CONIVEAU SPECTRAL SEQUENCES OF CLASSIFYING SPACES FOR EXCEPTIONAL AND SPIN GROUPS

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Abstract. Let k be an algebraically closed field of $ch(k) = 0$ and G be a simple simply connected algebraic group G over k. By using results of cohomological invariants, we compute the coniveau spectral sequence for classifying spaces BG.

1. Introduction

Let G be a simple simply connected algebraic group over an algebraically closed field k in $\mathbb{C}$. The cohomological invariant $Inv^*(G;\mathbb{Z}/p)$ is (roughly speaking) the ring of natural maps $H^1(F;G) \to H^*(F;\mathbb{Z}/p)$ for finitely generated field F over k. (For detailed definition and properties, see the book [Ga-Me-Se] by Garibaldi, Merkurjev and Serre.)

Let BG be the classifying space of G. Totaro showed that

$$Inv^*(G;\mathbb{Z}/p) \cong H^0(BG; H^*_{\mathbb{Z}/p})$$

where $H^*(X; H^{*'}_{\mathbb{Z}/p})$ is the cohomology of the Zarisky sheaf induced from the presheaf $H^*_{et}(V;\mathbb{Z}/p)$ for open subsets V of X. This sheaf cohomology is also the E_2-term

$$E_2^{*,*'} \cong H^*(BG; H^{*'}_{\mathbb{Z}/p}) \Longrightarrow H^*(BG;\mathbb{Z}/p)$$

of the coniveau spectral sequence by Bloch-Ogus [Bl-Og].

We restrict to consider a group G such that it has only one conjugacy class of nontoral maximal elementary abeilan p-group A. For exceptional cases, $G = G_2, F_4, E_6$ for $p = 2$, $G = F_4, E_6, E_7$ for $p = 3$, and $G = E_8$ for $p = 5$. We also consider groups $Spin_n$, $n \geq 7$.

Let $W_G(A)$ be the Weyl group of G for A. Then by using Rost, Serre and Garibaldi's results [Ga], we easily see that

$$Res_{Inv} : Inv^*(G;\mathbb{Z}/p) \cong Inv^*(A;\mathbb{Z}/p)^{W_G(A)}$$

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for cases of the above groups except for $(E_6, p=2)$ and $(Spin_n, p=2)$, $n \geq 10$.

Let Q_i be the Milnor operation and let

$$Q(n) = \Lambda(Q_0, ..., Q_n).$$

We easily see that operations Q_i can extend on $H^*(BG; H^*_{\mathbb{Z}/p})$ for these fields k. In particular, $H^*(BA; H^{*'}_{\mathbb{Z}/p})^{W_G(A)}$ has also the $Q(\infty)$-module structure. We can prove that for the above cases except for $(E_7, p=3)$, the invariant is generated as $Q(\infty)$-algebras by elements in $Res(Inv^{*'}(G; \mathbb{Z}/p))$ and $Res(H^*(BG; H^*_{\mathbb{Z}/p})) = Res(CH^*(BG)/p)$. (Moreover it is a direct sum of free $Q(n)$-modules.)

These facts imply the following theorem.

Theorem 1.1. *Let $G = G_2, Spin_n (7 \leq n \leq 9), F_4$ for $p=2$, $G = F_4, E_6$ for $p=3$, or $G = E_8$, $p=5$. Then the following restriction map*

$$Res_{E_2} : H^*(BG; H^{*'}_{\mathbb{Z}/p}) \to H^*(BA; H^{*'}_{\mathbb{Z}/p})^{W_G(A)}$$

is an epimorphism.

For $(E_7, p=3)$, the map Res_{E_2} is not epic, while Res_{Inv} is epic.

We note that the restriction map

$$Res_{H\mathbb{Z}/p} : H^*(BG; \mathbb{Z}/p) \to H^*(BA; \mathbb{Z}/p)^{W_G(A)}$$

is not an epimorphism for $p \geq 3$, while $H^*(BA; \mathbb{Z}/p) \cong H^*(BA; H^{*'}_{\mathbb{Z}/p})$ as algebras. (Indeed, BG is 3-connected but $H^0(BG; H^3_{\mathbb{Z}/p}) \neq 0$.) Note also that the right hand side invariant and $Im(Res_{H\mathbb{Z}/p})$ are computed by Kameko-Mimura [Ka-Mi] for odd primes p.

When $p=2$, the maps $Res_{H\mathbb{Z}/2}$ are even isomorphic except for the case E_6. However note ([Or-Vi-Vo]))

$$H^*(BA; H^{*'}_{\mathbb{Z}/2}) \cong grH^*(BA; \mathbb{Z}/2) \cong \mathbb{Z}/2[y_1, ..., y_n] \otimes \Lambda(x_1, ..., x_n)$$

with $\beta x_i = y_i$ but $y_i \neq x_i^2 = 0$ as the cases $p = odd$. So we know

$$Res_{H\mathbb{Z}/2} \cong gr(H^*(BA; \mathbb{Z}/2)^{W_G(A)}) \overset{\subset}{\neq} H^*(BA; H^{*'}_{\mathbb{Z}/2})^{W_G(A)}$$

for the above groups.

The arguments seem something subtle and we give here an example, the case $G = G_2$ and $p=2$. Then $A \cong (\mathbb{Z}/2)^3$ and $W_G(A) \cong GL_3(\mathbb{Z}/2)$, moreover

$$H^*(BG_2; \mathbb{Z}/2) \cong H^*(BA; \mathbb{Z}/2)^{W_G(A)} \cong \mathbb{Z}/2[w_4, w_6, w_7] \quad |w_i| = i.$$

The cohomological invariant is known by Rost and Serre

$$Inv^*(G_2; \mathbb{Z}/2) \cong \mathbb{Z}/2\{1, u_3\} \quad |u_3| = 3$$

where $u_3 = x_1x_2x_3$ in $H^0(BA; H^3_{\mathbb{Z}/2})$. From Mui and Kameko-Mimura results [Ka-Mi], we can show

$$H^*(BA; H^{*'}_{\mathbb{Z}/2})^{W_G(A)} \cong \mathbb{Z}/2[c_4, c_6] \otimes (\mathbb{Z}/2\{1\} \oplus \mathbb{Z}/2[c_7] \otimes Q(2)\{u_3\})$$

where $Q_0Q_1Q_2(u_3) = c_7$, $deg(c_i) = (i, i)$ (and $w_i^2 = c_i$ in $H^*(BA; \mathbb{Z}/2)$). These c_i are represented by Chern classes, and hence Res_{E_2} is an epimorphism.

Of course $u_3 \notin Res_{H\mathbb{Z}/2}$, and moreover we see

$$H^*(BA; H^{*'}_{\mathbb{Z}/2})^{W_G(A)}/Res_{H\mathbb{Z}/2} \cong \mathbb{Z}/2[c_4, c_6]\{u_3\}.$$

For example, we have

$$Q_0(u_3) = w_4, \quad Q_1(u_3) = w_6, \quad Q_0Q_1(u_3) = w_7, \quad Q_2(u_3) = w_4w_6.$$

Here u_3 does not exist in $H^*(BG_2; \mathbb{Z}/2)$, and hence $d_r(u_3) = y \neq 0$ for some $r \geq 2$ and $y \in H^*(BG_2; H^*_{\mathbb{Z}/2})$ in the coniveau spectral sequence. We can see this $r = 2$.

Theorem 1.2. *We have the epimorphism (as bidegree $Q(2)$-modules) from $H^*(BG_2; H^{*'}_{\mathbb{Z}/2})$ onto*

$$H^*(BA; H^{*'}_{\mathbb{Z}/2})^{W_G(A)} \oplus (H^*(BA; H^{*'}_{\mathbb{Z}/2})^{W_G(A)}/Res_{H\mathbb{Z}/2})(-1)[2]$$
$$\cong \mathbb{Z}/2[c_4, c_6] \otimes (\mathbb{Z}/2\{1, y\} \oplus \mathbb{Z}/2[c_7] \otimes Q(2)\{u_3\})$$

where $(-1)[2]$ is the degree shift operation so that $deg(y) = (2, 2)$. Moreover $d_2(u_3) = y$ in the coniveau spectral sequence.

Moreover if the Gottlieb transfer exists in the motivic cohomology, then the above epimorphism is indeed isomorphism. The similar fact also holds for $G = Spin_7$ and $p = 2$.

Note that y in the above theorem, is a ($mod(p)$) Griffith element, namely,

$$y \in Ker(cycle\ map : CH^*(BG)/p \to H^{2*}(BG; \mathbb{Z}/p)).$$

Each non zero element in $(H^*(BA; H^{*'}_{\mathbb{Z}/p})^{W_G(A)}/Res_{H\mathbb{Z}/p})$ of $deg = (* - 2, * + 1)$ corresponds to a Griffith element of $deg = (*, *)$. So we can construct many Griffith elements in $CH^*(BG)/p$ for the above groups G.

An outline of this paper is following. In §2, we recall the relation between the motivic cohomology $H^{*,*'}(X; \mathbb{Z}/p)$ and the sheaf cohomology $H^*(X; H^{*'}_{\mathbb{Z}/p})$. In §3, we show that $H^*(X; H^{*'}_{\mathbb{Z}/p})$ has the Q_i-action. In §4, we recall the cohomological invariant and give a sufficient condition such that Res_{E_2} is epic when $Inv^*(G; \mathbb{Z}/p)$ is known. In §5, we study the Dickson invariant for $H^*(BA; H^{*'}_{\mathbb{Z}/p})$ using Q_i actions by Kameko-Mimura. In §6 – §8, we compute $H^*(BG; H^{*'}_{\mathbb{Z}/p})$ for concrete cases, e.g., $(G_2, p = 2)$ is studied in §6. In §9, we study the relation between $H^*(BG; H^{*'}_{\mathbb{Z}/p})$ and the Brown-Peterson theory $BP^*(BG)$. In the last section, we study the image of Griffith elements to $BP^*(BG) \otimes_{BP^*} \mathbb{Z}/p$, in particular, for $(Spin_9, p = 2)$.

2. MOTIVIC COHOMOLOGY

Let X be a smooth (quasi projective) variety over a field $k \subset \mathbb{C}$. Let $H^{*,*'}(X; \mathbb{Z}/p)$ be the $mod(p)$ motivic cohomology defined by Voevodsky and Suslin ([Vo1-3]).

Recall that the ($mod\ p$) $B(n, p)$ condition holds if

$$H^{m,n}(X; \mathbb{Z}/p) \cong H^m_{et}(X; \mu_p^{\otimes n}) \qquad for\ all\ m \leq n.$$

Recently M.Rost and V.Voevodsky ([Vo5],[Su-Jo],[Ro]) proved that $B(n, p)$ condition holds for each p and n. Hence the Bloch-Kato conjecture also holds. Therefore in this paper, we *always assume* the $B(n, p)$-*condition* and also the Bloch-Kato conjecture for all n, p.

Moreover we always assume that k contains a primitive p-th *root of unity*. For these cases, we see the isomorphism $H^m_{et}(X; \mu_p^{\otimes n}) \cong H^m_{et}(X; \mathbb{Z}/p)$. Let τ be a generator of $H^{0,1}(Spec(k); \mathbb{Z}/p) \cong \mathbb{Z}/p$, so that

$$colim_i \tau^i H^{*,*'}(X; \mathbb{Z}/p) \cong H^*_{et}(X; \mathbb{Z}/p).$$

Let $H^*(X; H^{*'}_{\mathbb{Z}/p})$ be the sheaf cohomology where $H^n_{\mathbb{Z}/p}$ is the Zarisky sheaf induced from the presheaf $H^n_{et}(V; \mathbb{Z}/p)$ for open subset V of X.

Let $X = \bigcup U_\lambda$ for Zarisky open sets U_λ. The sheaf cohomology $H^*(X; H^{*'}_{\mathbb{Z}/p})$ is defined as the colimit of the cohomology of the following $\check{C}$eck complex

$$(2.1) \qquad \to \prod \Gamma_{(i_1,...,i_n)} \xrightarrow{\delta} \prod \Gamma_{(j_1,...,j_{n+1})} \to$$

$$where \quad \Gamma_{(i_1,...,i_n)} = \Gamma(U_{i_1} \cap ... \cap U_{i_n}; H^*(U_{i_1} \cap ... \cap U_{i_n}; \mathbb{Z}/p)^a)$$

and where $H^*(-; \mathbb{Z}/p)^a$ is a sheaficication of the presheaf $H^*(-; \mathbb{Z}/p)$. Here δ is induced map from the inclusions $U_i \cap U_j \subset U_i$, $U_i \cap U_j \subset U_j$.

The Beilinson and Lichtenbaum conjecture (hence $B(n, p)$-condition) (see [Vo2,5]) implies the exact sequences of cohomology theories

Theorem 2.1. *([Or-Vi-Vo], [Vo5]) There is a long exact sequence*

$$\to H^{m,n-1}(X; \mathbb{Z}/p) \xrightarrow{\times\tau} H^{m,n}(X; \mathbb{Z}/p)$$
$$\to H^{m-n}(X; H^n_{\mathbb{Z}/p}) \to H^{m+1,n-1}(X; \mathbb{Z}/p) \xrightarrow{\times\tau} .$$

In particular, we have

Corollary 2.2. *We have the additive isomorphism*

$$H^{m-n}(X;H^n_{\mathbb{Z}/p})\cong H^{m,n}(X;\mathbb{Z}/p)/(\tau)\oplus Ker(\tau)|H^{m+1,n-1}(X;\mathbb{Z}/p)$$

where $H^{m,n}(X;\mathbb{Z}/p)/(\tau)=H^{m,n}(X;\mathbb{Z}/p)/(\tau H^{m,n-1}(X;\mathbb{Z}/p))$.

Note that the long exact sequence in Theorem 2.1 induces the τ-Bockstein spectral sequence

$$E(\tau)_1=H^{m-n}(X;H^n_{\mathbb{Z}/p})\Longrightarrow colim_i\tau^iH^{*,*'}(X;\mathbb{Z}/p)\cong H^*_{et}(X;\mathbb{Z}/p).$$

On the other hand, the filtration *coniveau* is given by

$$N^cH^m_{et}(X;\mathbb{Z}/p)=\cup_Z Ker\{H^m_{et}(X;\mathbb{Z}/p)\to H^m_{et}(X-Z;\mathbb{Z}/p)\}$$

where Z runs in the set of closed subschemes of X of $codim=c$. The induced spectral sequence is called the coniveau spectral sequence. Bloch-Ogus [Bl-Og] proved that its E_2-term is given by

$$E(c)_2^{c,m-c}\cong H^c(X,H^{m-c}_{\mathbb{Z}/p}).$$

By Deligne (foot note (1) in Remark 6.4 in [Bl-Og]) and Paranjape (Corollary 4.4 in [Pj]), it is proven that there is an isomorphism of the coniveau spectral sequence with the Leray spectral sequence for the natural map of the sites. Hence we have ;

Theorem 2.3. *(Deligne, Parajape) There is the isomorphism* $E(c)_r^{c,m-c}\cong E(\tau)_{r-1}^{m,m-c}$ *for* $r\geq 2$ *of spectral sequences. Hence the filtrations are the same* $N^cH^m_{et}(X;\mathbb{Z}/p)=F_\tau^{m,m-c}$ *where*

$$F_\tau^{m,m-c}=Im(\times\tau^c:H^{m.m-c}(X;\mathbb{Z}/p)\to H^{m,m}(X;\mathbb{Z}/p)).$$

3. cohomology operation

Let $t_{\mathbb{C}}:H^{*,*'}(X;\mathbb{Z}/p)\to H^*(X(\mathbb{C});\mathbb{Z}/p)$ be the realization map ([Vo1]) for the inclusion $k\subset\mathbb{C}$. The motivic cohomology has (Bockstein, reduced powered) cohomology operations ([Vo2,4])

$$\beta:H^{*,*'}(X;\mathbb{Z}/p)\to H^{*+1,*'}(X;\mathbb{Z}/p)$$

$$P^i:H^{*,*'}(X;\mathbb{Z}/p)\to H^{*+2i(p-1),*'+i(p-1)}(X;\mathbb{Z}/p)$$

which are compatible with the usual (topological) cohomology operations by the realization map $t_{\mathbb{C}}$. Voevodsky defines the Milnor operation Q_i also in the mod p motivic cohomology

$$Q_i:H^{*,*'}(-;\mathbb{Z}/p)\to H^{*+2p^i-1,*'+p^i-1}(-;\mathbb{Z}/p).$$

Here we define the weight degree by

$$w(x)=2n-m\ (resp.\ =n'-m')$$

for $0 \neq x \in H^{m,n}(X;\mathbb{Z}/p)$ (resp. $H^{m'}(X;H^{n'}_{\mathbb{Z}/p})$). Similarly, we also define the weight degree for cohomology operations and differentials of spectral sequences,e.g.,

$$w(\tau) = 2, \quad w(P^i) = 0, \quad w(Q_j) = -1.$$

Let $\rho_p = (\xi_p) \in k^*/(k^*)^p = H^{1,1}(Spec(k);\mathbb{Z}/p)$ where ξ_p is the primitive p-th root of unity. The Q_i operation has the same property as the topological case only with $mod(\rho_2)$. For example, Q_i is a derivative only $mod(\rho_2)$.

Let A_p be the mod p Steenrod algebra generated by all cohomology operations on $H^{*,*'}(X;\mathbb{Z}/p)$. (Voevodsky proved that A_p is multiplicatively generated by elements in $H^{*,*'}(Spec(k);\mathbb{Z}/p)$, P^j and Q_i.)

Lemma 3.1. *Suppose $\rho_p = 0$. Then the Steenrod algebra A_p acts on the etale cohomolgy $H^*(X;\mathbb{Z}/p)$.*

Proof. In $H^{*,*'}(Spec(k);\mathbb{Z}/p)$, we know

$$P^i(\tau) = 0 \ for\ i > 0, \quad and \quad \beta(\tau) = \rho_p = 0.$$

When $p \geq 3$, the Cartan formula holds in the motivic cohomology (Proposition 9.6 in [Vo4]), and we have

$$P^i(\tau x) = \tau P^i(x) \ for\ i > 0, \quad and \quad \beta(\tau x) = \tau\beta(x).$$

From the $B(n,p)$ condition, $H^*_{et}(X;\mathbb{Z}/p) = colim_i \tau^i H^{*,*'}(X;\mathbb{Z}/p)$, which implies the lemma.

For $p = 2$, we also know from Proposition 9.6 in [Vo4],

$$Sq^{2*}(xy) = \sum_i Sq^{2i}(x)Sq^{2*-2i}(y) + \tau \sum_i Sq^{2i+1}(x)Sq^{2*-2i-1}(y),$$

$$Sq^{2*+1}(xy) = \sum_j Sq^j(x)Sq^{2*+1-j}(y) + \rho_2 \sum_i Sq^{2i+1}(x)Sq^{2*-2i-1}(y).$$

Since $\rho_2 = 0$, we see $Sq^{2i+1}(\tau) = 0$, and so $Sq^*(\tau x) = \tau Sq^*(x)$. This also induces the lemma. □

Theorem 3.2. *Suppose $\rho_p = 0$. Then the cohomology operation Q_i and P^i can be extended on the τ-Bockstein spectral sequence and so on the coniveau spectral sequence E_r, $r \geq 2$ (e.g., on $H^*(X;H^{*'}_{\mathbb{Z}/p})$).*

Proof. In the stable $\mathbb{A}^1$-homotopy category $SHot$, let $H\mathbb{Z}/p$ be the Eilenberg-MacLane spectrum representing the mod p motivic cohomology

$$H^{*,*'}(X;\mathbb{Z}/p) \cong Hom_{SHot}(X, S^{*,*'} \wedge H\mathbb{Z}/p)$$

where $S^{*,*'}$ is the sphere of bidegree $(*,*')$.

Let $op. = Q_i$ or P^i of bidegree (m,n). Consider the diagram

$$\begin{array}{ccccc} S^{0,1}\wedge H\mathbb{Z}/p & \xrightarrow{\times\tau} & H\mathbb{Z}/p & \xrightarrow{\rho} & cone \\ {\scriptstyle op.}\downarrow & & {\scriptstyle op.}\downarrow & & \\ S^{m,n+1}\wedge H\mathbb{Z}/p & \xrightarrow{\times\tau} & S^{m,n}\wedge H\mathbb{Z}/p & \xrightarrow{\rho} & S^{m,n}\wedge cone. \end{array}$$

Here *cone* is the mapping cone of τ so that

$$H^{*+*'}(X;H^{*'}_{\mathbb{Z}/p})\cong Hom_{SHot}(X,S^{*,*'}\wedge cone).$$

In the above diagram, we see $\rho\cdot op.\cdot\tau=0$. Hence there is a map $op'.: cone\to S^{m,n}\wedge cone$ such that $\rho\cdot op. = op'.\cdot\rho$. □

Here we do not see yet that A_p acts on E_r, e.g., we do not see that Q_i generates the exterior algebra $Q(\infty)$. However when $r=2$, the following theorem holds.

Lemma 3.3. *Let k be an algebraically closed field. Then the Steenrod algebra A_p acts on $H^*(X;H^{*'}_{\mathbb{Z}/p)})$.*

Proof. Recall that $H^*(X;H^{*'}_{\mathbb{Z}/p})$ is defined as the cohomology of the Čeck complex. Given $op.\in A_p$, by the universality of sheaficication, the following diagram from (2.1) is commutative

$$\begin{array}{ccc} \prod\Gamma_{(i_1,...,i_n)} & \xrightarrow{\delta} & \prod\Gamma_{(j_1,...,j_{n+1})} \\ {\scriptstyle op.}\downarrow & & {\scriptstyle op.}\downarrow \\ \prod\Gamma_{(i_1,...,i_n)} & \xrightarrow{\delta} & \prod\Gamma_{(j_1,...,j_{n+1})}. \end{array}$$

Thus we have the desired result. □

Let us write $H^{*,*'}=H^{*,*'}(Spec(k);\mathbb{Z}/p)$ and $H^*=K^*_M(k)/p$ so that $H^{*,*'}\cong H^*[\tau]$. (Note if k is algebraically closed, $H^{*,*'}\cong\mathbb{Z}/p[\tau]$.) For an elementary abelian p-group $A=A_n\cong(\mathbb{Z}/p)^n$, the $mod(p)$ motivic cohomology is given by Voevodsky ([Vo2,4])

$$H^{*,*'}(BA;\mathbb{Z}/p)\cong H^{*,*'}[y_1,...,y_n]\otimes\Delta(x_1,..,x_n)$$

with $x_i^2=y\tau+x\rho_2$ for $p=2$ and $x_i^2=0$ otherwise.

Since $Ker(\tau)|H^{*,*'}(BA;\mathbb{Z}/p)=0$, from Corollary 2.2, we have

$$H^*(BA;H^{*'}_{\mathbb{Z}/p})\cong H^{*,*}(BA;\mathbb{Z}/p)/(\tau H^{*,*-1}(BA;\mathbb{Z}/p))$$
$$\cong H^*[y_1,...,y_n]\otimes\Lambda(x_1,...,x_n)\quad (mod(\rho_2))$$

for all primes p. Each Q_i is a derivation $mod(\rho_2)$, and hence

$$Q_0...Q_{s-1}(x_1...x_s)=\sum sgn(j_1,...,j_s)y_1^{p^{j_1}}y_2^{p^{j_2}}...y_s^{p^{j_s}}\neq 0\quad mod(\rho_2)$$

where $(j_1, ..., j_s)$ are permutations of $(0, ..., s-1)$.

Let us write

$$Q(n) = \Lambda(Q_0, ..., Q_n),$$
$$\bar{Q}(n) = Q(n) - \mathbb{Z}/p\{Q_0...Q_n\} = \mathbb{Z}/p\{Q_{i_0}...Q_{i_s} | 0 \le i_k \le n,\ s < n\}.$$

Let $u_i = x_1...x_i \in H^0(BA; H^i_{\mathbb{Z}/p})$. For example, we have

$$H^*(BA; H^{*'}_{\mathbb{Z}/p}) \supset H^*[y_1, ..., y_n] \otimes (\oplus_i \bar{Q}(i-1)\{u_i\})$$
$$\supset \oplus_i H^*[y_1, ..., y_i] \otimes Q(i-1)\{u_i\} \quad (*)$$

since $Q_0...Q_{i-1}(u_i) \in H^*[y_1, ..., y_i]\{y_1...y_i\}$. In sections bellow, we show that the last sum $(*)$ of free $Q(i-1)$-modules contains $H^*(BG; H^{*'}_{\mathbb{Z}/p})^{W_G(A)}$, as a direct summand , for many cases of G. (See Assumption (1) in §4.)

4. COHOMOLOGICAL INVARIANT

Let G be a linear algebraic group over k. Recall that $H^1(k; G)$ is the first non abelian Galois cohomology set of G, which represents the set of G-torsors over k. The cohomology invariant is defined by

$$Inv^i(G, \mathbb{Z}/p) = Func(H^1(F; G) \to H^i(F; \mathbb{Z}/p))$$

where $Func$ means natural functions for each field F which is finitely generated over k. (For details for the definition or properties, see the book [Ga-Me-Se].)

Totaro proved [Ga-Me-Se] the following theorem in the letter to Serre.

Theorem 4.1. *(Totaro)* $Inv^*(G; \mathbb{Z}/p) \cong H^0(BG; H^*_{\mathbb{Z}/p})$.

Hereafter (throughout this paper), we assume that k is an *algebraically closed* field in $\mathbb{C}$. Moreover, in this paper, we only consider *simple simply connected* groups G which have the following property. we assume that the algebraic group G has only *one* conjugacy class A of non toral maximal elementary abelian p-subgroups. Exceptional groups are

$$G = \begin{cases} G_2, F_4, E_6 & for\ p = 2 \\ F_4, E_6, E_7 & for\ p = 3 \\ E_8 & for\ p = 5. \end{cases}$$

For spin groups $Spin_n$, we consider the cases $n \le 9$ only in this paper.

We consider the restriction maps (of cohomology) to A and the maximal torus T_G

$$Res_{H\mathbb{Z}/p} : H^*(BG; \mathbb{Z}/p) \xrightarrow{i^*} H^*(BT_G; \mathbb{Z}/p) \times H^*(BA; \mathbb{Z}/p)$$
$$\xrightarrow{pr.} H^*(BA; \mathbb{Z}/p)^{W_G(A)}.$$

By the Quillen's theorem the above i^* has nilpotent kernel. More strongly, Toda, Kono, Tezuka and Kameko show i^* is really injective, namely, $H^*(BG;\mathbb{Z}/p)$ is detected by A and T_G. Moreover when $p=2$, $Res_{H\mathbb{Z}/2}$ are isomorphic except the case E_6. However $Res_{H\mathbb{Z}/p}$ is not epic for $p\geq 3$.

On the other hand, by Serre, Rost and Garibaldi([Ga-Me-Se],[Ga]), $Inv^*(G;\mathbb{Z}/p)$ are computed for these groups, e.g.,

$$Inv^*(G;\mathbb{Z}/p)\cong\begin{cases}\mathbb{Z}/p\{1,u_3\} \quad for\ (G_2,E_6,p=2),\ (F_4,E_7,p=3),\\ \qquad\qquad (E_8,p=5)\\ \mathbb{Z}/p\{1,u_3,u_4\} \quad for\ (Spin_7,p=2),\ (E_6,p=3)\\ \mathbb{Z}/p\{1,u_3,u_4,u_5\} \quad for\ (Spin_9,p=2)\\ \mathbb{Z}/p\{1,u_3,u_4,u_4',u_5\} \quad for\ (Spin_8,p=2)\\ \mathbb{Z}/p\{1,u_3,u_5\} \quad for\ (F_4,p=2).\end{cases}$$

(Moreover Rost and Garibaldi determined $Inv^*(Spin_n;\mathbb{Z}/2)$ for $n\leq 12$).

For these groups, we note (Ga-Me-Se],[Ga]) the the restriction

$$Res_{Inv}:Inv^*(G;\mathbb{Z}/p)\to Inv^*(A;\mathbb{Z}/p)\cong\Lambda(x_1,...,x_n).$$

is injective (identifying $u_i=x_1...x_i$ and $u_4'=x_1x_2x_3x_5$). We will show the following theorem in §$6-8$ bellow (by computations of concrete cases)

Theorem 4.2. *Let G be an above type except for $G=E_6$ and $p=2$. Then*

$$Res_{Inv}:Inv^*(G;\mathbb{Z}/p)\cong Inv^*(A;\mathbb{Z}/2)^{W_G(A)}.$$

Remark. When $G=E_6$ and $p=2$, the above Res_{Inv} is not epic.

We want to extend above isomorphism in the theorem to say that

$$Res_{E_2}:H^*(BG;H^{*'}_{\mathbb{Z}/p})\to H^*(BA;H^{*'}_{\mathbb{Z}/p})^{W_G(A)}$$

is an epimorphism. (Of course for $p\geq 3$ the above map is not injective.) We will prove the following assumption (in the sections bellow) for the above groups except for $(E_6,p=2)$ and $(E_7,p=3)$. (When $G=Spin_8$, some modification of Assumption (1) holds.)

Assumption When $Inv^*(G;\mathbb{Z}/p)\cong\mathbb{Z}/p\{1,u_{i_1},...,u_{i_m}\}$, there is a bidegree isomorphism

$$(1)\quad H^*(BA;H^{*'}_{\mathbb{Z}/p})^{W_G(A)}\cong\oplus_{s=1}^m\mathbb{Z}/p[f_{s1},...,f_{sk_s}]\otimes Q(i_s-1)\{u_{i_s}\}$$

$$(2)\quad f_{st}\in Res_{H\mathbb{Z}/p}(H^{2*,*}(BG;\mathbb{Z}/p))=Res(CH^*(BG)/p)$$
$$for\ all\ 1\leq s\leq m,\ 1\leq t\leq k_s.$$

If this assumption is satisfied then $H^*(BA;H^{*'})^{W_G(A)}$ is generated as bidegree $Q(\infty)$-algebra by u_{i_s} and $Res(CH^*(BG)/p)$. Hence the surjectivity of Res_{E_2} is immediate.

Lemma 4.3. *If Assumption (1),(2) are satisfied, then*

$$Res_{E_2} : H^*(BG; H^{*'}_{\mathbb{Z}/p}) \to H^*(BA; H^{*'}_{\mathbb{Z}/p})^{W_G(A)}$$

is an epimorphism.

Thus we can prove Theorem 1.1 in the introduction. As for the statements of differential and (Griffith elements), the following lemma is useful.

Lemma 4.4. *Let $Res_{Inv}(a) \neq 0$ for $a \in Inv^i(G;\mathbb{Z}/p) = H^0(BG; H^*_{\mathbb{Z}/p})$. (Namely, the above element is a permanent cycle in the coniveau spectral sequence.) Moreover let*

$$Q_{j_1}...Q_{j_{i-3}}(a) \notin H^*(BG;\mathbb{Z}/p).$$

Then $d_2(a) = y \neq 0 \in H^2(BG; H^{i-1}_{\mathbb{Z}/p})$ in the coniveau spectral sequence, and elements

$$Q_{j_1}...Q_{j_{i-3}}(y) \neq 0 \in CH^*(BG)/p = H^{2*,*}(BG;\mathbb{Z}/p)$$

are Griffith elements (i.e., in the kernel of $CH^{2*}(BG)/p \to H^{2*}_{et}(BG;\mathbb{Z}/p)$).

Proof. Take $q = Q_{j_1}...Q_{j_{i-3}}(a)$. Since q does not exist in $H^*(BG;\mathbb{Z}/p)$, we see $d_r(q) \neq 0$ in the spectral sequence for some r.

This $r = 2$ because the following reason of weight degree. First note

$$w(d_r) = wt(1, 1-r) = 2(1-r) - 1 = 1 - 2r.$$

Since $w(q) = w(a) - (i-3) = 3$, we have

$$w(d_r(q)) = 3 + 1 - 2r = 4 - 2r.$$

If $r \geq 3$, then the above weight is negative and $d_r(q) = 0$.

This implies that $d_2(a) \neq 0$. Otherwise

$$d_2(q) = d_2(Q_{j_1}...Q_{j_{i-3}}a) = Q_{j_1}...Q_{j_{n-3}}(d_2(a)) = 0,$$

which is a contradiction. □

5. Dickson invariant

At first we assume $p \geq 3$. Dickson computed the ring of invariants of $\mathbb{Z}/p[y_1, ..., y_n]$ with respect to the action of $GL_n(\mathbb{Z}/p)$. The ring of invariants is a polynomial algebra

$$D_n = \mathbb{Z}/p[y_1, ..., y_n]^{GL_n(\mathbb{Z}/p)} \cong \mathbb{Z}/p[c_{n,0}, ..., c_{n,n-1}]$$

where the generators are given by the equation

$$\mathcal{O}_n(X) = \prod_{y \in \mathbb{Z}/p\{y_1,\cdots,y_n\}} (X+y) = X^{p^n} + \sum_{j=0}^{n-1} (-1)^{n-j} c_{n,j} X^{p^j}.$$

Let $reg : A \to GL_n(\mathbb{C})$ be the regular representation and $c(reg)$ the total Chern class. Then it is well known that

$$c(reg) = \mathcal{O}_n(1) = 1 - c_{n,n-1} + ... + (-1)^n c_{n,0}.$$

We also note the following lemma.

Lemma 5.1. *(Lemma 2.3,2.4 in [Ka-Ya2]) Let $\rho : A_n \to GL_m(\mathbb{C})$ be an representation such that $c(\rho) \in H^*(BA;\mathbb{Z}/p)^{SL_n(\mathbb{Z}/p)}$. Then $c(\rho) = c(reg)^a$ for some $a \geq 0$.*

For the invariant ring SD_n under $SL_n(\mathbb{Z}/p)$, we have

$$\begin{aligned} SD_n &= \mathbb{Z}/p[y_1, ..., y_n]^{SL_n(\mathbb{Z}/p)} \\ &\cong D_n\{1, e_n, ..., e_n^{p-2}\} \quad with\ e_n^{p-1} = c_{n,0} \\ &\cong D_n' \otimes \mathbb{Z}/p[e_n] \quad with\ D_n' = \mathbb{Z}/p[c_{n,1}, ..., c_{n,n-1}]. \end{aligned}$$

Mui computed the ring of invariants of

$$H^*(BA; H^{*\prime}_{\mathbb{Z}/p}) \cong \mathbb{Z}/p[y_1, ..., y_n] \otimes \Lambda(x_1, ...x_n)$$

with respect to the action of $SL_n(\mathbb{Z}/p)$. (In fact, Mui studied $H^*(BA;\mathbb{Z}/p)$ for odd prime p, however we study $H^*(BA; H^{*\prime}_{\mathbb{Z}/p})$ for all primes.) Of course $u_n = x_1...x_n$ is invariant under $SL_n(\mathbb{Z}/p)$. In terms of Milnor's operation, we may state Mui's result in the following form.

Theorem 5.2. *(Mui[Mu], Kameko-Mimura [Ka-Mi])*

$$\begin{aligned} H^*(BA; H^{*\prime}_{\mathbb{Z}/p})^{SL_n(\mathbb{Z}/p)} &\cong \mathbb{Z}/p[e_n, c_{n,1}, ..., c_{n,n-1}] \otimes (\mathbb{Z}/p\{1\} \oplus \bar{Q}(n-1)\{u_n\}) \\ &\cong D_n' \oplus SD_n \otimes Q(n-1)\{u_n\} \end{aligned}$$

where $Q_0...Q_{n-1}u_n = e_n$.

The Q_i-operation acts on u_n as follows.

Lemma 5.3. *(Kameko-Mimura [Ka-Mi],[Ka-Ya1]) For $x \in H^*(BA; H^{*\prime}_{\mathbb{Z}/p})$, it holds*

$$(Q_n + \sum_{i=0}^{n-1} (-1)^{n-i} c_{n,i} Q_i)(x) = 0, \quad Q_0...\hat{Q}_i...Q_n(u_n) = c_{n,i} e_n.$$

Let $U_n \subset SL_n(\mathbb{Z}/p)$ be the maximal unipotent subgroup generated by upper triangular matrices with diagonals 1, so that U_n is a Sylow p-subgroup of $SL_n(\mathbb{Z}/p)$. The invariant under this group is given by Mui, Kameko-Mimura.

Theorem 5.4. *(Kameko-Mimura Theorem 4.2 in [Ka-Mi]) Let* $G' \subset GL_n(\mathbb{Z}/p)$ *such that* $\mathbb{Z}/p[y_1,...,y_n]^{G'} \cong \mathbb{Z}/p[f_1,...,f_n]$ *and*

$$H^*(BA_n; H^{*'}_{\mathbb{Z}/p})^{G'} \cong \mathbb{Z}/p[f_1,...,f_n]\{v_1 = 1,...,v_{2^n}\}.$$

Then the invariant under $G = \langle G', U_{n+1}\rangle \subset GL_{n+1}(\mathbb{Z}/p)$ *is given by*

$$(1) \quad \mathbb{Z}/p[y_1,...,y_{n+1}]^G \cong \mathbb{Z}/p[f_1,...,f_n,\mathcal{O}_n(y_{n+1})]$$

$$(2) \quad H^*(BA_{n+1}; H^{*'}_{\mathbb{Z}/p})^G \cong$$

$$\mathbb{Z}/p[f_1,...,f_n,\mathcal{O}_n(y_{n+1})] \otimes (\mathbb{Z}/p\{v_1,...,v_{2^n}\} \oplus Q(n-1)\{u_{n+1}\}).$$

Corollary 5.5.

$$H^*(BA; H^{*'}_{\mathbb{Z}/p})^{U_n} \cong \oplus_{i=0}^n \mathbb{Z}/p[\mathcal{O}_0(y_1),...,\mathcal{O}_{i-1}(y_i)] \otimes Q(i-1)\{u_i\}.$$

Corollary 5.6. *(Lemma 5.8 in [Ka-Ya])*

$$\mathcal{O}(y_{n+1})u_n = (Q_n + \sum_{i=0}^{n-1}(-1)^{n-i}c_{n,i}Q_i)(u_{n+1}).$$

Hereafter this section, we assume $p = 2$. Of course we have the isomorphism $H^*(BA;\mathbb{Z}/2) \cong \mathbb{Z}/2[x_1,...,x_n]$ and its invariant under $GL_n(\mathbb{Z}/2) = SL_n(\mathbb{Z}/2)$ is

$$\mathbb{Z}/2[x_1,...,x_n]^{GL_n(\mathbb{Z}/2)} \cong \mathbb{Z}/2[d_{n,0},...,d_{n,n-1}]$$

where the generators are given by the equation

$$\mathcal{O}'_n(X) = \prod_{x\in\mathbb{Z}/p\{x_1,\cdots,x_n\}}(X+x) = X^{2^n} + \sum_{j=0}^{n-1} d_{n,j}X^{2^j}.$$

Here $d_{n,i}^2 = c_{n,i}$ in $H^*(BA;\mathbb{Z}/2)$ identifying $y_i = x_i^2$. The Milnor Q_i-operations (see (2.6) in Schuster-Yagita [Sc-Ya]) are given as the case p odd. (Hereafter let us write $d_{n,i}$ by d_i simply.)

$$d_0 = Q_0...Q_{n-2}(u_n), \qquad d_i = Q_0...\hat{Q}_{i-1}...Q_{n-1}(u_n)/d_0.$$

From Lemma 2.1 in [Sc-Ya], we have

$$(*) \quad Q_{n-1}(d_i) = d_0d_i, \quad Q_{i-1}(d_i) = d_0.$$

In $H^*(BA; H^{*'}_{\mathbb{Z}/2})$, we can get more strong result. Let us write simply

$$I(GL_n) = gr(H^*(BA_n;\mathbb{Z}/2)^{GL_n(\mathbb{Z}/2)}) \subset H^*(BA; H^{*'}_{\mathbb{Z}/2})$$

$$Igr(GL_n) = H^*(BA_n; H^{*'}_{\mathbb{Z}/2})^{GL_n(\mathbb{Z}/2)}.$$

By Kameko-Mimura theorem, we have showed

$$Igr(GL_n) \cong D'_n \oplus D_n \otimes Q(n-1)\{u_n\})$$

where $D_n = \mathbb{Z}/2[c_{n,n-1},...,c_{n,0}]$ *and* $D'_n = \mathbb{Z}/2[c_{n,n-1},...,c_{n,1}]$.

Lemma 5.7. *In $H^*(BA; H^{*'}_{\mathbb{Z}/2})$, we have*

$$d_i = Q_0...\hat{Q}_{i-1}...\hat{Q}_{n-1}(u_n), \quad d_id_0 = Q_0...\hat{Q}_{i-1}...Q_{n-1}(u_n).$$
$$d_id_j = Q_0...\hat{Q}_{i-1}...\hat{Q}_{j-1}...Q_{n-1}(u_n) \quad i \neq j.$$

Proof. Consider the element

$$a_i = Q_0...\hat{Q}_{i-1}...Q_{n-2}\hat{Q}_{n-1}(u_n) \in Igr(GL_n).$$

Then $Q_{i-1}(a_i) = d_0$ and $Q_{n-1}a_i = d_0d_i$.

Using property $(*)$, we see $Q_j(a_i - d_i) = 0$ for all j. Of course $a_i - d_i \in Igr(GL_n)$. From Kameko-Mimura theorem, we still know

$$Igr(GL_n) \cap \cap_j Ker(Q_j) = D_n.$$

This means $d_i = a_i \in Igr(GL_n)$. (In fact $d_i = a_i \; mod(D_n)$ in $H^*(BA_n; \mathbb{Z}/2)$, but $d_i = a_i$ exactly in $H^*(BA_n; H^{*'}_{\mathbb{Z}/2})$.)

Therefore we have

$$d_i = Q_0...\hat{Q}_{i-1}...\hat{Q}_{n-1}(u_n), \quad d_id_0 = Q_0...\hat{Q}_{i-1}...Q_{n-1}(u_n).$$

By the similar arguments, we have

$$d_id_j = Q_0...\hat{Q}_{i-1}...\hat{Q}_{j-1}...Q_{n-1}(u_n).$$

□

Of course $I(GL_n) \subset Igr(GL_n)$, but this injection is not an isomorphism for $n \geq 3$.

Lemma 5.8. *Let $n \geq 3$. Then we have*

$$Igr(GL_n)/I(GL_n) \supset Q(n-1)/(Q(n-1)^+)^{n-2}\{u_n\}.$$

Proof. Consider the element

$$x = Q_0...\hat{Q}_{i-1}...\hat{Q}_{j-1}...\hat{Q}_{k-1}...Q_{n-1}(u_n).$$

Its image $Q_{i-1}(x) = d_jd_k$ or d_j (when $k = n$.) Hence x is not in $I(GL_n)$ because Q_i maps n-product elements into also n-product elements. (If $x \in I(GL_n)$, then x must be a sum of d_i or d_id_j, but it still appeared in $(Q(n-1)^+)^{n-2}(u_n)$.) Thus we get the lemma. □

Let $A = A_n$ be a maximal elementary abelian p-subgroup of G and $W_G(A)$ its Weyl group.

Lemma 5.9. *If $Res_{Inv} : Inv^*(G; \mathbb{Z}/p) \to Inv^*(A_n; \mathbb{Z}/p)^{W_G(A)}$ is an epimorphism, then*

$$(Q(n-1)^+)^{n-2}\{u_n\} \subset Res_{H_{\mathbb{Z}/p}}(H^*(BG; \mathbb{Z}/p) \to H^*(BA_n; \mathbb{Z}/p))$$

(e.g., $d_1, ..., d_{n-1}$ for $p = 2$ are in $Res_{H_{\mathbb{Z}/2}}$).

Proof. Let x be an element in $H^*(BG;H^{*\prime}_{\mathbb{Z}/p})$ with

$$Res_{E_2}(x)=Q_{i_1}...Q_{i_{n-2}}(u_n).$$

Then $w(x)=n-(n-2)=2$. However the weight of the differential d_r of the coniveau spectral sequence is $w(d_r)=1-2r$ (see the proof of Lemma 4.4). Hence $d_r(x)=0$ for $r\geq 2$, namely, x is a permanent cycle and is an element in $H^*(BG;\mathbb{Z}/p)$. □

6. $W_G(A)\cong SL_3(\mathbb{Z}/p)$

We consider the cases of $A\cong(\mathbb{Z}/p)^3$ and $W_G(A)\cong SL_3(\mathbb{Z}/p)$, namely, $(G_2,2)$, $(F_4,3)$ and $(E_8,5)$. These cases

$$Inv^*(G;\mathbb{Z}/p)\cong\mathbb{Z}/p\{1,u_3\}.$$

(These u_3 are called the Rost invariants.)

Let $G=G_2$ and $p=2$. It is well known that

$$H^*(BG;\mathbb{Z}/2)\cong I(GL_3)\cong\mathbb{Z}/2[w_4,w_6,w_7]$$

where w_i is the Stiefel-Whitney class of $G_2\subset SO_7$. We can identify

$$w_4=d_{3,2},\quad w_6=d_{3,1},\quad w_7=d_{3,0}.$$

On the other hand, Kameko-Mimura theorem implies

$$Igr(GL_3)\cong\mathbb{Z}/2[c_4,c_6]\otimes(\mathbb{Z}/2\{1\}\oplus\mathbb{Z}/2[c_7]\otimes Q(2)\{u_3\}).$$

Also by using Lemma 5.8, we can show

$$Igr(GL_3)/I(GL_3)\cong\mathbb{Z}/2[c_4,c_6]\{u_3\}.$$

In fact (from Lemma 5.7 or from dimensional reason), we have

$$Q_0(u_3)=w_4,\ Q_1(u_3)=w_6,\ Q_0Q_1(u_3)=w_7,\ Q_2(u_3)=w_4w_6.$$

Moreover we note

$$c_7u_3=w_4w_6w_7$$

because both above elements are same after acting Q_i, e.g. $Q_0(c_7u_3)=c_7w_4=Q_0(w_4w_6w_7)$ (see the proof of Lemma 5.7 or see [Ya2]).

Therefore Assumption (1),(2) are satisfied. Moreover from Lemma 4.4, we see $d_2(u_3)=y\neq 0$. Therefore we have the following theorem.

Theorem 6.1. *There is a $Q(2)$ bidegree module epimorphism from $H^*(BG_2;H^{*\prime}_{\mathbb{Z}/2})$ to*

$$Igr(GL_3)\oplus\mathbb{Z}/2[c_4,c_6]\{y\}\cong\mathbb{Z}/2[c_4,c_6]\otimes(\mathbb{Z}/2\{1,y\}\oplus\mathbb{Z}/2[c_7]\otimes Q(2)\{u_3\}).$$

Remark. If there is a Gottlib transfer in the motivic theory $H^{*,*'}(-;\mathbb{Z}/2)$ or the sheaf theory $H^*(-;H^*_{\mathbb{Z}/2})$, then the above epimorphism is in fact an isomorphism.

Next we consider the odd prime cases i.e., $(G,p)=(F_4,3)$ or $(E_8,5)$. From Kameko-Mimura theorem, we also have

$$Igr(SL_3)\cong D_3'\oplus SD_3\otimes Q(2)\{u_3\}.$$

Moreover from Kameko (Lemma 5.2 in [Ka-Ya1]), it is known that

$$Igr(SL_3)/Res_{H\mathbb{Z}/p}\cong SD_3/(e)\{u_3\}$$

as the case $(G_2,2)$. Hence Assumption (1) satisfied and Lemma 4.4 can be applied so that $d_2(u_3)=y$.

To see Assumption (2), we consider the representations. We consider the case $(E_8,5)$. (The case $(F_4,3)$ is similar.) It is known that there is a non trivial representation ([Ad], [Ka-Ya2])

$$\rho: E_8\to SO(248).$$

We consider the total Chern class of the representation $\rho|A$ for $A\cong(\mathbb{Z}/5)^3$,

$$c(\rho|A)=(1-c_{3,2}+c_{3,1}-c_{3,0})^a\quad for\ a\ge 0$$

from Lemma 5.1. Since $\rho|A$ is non trivial, $a\ge 1$. Moreover

$$|c_{3,0}|=2(5^3-1)=248.$$

So $a=1$. This means that $c_{3,i}$ are represented by Chern classes. (We also note $c_{3,1}=P^1c_{3,2}$ for the reduced power operation P^1.) Hence $w(c_{3,1})=w(c_{3,2})=0$. Thus we can see Assumption (2).

Theorem 6.2. *Let $(G,p)=(F_4,3)$ or $(E_8,5)$. Then there is an epimorphism of $Q(2)$-bidegree modules from $H^*(BG;H^{*'}_{\mathbb{Z}/p})$ to*

$$\mathbb{Z}/p[c_{3,2},c_{3,1}]\otimes(\mathbb{Z}/p\{1,y\}\oplus\mathbb{Z}/p[e_3]\otimes Q(2)\{u_3\}).$$

7. $W_G(A)\cong\langle U_4,SL_3(\mathbb{Z}/p)\rangle$

We consider the cases of $A\cong(\mathbb{Z}/p)^4$ and

$$W_4=W_G(A)\cong\langle U_4,SL_3(\mathbb{Z}/p)\rangle,$$

namely, $(Spin_7,2)$, $(E_6,3)$. For these cases, we have the isomorphism

$$Inv^*(G;\mathbb{Z}/p)\cong\mathbb{Z}/p\{1,u_3,u_4\}.$$

We also study the case $(E_7,3)$, while the above facts do not satisfied.

Let $G=Spin_7$ and $p=2$. It is well known that

$$H^*(BG;\mathbb{Z}/2)\cong I(W_4)=\mathbb{Z}/2[x_1,...,x_4]^{W_4}$$

$$\cong \mathbb{Z}/2[w_4, w_6, w_7, w_8]$$

where w_8 is the Stiefel-Whitney class of some spin representation. We can identify $w_8 = \mathcal{O}_3'(x_4)$.

Lemma 7.1.

$$Igr(W_4) \cong Igr(GL_3) \oplus \mathbb{Z}/2[c_4, c_6] \otimes (\mathbb{Z}/2[c_8]\{c_8\} \oplus \mathbb{Z}/2[c_7, c_8] \otimes Q(3)\{u_4\}).$$

Proof. Recall that using $\bar{Q}(2)$, we have $(Q_2Q_1Q_0(u_3) = c_7)$

$$Igr(GL_3) \cong \mathbb{Z}/2[c_4, c_6, c_7] \otimes (\mathbb{Z}/2\{1\} \oplus \bar{Q}(2)\{u_3\}).$$

By Theorem 5.4, we have

$$Igr(W_4) \cong \mathbb{Z}/2[c_4, c_6, c_7, c_8] \otimes (\mathbb{Z}/2\{1\} \oplus \bar{Q}(2)\{u_3\} \oplus Q(2)\{u_4\})$$

$$\cong \mathbb{Z}/2[c_4, c_6] \otimes (\mathbb{Z}/2[c_8]\{1\} \oplus \mathbb{Z}/2[c_7] \otimes Q(2)\{u_3\} \oplus \mathbb{Z}/2[c_7, c_8] \otimes Q(3)\{u_4\}).$$

The last isomorphism is shown by using the following facts. From Lemma 5.6, we can see (Lemma 5.8 in [Ka-Ya2])

$$\mathcal{O}_3(y_4)u_3 = (Q_3 + c_{3,2}Q_2 + c_{3,1}Q_1 + c_{3,0}Q_0)(u_4),$$

namely, $Q_3(u_4) = c_8u_3 + c_4Q_2(u_4) + c_6Q_1(u_4) + c_7Q_0(u_4)$. Hence

$$Q(2)\{u_4\} \oplus Q(2)\{c_8u_3\} \cong Q(3)\{u_4\}.$$

Using $Q_0Q_1Q_2Q_3(u_4) = c_7c_8$, we get the last isomorphism. □

Lemma 7.2.

$$Igr(W_4)/I(W_4) \cong Igr(GL_3)/I(GL_3) \oplus$$
$$\mathbb{Z}/2[c_4, c_6, c_8] \otimes \{1, Q_0, ..., Q_3\}\{u_4\}.$$

Proof. At first, we see

$$Q_1Q_0(u_4) = Q_1Q_0(u_3x_4) = Q_1(w_4x_4 + u_3y_4)$$
$$= w_7x_4 + w_4y_4^2 + w_6y_4 = w_8 \quad in\ H^*(BA; H_{\mathbb{Z}/2}^{*'}).$$

(This fact also follows from $d_{4,3} = w_8$ and Lemma 5.7.) Similarly, we can compute the Q_i action on u_4, which is given as $Q_0Q_1(u_4) = w_8$, $Q_0Q_2(u_4) = w_4w_8$, $Q_1Q_2(u_4) = w_6w_8$, $Q_0Q_3(u_4) = c_8w_4$, $Q_1Q_3(u_4) = c_8w_6$, $Q_2Q_3(u_4) = c_8w_4w_6$.

Moreover we have

$$c_7u_4 = c_7u_3x_4 = w_4w_6w_7x_4 = w_4w_6w_8.$$

Therefore $Q_iu_3 \notin I(W_4)$ but $Q_iQ_j(u_3) \in I(W_4)$. Thus we have

$$Igr(W_4)/I(W_4) \cong \mathbb{Z}/2[c_4, c_6] \otimes (\mathbb{Z}/2\{u_3\} \oplus \mathbb{Z}/2[c_8]\{1, Q_0, ..., Q_3\}\{u_4\}).$$

□

Therefore Assumption (1),(2) are satisfied. Therefore we can compute ;

Theorem 7.3. *There is a $Q(2)$ bidegree module epimorphism from $H^*(BSpin_7; H^{*'}_{\mathbb{Z}/2})$ to*

$$H^*(BG_2; H^{*'}_{\mathbb{Z}/2}) \oplus \mathbb{Z}/2[c_4, c_6, c_8] \otimes (\mathbb{Z}/2\{c_8, y_2', Q_0y_2', ..., Q_3y_2'\} \oplus \mathbb{Z}/2[c_7] \otimes Q(3)\{u_4\})$$

where $Q_0Q_1Q_2Q_3u_4 = c_7c_8$. The differentials $d_2(u_3) = y_2$, $d_2(u_4) = y_2'$ in the coniveau spectral sequence.

Remark. If the epimorphism in Theorem 6.1 is an isomorphism, then that in the above theorem is also an isomorphism.

Remark. The notations in [Ya3] are given : $a' = u_4$ as a virtual element and

$$\xi_3 = Q_0y_2', \ \xi_4 = Q_1y_2', \ \xi_6 = Q_2y_2', \ c_8y_2 = Q_3y_2'.$$

Next we consider the odd prime cases i.e., $(G, p) = (E_6, 3)$. Let us denote by $\mathcal{O}$ simply, $\mathcal{O}_3(x_4)$ so that $e_4 = Q_0Q_1Q_2Q_3(u_4) = e_3\mathcal{O}$. Then from Kameko-Mimura lemma, we also have ([Ka-Mi],

$$Igr(W_4) \cong SD/(e)[\mathcal{O}] \oplus SD_3[\mathcal{O}] \otimes Q(2)\{u_3, u_4\}$$

$$\cong SD_3/(e)[\mathcal{O}] \oplus SD_3 \otimes Q(2)\{u_3\} \oplus SD_3[\mathcal{O}] \otimes Q(3)\{u_4\}.$$

Moreover from Kameko (Lemma 5.2 in [Ka-Ya1]), it is known that

$$Igr(W_4)/Res_{H\mathbb{Z}/p} \cong SD_3/(e) \otimes (\mathbb{Z}/3\{u_3\} \oplus \mathbb{Z}/3[\mathcal{O}]\{u_4, Q_0u_4, ..., Q_3u_4\})$$

as the case $(Spin_7, 2)$. Hence Assumption (1) satisfied and Lemma 4.4 can be applied so that $d_2(u_3) = y$.

To see Assumption (2), we consider the representations. It is known that there is a non trivial representation $E_6 \to SO(26)$. Hence we know that $c_{3,i}$ is represented by Chern classes by the arguments similar to the case $(F_4, 3)$. As for the element $\mathcal{O}$, we consider the restriction

$$\langle a_4 \rangle \subset A \subset E_6 \xrightarrow{\rho} SO(26).$$

Here $a_4 \in A$ is the dual of $x_4 \in Hom(A; \mathbb{Z}/3)$. We see $\mathcal{O}|\langle a_4 \rangle \neq 0$ but $SD_3|\langle a_4 \rangle = 0$. Hence the fact

$$c_{26}(\rho|\langle a_4 \rangle) = y_4^{3^3-1} \neq 0$$

implies $\mathcal{O}$ (modulo elements in SD_3) can be represented by a Chern class. Hence Assumption (2) is also satisfied.

Theorem 7.4. *There is an epimorphism of $Q(3)$-bimodules from $H^*(BE_6; H^{*'}_{\mathbb{Z}/3})$ to*

$$H^*(BF_4; H^{*'}_{\mathbb{Z}/3}) \oplus \mathbb{Z}/3[c_{3,2}, c_{3,1}, \mathcal{O}] \otimes (\mathbb{Z}/3\{\mathcal{O}, y_2', Q_0y_2', ..., Q_3y_2'\}$$

$$\oplus \mathbb{Z}/2[e_3] \otimes Q(3)\{u_4\}).$$

At last of this section, we consider the case $(E_7, 3)$. This case

$$W_G(A) = W_4' = \langle W_4, diag(1,1,1,-1)\rangle \subset GL_4(\mathbb{Z}/3).$$

The invariant is also computed by Kameko-Mimura

$$Igr(W_4') \cong SD/(e)[\mathcal{O}^2] \oplus SD_3[\mathcal{O}^2] \otimes Q(2)\{u_3, \mathcal{O}u_4\}$$

Moreover from Kameko (page 2279 in [Ka-Ya1]), it is known that

$$Igr(W_4')/Res_{H\mathbb{Z}/3} \cong SD_3/(e)[\mathcal{O}^2] \otimes (\mathbb{Z}/3\{u_3\} \oplus Q(2)\{\mathcal{O}u_4\}).$$

It is also known $Inv^*(E_7; \mathbb{Z}/3) \cong \mathbb{Z}/3\{1, u_3\}$ and hence

$$Inv^*(E_7; \mathbb{Z}/3) \cong Inv^*(A; \mathbb{Z}/3)^{W_4'}$$

from the above result.

There is the natural representation $\rho : E_7 \to SO(52)$. Hence we see that $\mathcal{O}^2$ can be represented by a Chern class. So Assumption (2) is satisfied. However Assumption (1) is not.

Theorem 7.5. *The following restriction map*

$$Res_{E_2} : H^*(BE_7, H^{*'}_{\mathbb{Z}/3}) \to H^*(BA, H^{*'}_{\mathbb{Z}/3})^{W_4'}$$

is not an epimorphism.

Proof. Recall arguments in the proof of Lemma 5.9. Suppose $\mathcal{O}u_4 \in Res_{E_2}$ and $Res(x) = \mathcal{O}u_4$ for $x \in H^*(BE_7; H^{*'}_{\mathbb{Z}/3})$.

Of course $w(x) = w(\mathcal{O}u_4) = 4$. The weight $Q_0Q_1(x) = 2$. Recall $w(d_r) = 1 - 2r$ and

$$d_r(Q_0Q_1(x)) = 0 \quad for\ r \geq 2.$$

Hence $Q_0Q_1(x) \in H^*(BE_3; \mathbb{Z}/3)$ from the coniveau spectral sequence. So $Q_0Q_1(\mathcal{O}u_4) \in Res_{H\mathbb{Z}/3}$, which contradicts to the result above

$$Igr(W_4')/Res_{H\mathbb{Z}/3} \supset Q(2)\{\mathcal{O}u_4\}.$$

□

8. $Spin_9$ FOR $p = 2$

In this section, we consider the groups $Spin_8, Spin_9, F_4, E_6$ for $p = 2$. At first we consider the case $G = Spin_9$. Then the maximal elementary abelian 2-group is $rank_2 = 5$, and the Weyl group is

$$W_G(A) = W_5 \cong \langle U_5, SL_3\rangle \subset SL_5(\mathbb{Z}/2).$$

The cohomology is known that

$$H^*(BG; \mathbb{Z}/2) \cong H^*(BA; \mathbb{Z}/2)^{W_G(A)} \cong \mathbb{Z}/2[w_4, w_6, w_7, w_8, w_{16}]$$

where w_{16} is the Stiefel-Whitney class of some spin representation. We can also identify

$$w_{16} = \mathcal{O}'_4(x_5) = x_5^{16} + d_3x_5^8 + d_2x_5^4 + d_1x_5^2 + d_0x_5$$

where $d_3 = w_8,\ d_2 = w_8w_4,\ d_1 = w_8w_6,\ d_0 = w_8w_7$ (see the proof of Lemma 7.2 or p1051 in [Sc-Ya]). As the case $(*)$ in §5, we know ([Sc-Ya])

$$Q_3w_{16} = d_0w_{16}, \qquad Q_4(d_0w_{16}) = d_0^2w_{16}^2.$$

We can prove from Kameko-Mimura theorem

Lemma 8.1.

$$Igr(W_5) \cong Igr(W_4) \oplus \mathbb{Z}/2[c_4, c_6, c_{16}] \otimes (\mathbb{Z}/2[c_8]\{c_{16}\} \oplus$$
$$\mathbb{Z}/2[c_7] \otimes Q(2)\{c_{16}u_3\} \oplus \mathbb{Z}/2[c_7, c_8] \otimes Q(4)\{u_5\})$$

Proof. Recall that

$$Igr(W_4) \cong \mathbb{Z}/2[c_4, c_6, c_7, c_8] \otimes (\mathbb{Z}/2\{1\} \oplus \bar{Q}(2)\{u_3\} \oplus Q(2)\{u_3\}).$$

From Kameko-Mimura theorem (Theorem 5.4) we see that

$$Igr(W_5) \cong \mathbb{Z}/2[c_4, c_6, c_7, c_8, c_{16}] \otimes (\mathbb{Z}/2\{1\} \oplus \bar{Q}(2)\{u_3\} \oplus Q(2)\{u_4\} \oplus Q(3)\{u_5\}).$$

Using $Q_2Q_1Q_0(u_3) = c_7$ and $Q_3(u_4) = c_8u_3 + ...$ as the case for $Spin_7$, we have the isomorphism

$$Q(2)\{u_3\} \cong \mathbb{Z}/2\{c_7\} \oplus \bar{Q}(2)\{u_3\}, \quad Q(3)\{u_4\} \cong Q(2)\{c_8u_3\} \oplus Q(2)\{u_4\}.$$

Hence $Igr(W_5)$ is isomorphic to

$$\mathbb{Z}/2[c_4, c_6, c_{16}] \otimes (\mathbb{Z}/2[c_8] \oplus \mathbb{Z}/2[c_7] \otimes Q(2)\{u_3\} \oplus$$
$$\mathbb{Z}/2[c_7, c_8] \otimes Q(3)\{u_4\} \oplus \mathbb{Z}/2[c_7, c_8] \otimes Q(3)\{u_5\})$$

Using the fact $Q_4(u_5) = c_{16}u_4 + ...$ from Lemma 5.6, we have the isomorphism

$$Q(4)\{u_5\} \cong Q(3)\{c_{16}u_4\} \oplus Q(3)\{u_5\}.$$

This induces the isomorphism

$$Igr(W_5) \cong \mathbb{Z}/2[c_4, c_6] \otimes (\mathbb{Z}/2[c_8, c_{16}] \oplus \mathbb{Z}/2[c_7, c_{16}] \otimes Q(2)\{u_3\}$$
$$\oplus \mathbb{Z}/2[c_7, c_8] \otimes Q(3)\{u_4\} \oplus \mathbb{Z}/2[c_7, c_8, c_{16}] \otimes Q(4)\{u_5\}).$$

Hence we have the desired isomorphism. □

The cohomological invariant is known

$$Inv^*(Spin_9, \mathbb{Z}/2) \cong \mathbb{Z}/2\{1, u_3, u_4, u_5\}.$$

Hence Assumption (1),(2) are also satisfied for $(Spin_9, 2)$ (from the last isomorphism in the above proof).

Lemma 8.2.

$$Igr(W_5)/I(W_5) \cong Igr(W_4)/(W_4) \oplus \mathbb{Z}/2[c_4, c_6, c_{16}] \otimes (\mathbb{Z}/2\{c_{16}u_3\}\oplus$$
$$\oplus\mathbb{Z}/2[c_8] \otimes ((Q(4)/(Q(4)^+)^3\{u_5\} \oplus \mathbb{Z}/2\{c_7u_5\})).$$

Proof. Since

$$Q_0(u_5) = Q_0(u_4x_5) = Q_0(u_4)x_5 + u_4y_5,$$

we can compute

$$Q_2Q_1Q_0(u_5) = Q_2Q_1Q_0(u_4)x_5+Q_1Q_0(u_4)y_5^4+Q_2Q_0(u_4)y_5^2+Q_2Q_1(u_4)y_5$$
$$= w_7w_8x_5 + w_8y_5^4 + w_4w_8y_5^2 + w_6w_8y_5 = w_{16}.$$

(This fact also follows from $d_{5,4} = w_{16}$.) Let us write $Q_{i_1,\dots,i_j}(u_5) = Q_{i_1,\dots,i_j}$ simply. Similarly we can compute

$$Q_{012} = w_{16}, \quad Q_{013} = w_{16}w_8, \quad Q_{023} = w_{16}w_4w_8, \quad Q_{123} = w_{16}w_6w_8,$$
$$Q_{014} = c_{16}w_8, \quad Q_{024} = c_{16}w_8w_4, \quad Q_{034} = c_{16}c_8w_4,$$
$$Q_{124} = c_{16}w_6w_8, \quad Q_{134} = c_{16}c_8w_6, \quad Q_{234} = c_{16}c_8w_4w_6.$$

We can compute

$$Q_0(c_7u_5) = Q_0(c_7u_4x_5) = Q_0(w_4w_6w_8x_5)$$
$$= w_4w_7w_8x_5 + w_4w_6w_8y_5 + \dots = w_4w_{16}.$$

Let us write $Q_{i_1,\dots,i_j}(c_7u_5) = Q'_{i_1,\dots,i_j}$ simply. Then we can compute

$$Q'_0 = w_{16}w_4, \quad Q'_1 = w_{16}x_6, \quad Q'_2 = w_{16}w_4w_6, \quad Q'_3 = w_{16}w_4w_6w_8,$$
$$Q'_4 = c_{16}w_4w_6w_8, \quad Q'_{01} = w_{16}w_7, \quad Q'_{12} = w_{16}w_6w_7, \quad Q'_{04} = c_{16}w_7w_4w_8.$$

Moreover

$$c_7^2u_5 = c_7w_4w_6w_8x_5 = w_4w_6w_7w_{16}.$$

There appear all generators of the $\mathbb{Z}/2[c_4, c_6, c_8, c_{16}]$-module with modulo $Ideal(c_7, w_7)$. Thus we can see

$$Igr(W_5)/I(W_5) \cong \mathbb{Z}/2[c_4, c_6]\otimes(\mathbb{Z}/2[c_{16}]\{u_3\}\oplus\mathbb{Z}/2[c_8](Q(3)/(Q(3)^+)^2\{u_4\}$$
$$\oplus\mathbb{Z}/2[c_8, c_{16}] \otimes ((Q(4)/(Q(4)^+)^3\{u_5\} \oplus \mathbb{Z}/2\{c_7u_5\})).$$

□

Theorem 8.3. *There is an epimorphism of $Q(4)$-modules from $H^*(BSpin_9; H^{*'}_{\mathbb{Z}/2})$ to*

$$H^*(BSpin_7; H^{*'}_{\mathbb{Z}/3})\oplus\mathbb{Z}/p[c_4, c_6, c_{16}]\otimes(\mathbb{Z}/2[c_7]\otimes Q(2)\{c_{16}u_3\}\oplus\mathbb{Z}/2\{c_{16}y\}$$
$$\oplus\mathbb{Z}/2[c_8, c_7] \otimes Q(4)\{u_5\} \oplus \mathbb{Z}/2[c_8] \otimes (Q(4)/(Q(4)^+)^3\{y''\})$$

where $d_2u_3 = y$ and $d_2(u_5) = y''$.

Remark. However note that we can not see $d_2(c_7u_5) \neq 0$ or not.

From Lemma 4.4, we know $d_2(u_5) = y'' \neq 0$ and we get many Griffith elements

$$Q_iQ_j(y'') \quad for\ 0 \leq i < j \leq 4.$$

We study the image of the cycle map $\tilde{cl}$ to $BP^*(BSpin_9) \otimes_{BP^*} \mathbb{Z}/2$ in the last section (indeed $\tilde{cl}(Q_iQ_j(y'') \neq 0)$.

Next we consider the case $(Spin_8, 2)$. The Weyl group is

$$W_4 \subset W_G(A) = W_5'' = \{(w_{ij}) \in GL_5(\mathbb{Z}/2) | w_{5,4} = 0\} \subset W_5.$$

We can compute

$$Igr(W_5'') \cong Igr(W_4) \oplus \mathbb{Z}/2[c_4, c_6, c_8', c_8] \otimes$$
$$(\mathbb{Z}/2\{c_8'\} \oplus \mathbb{Z}/3[c_7] \otimes Q(3)\{u_4'\} \otimes \mathbb{Z}/2[c_7] \otimes Q(4)\{u_5\}).$$

Hence Assumption (1) (with some modification for u_4') and (2) are satisfied.

We consider the case $(F_4, 2)$ also . This case $A \cong (\mathbb{Z}/2)^5$ but

$$W_G(A) = W_5' = \langle U_5, GL_3(\mathbb{Z}/2) \oplus GL_2(\mathbb{Z}/2) \rangle \subset GL_5(\mathbb{Z}/2).$$

The cohomology is given by

$$H^*(BG; \mathbb{Z}/2) \cong H^*(BA; \mathbb{Z}/2)^{W_G(A)} \cong \mathbb{Z}/2[w_4, w_6, w_7, x_{16}, x_{24}]$$

where $x_{16} = w_8^2 + w_{16}$ and $x_{24} = w_8w_{16}$. We consider the representation

$$\langle a_4, a_5 \rangle \subset A \subset F_4 \xrightarrow{\rho} SO(26)$$

where a_4, a_5 are dual of x_4, x_5. Then the total Chern class is

$$c(\rho | \langle a_4, a_5 \rangle) = (1 + c_{2,1} + c_{2,0})^a$$

from Lemma 5.1. By dimensional reason, $a \leq 8$. So we see x_{16}^2 and x_{24}^2 are represented by Chern classes. Thus we can write

$$Igr(W_5') \cong \mathbb{Z}/2[c_4, c_6] \otimes (\mathbb{Z}/2[c_{16}, c_{24}] \oplus \mathbb{Z}/2[c_7, c_{24}] \otimes Q(2)\{u_3\}$$
$$\oplus \mathbb{Z}/2[c_7, c_{16}, c_{24}] \otimes Q(4)\{u_5\}).$$

Hence Assumption (1),(2) are also satisfied. So Res_{E_2} is an epimorphism for $(F_4, 2)$.

At last of examples, we consider the case $(E_6, 2)$. This case $W_G(A) \cong W_5'$ same as the case F_4. But it is known ([Ga-Me-Se]) that

$$Inv^*(E_6, \mathbb{Z}/2) \cong \mathbb{Z}/2\{1, u_3\}.$$

Therefore we see

Lemma 8.4. *When $G = E_6$ and $p = 2$, the restriction map*

$$Res_{E_2} : H^*(BG; H_{\mathbb{Z}/2}^{*'}) \to H^*(BA; H_{\mathbb{Z}/2}^{*'})^{W_G(A)}$$

is not an epimorphism.

The above fact also proved by using the cohomology $H^*(BE_6;\mathbb{Z}/2)$ and Lemma 5.9 as follows.

Theorem 8.5. *The restriction maps*

$$Res_{Inv} : Inv^*(G;\mathbb{Z}/2) \to Inv^*(A_5;\mathbb{Z}/2)^{W_G(A_5)}$$

are epimorphisms for $G = E_6$ *and* $G = Spin_n$, $n \geq 10$.

Proof. Of course there is the embedding $i : Spin_9 \subset Spin_{10}$. From Kono-Mimura [Ko-Mi], we know

$$H^*(BSpin_{10};\mathbb{Z}/2) \cong H^*(BSpin_7;\mathbb{Z}/2) \otimes \mathbb{Z}/2[x_{10}, x_{32}]/(w_7x_{10}).$$

Hence $H^*(BE_6;\mathbb{Z}/2)$ does not contain an element x with $i^*(x) = w_{16}$ for $w_{16} = d_{5,4} = Q_0Q_1Q_2(u_5)$. From Lemma 5.9, we see that Res_{Inv} is not an epimorphism.

There is the embedding $F_4 \subset E_6$. From also Kono-Mimura [Ko-Mi], we know

$$H^*(BE_6;\mathbb{Z}/2) \cong H^*(BSpin_7;\mathbb{Z}/2)\otimes\mathbb{Z}/2[x_{10}, x_{18}, x_{32}, x_{48}]/(relations).$$

Hence Res_{Inv} is not epic from the lack of element x_{16}. □

9. BP-theory and Griffith elements

In this section, we recall the results in §5 in [Ya1] and consider the relation between BP-theory and results in the preceding sections. We always assume $k = \mathbb{C}$ in this section. Let $BP^*(-)$ be the Brown-Peterson theory with the coefficient ring $BP^* = \mathbb{Z}_{(p)}[v_1, ...]$, $|v_i| = -2(p^i - 1)$. The Thom map induces $\rho : BP^*(X) \otimes_{BP^*} \mathbb{Z}_{(p)} \to H^*(X;\mathbb{Z}_{(p)})$. Totaro constructs [To1] the map

$$\tilde{cl} : CH^*(X)_{(p)} \to BP^*(X) \otimes_{BP^*} \mathbb{Z}_{(p)}$$

such that the composition $\rho \cdot \tilde{cl}$ is the usual cycle map $cl = t_{\mathbb{C}}$ which is also the realization map. Totaro conjectured that this map is isomorphic for $X = BG$.

Let us write by

$$P(n)^* = BP^*/(p, v_1, ..., v_{n-1}),$$

e.g., $P(0)^* = BP^*$, $P(1)^* = BP^*/p$ and $P(\infty)^* = \mathbb{Z}/p$.

Many cases of X([Te-Ya2], [Ko-Ya]), $BP^*(X)$ are computed by the Atiyah-Hirzebruch spectral sequences

$$E_2^{*,*'} = H^*(X) \otimes BP^{*'} \Longrightarrow BP^*(X).$$

It is known that $d_{2p^i-1}(x) = v_i \otimes Q_i(x)\ mod(M_i)$ where M_i is the ideal of $E_{2p^i-1}^{*,*'}$ generated by elements in $(p, ..., v_{i-1})E_2^{*,*'}$.

We assume that $H^*(X)$ has no higher p-torsion and all non zero differentials are of form

$$(9.1) \quad d_{2p^i-1}(x) = v_i \otimes Q_i(x)\ mod(M_i).$$

Let us write

$$(9.2) \quad grBP^*(X) \cong E_\infty^{*,*} \cong A \oplus B$$

where A (resp. B) is a BP^*-module generated by elements in $H^*(X)/p$ (resp. $pH^*(X) \oplus E_\infty^{*,minus}$) so that $B \subset Ker(\rho_p)$. Then we can write

$$A \cong \oplus_{n=-1} P(n+1)^* \tilde{G}_n$$

by the prime invariant ideal theorem of Landweber ; if $P(n)^*/(a)$ is a $BP^*(BP)$-module, then $a = v_n^s$ for some $s \geq 1$.

Lemma 9.1. *(Lemma 5.1 in [Ya1]) Let $H^*(X)_{(p)}$ has no higher p-torsion. Suppose (9.1) and $A = \oplus_{n=-1} P(n+1)^* \tilde{G}_n$ in (9.2). Then there is a injection of $Q(\infty)$-modules*

$$H^*(X;\mathbb{Z}/p) \hookrightarrow \oplus_{n=-1} Q(n) G_n \quad \textit{with } Q_0 ... Q_n G_n = \tilde{G}_n.$$

It is proved ([Ko-Ya],[Ka-Ya],[Ya1]) that all $X = BG$ in Theorem 1.2 satisfy the assumption in the above lemma. Hence

$$H^*(BG;\mathbb{Z}/p) \hookrightarrow \oplus_{n=-1}^{\infty} Q(n) G_n.$$

Moreover when $n \neq -1$, we still know

$$w(G_n) = n.$$

Indeed if $n \geq 0$, then $G_n = G_n'\{u_n\}$ where G_n' is represented by elements in $CH^*(BG)/p$ (see Assumption (1),(2)). From Theorem 1.2, we also know

Corollary 9.2. *Let G be a group in Theorem 1.2. Then there is a bidegree $Q(n)$-module injection*

$$\oplus_{i \geq 0} Q(n) G_n \subset H^*(BG; H_{\mathbb{Z}/p}^{*'}).$$

Lemma 9.3. *(Lemma 5.2 in [Ya]) Let $H^*(X)_{(p)}$ has no higher p-torsion.*

(1) If (9.1) is satisfied and in (9.2),

$$A \cong \oplus_{n=-1} P(n+1)^* \tilde{G}_n \quad \textit{and } B \cong \oplus_{s=0} BP^*\{p, v_1, .., v_s\} \tilde{K}_s,$$

then we have the isomorphism

$$H^*(X;\mathbb{Z}/p) \cong (\oplus_{n=-1} Q(n) G_n) - (\oplus_s \bar{Q}(s) K_s)$$

with $Q_0 ... Q_n G_n = \tilde{G}_n$, and $Q_1 ... Q_s K_s = \tilde{K}_s$.

(2) If $Q_0 ... Q_n G_n \in Im(\rho)$ and $|Q_1 ... Q_s K_s| = even$, then the converse also holds.

Corollary 9.4. *Let G be a group in Theorem 1.2 so that Assumption (1),(2) are satisfied. Then there are epimorphisms from $grBP^*(BG) \cong E_\infty^{*,*'}$ to*

$$(9.3) \quad \oplus_s P(i_s)^*[f_{s1},...,f_{sk_s}]\{\tilde{u}_{i_s}\} \oplus \oplus_t BP^*(p,...,v_t)\{\tilde{K}_t\}$$

and from $BP^*(BG) \otimes_{BP^*} \mathbb{Z}/p$ to

$$\oplus_s \mathbb{Z}/p[f_{s1},...,f_{sk_s}]\{\tilde{u}_{i_s}\} \oplus \oplus_t \mathbb{Z}/p\{p,...,v_t\}\{\tilde{K}_t\}$$

where $\tilde{u}_{i_s} = Q_{i_s-1}...Q_0(u_{i_s})$ and $Igr(W)/Res_{H\mathbb{Z}/p} \cong \oplus_t \bar{Q}(t)K_t$.

We give examples. At first we recall the Atiyah-Hirzebruch spectral sequence for BG_2 in [Ko-Ya]. Since $H^*(BG)$ has no higher torsin, we have

$$H^*(BG_2)_{(2)} \cong \mathbb{Z}_{(2)}[w_4, c_6] \otimes (\mathbb{Z}_{(2)}\{1\} \oplus \mathbb{Z}/2[w_7]\{w_7\}).$$

Let us write $B_{i_1,...,i_j} = \mathbb{Z}_{(2)}[c_{i_1},...,c_{i_j}]$, e.g., $B_{4,6} = \mathbb{Z}_{(2)}[c_4, c_6]$.

Since $Q_1(w_4) = w_7$, we have $d_3(w_4) = v_1 \otimes w_7$. Hence the E_4-term of the spectral sequence is

$$E(G_2)_4^{*,*} \cong B_{4,6} \otimes (BP^*\{1, 2w_4\} \oplus P(2)^*[c_7]\{c_7, w_7\}).$$

Next differential is $d_7(w_7) = v_2 \otimes Q_2(w_7) = v_2 c_7$ and

$$E(G_2)_8^{*,*} \cong B_{4,6} \otimes (BP^*\{1, 2w_4\} \oplus P(3)^*[c_7]\{c_7\}).$$

which is isomorphic to $E(G_2)_\infty^{*,*}$. In particular

$$BP^*(BG) \otimes_{BP^*} \mathbb{Z}/2 \cong \mathbb{Z}/2[c_4, c_6] \otimes (\mathbb{Z}/2\{1, 2w_4\} \oplus \mathbb{Z}/2[c_7]\{c_7\}).$$

This result is also immediate from Corollary 9.4 and Theorem 6.1, in fact, we have the epimorphism

$$H^*(BG_2; H^{*'}_{\mathbb{Z}/2}) \to \mathbb{Z}/2[c_4, c_6] \otimes (\mathbb{Z}/2\{1, y\} \oplus \mathbb{Z}/2[c_7] \otimes Q(2)\{u_3\}).$$

Here $\tilde{u}_3 = Q_0Q_1Q_2(u_3) = c_7$ and $d_2(u_3) = y$ in the coniveau spectral sequence, and we have

$$\bar{Q}(0)K_0 = K_0 = \mathbb{Z}/2[c_4, c_6]\{u\}.$$

Hence the cycle map $\tilde{cl}$ is epimorphism and $\tilde{cl}(y) = \{2w_4\}$ (which is represented by a Chern class c_2). Moreover we know [Ya2] that this $\tilde{cl}$ is a really isomorphism.

Next consider the case $Spin_7$. From [Ko-Ya], we can compute

$$E(Spin_7)_{16}^{*,*} \cong B_{4,6} \otimes P(3)^*[c_7]\{c_7\} \oplus$$

$$B_{4,6,8} \otimes (BP^*\{1, 2w_4, 2w_4w_8, 2w_8, v_1w_8\} \oplus P(4)^*[c_7]\{c_7c_8\}).$$

This term is also the infinity term. Hence we have

$$BP^*(BG) \otimes_{BP^*} \mathbb{Z}/2 \cong BP^*(BG_2) \otimes_{BP^*} \mathbb{Z}/2 \oplus$$

$$\mathbb{Z}/2[c_4,c_6,c_8]\otimes(\mathbb{Z}/2\{c_8,2c_8w_4,2w_4w_8,2w_8,v_1w_8\}\oplus\mathbb{Z}/2[c_7]\{c_7c_8\})$$

This result is also get from Corollary 9.4 and Theorem 7.3, indeed, we have the epimorphsm

$$H^*(BSpin_7;H^{*'}_{\mathbb{Z}/2})\to H^*(BG_2;H^{*'}_{\mathbb{Z}/2})\oplus$$
$$\mathbb{Z}/2[c_4,c_6,c_8]\otimes(\mathbb{Z}/2\{c_8,y_2',Q_0y_2',...,Q_3y_2'\}\oplus\mathbb{Z}/2[c_7]\otimes Q(3)\{u_4\})$$

This case $d_2(u_3)=y, d_2(u_4)=y'$ in the coniveau spectral sequence. Recall

$$\oplus_s\bar{Q}(s)K_s\cong Igr(W_4)/I(W_4)\cong I(GL_3)/I(GL_3)\oplus$$
$$\mathbb{Z}/2[c_4,c_6,c_8]\{1,Q_0,...,Q_3\}\{u_4\}.$$

We can take

$$\bar{Q}(1)K_1\cong\mathbb{Z}/2[c_4,c_6,c_8]\{1,Q_0,Q_1\}\{u_4\}$$
$$\bar{Q}(0)K_0\cong\mathbb{Z}/2[c_4,c_6]\otimes(\mathbb{Z}/2\{u_3\}\oplus\mathbb{Z}/2\{Q_2u_4,Q_3u_4\}).$$

The cycle map $\tilde{cl}$ is given by

$$Q_0(y')\mapsto v_1w_8\quad ,Q_1y'\mapsto 2w_8\quad ,Q_2y'\mapsto 2w_4w_8\quad ,Q_3y'\mapsto 2c_8w_4.$$

Of course the cycle map $\tilde{cl}$ is isomorphic. This fact is still proved by P.Guillot [Gu1], and the case $G=Spin_8$ is computed by Molina [Mo].

10. $BP^*(BSpin_9)\otimes_{BP^*}\mathbb{Z}/2$

At last of this paper, we consider the case $Spin_9$. In [Ko-Ya], we can compute (which is quite complicated)

$$E(Spin_9)^{*,*}_{32}\cong B_{4,6,8,16}\otimes(BP^*\{1,2w_4,2w_4w_8,2w_8,v_1w_8,$$
$$2w_4w_{16},2w_4w_8w_{16},2w_8w_{16},v_1w_8w_{16},2w_{16},v_1w_{16}.v_2w_{16}\}$$
$$\oplus P(5)^*[c_7]\{c_7c_8c_{16}\})\oplus B_{4,6,7,8}\otimes P(4)^*[c_7]\{c_7c_8\}\oplus B_{4,6,16}\otimes P(3)^*[c_7]\{c_7\}.$$

This term is the infinite term. (See Theorem 8.3 also.)

Lemma 10.1. *([Ko-Ya])*

$$BP^*(BSpin_9)\otimes_{BP^*}\mathbb{Z}/2\cong BP^*(BSpin_7)\otimes_{BP^*}\mathbb{Z}/2\oplus$$
$$\mathbb{Z}/2[c_4,c_6,c_8,c_{16}]\otimes(\mathbb{Z}/2\{c_{16},2w_4c_{16},2w_4c_8c_{16},(2,v_1)w_8c_{16},2w_4w_8c_{16},$$
$$(2,v_1,v_2)w_{16},2w_4w_{16},(2,v_1)w_8w_{16},2w_4w_8w_{16}\}\oplus\mathbb{Z}/2[c_7]\{c_{16}c_7\}).$$

We still know there is an epimorphism from $H^*(BSpin_9;H^{*'}_{\mathbb{Z}/2})$ to

$$H^*(BSpin_7;H^{*'}_{\mathbb{Z}/2})\oplus\mathbb{Z}/2[c_4,c_6,c_{16}]\otimes(\mathbb{Z}/2[c_7]\otimes Q(2)\{c_{16}u_3\}\oplus\mathbb{Z}/2\{c_{16}y\}$$
$$\oplus\mathbb{Z}/2[c_8,c_7]\otimes Q(4)\{u_5\}\oplus\mathbb{Z}/2[c_8]\otimes(Q(4)/(Q(4)^+)^3\{y''\})$$

where $d_2u_3=y$ and $d_2(u_5)=y''$.

We can see the following lemma;

Proposition 10.2. *When* $(G,p)=(Spin_9,2)$*, the cycle map*

$$\tilde{cl}: CH^*(BG)/2 \to (BP^*(BG)\otimes_{BP^*}\mathbb{Z}/2)$$

is an epimorphism with $mod(\mathbb{Z}/2[c_4,c_6,c_8,c_{16}]\{2w_4w_{16}\})$.

The image of the cycle map $\tilde{cl}$ is given as follows. By arguments for G_2, we see

$$d_2(c_{16}u_3)\mapsto 2w_4c_{16}$$

If we can see

$$d_3(c_7u_5)\mapsto 2w_4w_{16},$$

then the Totaro's map $\tilde{cl}$ is an epimorphism. Unfortunately, we do not see even $d_2(c_7u_5)=c_7y''=0$ yet.

Let us write $Q_{ij}=Q_iQ_j(d_2u_5)=Q_iQ_j(y'')$. Then the cycle map $\tilde{cl}$ is written as

$$Q_{01}\mapsto v_2w_{16},\quad Q_{02}\mapsto v_1w_{16},\quad Q_{12}\mapsto 2w_{16},$$

$$Q_{03}\mapsto v_1w_8w_{16},\quad Q_{13}\mapsto 2w_8w_{16},\quad Q_{23}\mapsto 2w_4w_8w_{16},$$

$$Q_{04}\mapsto v_1w_8c_{16},\quad Q_{14}\mapsto 2w_8c_{16},\quad Q_{24}\mapsto 2w_4w_8c_{16},$$

$$Q_{34}\mapsto 2w_4c_8c_{16}.$$

By the arguments for $Spin_7$, we still know, for example $Q_1(d_2u_4)\mapsto 2w_8$, so we get the $Q_4Q_1(d_2u_5)\mapsto 2w_8c_{16}$ using $Q_4(u_5)=c_{16}u_4+....$ Similarly we get the maps for Q_{*4} from that for $Spin_7$.

For other maps, we use the Quillen operation in $BP^*(-)$ theory. For a sequence $\alpha=(\alpha_1,...,\alpha_m)$, $\alpha_i\geq 0$, we have the Quillen cohomology operation in $BP^*(X)$ (and also in $ABP^*(X)$) (see [Ra], [Ha], [Ya2])

$$r_\alpha: BP^*(X)\to BP^{*+|\alpha|}(X)\quad |\alpha|=\sum 2p^i\alpha_i$$

such that $\rho(r_\alpha)=P^\alpha$ the fundamental basis of the reduced power operations (see [Ha]) and $r_\alpha(v_i)\in Ideal(p,...,v_i)$. Hence r_α acts also on $BP^*(X)\otimes_{BP^*}\mathbb{Z}/p$.

Let us write by $\bar{S}q^{even}$ the Quillen operation corresponding Sq^{even}. By the definition of Q_i, se see the equation $Sq^{16}Sq^8(Q_2Q_1(u_5))=Q_4Q_1(u_5)$ in $H^*(BG;\mathbb{Z}/2)$. We still know the image of the cycle map

$$Sq^{16}Sq^8(Q_{12})=Q_{14}\mapsto 2w_8c_{16}\in BP^*(BG)\otimes_{BP^*}\mathbb{Z}/2.$$

Let $\tilde{cl}(Q_{12})=x$. Then

$$\bar{S}q^{16}\bar{S}q^8(x)=2w_8c_{16}\quad in\ BP^*(BG)\otimes_{BP^*}\mathbb{Z}/2.$$

So x is non zero. The equation $Sq^{16}Sq^8(w_{16})=w_8c_{16}$ in $H^*(BG;\mathbb{Z}/2)$ implies

$$\bar{S}q^{16}\bar{S}q^8(2w_{16})=2w_8c_{16}\quad mod(v_1,...)\ \ in\ BP^*(BG).$$

Hence we can take $x = 2w_{16}$.

Using $\bar{S}q^4\bar{S}q^2$ and dimensional reason, we have the first map $Q_{01} \mapsto v_2w_{16}$. The other cases are also proved similarly.

The above Q_{ij} are all Griffith elements. From Corollary 9.4, we can write

$$Q(4)\{u_5\}/(Q(4)^+)^3\{u_5\} \cong \oplus_{t=1}^2 \bar{Q}(t)K_t'.$$

In fact, we can take

$$\bar{Q}(2)K_2' = \mathbb{Z}/2\{u_5, Q_0, Q_1, Q_2, Q_{01}, Q_{02}, Q_{12}\},$$
$$\bar{Q}(1)K_1' = \mathbb{Z}/2\{Q_3, Q_{03}, Q_{13}, \quad Q_4, Q_{04}, Q_{14}\},$$
$$\bar{Q}(0)K_0' = \mathbb{Z}/2\{Q_{23}, Q_{24}, Q_{34}\}.$$

Recall Corollary 9.4 and $Igr(W)/Res_{H\mathbb{Z}/p} \cong \oplus_t \bar{Q}(t)K_t$. Let $k \in K_t$. Then we can identify $k \in H^*(BG; H^{*'}_{\mathbb{Z}/p})$ and

$$k(i) = Q_0...\hat{Q}_i...Q_t(k) \in \bar{Q}(t)K_t \subset Igr(W)/Res_{H\mathbb{Z}/p}.$$

Moreover suppose $w(k) = t+3$. Since $w(k(i)) = 3$ and $w(d_r) = 1-2r$, we see

$$d_2(k(i)) \neq 0 \quad (hence\ d_2(k) \neq 0).$$

Let us consider the projection map

$$pr. : BP^*(X) \otimes_{BP^*} \mathbb{Z}/p \to (9.3) \otimes_{BP^*} \mathbb{Z}/p \to (p, v_1, ..., v_t)\{\tilde{k}\}$$

where $\tilde{k} = Q_0...Q_t(k)$. For $G = G_2, Spin_7$ and $Spin_9$, it holds that $pr.\tilde{cl}(d_2(k(i))) = v_i\tilde{k}$, that is,

$$pr.\tilde{cl}(d_2(Q_0...\hat{Q}_i...Q_t(k))) = v_iQ_0...Q_t(k),$$

while we do not show it for general cases.

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