ is split since $\mathsf{gr}(L) = \mathsf{gr}(F) \otimes_{\overline{F}} L'$. Therefore, Theorem 10 shows that q_L is hyperbolic. □

As an example, consider the field $F = \mathbb{Q}_2((t))$ of Laurent series in one indeterminate over the field of 2-adic numbers. The composite of the 2-adic valuation on $\mathbb{Q}_2$ and the t-adic valuation on F is a Henselian valuation v on F with value group $\mathbb{Z}^2$ and residue field $\mathbb{F}_2$. We have $I_q(\mathbb{F}_2) \simeq \mathbb{Z}/2\mathbb{Z}$, and the unique nontrivial Witt class is represented by the norm form of $\mathbb{F}_4$. The unique inertial quadratic extension of F is $F(\sqrt{5})$, and it follows from Theorem 2, Corollary 11, Corollary 17 and Theorem 18 that

$$I_{qt}(F) \simeq (\mathbb{Z}/2\mathbb{Z})^4,$$

with generators the Witt classes of the following forms: $\langle 1, -5 \rangle$, $\langle 2 \rangle \langle 1, -5 \rangle$, $\langle t \rangle \langle 1, -5 \rangle$, $\langle 2t \rangle \langle 1, -5 \rangle$.

By contrast, the full Witt groups $W(F)$, $I_q(F)$ may also be determined by using Springer's theorem for the t-adic valuation, since the Witt group of $\mathbb{Q}_2$ is known (see for instance [6, Ch. VI, Remark 2.31]); we thus get

$$W(F) \simeq (\mathbb{Z}/8\mathbb{Z})^2 \oplus (\mathbb{Z}/2\mathbb{Z})^4,$$

with generators the Witt classes of $\langle 1\rangle$, $\langle t\rangle$, $\langle 1,-2\rangle$, $\langle t\rangle\langle 1,-2\rangle$, $\langle 1,-5\rangle$, and $\langle t\rangle\langle 1,-5\rangle$. Note that $4\langle 1\rangle \simeq \langle 1,-2,-5,10\rangle$ over $\mathbb{Q}_2$ (see [6, *loc. cit.*]), hence $\langle 2\rangle\langle 1,-5\rangle$ is Witt-equivalent to $\langle 1,-5\rangle - 4\langle 1\rangle$. Therefore,

$$I_q(F)/I_{qt}(F) \simeq (\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})^3$$

with generators represented by $\langle 1,t\rangle$, $\langle 1,1\rangle$, $\langle 1,-2\rangle$, and $\langle t\rangle\langle 1,-2\rangle$.

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