# ORTHOGONAL SYMMETRIES AND CLIFFORD ALGEBRAS

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Abstract. Involutions of the Clifford algebra of a quadratic space induced by orthogonal symmetries are investigated.

*2000 Mathematics Subject Classification:* 16W10, 11E39

*Key words:* Orthogonal symmetry, reflection, Clifford algebra, Involution, quadratic form, even Clifford algebra, multiquaternion algebra, universal property of Clifford algebra, Clifford map

## 1. Introduction

Clifford algebra is one of the important algebraic structures which can be associated to a quadratic form. These algebras are among the most fascinating algebraic structures. Not only they have many applications in algebra and other branches of Mathematics, but also they have wide applications beyond Mathematics, e.g., Physics, Computer Science and Engineering [6], [18], [14], [15].

A detailed historical account of Clifford algebras from their genesis can be found in [21]. See also [7] and [14] for an interesting brief historical account of Clifford algebras.

Many familiar algebras can be regarded as special cases of Clifford algebras. For example the algebra of Complex numbers is isomorphic to the Clifford algebra of any one-dimensional negative definite quadratic form over $\mathbb{R}$. The algebra of Hamilton quaternions is isomorphic to the Clifford algebras of a two-dimensional negative definite quadratic form over $\mathbb{R}$. More generally, as shown by D. Lewis in [13, Proposition 1], a multiquaternion algebra, i.e., an algebra like $Q_1 \otimes \cdots \otimes Q_n$ where $Q_i$ is a quaternion algebra over a field $K$, can be regarded as the Clifford algebra of a suitable nondegenerate quadratic form $q$ over the base field $K$. In [13], such a form $q$ is also explicitly constructed. The Grassmann algebra (or the exterior algebra) may also be regarded as the Clifford algebra of the null (totally isotropic) quadratic form.

The word *'involution'* appears in different contexts in Mathematics. In group theory, an involution is an element of a group whose order is two. In analysis, an involution of a (normed) Banach algebra $A$ is a map $*$ from $A$ into itself such that for every $x,\ y \in A$: $x^{**} = x$, $(xy)^* = y^*x^*$ and $\|x^*x\| = \|x\|^2$. More generally in the theory of algebras, an involution of an algebra $A$ is a self inverse map $\varphi$ from $A$ to itself such that for every $x,\ y \in A$, we have $\varphi(x+y) = \varphi(x) + \varphi(y)$ and $\varphi(xy) = \varphi(y)\varphi(x)$, i.e., $\varphi$ is an anti-automorphism of order 2. Many algebras are naturally equipped with an involution, for example the full matrix algebra $\mathbb{M}_n(K)$, consisting of all $n \times n$ matrices over a field $K$ is equipped with the *transposition* involution, i.e., the involution which maps a matrix $A \in \mathbb{M}_n(K)$ to its transpose, $A^t$. If $q$ is a quadratic form defined on a vector space $V$, then the endomorphism algebra $\mathrm{End}(V)$ is equipped with the adjoint involution $\sigma_q : \mathrm{End}(V) \to \mathrm{End}(V)$ characterized by the property $b(x, fy) = b(\sigma_q(f)x, y)$ for every $x,\ y \in V$ and for every $f \in \mathrm{End}(V)$, here $b$ is the bilinear form associated to $q$. This involution, which determines $q$ up to similarity, reflects many properties of the quadratic form $q$.

For example we have the following assertions: (1) $q$ is isotropic if and only if there exists $0 \neq a \in \mathrm{End}(V)$ such that $\sigma_q(a)a = 0$. (2) $q$ is hyperbolic if and only if there exists an idempotent $e \in \mathrm{End}(V)$ such that $\sigma_q(e) = 1 - e$, see [1] or [9, Ch. II].

Clifford algebras have also many natural involutions. In fact every orthogonal symmetry $s$ (i.e., a self-inverse isometry) of a quadratic space $(V, q)$ can be extended to an involution $J^s$ of both the Clifford algebra and the even Clifford algebra of $q$. In particular the Clifford algebra of $q$ has two natural involutions which are induced by the maps $\mathrm{id} : V \to V$ and $-\mathrm{id} : V \to V$. The involutions $J^{\mathrm{id}}$ and $J^{-id}$ also reflect certain properties of $q$; for instance their hyperbolicity is equivalent to the existence of particular subforms of $q$. (see [16]).

Finite dimensional simple algebras with involution form an important class of algebras with involution whose properties are relatively well understood. By a theorem due to Albert, a central simple $K$-algebra $A$ carries an involution fixing $K$ if and only if the order of the class of $A$ in its Brauer group of $K$ is at most two (see [20, Ch. 8, 8.4]). As a consequence of a theorem due to Merkurjev, every central simple algebra whose class in the Brauer group of $K$ is at most two is equivalent to a tensor product of quaternion algebras. An important class of algebras with involution is tensor products of quaternion algebras with involution which are extensively studied in the literature. There are some close analogies between the properties of tensor products of quaternion algebras with involution on one hand and on the other hand the properties of multiples of Pfister forms. See [2], where it is for example proved that if $(A, \sigma)$ is a tensor product of quaternion algebras with involution such that $A$ is Brauer equivalent to a quaternion algebra over field $K$ then for every field extension $L/K$, $(A, \sigma)_L$ is either anisotropic or hyperbolic. This property is one of the characteristic properties of the multiples of Pfister forms.

One can find tensor product of quaternion algebras with involution which cannot occur as $(C(q), J^{\pm \mathrm{id}})$ for any form $q$ (see comment after Proposition 3 in [13]). We however show that for every tensor product of quaternion algebras with involution $(A, \sigma) = (Q_1, \sigma_1) \otimes \cdots \otimes (Q_n, \sigma_n)$, there exists a quadratic space $(V, q)$ of dimension $2n$ and an orthogonal symmetry $\sigma : V \to V$ such that $(A, \sigma)$ is isomorphic to $(C(q), J^\sigma)$, i.e., the Clifford algebra of $q$ equipped with the involution induced by $\sigma$ (see Theorem 6.3). We then provide more detailed statements; it turns out that when $\sigma$ is of the first kind, the orthogonal symmetry $\sigma$ can be chosen to be either $\pm\mathrm{id}$ or a reflection (see Proposition 6.8 and Proposition 6.9). For the case where $\sigma$ is of the second kind, a similar result is proved (see Proposition 6.11). Some results in this direction were already available in the literature, see [8, §3] and [19, Lemma 10.6]. In order to prove these results, we need to provide involutorial versions of some of the main structure theorems of Clifford algebras (i.e., the results which link the Clifford algebra of an orthogonal sum of two quadratic space to the Clifford algebras of summands, see [3, Ch. II] or [11, Ch. V]). Section 5 is devoted to prove these results. Some results in this direction were already been obtained in the literature, see Proposition 2 of [12].

In the literature, the even Clifford algebra (i.e., the even sub-algebra of the Clifford algebra) of a quadratic space $(V, q)$ is generally defined as a sub-algebra of the Clifford algebra which is generated by products of an even numbers of vectors in $V$. As we will observe in section 5, having a definition of the even Clifford algebra, as an individual mathematical object by means of a universal property, would be a handy tool at our disposal. Especially in proving isomorphisms of algebras with involution involving even Clifford algebras, the universal property can slightly shorten the proofs. This was our motivating reason to find a universal property of the even Clifford algebra in section 3. We also hope that this approach will be useful from a pedagogical point of view.

In sections 4 and 7, we make some general observations about involutions of a Clifford algebra which are induced by an orthogonal symmetry, for instance as an applications of results of section 5 the type of such involutions is determined. The type of natural involutions of the Clifford algebra $C(V,q)$ induced by $\pm\mathrm{id}$ are known, see [13] and [19, pp. 116-118].

## 2. Preliminaries

Let $K$ be a field of characteristic different from 2. A quadratic space $(V,q)$ over $K$ is a pair, consisting of a finite dimensional vector space $V$ over $K$ and a quadratic form $q: V \to K$, i.e., a map $q: V \to K$ which satisfies $q(\lambda x) = \lambda^2 q(x)$ for all $\lambda \in K$ and $x \in V$ so that map $b_q : (x,y) \mapsto \frac{1}{2}(q(x+y) - q(x) - q(y))$ is a bilinear map from $V \times V$ to $K$.

A $K$-linear map from a quadratic space $(V,q)$ to a $K$-algebra $A$ with the unity element $1_A$ is called a *Clifford map* if for every $x \in V$:

$$\varphi(x)^2 = q(x) \cdot 1_A. \tag{1}$$

By replacing $x$ by $x+y$ in the relation (1), we obtain

$$\varphi(x)\varphi(y) + \varphi(y)\varphi(x) = 2b_q(x,y) \cdot 1_A, \tag{2}$$

where $b_q(x,y) = \frac{1}{2}(q(x+y) - q(x) - q(y))$ is the bilinear form associated to the quadratic form $q$. In particular the vectors $x$ and $y$ are orthogonal with respect to the bilinear form $b_q$ if and only if their images under the Clifford map $\varphi$ anticommute.

We recall that the *Clifford algebra* of a quadratic space $(V,q)$ over a field $K$ is an algebra $C = C(V,q)$ over $K$ with the unity element $1_C$, endowed with a Clifford map $i_q$ from $(V,q)$ to $C$ such that:

( C1) the $K$-algebra generated by $1_C$ and $i_q(V)$ is $C$;
(C2) for every Clifford map $\varphi : (V,q) \to A$, there exists an algebra homomorphism $\Phi : C \to A$ such that: $\varphi = \Phi \circ i_q$:

$$\begin{tikzcd} V \arrow[rr, "\varphi"] \arrow[dr, "i_q"'] & & A \\ & C \arrow[ur, "\Phi"'] & \end{tikzcd} \tag{3}$$

Note that according to the properties ($C1$) and ($C2$), the map $\Phi$ is unique and $i_q$ is also injective. The quotient of the tensor algebra $T(V)$ by the ideal generated by all elements of the form $x \otimes x - q(x)\cdot 1$ where $x \in V$, satisfies this universal property. The Clifford algebra of a quadratic space $(V,q)$ which is denoted by $C(V,q)$ therefore exists and it is uniquely determined (up to $K$-algebra isomorphism) by the universal property ($C2$). Fixing the injection map $i_q$, one may identify $V$ with a subspace of $C(V,q)$. The algebra $C(V,q)$ is also denoted by $C(q)$ if this leads to no confusion.

Let $i_q$ be the injection map in the definition of the Clifford algebra $C = C(V,q)$. The map $-i_q$ defined by $(-i_q)(x) = -i_q(x)$ is also a Clifford map from $V$ to $C$, and thus there exists an algebra homomorphism $\gamma : C \to C$ such that the following diagram commutes:

$$
\begin{tikzcd}
V \arrow[rr, "-i_q"] \arrow[dr, "i_q"'] & & C \\
& C \arrow[ur, "\gamma"'] &
\end{tikzcd}
\qquad
\begin{array}{l}
\gamma(1_C) = 1_C, \\
\gamma \circ i_q = -i_q, \\
\gamma(i_q(x)) = -i_q(x).
\end{array}
\tag{4}
$$

As $\gamma^2(i_q(x)) = i_q(x)$, the map $\gamma^2$ coincides with the identity automorphism of $C$.

Let $C^+$ be the set of all elements $x \in C$ such that $\gamma(x) = x$. The set $C^+$ which is also denoted by $C_0(V,q)$ or $C_0(q)$, is a subalgebra of $C$ which is called the *even subalgebra* of $C$. This subalgebra contains every element of $C(V,q)$ which is a product of an even number of vectors of $V$, i.e., the products like $i_q(x_1)i_q(x_2)\cdots i_q(x_{2p})$ where $x_1,\cdots,x_{2p} \in V$.

Let $C^-$ be the set of all elements $x \in C$ such that $\gamma(x) = -x$. The set $C^-$ is a subvector space $C(V,q)$ which is called the odd part of the Clifford algebra $C(V,q)$. This subspace contains contains every products of odd numbers of vectors of $V$.

Let $C^{op}$ be the opposite algebra of $C = C(V,q)$. The map $i_q^* : V \to C^{op}$ defined by $i_q^*(x) = i_q(x)$ is also a Clifford map. There exists so a $K$-algebra homomorphism $J : C \to C^{op}$ such that $J \circ i_q = i_q^*$. As $i_q(V) = i_q^*(V)$, the map $J$ is a bijection. The map $J$ is the unique antiautomorphism of $C$ fixing $i_q(V)$ pointwise. The image of a product $i_q(x_1)\cdots i_q(x_p)$ under the map $J$ is the product of the same terms in the inverse order. The map $J$ is an important involution of $C$ which is sometimes called the *reversion* of $C(V,q)$ (cf. [7, p. 107]). In general, an involution of a ring $R$ is an antiautomorphism of $R$ of order 2. If $S$ is the subring of $R$ and if $S$ is a subset of $\sigma$-invariant elements of $R$ then $\sigma$ is called an $S$-involution. In particular, if $A$ is a $K$-algebra, then a $K$-involution of $A$ is an involution of $A$ fixing $K$ elementwise.

## 3. A universal property of the even Clifford algebra

The aim of this section is to prove that the even Clifford algebra of a quadratic space satisfies a certain universal property.

**Definition 3.1.** Let $(V,q)$ be a quadratic space over a field $K$ and let $A$ be an algebra over $K$ with the unity element $1_A$. An *even Clifford map* is a bilinear map $\psi$ from $V \times V$ to $A$ such that for all $x,\ y,\ z \in V$

1) $\psi(x,y)\psi(y,z) = q(y)\cdot\psi(x,z)$
2) $\psi(x,x) = q(x)\cdot 1_A$

**Remark 3.2.** The conditions (1) and (2) given in the previous definition are respectively equivalent to the following conditions:

1') $\psi(x,y)\psi(x,z) = 2b_q(x,y)\cdot\psi(x,z) - q(x)\cdot\psi(y,z)$
2') $\psi(x,y) + \psi(y,x) = 2b_q(x,y)\cdot 1_A$,

where $b_q$ is the associated quadratic form to $q$. In particular if $x,y$ are orthogonal with respect to the bilinear form $b_q$ then $\psi(x,y)\psi(x,z) = -q(x)\cdot\psi(y,z)$.

**Theorem 3.3.** *Let $C_0 = C_0(V,q)$ be the even Clifford algebra of the nondegenerate quadratic space $(V,q)$ over a field $K$ with the unity element $1_{C_0}$. There exists an even Clifford map $j$ from $V \times V$ to $C_0$ which satisfies the following conditions:*

a) *as a $K$-algebra $C_0$ is generated by $1_{C_0}$ and $\{j(x,y),\ x,\ y \in V\}$.*

b) *for every even Clifford map $\psi : V \times V \to A$, there exists a unique algebra homomorphism $\Psi : C_0 \to A$ such that $\psi = \Psi \circ j$, i.e., the following diagram commutes:*

(5)
$$\begin{tikzcd} V \times V \arrow[rr, "\psi"] \arrow[dr, "j"'] & & A \\ & C_0 \arrow[ur, "\Psi"'] & \end{tikzcd}$$

**Proof.** Let $C = C(V,q)$ be the Clifford algebra of $(V,q)$ and let $i_q : V \to C$ be the canonical injection of $C(V,q)$. We claim that the map $j : V \times V \to C_0$ defined by $j(x,y) = i_q(x)i_q(y)$ is an even Clifford map which satisfies the universal property described in the statement of the theorem.

The fact that $i_q$ is a Clifford map readily implies that $j$ is an even Clifford map. Let $\psi : V \times V \to A$ be an arbitrary even Clifford map satisfying:

$$\begin{cases} \psi(x,y)\psi(y,z) = q(y) \cdot \psi(x,z) \\ \psi(x,x) = q(x) \cdot 1_A \end{cases}$$

We have to find a $K$-algebra homomorphism $\Psi : C_0 \to A$ such that the diagram (5) commutes.

As $(V,q)$ is nondegenerate, there exists a vector $v \in V$ such that $q(v) \neq 0$. Take $d := b_q(v,v) = q(v) \neq 0$ and let $V_0 = v^\perp = \{x \in V : b_q(x,v) = 0\}$. The Witt decomposition $q \simeq \langle d \rangle \perp q_0$ where $q_0$ is a subform of $q$ with $\dim q_0 = \dim q - 1$ induces a decomposition of vector spaces $V = (K \cdot v) \perp V_0$ where $V_0$ is a subspace of $V$ of codimension 1.

Let $f : V_0 \to A$ and $g : V_0 \to C_0$ the maps defined by $f(w) = \psi(v,w)$ and $g(w) = i_q(v)i_q(w)$ for every $w \in V_0$. We have $f(w)^2 = \psi(v,w)^2 = -q(v)q(w) \cdot 1_A = -dq_0(w) \cdot 1_A$ and $g(w)^2 = (i_q(v)i_q(w))^2 = -dq_0(v) \cdot 1_C = -dq_0(v) \cdot 1_{C_0}$. Thus $f : (V_0, -dq_0) \to A$ and $g : (V_0, -dq_0) \to C_0$ are two Clifford maps. It follows that there exist unique homomorphisms $F : C(V_0, -dq_0) \to A$ and $G : C(V_0, -dq_0) \to C_0$ such that the following diagram commutes:

(6)
$$\begin{tikzcd} & C_0 & \\ V_0 \arrow[ur, "g"] \arrow[rr, "i_{-dq_0}"] \arrow[dr, "f"'] & & C(V_0, -dq_0) \arrow[ul, "G"'] \arrow[dl, "F"] \\ & A & \end{tikzcd}$$

For all $w,\ w' \in V_0$ we have $g(w)g(w') = -d i_q(w)i_q(w')$. Therefore for all $w,\ w' \in V_0$, the image of $G$ contains the elements $i_q(w)i_q(w')$. But $C_0$ is generated by the elements $i_q(x)i_q(y)$, $x,\ y \in V$ and $1_{C_0}$ and the element $i_q(x)i_q(y)$ can be written as a linear combination of the elements $i_q(w)i_q(w')$, $i_q(v)i_q(w)$, $1_C$ for suitable $w,\ w' \in V_0$. These elements are in the image of $G$. Therefore the homomorphism $G$ is surjective. For dimension reasons it is also injective, it is therefore an isomorphism. Consider the map $\Psi : C_0 \to A$ defined by

$$\Psi = F \circ G^{-1}.$$

We claim that the diagram (5) commutes. Let $(v_1, v_2) \in V \times V$. We can write $(v_1, v_2) = (\lambda_1 v + w_1, \lambda_2 v + w_2)$ where $\lambda_1,\ \lambda_2 \in K$ and $w_1,\ w_2 \in V_0$. We have:

$$\begin{aligned}
\Psi(j(v_1,v_2)) &= F \circ G^{-1}(j(v_1,v_2)) \\
&= F \circ G^{-1}(i_q(v_1) i_q(v_2)) \\
&= F \circ G^{-1}(i_q(\lambda_1 v + w_1) i_q(\lambda_2 v + w_2)) \\
&= F \circ G^{-1}(\lambda_1 \lambda_2 q(v) + \lambda_2 i_q(w_1) i_q(v) \\
&\qquad + \lambda_1 i_q(v) i_q(w_2) + i_q(w_1) i_q(w_2)) \\
&= F \circ G^{-1}(\lambda_1 \lambda_2 q(v) - \lambda_2 i_q(v) i_q(w_1) \\
&\qquad + \lambda_1 i_q(v) i_q(w_2) + i_q(w_1) i_q(w_2)) \\
&= F \circ G^{-1}(\lambda_1 \lambda_2 q(v) - \lambda_2 g(w_1) \\
&\qquad + \lambda_1 g(w_2) - d^{-1} g(w_1) g(w_2)) \\
&= F(\lambda_1 \lambda_2 q(v) - \lambda_2 i_{-dq_0}(w_1) \\
&\qquad + \lambda_1 i_{-dq_0}(w_2) - d^{-1} i_{-dq_0}(w_1) i_{-dq_0}(w_2)) \\
&= \lambda_1 \lambda_2 q(v) - \lambda_2 \psi(v, w_1) \\
&\qquad + \lambda_1 \psi(v, w_2) - d^{-1} \psi(v, w_1) \psi(v, w_2) \\
&= \lambda_1 \lambda_2 q(v) + \lambda_2 \psi(w_1, v) + \lambda_1 \psi(v, w_2) + \psi(w_1, w_2) \\
&= \psi(\lambda_1 v + w_1, \lambda_2 v + w_2) \\
&= \psi(v_1, v_2).
\end{aligned}$$

□

**Remark 3.4.** As an immediate consequence, we obtain that the algebra $C_0 = C_0(V, q)$ is uniquely determined by the universal property described in the statement of the previous theorem. In fact if $(C_0, j)$ and $(C_0', j')$ are two pairs satisfying the universal property proved in the previous theorem, then there exist two $K$-algebra homomorphisms $J$ and $J'$ such that the following diagram commutes:

(7)
$$\begin{tikzcd}
V \times V \arrow[rr, "j'"] \arrow[dr, "j"'] & & C_0' \arrow[dl, shift left, "J'"] \\
& C_0 \arrow[ur, shift left, "J"] &
\end{tikzcd}$$

The relations $J \circ j = j'$ and $J' \circ j' = j$ imply that: $J' \circ J \circ j = J' \circ j' = j$ and $J \circ J' \circ j' = J \circ j = j'$. As $C_0$ is generated, as a $K$-algebra, by $1_{C_0}$ and $j(V \times V)$ and $C_0'$ is generated, as a $K$-algebra, by $1_{C_0'}$ and $j'(V \times V)$, we obtain $J' \circ J = \mathrm{id}_{C_0}$ and $J \circ J' = \mathrm{id}_{C_0'}$.

## 4. Involutions induced by an orthogonal symmetry

**Definition 4.1.** Let $V$ be a finite dimensional vector space over a field $K$ endowed with a nondegenerate (symmetric or anti-symmetric) bilinear form $b$. An *isometry* of $(V, q)$, i.e., an element $\sigma \in \mathrm{End}(V)$ such that $b(\sigma x, \sigma y) = b(x, y)$ for all $x,\ y \in V$ is said to be a *orthogonal symmetry* if $\sigma^2 = \mathrm{id}$. In the literature, such maps are sometimes also called "orthogonal involutions" (cf. [4, Ch.III, §5]). We have, however, preferred to use the former term in order to avoid any possible confusion with already well-established notions of orthogonal, symplectic and unitary involutions (see [9]). A *reflection* $\tau$ of $(V, q)$ is an orthogonal symmetry whose invariant subspace $V^+ = \{x \in V :\ \tau(x) = x\}$ is a hyperplane of $V$.

We recall the following result from linear algebra:

**Proposition 4.2.** *Let $(V, b)$ be a nondegenerate symmetric or anti-symmetric bilinear space over a field $K$. Let $\sigma \in \mathrm{End}(V)$ be an element with $\sigma^2 = \mathrm{id}$. Let $V^+$ and $V^-$ be respectively the eigenspaces of $\sigma$ associated to the eigenvalues $+1$ and $-1$. Then:*

1) *the space $V$ is the direct sum of $V^+$ and $V^-$.*
2) *the map $\sigma$ is an orthogonal symmetry if and only if $V^+$ and $V^-$ are orthogonal with respect to the form $b$.*

*See* [4, Ch. III, §5] or [5, Ch. I, §3].

**Proposition 4.3.** *Let $(V,q)$ be a quadratic space over a field $K$ and let $s \in \operatorname{End} V$. In order that there exists a $K$-involution $\sigma$ of $C(V,q)$ such that for every $x \in V$, $\sigma(i_q(x)) = i_q(s(x))$, it is necessary and sufficient that $s$ be an orthogonal symmetry of $(V,q)$.*

**Proof.** Suppose that there exists a $K$-involution $\sigma$ of $C(V,q)$ such that for every $x \in V$, $\sigma(i_q(x)) = i_q(s(x))$. As $q(x) = \sigma(q(x)) = \sigma(i_q(x)^2) = i_q(s(x))^2 = q(s(x))$, the map $s$ is an isometry of $(V,q)$. We also have $i_q(s^2(x)) = \sigma(i_q(s(x))) = \sigma(\sigma(i_q(x))) = i_q(x)$. The injectivity of $i_q$ implies that $s^2 = \mathrm{id}$. Therefore $s$ is an orthogonal symmetry.

Conversely suppose that $s$ is an orthogonal symmetry. Consider the map $\varphi : V \to C(V,q)$ defined by $\varphi(x) = i_q(s(x))$ for every $x \in V$. As $\varphi(x)^2 = i_q(s(x))^2 = q(s(x)) = q(x)$, $\varphi$ is a Clifford map. According to the universal property of the Clifford algebra, there exists a unique homomorphism $\Phi : C(V,q) \to C(V,q)$ such that $\Phi(i_q(x)) = \varphi(x)$ for every $x \in V$. Note that the image of $\Phi$ contains $i_q(V)$. As $C(V,q)$ is generated by $i_q(V)$, the homomorphism $\Phi$ is surjective, hence it is an isomorphism.

Let $j$ be the reversion involution of $C(V,q)$ as defined in §2. Consider the map $\sigma : C(V,q) \to C(V,q)$ defined by $\sigma = j \circ \Phi$. The map $\sigma$ is an anti-automorphism because $j$ is an anti-automorphism and $\Phi$ is an isomorphism. We have $\sigma^2 = \mathrm{id}$, in fact let $x \in V$, we obtain $\sigma^2(i_q(x)) = j\circ\Phi\circ j\circ\Phi(i_q(x)) = j\circ\Phi\circ j(\varphi(x)) = j\circ\Phi(\varphi(x)) = j\circ\Phi(i_q(s(x))) = j(\varphi(s(x))) = \varphi(s(x)) = i_q(s^2(x)) = i_q(x)$. Thus we have shown that $\sigma$ is a $K$-involution of $C(V,q)$. We also have $\sigma(i_q(x)) = j\circ\Phi(i_q(x)) = j(\varphi(x)) = i_q(s(x))$. This completes the proof. □ J. Pure Appl. Algebra 147 (2000), no. 3, 303–309.

**Notation 4.4.** Let $s$ be an orthogonal symmetry of a quadratic space $(V,q)$. From now on, the unique involution of the Clifford algebra $C(V,q)$ which maps $i_q(x)$ to $i_q(s(x))$, for every $x \in V$, is denoted by $J_q^s$. Compare with [19, Ch. 3, 3.15; Ch. 4, 4.3]

**Remark 4.5.** We recall that the *canonical involution* $\gamma$ of a quaternion algebra $Q = (a,b)_K$ is the involution $\gamma : Q \to Q$, defined by $\gamma(a+bi+cj+dk) = a-bi-cj-dk$ where $a$, $b$, $c$, $d \in K$ and $i$, $j$, $k$ are the generators of $Q$ with $i^2 = a$, $j^2 = -b$ and $ij = -ji = k$. When $q$ is a nondegenerate quadratic form of dimension 2, $C(V,q)$ is a quaternion algebra. In this case, the canonical involution of $C(V,q)$ coincides with the involution $J_q^{-\mathrm{id}}$.

**Corollary 4.6.** *Let $(V,q)$ be a quadratic space over a field $K$, let $s$ be an orthogonal symmetry of $V$ and let $j_q : V\times V \to C_0(V,q)$ be the even Clifford map which satisfies the universal property described in the statement of theorem 3.3. Then there exists a unique $K$-involution of $C_0(V,q)$, again denoted by $J_q^s$, such that $J_q^s(j_q(x,y)) = j_q(s(y),s(x))$ for all $x$, $y \in V$.*

**Proof.** The involution $J_q^s$ of $C_0(V,q)$ is exactly the restriction of the involution $J_q^s$ of $C(V,q)$. One can also give a direct proof using the universal property of $C_0(V,q)$. □

## 5. Decomposition of involutions induced by an orthogonal symmetry

We recall the following well known result:

**Lemma 5.1.** *Let $s$ be an orthogonal symmetry of a quadratic space $(V,q)$. Let $\{e_1,\cdots,e_r\}$ be an orthogonal basis of $V^+$ and let $\{f_1,\cdots,f_s\}$ be an orthogonal basis of $V^-$ (cf. 4.2). Consider the element $z = i_q(e_1)\cdots i_q(e_r)\cdot i_q(f_1)\cdots i_q(f_s) \in C(V,q)$. Then:*

1) *we have* $J_q^s(z) = (-1)^s(-1)^{\frac{n(n-1)}{2}}z$, *where* $n = r+s = \dim V$, *in particular if* $s$ *is a reflection we have* $J_q^s(z) = (-1)^{\frac{n(n+1)}{2}}z$.
2) *we have* $z^2 = \mathrm{d}_{\pm}q$.
3) *when* $n$ *is even,* $z$ *anti-commutes with every element* $i_q(x)$, *where* $x \in V$.
4) *when* $n$ *is odd,* $z$ *commutes with every element* $i_q(x)$, *where* $x \in V$.

**Proof.** The set $\{e_1,\cdots,e_r,f_1,\cdots,f_s\}$ is an orthogonal basis of $V$ because according to 4.2, $V^+$ and $V^-$ are orthogonal. We have:

$$\begin{aligned} J_q^s(z) &= ((-1)^s i_q(f_s)\cdots i_q(f_1))(i_q(e_r)\cdots i_q(e_1)) \\ &= (-1)^s(-1)^{\frac{n(n-1)}{2}} i_q(e_1)\cdots i_q(e_r) i_q(f_1)\cdots i_q(f_s) \\ &= (-1)^s(-1)^{\frac{n(n-1)}{2}} z. \end{aligned}$$

The assertions (2), (3) and (4) are simple well known calculations, see for instance [11, Ch.V, §2 ]. □

**Proposition 5.2.** *Let* $(V_1,q_1)$ *be a quadratic space of even dimension over a field* $K$, *let* $(V,q)$ *be an arbitrary quadratic space, let* $\sigma_1$ *be an orthogonal symmetry of* $(V_1,q_1)$ *and let* $\sigma$ *be an orthogonal symmetry of* $(V,q)$. *Let* $V_1$ *be the vector space of* $\sigma_1$*-anti-symmetric elements of* $V_1$ *and suppose that* $\dim {V_1}^- = s$ *(cf. 4.2). Then:*

(a) *If* $\dim V_1 \equiv 1 \mod 4$ *then we have:*

$$(C_0(V_1 \perp V, q_1 \perp q), J_{q_1\perp q}^{\sigma_1\oplus\sigma}) \simeq (C_0(V_1,q_1)\otimes C(V,-\mathrm{d}_{\pm}q_1\cdot q), J_{q_1}^{\sigma_1}\otimes J_{-\mathrm{d}_{\pm}q_1\cdot q}^{(-1)^{s+1}\sigma}),$$

*in particular if* $\sigma_1 = \tau$ *is a reflection then we have:*

$$(C_0(V_1 \perp V, q_1 \perp q), J_{q_1\perp q}^{\tau\oplus\sigma}) \simeq (C_0(V_1,q_1)\otimes C(V,-\mathrm{d}_{\pm}q_1\cdot q), J_{q_1}^{\tau}\otimes J_{-\mathrm{d}_{\pm}q_1\cdot q}^{\sigma}).$$

(b) *If* $\dim V_1 \equiv 3 \mod 4$ *then we have:*

$$(C_0(V_1 \perp V, q_1 \perp q), J_{q_1\perp q}^{\sigma_1\oplus\sigma}) \simeq (C_0(V_1,q_1)\otimes C(V,-\mathrm{d}_{\pm}q_1\cdot q), J_{q_1}^{\sigma_1}\otimes J_{-\mathrm{d}_{\pm}q_1\cdot q}^{(-1)^{s}\sigma}),$$

*in particular if* $\sigma_1 = \tau$ *is a reflection then we have:*

$$(C_0(V_1 \perp V, q_1 \perp q), J_{q_1\perp q}^{\tau\oplus\sigma}) \simeq (C_0(V_1,q_1)\otimes C(V,-\mathrm{d}_{\pm}q_1\cdot q), J_{q_1}^{\tau}\otimes J_{-\mathrm{d}_{\pm}q_1\cdot q}^{-\sigma}).$$

**Proof.** Let $j_{q_1\perp q} : (V_1 \perp V)\times(V_1 \perp V) \to C_0(V_1 \perp V, q_1 \perp q)$ be a map which satisfies the universal property discussed in 3.3. Let $z \in C(V_1,q_1)$ be the element defined in 5.1. Let $\eta : (V_1 \perp V)\times(V_1 \perp V) \to C_0(V_1,q_1)\otimes C(V,-\mathrm{d}_{\pm}q_1\cdot q)$ be the map defined by

$$\eta(x_1\oplus x, y_1\oplus y) = (i_{q_1}(x_1)\otimes 1 + z^{-1}\otimes i_{-\mathrm{d}_{\pm}q_1\cdot q}(x))\times(i_{q_1}(y_1)\otimes 1 - z^{-1}\otimes i_{-\mathrm{d}_{\pm}q_1\cdot q}(y)).$$

It is easy to verify that $\eta$ is an even Clifford map. There exists so a homomorphism

$$H : C_0(V_1 \perp V, q_1 \perp q) \to C_0(V_1,q_1)\otimes C(V,-\mathrm{d}_{\pm}q_1\cdot q)$$

such that the following diagram commutes:

$$\begin{tikzcd} (V_1 \perp V)\times(V_1 \perp V) \arrow[r, "j_{q_1\perp q}"] \arrow[dr, "\eta"'] & C_0(V_1 \perp V, q_1 \perp q) \arrow[d, "H"] \\ & C_0(V_1,q_1)\otimes C(V,-\mathrm{d}_{\pm}q_1\cdot q) \end{tikzcd} \tag{8}$$

We now prove that $H$ is an isomorphism. As

$$\dim_K C_0(V_1 \perp V, q_1 \perp q) = \dim_K C_0(V_1,q_1)\otimes C(V,-\mathrm{d}_{\pm}q_1\cdot q),$$

the homomorphism $H$ is surjective if and only if it is injective. We show that $H$ is surjective. The algebra $C_0(V_1,q_1)\otimes C(V,-\mathrm{d}_{\pm}q_1\cdot q)$ is generated by all elements

$i_{q_1}(x_1)i_{q_1}(y_1)\otimes 1$ and $1\otimes i_{-\mathrm{d}_\pm q_1\cdot q}(y)$ where $x_1,\ y_1\in V_1$ and $y\in V$. It is enough to show that these elements lie in the image of $H$.

We have:

$$\eta(x_1\oplus 0, y_1\oplus 0)=i_{q_1}(x_1)i_{q_1}(y_1)\otimes 1,$$

thus the element $i_{q_1}(x_1)i_{q_1}(y_1)\otimes 1$ lie in the image of $H$.

On the other hand, $\eta(x_1\oplus 0, 0\oplus y)=i_q(x_1)z^{-1}\otimes i_{-\mathrm{d}_\pm q_1\cdot q}(y)$. By definition, the element $z$ is of the form $i_{q_1}(e_1)i_{q_1}(e_1)\cdots i_{q_1}(e_{2m+1})$ where $\{e_1,\, e_2,\, \cdots,\, e_{2m+1}\}$ is a basis of $V_1$. Therefore the element

$$i_q(e_{2m})^{-1}\cdots i_q(e_2)^{-1}i_q(e_1)^{-1}\otimes i_{-\mathrm{d}_\pm q_1\cdot q}(y)=i_q(e_{2m+1})z^{-1}\otimes i_{-\mathrm{d}_\pm q_1\cdot q}(y)$$

lies in the image of $H$. As $i_q(e_1)i_q(e_2)\cdots i_q(e_n)\otimes 1$ also lies to the image of $H$, we deduce that $1\otimes i_{-\mathrm{d}_\pm q_1\cdot q}(y)$ lies in the image of $H$.

Suppose that $\dim V_0\equiv 1 \mod 4$, to prove (a), it is enough to verify that the isomorphism

$$C_0(V_1\perp V, q_1\perp q)\simeq (C(V_0,q_0)\otimes C(V,-\mathrm{d}_\pm q_1\cdot q)$$

is compatible with the involutions $J^{\sigma_1\oplus\sigma}_{q_1\perp q}$ and $J^{\sigma_1}_{q_1}\otimes J^{(-1)^{s+1}\sigma}_{-\mathrm{d}_\pm q_1\cdot q}$. For all $x_1,\ y_1\in V_1$ and $x,\ y\in V$ we have:

$$\begin{aligned}
H\circ J^{\sigma_1\oplus\sigma}_{q_1\perp q}(j_{q_1\perp q}(x_1\oplus x, y_1\oplus y)) &= H(j_{q_1\perp q}(\sigma_1(y_1)\oplus\sigma(y),\sigma_1(x_1)\oplus\sigma(x)))\\
&= \eta(\sigma_1(y_1)\oplus\sigma(y),\sigma_1(x_1)\oplus\sigma(x)))\\
&= (i_{q_1}(\sigma(y_1))\otimes 1+z^{-1}\otimes i_{-\mathrm{d}_\pm q_1\cdot q}(\sigma(y)))\\
&\quad\times(i_{q_1}(\sigma(x_1))\otimes 1-z^{-1}\otimes i_{-\mathrm{d}_\pm q_1\cdot q}(\sigma(x)))
\end{aligned}$$

On the other hand, if $A=J^{\sigma_1}_{q_1}\otimes J^{(-1)^{s+1}\sigma}_{-\mathrm{d}_\pm q_1\cdot q}\circ H(j_{q_1\perp q}(x_1\oplus x, y_1\oplus y))$, using 5.1 we obtain

$$\begin{aligned}
&J^{\sigma_1}_{q_1}\otimes J^{(-1)^{s+1}\sigma}_{-\mathrm{d}_\pm q_1\cdot q}\circ H(j_{q_1\perp q}(x_1\oplus x, y_1\oplus y))\\
={}& J^{\sigma_1}_{q_1}\otimes J^{(-1)^{s+1}\sigma}_{-\mathrm{d}_\pm q_1\cdot q}\Big((i_{q_1}(x_1)\otimes 1\\
&+z^{-1}\otimes i_{-\mathrm{d}_\pm q_1\cdot q}(x))\\
&\times(i_{q_1}(y_1)\otimes 1-z^{-1}\otimes i_{-\mathrm{d}_\pm q_1\cdot q}(y))\Big)\\
={}& (i_{q_1}(\sigma_1(y_1))\otimes 1\\
&-(-1)^s z^{-1}\otimes i_{-\mathrm{d}_\pm q_1\cdot q}((-1)^{s+1}\sigma(y)))\\
&\times(i_{q_1}(\sigma_1(x_1))\otimes 1\\
&+(-1)^s z^{-1}\otimes i_{-\mathrm{d}_\pm q_1\cdot q}((-1)^{s+1}\sigma(x)))\\
={}& (i_{q_1}(\sigma(y_1))\otimes 1+z^{-1}\otimes i_{-\mathrm{d}_\pm q_1\cdot q}(\sigma(y)))\\
&\times(i_{q_1}(\sigma(x_1))\otimes 1-z^{-1}\otimes i_{-\mathrm{d}_\pm q_1\cdot q}(\sigma(x))).
\end{aligned}$$

We thus have:

$$(J^{\sigma_1}_{q_1}\otimes J^{(-1)^{s+1}\sigma}_{-\mathrm{d}_\pm q_1\cdot q})\circ H=H\circ J^{\sigma_1\oplus\sigma}_{q_1\perp q}$$

It follows that $H$ is an automorphism of algebras with involution.

The proof of (b) is similar. □

**Remark 5.3.** The isomorphism

$$C_0(V_1\perp V, q_1\perp q)\simeq (C_0(V_1,q_1)\otimes C(V,-\mathrm{d}_\pm q_1\cdot q) \tag{9}$$

is known. (cf. [11, Ch.V, §2]). Proposition 5.2 is so an involutorial version of (9). The classical proof of (9) given in [11] is different from what we have provided and uses the properties of graded simple algebras and graded tensor product.

**Corollary 5.4.** *Keeping the same hypotheses as in 5.2:*

(a) *If* $\dim V_1 \equiv 1 \mod 4$ *then we have:*

$$(C_0(V_1,q_1)\otimes C(V,q), J_{q_1}^{\sigma_1}\otimes J_q^{\sigma}) \simeq (C_0(V_1 \perp V, q_1 \perp -\mathrm{d}_{\pm}q_1\cdot q), J_{q_1\perp -\mathrm{d}_{\pm}q_1\cdot q}^{\sigma_1\oplus(-1)^{s+1}\sigma}),$$

*in particular if* $\sigma_1 = \tau$ *is a reflection then we have:*

$$(C_0(V_1,q_1)\otimes C(V,q), J_{q_1}^{\tau}\otimes J_q^{\sigma}) \simeq (C_0(V_1 \perp V, q_1 \perp -\mathrm{d}_{\pm}q_1\cdot q), J_{q_1\perp -\mathrm{d}_{\pm}q_1\cdot q}^{\tau\oplus\sigma}).$$

(b) *If* $\dim V_1 \equiv 3 \mod 4$ *then we have:*

$$(C_0(V_1,q_1)\otimes C(V,q), J_{q_1}^{\sigma_1}\otimes J_q^{\sigma}) \simeq (C_0(V_1 \perp V, q_1 \perp -\mathrm{d}_{\pm}q_1\cdot q), J_{q_1\perp -\mathrm{d}_{\pm}q_1\cdot q}^{\sigma_1\oplus(-1)^{s}\sigma}),$$

*in particular if* $\sigma_1$ *is a reflection then we have:*

$$(C_0(V_1,q_1)\otimes C(V,q), J_{q_1}^{\tau}\otimes J_q^{\sigma}) \simeq (C_0(V_1 \perp V, q_1 \perp -\mathrm{d}_{\pm}q_1\cdot q), J_{q_1\perp -\mathrm{d}_{\pm}q_1\cdot q}^{\tau\oplus-\sigma}).$$

**Corollary 5.5.** *For every quadratic form $q$ over a field $K$ and for every $d \in K^*$, there exists an isomorphism of algebras with involution*

$$(C_0(\langle -d\rangle \perp q), J^{\pm\mathrm{id}}) \simeq (C(d\cdot q), J^{-\mathrm{id}}). \tag{10}$$

*In particular,*

$$(C_0(\langle -1\rangle \perp q), J^{\pm\mathrm{id}}) \simeq (C(q), J^{-\mathrm{id}}). \tag{11}$$

**Corollary 5.6.** *Keeping the notation of 5.5, let $V_1$ and $V$ be respectively the underlying vector spaces of $\langle -d\rangle$ and $q$. Consider the reflection $\tau = -\mathrm{id}_{V_1}\oplus\mathrm{id}_V$. Then we have an isomorphism of algebras with involution*

$$(C_0(\langle -d\rangle \perp q), J^{\tau}) \simeq (C(d\cdot q), J^{\mathrm{id}}). \tag{12}$$

*In particular,*

$$(C_0(\langle -1\rangle \perp q), J^{\tau}) \simeq (C(q), J^{\mathrm{id}}). \tag{13}$$

**Corollary 5.7.** *Let $(V,q)$ be a quadratic space over a field $K$, let $\sigma$ be an orthogonal symmetry of $V$ and let $v \in V$ be an anisotropic vector such that $\sigma(v) = \varepsilon v$ where $\varepsilon = 1$ ou $-1$. Let $V' = \{x \in V : b_q(x,v) = 0\}$ the orthogonal complement of the subspace generated by $v$ and let $d = q(v)$. Then $V'$ is stable under $\sigma$ and we have an isomorphism of algebras with involution*

$$(C_0(V,q), J_q^{\sigma}) \simeq (C(V', -d\cdot q'), J_{-d\cdot q'}^{\sigma'}), \tag{14}$$

*where $\sigma' = -\varepsilon\sigma|_{V'}$ and $q' = q|_{V'}$.*

**Proof.** It is enough to note that, $V'$ is stable under $\sigma$. We have the decomposition $(V,\sigma) = (v\cdot K \oplus V', \pm\mathrm{id}\oplus\sigma|_{V'})$ and we can use Proposition 5.2. □

**Corollary 5.8.** *Let $(V,q)$ be a nondegenerate quadratic space over a field $K$ and let $a \in K^*$. Then for every orthogonal symmetry $\sigma$ of $(V,q)$ we have an isomorphism of algebras with involution*

$$(C_0(V,q), J_q^{\sigma}) \simeq (C_0(V, a\cdot q), J_{a\cdot q}^{\sigma}). \tag{15}$$

**Proof.** Let $V^+$ and $V^-$ be respectively the subspaces of the symmetric and anti-symmetric elements of $V$ (cf. 4.2). As $q$ is nondegenerate, it follows that either $V^+$ or $V^-$ contain an anisotropic vector $v$. Thus there exists $v \in V$ such that $\sigma(v) = \varepsilon v$ where $\varepsilon = \pm 1$ and $b = q(v) \in K^*$. Let $V'$ be the subspace of $V$ consisting of all elements, orthogonal to $v$ and let $q' = q|_{V'}$. Using Proposition 5.7 we obtain

$$(C_0(V, a\cdot q), J_q^{\sigma}) \simeq (C(V', -a^2b\cdot q'), J_{-a^2b\cdot q'}^{-\varepsilon\sigma}),$$

$$(C_0(V,q), J_q^{\sigma}) \simeq (C(V', -b\cdot q'), J_{-b\cdot q'}^{-\varepsilon\sigma}).$$

The quadratic spaces $(V',-b\cdot q')$ and $(V',-ba^2\cdot q')$ are isometric. We thus have an isomorphism: $C(V',-b\cdot q') \simeq C(V',-a^2b\cdot q')$. This isomorphism which is induced by the isometry between $(V',-b\cdot q')$ and $(V',-ba^2\cdot q')$ is compatible with $J^{\varepsilon\sigma}_{-ba^2\cdot q'}$ and $J^{\varepsilon\sigma}_{-b\cdot q'}$, the proof is thus achieved. □

The analogue of Proposition 5.2 for the quadratic forms of even dimension in the particular case where $\sigma = \mathrm{id}$ or $-\mathrm{id}$ is known: in [12], David Lewis proved that for a quadratic space $(V_0, q_0)$ of even dimension and for any quadratic form $(V, q)$, there exists an isomorphism of algebras with involution

$$(C(V_0 \perp V), J^{\pm\mathrm{id}}_{q_0\perp q}) \simeq (C(V_0, q_0), J^{\pm\mathrm{id}}_{q_0}) \otimes (C(V, -\mathrm{d}_\pm q_0\cdot q), J^{\mp\mathrm{id}}_{-\mathrm{d}_\pm q_0\cdot q}). \tag{16}$$

This result can be generalized without difficulty to the case of an arbitrary orthogonal symmetry:

**Proposition 5.9.** ([12, Prop. 2]) *Let $(V_0, q_0)$ be a quadratic space of even dimension, let $(V, q)$ be an arbitrary quadratic space, let $\sigma_0$ be an orthogonal symmetry of $(V_0, q_0)$ and let $\sigma$ be an orthogonal symmetry of $(V, q)$. Suppose that* $\dim V_0^- = s$*. Then:*

(a) *If* $\dim V_0 \equiv 2 \mod 4$ *then we have:*

$$(C(V_0 \perp V, q_0 \perp q), J^{\sigma_0\oplus\sigma}_{q_0\perp q}) \simeq (C(V_0, q_0)\otimes C(V, \mathrm{d}_\pm q_0\cdot q), J^{\sigma_0}_{q_0} \otimes J^{(-1)^{s+1}\sigma}_{\mathrm{d}_\pm q_0\cdot q}),$$

*in particular if $\sigma_0 = \tau$ is a reflection then we have:*

$$(C(V_0 \perp V, q_0 \perp q), J^{\tau\oplus\sigma}_{q_0\perp q}) \simeq (C(V_0, q_0)\otimes C(V, \mathrm{d}_\pm q_0\cdot q), J^{\tau}_{q_0} \otimes J^{\sigma}_{\mathrm{d}_\pm q_0\cdot q}).$$

(b) *If* $\dim V_0 \equiv 0 \mod 4$ *then we have:*

$$(C(V_0 \perp V, q_0 \perp q), J^{\sigma_0\oplus\sigma}_{q_0\perp q}) \simeq (C(V_0, q_0)\otimes C(V, \mathrm{d}_\pm q_0\cdot q), J^{\sigma_0}_{q_0} \otimes J^{(-1)^{s}\sigma}_{\mathrm{d}_\pm q_0\cdot q}),$$

*in particular if $\sigma_0 = \tau$ is a reflection then we have:*

$$(C(V_0 \perp V, q_0 \perp q), J^{\tau\oplus\sigma}_{q_0\perp q}) \simeq (C(V_0, q_0)\otimes C(V, \mathrm{d}_\pm q_0\cdot q), J^{\tau}_{q_0} \otimes J^{-\sigma}_{\mathrm{d}_\pm q_0\cdot q}).$$

**Proof.** Let $\{e_1, \cdots, e_r\}$ be an orthogonal basis of $V_0{}^+$ and let $\{f_1, \cdots, f_s\}$ be an orthogonal basis of $V_0{}^-$. Consider the element

$$z = i_{q_0}(e_1)\cdots i_{q_0}(e_r) i_{q_0}(f_1)\cdots i_{q_0}(f_s) \in C(V_0 \perp V, q_0 \perp q).$$

Let $\varphi : V_0 \perp V_1 \to C(V_0, q_0)\otimes C(V, \mathrm{d}_\pm q_0\cdot q)$ be the map defined by

$$\varphi(x_0 \oplus x) = i_{q_0}(x_0)\otimes 1 + z^{-1}\otimes i_{\mathrm{d}_\pm q_0\cdot q}(x). \tag{17}$$

We note that $\varphi$ is a Clifford map, because:

$$\begin{aligned}
\varphi(x_0\oplus x)^2 &= (i_{q_0}(x_0)\otimes 1 + z^{-1}\otimes i_{\mathrm{d}_\pm q_0\cdot q}(x))^2 \\
&= q_0(x_0)\otimes 1 + (\mathrm{d}_\pm q_0)^{-1}\otimes(\mathrm{d}_\pm q_0\cdot q(x)) \\
&\quad + i_{q_0}(x_0)z^{-1}\otimes i_{\mathrm{d}_\pm q_0\cdot q}(x) + z^{-1} i_{q_0}(x_0)\otimes i_{\mathrm{d}_\pm q_0\cdot q}(x) \\
&= (q_0 \perp q)(x_0\oplus x)\cdot(1\otimes 1).
\end{aligned}$$

The map $\varphi$ can therefore be extended to a homomorphism

$$\Phi : C(V_0 \perp V) \to C(V_0, q_0)\otimes C(V, \mathrm{d}_\pm q_0\cdot q).$$

The definition of $\varphi$ in (17) implies that $\varphi$ is surjective. As

$$\dim_K C(q_0 \perp q) = \dim_K (C(V_0, q_0)\otimes C(V, \mathrm{d}_\pm q_0\cdot q)),$$

$\Phi$ is injective hence it is an isomorphism. It is enough to show that $\Phi$ is compatible with the indicated involutions.

If $\dim V_0 \equiv 2 \mod 4$, using 5.1 we obtain

$$\begin{aligned}(J_q^{\sigma_0}\otimes J_{\mathrm{d}_\pm q_0\cdot q}^{(-1)^{s+1}\sigma})\circ\Phi(i_{q_0\perp q}(x_0\oplus x)) &= (J_q^{\sigma_0}\otimes J_{\mathrm{d}_\pm q_0\cdot q}^{(-1)^{s+1}\sigma})(i_{q_0}(x_0)\otimes 1+z^{-1}\otimes i_{\mathrm{d}_\pm q_0\cdot q}(x))\\ &= i_{q_0}(\sigma_0(x_0))\otimes 1+J_{q_0}^{\sigma_0}(z^{-1})\otimes i_{\mathrm{d}_\pm q_0\cdot q}((-1)^{s+1}\sigma(x))\\ &= i_{q_0}(\sigma_0(x_0))\otimes 1+z^{-1}\otimes i_{\mathrm{d}_\pm q_0\cdot q}(\sigma(x)).\end{aligned}$$

On the other hand, $\Phi\circ J_{q_0\perp q}^{\sigma_0\oplus\sigma}(i_{q_0\perp q}(x_0\oplus x)) = i_{q_0}(\sigma_0(x_0))\otimes 1+z^{-1}\otimes i_{\mathrm{d}_\pm q_0\cdot q}(\sigma(x))$. We therefore have:

$$\Phi\circ J_{q_0\perp q}^{\sigma_0\oplus\sigma} = (J_q^{\sigma_0}\otimes J_{\mathrm{d}_\pm q_0\cdot q}^{(-1)^{s+1}\sigma})\circ\Phi.$$

Consequently $\Phi$ is an isomorphism of algebras with involution.
The proof for the case where $\dim V_0\equiv 0 \mod 4$ is similar. □

**Corollary 5.10.** *Keeping the same hypotheses as in 5.9 :*

(a) If $\dim V_0\equiv 2 \mod 4$ then we have:

$$(C(V_0,q_0)\otimes C(V,q), J_{q_0}^{\sigma_0}\otimes J_q^{\sigma})\simeq (C(V_0\perp V, q_0\perp \mathrm{d}_\pm q_0\cdot q), J_{q_0\perp \mathrm{d}_\pm q_0\cdot q}^{\sigma_0\oplus(-1)^{s+1}\sigma}),$$

in particular if $\sigma_0=\tau$ is a reflection then we have:

$$(C(V_0,q_0)\otimes C(V,q), J_{q_0}^{\tau}\otimes J_q^{\sigma})\simeq (C(V_0\perp V, q_0\perp \mathrm{d}_\pm q_0\cdot q), J_{q_0\perp \mathrm{d}_\pm q_0\cdot q}^{\tau\oplus\sigma}).$$

(b) If $\dim V_0\equiv 0 \mod 4$ then we have:

$$(C(V_0,q_0)\otimes C(V,q), J_{q_0}^{\sigma_0}\otimes J_q^{\sigma})\simeq (C(V_0\perp V, q_0\perp \mathrm{d}_\pm q_0\cdot q), J_{q_0\perp \mathrm{d}_\pm q_0\cdot q}^{\sigma_0\oplus(-1)^{s}\sigma}),$$

in particular if $\sigma_0=\tau$ is a reflection then we have:

$$(C(V_0,q_0)\otimes C(V,q), J_{q_0}^{\tau}\otimes J_q^{\sigma})\simeq (C(V_0\perp V, q_0\perp \mathrm{d}_\pm q_0\cdot q), J_{q_0\perp \mathrm{d}_\pm q_0\cdot q}^{\tau\oplus-\sigma}).$$

**Remark 5.11.** Anne Cortella has pointed out to me that using the notion of determinant can lead to a simplification of the statements of 5.2 and 5.9. For an orthogonal symmetry $s$ of a quadratic space $(V,\varphi)$ of dimension $n$, we define $\mathrm{d}_\pm s=(-1)^{n(n-1)/2}\det(s)$, then one can express the isomorphisms 5.2 and 5.9 in the following way:

$$(C_0(V_1\perp V, q_1\perp q), J_{q_1\perp q}^{\sigma_1\oplus\sigma})\simeq (C_0(V_1,q_1)\otimes C(V,-\mathrm{d}_\pm q_1\cdot q), J_{q_1}^{\sigma_1}\otimes J_{-\mathrm{d}_\pm q_1\cdot q}^{-\mathrm{d}_\pm\sigma_1\cdot\sigma}),$$

$$(C(V_0\perp V, q_0\perp q), J_{q_0\perp q}^{\sigma_0\oplus\sigma})\simeq (C(V_0,q_0)\otimes C(V,\mathrm{d}_\pm q_0\cdot q), J_{q_0}^{\sigma_0}\otimes J_{\mathrm{d}_\pm q_0\cdot q}^{\mathrm{d}_\pm\sigma_0\cdot\sigma}),$$

because for every orthogonal symmetry $s$ of a quadratic space $(V,\varphi)$ we have: $\det(s)=(-1)^m$ where $m$ is the dimension of the subspace of the anti-symmetric elements of $V$ with respect to $s$.

## 6. Tensor products of quaternion algebras with involution

**Proposition 6.1.** *Let $(Q,J)$ be a quaternion algebra with involution over a field $K$. Suppose that $J$ is of the first kind. Then there exists a nondegenerate quadratic space $(V,q)$ of dimension 2 and an orthogonal symmetry $\sigma: V\to V$ such that $(Q,J)\simeq(C(V,q),J_q^{\sigma})$. More precisely*

(a) *if $J$ is symplectic, one can choose a quadratic space $(V,q)$ of dimension 2 such that $(Q,J)\simeq(C(V,q),J_q^{-\mathrm{id}})$.*
(b) *if $J$ is orthogonal, one can choose a quadratic space $(V,q)$ of dimension 2 such that $(Q,J)\simeq(C(V,q),J_q^{\mathrm{id}})$.*
(c) *if $J$ is orthogonal, one can also choose a quadratic space $(V,q)$ of dimension 2 and a reflection $\sigma$ of $(V,q)$ such that $(Q,\sigma)\simeq(C(V,q),J_q^{\sigma})$.*

**Proof.** Suppose that $Q = (a,b)_K$ is generated by the elements $i$ and $j$ with $i^2 = a \in K^*$, $j^2 = b \in K^*$ and $ij = -ji$.

First consider the case where $J$ is symplectic. We have $J(i) = -i$ and $J(j) = -j$. Consider the vector space $V = Ki \oplus Kj$ and the nondegenerate quadratic form $q : V \to K$ defined by $q(\lambda_1 i + \lambda_2 j) = \lambda_1^2 a + \lambda_2^2 b$ for all $\lambda_1,\ \lambda_2 \in K$. We have $(Q,J) \simeq (C(V,q), J_q^{-\mathrm{id}})$.

Now suppose that $J$ is orthogonal. Let $u$ be an anti-symmetric invertible element of $Q$ with respect to $J$. The involution $J' = \mathrm{Int}(u) \circ J$ is of symplectic type. According to [20, Ch. 8, 10.1], we have:

$$xu^{-1}J(x)u \in K \text{ for every } x \in Q \tag{18}$$

By putting $x = u$ in (18) we obtain $u^2 \in K$. Consider the quadratic extension $K(u)/K$. The restriction $J|_{K(u)}$ is the nontrivial automorphism of $K(u)/K$. According to Skolem-Noether's Theorem, there exists an invertible element $v \in Q$ such that $uv = -vu$. As $v^2$ commutes with both $u$ and $v$, it is in the center, i.e., $v^2 \in K$.

By putting $x = v$ in (18) we obtain $vJ(v) \in K$. Thus there exists $\alpha \in K$ such that $J(v) = \alpha v$. We have $v = J^2(v) = \alpha^2 v$. Thus $\alpha = 1$ or $\alpha = -1$. The case $\alpha = -1$ is excluded, because $J$ is orthogonal (the dimension of the subspace of anti-symmetric elements of $Q$ is 1). Thus we have $J(v) = v$. The elements $u$ and $v$ satisfy: $u^2 = a' \in K$, $v^2 = b' \in K$, $J(u) = -u$, $J(v) = v$ and $uv + vu = 0$. Consider the vector space $V = Ku \oplus Kv$ and $q : V \to K$ defined by $q(\lambda_1 u + \lambda_2 v) = \lambda_1^2 a' + \lambda_2^2 b'$ for every $\lambda_1, \lambda_2 \in K$ and the orthogonal symmetry $\sigma : V \to V$ defined by $\sigma(u) = -u$ and $\sigma(v) = v$. Consider the $K$-linear map $f : V \to Q$, defined by $f(u) = -u$, $f(v) = v$. The map $f$ is a Clifford map and can be extended to an isomorphism between $C(V,q)$ and $Q$. The construction of $f$ implies that it is compatible with $J_q^\sigma$ and $J$. It is therefore an isomorphism of algebras with involution. In this case $\sigma$ is a reflection of $(V,q)$ because the dimension of the vector space of anti-symmetric elements of $V$ with respect to $\sigma$ is 1.

Let $w = uv$, we have $w^2 = -a'b'$ and $J(w) = w$. Consider the vector space $V = Kw \oplus Kv$, the quadratic form $q : V \to K$ defined by $q(\lambda_1 w + \lambda_2 v) = -\lambda_1^2 a'b' + \lambda_2^2 b'$ for all $\lambda_1, \lambda_2 \in K$ and the orthogonal symmetry $\sigma : V \to V$ defined by $\sigma(w) = w$ and $\sigma(v) = v$. Consider the $K$-linear map $f : V \to Q$ defined by $f = \mathrm{id}|_V$. As $f$ is a Clifford map, it can be extended to an isomorphism between $C(V,q)$ and $Q$. The construction of $f$ shows that $f$ is compatible with $J_q^\sigma$ and $J$. It is therefore an isomorphism of algebras with involution. In this case, $\sigma$ is the identity map. □

As a direct consequence of the proof of Proposition 6.1, we obtain

**Corollary 6.2.** *Let $a$ and $b$ be two invertible elements of a field $K$. Then we have an isomorphism*

$$(C(\langle a,b\rangle), J^{-+}) \simeq (C(\langle -ab, b\rangle), J^{++}),$$

*or equivalently*

$$(C(\langle a,b\rangle), J^{++}) \simeq (C(\langle -ab, b\rangle), J^{-+}),$$

*here $J^{-+}$ is the involution of $C(\langle a,b\rangle)$ induced by the reflection $\tau = -\mathrm{id}_{K\cdot x} \oplus \mathrm{id}_{K\cdot y}$, where $V = K\cdot x \perp K\cdot y$ is the underlying vector space of $q = \langle a,b\rangle$ with $q(x) = a$, $q(y) = b$ and $J^{++}$ is the involution of $q' = \langle -ab, b\rangle$ induced by the identity map on the underlying vector space of $q'$.*

**Theorem 6.3.** *Let $(Q_1,J_1), \cdots, (Q_n,J_n)$ be quaternion algebras over a field $K$ with involutions of the first kind. Then there exists a quadratic space $(V,q)$ over $K$ of dimension $2n$ and an orthogonal symmetry $\sigma : V \to V$ such that*

$$(C(V,q), J_q^\sigma) \simeq (Q_1,J_1) \otimes \cdots \otimes (Q_n,J_n).$$

**Proof.** We prove the result by induction on $n$. If $n = 1$, we use directly Proposition 6.1.

Assume that $n > 1$. By induction hypothesis, there exists a quadratic space $(W,q)$ of dimension $2(n-1)$ over $K$ and an orthogonal symmetry $\sigma : W \to W$ such that

$$(C(W,q),J_q^{\sigma}) \simeq (Q_2,J_2)\otimes\cdots\otimes(Q_{n-1},J_{n-1}).$$

According to Proposition 6.1, there also exists a quadratic space $(W_0,q_0)$ of dimension 2 over $K$ and an orthogonal symmetry $\sigma_0 : W_0 \to W_0$ such that

$$(C(W_0,q_0),J_{q_0}^{\sigma_0}) \simeq (Q_1,J_1).$$

Let $s$ be the dimension of the anti-symmetric elements of $(W_0,q_0)$. We have obviously $s = 2$ if $J_1$ is symplectic. Using 5.9, we obtain

$$\begin{aligned}(Q_1,J_1)\otimes\cdots\otimes(Q_n,J_n) &\simeq (C(W_0,q_0),J_{q_0}^{\sigma_0})\otimes(C(W,q),J_q^{\sigma})\\ &\simeq (C(W_0\perp W,h),J_h^{\tau}),\end{aligned}$$

where $h = q_0 \perp (\mathrm{d}_{\pm}q_0)^{-1}\cdot q$ and $\tau = \sigma_0 \oplus (-1)^{-(s+1)}\sigma$. □

According to a result due to Albert (cf. [9, 16.1]), every central simple algebra with involution $A$ of degree 4 can be decomposed as a tensor product of two quaternion algebras. In [10], it has been shown that a central simple algebra $A$ of degree 4 with an orthogonal involution $\sigma$ can be decomposed as a tensor product of two quaternion algebras with symplectic involution if and only if it can be decomposed as a tensor product of two quaternion algebras with orthogonal involution if and only if the discriminant of $\sigma$ is trivial. See also [9, 15.12]. Here we complement these results by showing that:

**Lemma 6.4.** ([10]) Let $(A,\sigma)$ be a central simple algebra of degree 4 over a field $K$. The following assertions are equivalent:

(i) $(A,\sigma)$ is isomorphic to the tensor product of two quaternion algebras with orthogonal involutions.

(ii) $(A,\sigma)$ is isomorphic to the tensor product of two quaternion algebras with symplectic involutions.

(iii) there exists a quadratic space $(V,q)$ of dimension 4 over $K$ and a reflection $\tau$ of $(V,q)$ such that $(A,\sigma)\simeq(C(V,q),J_q^{\tau})$.

**Proof.** For the sake of completeness, we also prove the equivalence of (i) and (ii) using 5.9.

(i)⇒(ii). Let $(A,\tau) = (Q_1,\tau_1)\otimes(Q_2,\tau_2)$ where $(Q_1,\tau_1)$ and $(Q_2,\tau_2)$ are quaternion algebras and $\tau_1$ and $\tau_2$ are orthogonal. According to 6.1, there exist quadratic spaces $(V_1,q_1)$ and $(V_2,q_2)$ of dimension 2 such that:

$$(Q_1,\tau_1)\simeq(C(V_1,q_1),J_{q_1}^{id}),$$

$$(Q_2,\tau_2)\simeq(C(V_2,q_2),J_{q_2}^{\mathrm{id}}).$$

We thus obtain

$$\begin{aligned}(Q_1,\tau_1)\otimes(Q_2,\tau_2) &\simeq (C(V_1,q_1),J_{q_1}^{\mathrm{id}})\otimes(C(V_2,q_2),J_{q_2}^{\mathrm{id}})\\ \text{using 5.10}\quad &\simeq (C(V_1\perp V_2,q_1\perp(\mathrm{d}_{\pm}q_1)\cdot q_2),J_{q_1\perp(\mathrm{d}_{\pm}q_1)\cdot q_2}^{\mathrm{id}\oplus-\mathrm{id}})\\ \text{using 5.9}\quad &\simeq (C(V_1,\mathrm{d}_{\pm}q_2\cdot q_1),J_{\mathrm{d}_{\pm}q_2\cdot q_1}^{-\mathrm{id}})\otimes(C(V_2,\mathrm{d}_{\pm}q_1\cdot q_2),J_{\mathrm{d}_{\pm}q_1\cdot q_2}^{-\mathrm{id}})\end{aligned}$$

The involutions $J_{\mathrm{d}_{\pm}q_2\cdot q_1}^{-\mathrm{id}}$ and $J_{\mathrm{d}_{\pm}q_1\cdot q_2}^{-\mathrm{id}}$ are both symplectic, the proof is therefore achieved.

The proof of (ii)⇒(i) is similar.

(i)⇒(iii). According to 6.1, there exist quadratic spaces $(V_1,q_1)$ and $(V_2,q_2)$ and a reflection $\rho$ of $(V_2,q_2)$ such that

$$(Q_1,\tau_1)\simeq(C(V_1,q_1),J_{q_1}^{id}),$$

$$(Q_2,\tau_2)\simeq(C(V_2,q_2),J_{q_2}^{\rho}).$$

We therefore obtain

$$\begin{array}{lcl}(Q_1,\tau_1)\otimes(Q_2,\tau_2) & \simeq & (C(V_1,q_1),J_{q_1}^{\mathrm{id}})\otimes(C(V_2,q_2),J_{q_2}^{\rho})\\ \text{using 5.10} & \simeq & (C(V_1\perp V_2,q_1\perp(\mathrm{d}_{\pm}q_1)\cdot q_2),J_{q_1\perp(\mathrm{d}_{\pm}q_1)\cdot q_2}^{\mathrm{id}\oplus-\rho})\end{array}$$

It suffices to put $V=V_1\perp V_2$, $q=q_1\perp(\mathrm{d}_{\pm}q_1)\cdot q_2$ and $\tau=\mathrm{id}\oplus-\rho$, we then have $(A,\sigma)\simeq(C(V,q),J_q^{\tau})$.

The implication (i)⇒(iii) follows from 6.1 and 5.9. □

In [17], it has been shown that every division algebra of degree 4 with symplectic involution can be decomposed as a tensor product of two quaternion algebras with involution. More generally if $A$ is a central simple algebra with a symplectic involution can be decomposed as a tensor product of two quaternion algebras with involution, see [19, Thm. 10.5, Prop. 10.21]. We complement these results by showing that:

**Lemma 6.5.** (Compare with [19, Lemma 10.6]) *Let $(A,\sigma)$ be a central simple algebra of degree 4 over a field $K$. The following assertions are equivalent:*

(i) *$(A,\sigma)$ is isomorphic to $(Q_1,\sigma_1)\otimes(Q_2,\sigma_2)$, where $Q_1$ and $Q_2$ are quaternion algebras over $K$, $\sigma_1$ is symplectic and $\sigma_2$ is orthogonal.*

(ii) *there exists a quadratic space $(V,q)$ of dimension 4 over $K$ such that $(A,\sigma)\simeq(C(V,q),J_q^{\mathrm{id}})$.*

(iii) *there exists a quadratic space $(V,q)$ of dimension 4 over $K$ such that $(A,\sigma)\simeq(C(V,q),J_q^{-\mathrm{id}})$.*

**Proof.** According to 6.1, there exists quadratic spaces $(V_1,q_1)$ and $(V_2,q_2)$ such that $(Q_1,\sigma_1)\simeq(C(V_1,q_1),J_{q_1}^{-\mathrm{id}})$ and $(Q_2,\sigma_2)\simeq(C(V_2,q_2),J_{q_2}^{\mathrm{id}})$. We thus obtain

$$\begin{array}{lcl}(Q_1,\sigma_1)\otimes(Q_2,\sigma_2) & \simeq & (C(V_1,q_1),J_{q_1}^{-\mathrm{id}})\otimes(C(V_2,q_2),J_{q_2}^{\mathrm{id}})\\ \text{using 5.10} & \simeq & (C(V_1\perp V_2,q_1\perp(\mathrm{d}_{\pm}q_1)\cdot q_2),J_{q_1\perp(\mathrm{d}_{\pm}q_1)\cdot q_2}^{-\mathrm{id}\oplus-\mathrm{id}}).\end{array}$$

Similarly we have:

$$\begin{array}{lcl}(Q_1,\sigma_1)\otimes(Q_2,\sigma_2) & \simeq & (C(V_1,q_1),J_{q_1}^{-\mathrm{id}})\otimes(C(V_2,q_2),J_{q_2}^{\mathrm{id}})\\ \text{using 5.10} & \simeq & (C(V_1\perp V_2,(\mathrm{d}_{\pm}q_2)\cdot q_1\perp q_2),J_{(\mathrm{d}_{\pm}q_2)\cdot q_1\perp q_2}^{\mathrm{id}\oplus\mathrm{id}}).\end{array}$$

We thus have the implications (i)⇒(ii) and (i)⇒(iii). The implications (ii)⇒(i) and (iii)⇒(i) follow from 5.9. □

**Lemma 6.6.** *Let $(Q_1,\sigma_1)$, $(Q_2,\sigma_2)$ and $(Q_3,\sigma_3)$ be quaternion algebras with involutions of the first kind. Let $(A,\sigma)=(Q_1,\sigma_1)\otimes(Q_2,\sigma_2)\otimes(Q_3,\sigma_3)$.*

(a) *if $\sigma$ is symplectic then there exists a quadratic space $(V,q)$ of dimension 6 over $K$ such that $(A,\sigma)\simeq(C(V,q),J_q^{\mathrm{id}})$.*

(b) *if $\sigma$ is orthogonal then there exists a quadratic space $(A,\sigma)$ of dimension 6 such that $(A,\sigma)\simeq(C(V,q),J_q^{-\mathrm{id}})$.*

**Proof.** If $\sigma$ is symplectic then we may assume that either all $\sigma_i$, $i=1,2,3$, are symplectic or, $\sigma_1$ is symplectic and $\sigma_2$ and $\sigma_3$ are orthogonal. Thanks to 6.4, the first case is reduced to the second case. In the second case, there exist quadratic spaces $(V_1,q_1)$, $(V_2,q_2)$ and $(V_3,q_3)$ of dimension 2 over $K$ such that $(C(V_1,q_1),J_{q_1}^{-\mathrm{id}})\simeq(Q_1,\sigma_1)$, $(C(V_2,q_2),J_{q_2}^{\mathrm{id}})\simeq(Q_2,\sigma_2)$ and $(C(V_3,q_3),J_{q_3}^{\mathrm{id}})\simeq(Q_3,\sigma_3)$. According to 6.5, there exists a quadratic space $(W,h)$ of dimension 4 over $K$ such that

$$(C(W,h),J_h^{\mathrm{id}})\simeq(Q_1,\sigma_1)\otimes(Q_2,\sigma_2).$$

According to 6.1, there exists a quadratic space $(W',h')$ of dimension 2 over $K$ such that:

$$(C(W',h'),J_{h'}^{\mathrm{id}}) \simeq (Q_3,\sigma_3).$$

We thus obtain

$$\begin{array}{lcl}(Q_1,\sigma_1)\otimes(Q_2,\sigma_2)\otimes(Q_3,\sigma_3) & \simeq & (C(W,h),J_h^{\mathrm{id}})\otimes(C(W',h'),J_h^{\mathrm{id}}) \\ \text{using 5.10 (b)} & \simeq & (C(W\perp W',h\perp(\mathrm{d}_\pm h)\cdot h'),J_{h\perp(\mathrm{d}_\pm h)\cdot h'}^{\mathrm{id}\oplus\mathrm{id}}).\end{array}$$

If $\sigma$ is orthogonal then we may assume that either all $\sigma_i$, $i=1,2,3$, are orthogonal or, $\sigma_1$ is orthogonal and $\sigma_2$ and $\sigma_3$ are symplectic. But thanks to 6.4, the first case is reduced to the second one. In the second case, according to 6.5, there exists a quadratic space $(W,h)$ of dimension 4 over $K$ such that

$$(C(W,h),J_h^{-\mathrm{id}}) \simeq (Q_1,\sigma_1)\otimes(Q_2,\sigma_2).$$

According to 6.1, there exists a quadratic space $(W',h')$ of dimension 2 over $K$ such that:

$$(C(W',h'),J_{h'}^{-\mathrm{id}}) \simeq (Q_3,\sigma_3).$$

We thus obtain

$$\begin{array}{lcl}(Q_1,\sigma_1)\otimes(Q_2,\sigma_2)\otimes(Q_3,\sigma_3) & \simeq & (C(W,h),J_h^{-\mathrm{id}})\otimes(C(W',h'),J_{h'}^{-\mathrm{id}}) \\ \text{using 5.10 (b)} & \simeq & (C(W\perp W',h\perp(\mathrm{d}_\pm h)\cdot h'),J_{h\perp(\mathrm{d}_\pm h)\cdot h'}^{-\mathrm{id}\oplus-\mathrm{id}}).\end{array}$$

The proof is thus achieved. □

**Lemma 6.7.** *Let $(Q_1,\sigma_1)$, $(Q_2,\sigma_2)$, $(Q_3,\sigma_3)$ and $(Q_4,\sigma_4)$ be quaternion algebras with involutions of the first kind. Let $(A,\sigma)=(Q_1,\sigma_1)\otimes(Q_2,\sigma_2)\otimes(Q_3,\sigma_3)\otimes(Q_4,\sigma_4)$.*

(a) *if $\sigma$ is orthogonal then there exist quadratic spaces $(V,q)$ and $(V',q')$ of dimension 8 over K such that $(A,\sigma)\simeq(C(V,q),J_q^{\mathrm{id}})$ and $(A,\sigma)\simeq(C(V',q'),J_{q'}^{-\mathrm{id}})$.*

(b) *if $\sigma$ is symplectic then there exists a quadratic space $(A,\sigma)$ of dimension 8 over K and a reflection $\tau$ of $(V,q)$ such that $(A,\sigma)\simeq(C(V,q),J_q^{\tau})$.*

**Proof.** (a) As $\sigma$ is orthogonal, the numbers of $\sigma_i$, $i=1,\cdots,4$, which are symplectic, should be even. Thanks to 6.4, we are reduced to consider the case where two of $\sigma_i$, $i=1,\cdots,4$, are orthogonal and two of them are symplectic. Without loss of generality, it may be assumed that $\sigma_1$ and $\sigma_3$ are orthogonal and $\sigma_2$ and $\sigma_4$ are symplectic. There exist so quadratic spaces $(V_i,q_i)$, $i=1,\cdots,4$, such that $(Q_i,\sigma_i)\simeq(C(V_i,q_i),J_{q_i}^{\mathrm{id}})$ for $i=1$, 3 and $(Q_i,\sigma_i)\simeq(C(V_i,q_i),J_{q_i}^{-\mathrm{id}})$ for $i=2$, 4. Using 6.5, there exist quadratic spaces $(V_1,q_1)$ and $(V_2,q_2)$ of dimension 4 over $K$ such that:

$$(Q_1,\sigma_1)\otimes(Q_2,\sigma_2)\simeq(C(V_1,q_1),J_{q_1}^{\mathrm{id}}),$$

$$(Q_3,\sigma_3)\otimes(Q_3,\sigma_3)\simeq(C(V_2,q_2),J_{q_2}^{\mathrm{id}}).$$

We thus obtain

$$\begin{array}{rcl}(A,\sigma) & \simeq & (C(V_1,q_1),J_{q_1}^{\mathrm{id}})\otimes(C(V_2,q_2),J_{q_2}^{\mathrm{id}}) \\ \text{using 5.10 (b)} & \simeq & (C(V_1\perp V_2,q_1\perp \mathrm{d}_\pm q_1\cdot q_2),J_{q_1\perp \mathrm{d}_\pm q_1\cdot q_2}^{\mathrm{id}\oplus\mathrm{id}}).\end{array}$$

The proof of the other assertion of (a) is similar and is left to the reader.

(b) Using 6.4, we may assume that $\sigma_1$, $\sigma_2$ and $\sigma_3$ are symplectic and $\sigma_4$ is orthogonal. According to 6.4 and 6.6, there exist quadratic spaces $(V_1,q_1)$ and $(V_2,q_2)$, respectively, of dimension 6 and 2 over $K$, and a reflection $\tau'$ of $(V_2,q_2)$ such that:

$$(C(V_1,q_1),J_{q_1}^{\mathrm{id}})\simeq(Q_1,\sigma_1)\otimes(Q_2,\sigma_2)\otimes(Q_3,\sigma_3).$$

$$(C(V_2,q_2),J_{q_2}^{\tau'})\simeq(Q_1,\sigma_1)$$

We so obtain

$$\begin{array}{rcl} (A,\sigma) & \simeq & (C(V_1,q_1),J_{q_1}^{\mathrm{id}})\otimes(C(V_2,q_2),J_{q_2}^{\tau'}) \\ \text{using 5.10 (a)} & \simeq & (C(V_1\perp V_2,q_1\perp \mathrm{d}_\pm q_1\cdot q_2),J_{q_1\perp \mathrm{d}_\pm q_1\cdot q_2}^{\mathrm{id}\oplus-\tau'}). \end{array}$$

Therefore it suffices to put $(V,q)=(V_1\perp V_2,q_1\perp \mathrm{d}_\pm q_1\cdot q_2)$ and $\tau=\mathrm{id}|_{V_1}\oplus -\tau'|_{V_2}$. □

**Proposition 6.8.** *Let $n$ be an odd positive integer. Let $(Q_1,\sigma_1),\cdots,(Q_n,\sigma_n)$ be quaternion algebras with involution of the first kind over a field $K$ and let*

$$(A,\sigma)=(Q_1,\sigma_1)\otimes\cdots\otimes(Q_n,\sigma_n).$$

*Then:*

(a) *If $n\equiv 1 \mod 4$ and if $\sigma$ is symplectic then there exists a quadratic space $(V,q)$ of dimension $2n$ over $K$ such that $(A,\sigma)\simeq(C(V,q),J_q^{-\mathrm{id}})$.*
(b) *If $n\equiv 1 \mod 4$ and if $\sigma$ is orthogonal then there exists a quadratic space $(V,q)$ of dimension $2n$ over $K$ such that $(A,\sigma)\simeq(C(V,q),J_q^{\mathrm{id}})$.*
(c) *If $n\equiv 3 \mod 4$ and if $\sigma$ is symplectic then there exists a quadratic space $(V,q)$ of dimension $2n$ over $K$ such that $(A,\sigma)\simeq(C(V,q),J_q^{\mathrm{id}})$.*
(d) *If $n\equiv 3 \mod 4$ and if $\sigma$ is orthogonal then there exists a quadratic space $(V,q)$ of dimension $2n$ over $K$ such that $(A,\sigma)\simeq(C(V,q),J_q^{-\mathrm{id}})$.*

**Proof.** We prove all the assertions by induction on $n$. If $n=1$, the assertions (a) and (b) follow from 6.1. If $n=3$, the assertions (c) and (d) follow from 6.6. Suppose that the assertions (a) and (b) and the assertions (c) and (d) are, respectively, true for $n=4k+1$ and $n=4k+3$ where $k$ is a nonnegative integer. We show that the statements (a) and (b) and the statements (c) and (d) are, respectively, true for $n=4k+5$ and $n=4k+7$.

We first consider the assertion (a) for the case where $n=4k+5$. As $\sigma$ is symplectic we deduce that either all of $\sigma_i$, $i=1,\cdots,n$, are symplectic or, at least two of $\sigma_i$, $i=1,\cdots,n$, say $\sigma_1$ and $\sigma_2$, are orthogonal. Using 6.4, there exist two symplectic involution $\sigma_1'$ and $\sigma_2'$ such that $\sigma_1\otimes\sigma_2\simeq\sigma_1'\otimes\sigma_2'$. The second case is so reduced to the first one. We may thus assume that all of $\sigma_i$, $i=1,\cdots,n$, are symplectic. As by induction hypothesis, (c) is true for $n=4k+3$, there exists a quadratic space $(W,h)$ of dimension $8k+6$ over $K$ such that

$$(Q_3,\sigma_3)\otimes\cdots\otimes(Q_n,\sigma_n)\simeq(C(W,h),J_h^{\mathrm{id}}).$$

According to 6.1, there exist two quadratic spaces $(V_1,q_1)$ such that $(Q_1,\sigma_1)\simeq(C(V_1,q_1),J_{q_1}^{-\mathrm{id}})$ and $(Q_2,\sigma_2)\simeq(C(V_2,q_2),J_{q_2}^{-\mathrm{id}})$. We thus obtain

$$\begin{array}{lcl} (Q_1,\sigma_1)\otimes\cdots\otimes(Q_n,\sigma_n) & \simeq & (Q_1,\sigma_1)\otimes(Q_2,\sigma_2)\otimes(C(W,h),J_h^{\mathrm{id}}) \\ \text{using 5.10 (a)} & \simeq & (Q_1,\sigma_1)\otimes(C(V_2\perp W,q_2\perp \mathrm{d}_\pm q_2\cdot h),J_{q_2\perp \mathrm{d}_\pm q_2\cdot h}^{-\mathrm{id}\oplus-\mathrm{id}}) \\ \text{using 5.10 (b)} & \simeq & (C(V_1\perp V_2\perp W,\mathrm{d}_\pm q'\cdot q_1\perp q'),J_{\mathrm{d}_\pm q'\cdot q_1\perp q'}^{-\mathrm{id}\oplus-\mathrm{id}\oplus-\mathrm{id}}), \end{array}$$

where $q'=q_2\perp \mathrm{d}_\pm q_2\cdot h$. The quadratic space $(V,q)=(V_1\perp V_2\perp W,\mathrm{d}_\pm q'\cdot q_1\perp q')$ is indeed the one we were looking for.

We now prove the assertion (b) for the case where $n=4k+5$. By the same argument as before, we may assume that all of $\sigma_i$, $i=1,\cdots,n$, are orthogonal. As by induction hypothesis (d) is true for $n=4k+3$, there exists a quadratic space $(W,h)$ of dimension $8k+6$ over $K$ such that:

$$(Q_3,\sigma_3)\otimes\cdots\otimes(Q_n,\sigma_n)\simeq(C(W,h),J_h^{-\mathrm{id}}).$$

According to 6.1, there exist quadratic spaces $(V_1,q_1)$ and $(V_2,q_2)$ such that $(Q_1,\sigma_1)\simeq(C(V_1,q_1),J_{q_1}^{\mathrm{id}})$ and $(Q_2,\sigma_2)\simeq(C(V_2,q_2),J_{q_2}^{\mathrm{id}})$. We so obtain

$$\begin{array}{lcl}(Q_1,\sigma_1)\otimes\cdots\otimes(Q_n,\sigma_n) & \simeq & (Q_1,\sigma_1)\otimes(Q_2,\sigma_2)\otimes(C(W,h),J_h^{-\mathrm{id}})\\ \text{using 5.10 (a)} & \simeq & (Q_1,\sigma_1)\otimes(C(V_2\perp W,q_2\perp \mathrm{d}_\pm q_2\cdot h),J^{\mathrm{id}\oplus\mathrm{id}}_{q_2\perp\mathrm{d}_\pm q_2\cdot h})\\ \text{using 5.10 (b)} & \simeq & (C(V_1\perp V_2\perp W,\mathrm{d}_\pm q'\cdot q_1\perp q'),J^{\mathrm{id}\oplus\mathrm{id}\oplus\mathrm{id}}_{\mathrm{d}_\pm q'\cdot q_1\perp q'}),\end{array}$$

where $q'=q_2\perp \mathrm{d}_\pm q_2\cdot h$. The quadratic space $(V,q)=(V_1\perp V_2\perp W,\mathrm{d}_\pm q'\cdot q_1\perp q')$ is indeed the one we were looking for.

We now prove the assertion (c) for the case where $n=4k+7$. As $\sigma$ is symplectic, by the same argument as before, we may assume that all of $\sigma_i$ are symplectic. As we have already shown that the assertion (a) is true for $n=4k+5$, there exists so a quadratic space $(W,h)$ of dimension $8k+10$ such that

$$(Q_3,\sigma_3)\otimes\cdots\otimes(Q_n,\sigma_n)\simeq(C(W,h),J_h^{-\mathrm{id}}).$$

According to 6.1, there exist quadratic spaces $(V_1,q_1)$ and $(V_2,q_2)$ such that $(Q_1,\sigma_1)\simeq(C(V_1,q_1),J_{q_1}^{-\mathrm{id}})$ and $(Q_2,\sigma_2)\simeq(C(V_2,q_2),J_{q_2}^{-\mathrm{id}})$. We thus obtain

$$\begin{array}{lcl}(Q_1,\sigma_1)\otimes\cdots\otimes(Q_n,\sigma_n) & \simeq & (Q_1,\sigma_1)\otimes(Q_2,\sigma_2)\otimes(C(W,h),J_h^{\mathrm{id}})\\ \text{using 5.10 (a)} & \simeq & (Q_1,\sigma_1)\otimes(C(V_2\perp W,q_2\perp \mathrm{d}_\pm q_2\cdot h),J^{-\mathrm{id}\oplus-\mathrm{id}}_{q_2\perp\mathrm{d}_\pm q_2\cdot h})\\ \text{using 5.10 (b)} & \simeq & (C(V_1\perp V_2\perp W,\mathrm{d}_\pm q'\cdot q_1\perp q'),J^{-\mathrm{id}\oplus-\mathrm{id}\oplus-\mathrm{id}}_{\mathrm{d}_\pm q'\cdot q_1\perp q'}),\end{array}$$

where $q'=q_2\perp \mathrm{d}_\pm q_2\cdot h$.

The proof of (d) is similar and is left to the reader. □

**Proposition 6.9.** *Let $n$ be an even positive integer. Let $(Q_1,\sigma_1),\cdots,(Q_n,\sigma_n)$ be quaternion algebras with involution of the first kind over a field $K$ and let*

$$(A,\sigma)=(Q_1,\sigma_1)\otimes\cdots\otimes(Q_n,\sigma_n).$$

*Then:*

(a) *If $n\equiv 0 \mod 4$ and if $\sigma$ is orthogonal then there exist quadratic spaces $(V,q)$ and $(V',q')$ of dimension $2n$ over $K$ such that $(A,\sigma)\simeq(C(V,q),J_q^{\mathrm{id}})$ and $(A,\sigma)\simeq(C(V',q'),J_{q'}^{-\mathrm{id}})$.*

(b) *If $n\equiv 0 \mod 4$ and if $\sigma$ is symplectic then there exists a quadratic space $(V,q)$ of dimension $2n$ over $K$ and a reflection $\tau$ of $(V,q)$ such that $(A,\sigma)\simeq(C(V,q),J_q^{\tau})$.*

(c) *If $n\equiv 2 \mod 4$ and if $\sigma$ is symplectic then there exist quadratic spaces $(V,q)$ and $(V',q')$ of dimension $2n$ over $K$ such that $(A,\sigma)\simeq(C(V,q),J_q^{\mathrm{id}})$ and $(A,\sigma)\simeq(C(V',q'),J_{q'}^{-\mathrm{id}})$.*

(d) *If $n\equiv 2 \mod 4$ and if $\sigma$ is orthogonal then there exists a quadratic space $(V,q)$ of dimension $2n$ over $K$ and a reflection $\tau$ of $(V,q)$ such that $(A,\sigma)\simeq(C(V,q),J_q^{\tau})$.*

**Proof.** We prove all assertions by induction on $n$. According to 6.4, 6.5 and 6.7, the assertions (a) and (b) are true for $n=4$ and (c) and (d) are true for $n=2$. Suppose that the assertions (a) and (b) are true for $n=4k+4$ and (c) and (d) are true for $n=4k+2$ where $k$ is a nonnegative integer. We should prove that the statements (a) and (b) are true for $n=4k+8$ and (c) and (d) are true for $n=4k+6$.

We prove the assertion (a) for $n=4k+8$. As $\sigma$ is orthogonal, the number of the involutions $\sigma_i$, $i=1,\cdots,n$, which are symplectic are even. Using 6.4, we may suppose that all of $\sigma_i$, $i=1,\cdots,n$, are orthogonal. By induction hypothesis there exists a quadratic space $(V',q')$ of dimension $8k+8$ over $K$ such that

$$(Q_5,\sigma_5)\otimes\cdots\otimes(Q_n,\sigma_n)\simeq(C(V',q'),J_{q'}^{\mathrm{id}}).$$

Using 6.7, there exists a quadratic form $(V'', q'')$ of dimension 8 over $K$ such that

$$(Q_1, \sigma_1) \otimes \cdots \otimes (Q_4, \sigma_4) \simeq (C(V'', q''), J^{\mathrm{id}}_{q''}). \tag{19}$$

We thus obtain

$$\begin{array}{rcl} (A, \sigma) & \simeq & (C(V'', q''), J^{\mathrm{id}}_{q''}) \otimes (C(V', q'), J^{\mathrm{id}}_{q'}) \\ \text{using 5.10 (b)} & \simeq & (C(V'' \perp V', q'' \perp \mathrm{d}_{\pm} q'' \cdot q'), J^{\mathrm{id} \oplus \mathrm{id}}_{q'' \perp \mathrm{d}_{\pm} q'' \cdot q'}). \end{array}$$

It suffices to put $(V, q) = (V'' \perp V', q'' \perp \mathrm{d}_{\pm} q'' \cdot q')$. The proof of the second assertion of (a) is similar.

In order to prove the assertion (b) for $n = 4k + 8$, note that as $\sigma$ is symplectic, using 6.4 we may suppose that $\sigma_n$ is symplectic and all $\sigma_i$, for $i = 1, \cdots, n-1$, are orthogonal. By induction hypothesis, there exists a quadratic space $(V', q')$ of dimension $8k + 8$ over $K$ and a reflection $\tau'$ of $(V', q')$ such that

$$(Q_5, \sigma_5) \otimes \cdots \otimes (Q_n, \sigma_n) \simeq (C(V', q'), J^{\tau'}_{q'}).$$

The relation (19) is also satisfied. We thus obtain

$$\begin{array}{rcl} (A, \sigma) & \simeq & (C(V'', q''), J^{\mathrm{id}}_{q''}) \otimes (C(V', q'), J^{\tau'}_{q'}) \\ \text{using 5.10 (b)} & \simeq & (C(V'' \perp V', q'' \perp \mathrm{d}_{\pm} q'' \cdot q'), J^{\mathrm{id} \oplus \tau'}_{q'' \perp \mathrm{d}_{\pm} q'' \cdot q'}). \end{array}$$

It suffices to put $(V, q) = (V'' \perp V', q'' \perp \mathrm{d}_{\pm} q'' \cdot q')$ and $\tau = \mathrm{id}|_{V''} \oplus \tau'$.

The proof of the assertions (c) and (d) are similar and are left to the reader. □

### 6.1. **Involutions of the second kind.**

**Lemma 6.10.** (Albert). *Let $(Q, \sigma)$ be a quaternion algebra with involution over a field $K$. The following assertions are equivalent:*

(i) *The $\sigma$ involution $\sigma$ is of the second kind, in other words $\sigma|_K$ is a nontrivial automorphism of $K$.*
(ii) *There exists a subfield $k$ of $K$ with $[K : k] = 2$ and a quaternion algebra $Q_0$ over $k$ such that $(Q, \sigma) \simeq (Q_0 \otimes_k K, \gamma \otimes -)$ where $\gamma$ is the canonical involution of $Q_0$ and $- : K \to K$ is the nontrivial automorphism of $K/k$.*
(iii) *There exists a subfield $k$ of $K$ with $[K : k] = 2$ and a quaternion algebra $Q_0$ over such that $(Q, \sigma) \simeq (Q_0 \otimes_k K, \rho \otimes -)$ where $\rho$ is an orthogonal involution of $Q_0$ and $- : K \to K$ is the nontrivial automorphism of $K/k$.*
(iv) *There exists a quadratic space $(V, q)$ of dimension 3 over $K$ such that $(Q, \sigma) \simeq (C(V, q), J^{\mathrm{id}}_q)$.*

**Proof.** Our sole contribution is to prove the equivalence of (ii), (iii) and (iv). According to 6.1, there exists a quadratic space $(V_0, q_0)$ of dimension 2 with $q_0 \simeq \langle a, b \rangle$ such that $(Q_0, \gamma) \simeq (C(V_0, q_0), J^{-\mathrm{id}}_{q_0})$. We may assume that $K = k(\sqrt{c})$ where $c \in K$ in a non-square element. We have $(K, -) \simeq (C(\langle c \rangle), J^{-\mathrm{id}})$. We thus obtain

$$\begin{array}{rcl} (Q_0 \otimes_k K, \gamma \otimes -) & \simeq & (C(V_0, q), J^{-\mathrm{id}}_{q_0})) \otimes (C(\langle c \rangle), J^{-\mathrm{id}}) \\ \text{using 5.10 (a)} & \simeq & (C(\langle a, b, -abc \rangle), J^{--+}) \\ \text{using 5.9 (a)} & \simeq & (C(\langle c \rangle), J^{-}) \otimes (C(\langle b, -abc \rangle), J^{-+}) \\ \text{using 6.1, } \exists d,\ e \in k^{\times} & \simeq & (C(\langle c \rangle), J^{-}) \otimes (C(\langle d, e \rangle), J^{++}) \\ \text{using 5.10 (a)} & \simeq & (C(\langle -dec, d, e \rangle), J^{+++}) \end{array}$$

Thus it suffices to set $q = \langle -dec, d, e \rangle$. This implies the equivalence of (ii) and (iv). In order to prove the equivalence of (ii) and (iii), note that according to the above relations, we may take $(Q_0, \rho) := (C(\langle b, -abc \rangle), J^{-+})$ or $(Q_0, \rho) := (C(\langle d, e \rangle), J^{++})$ which are both the quaternion algebras with orthogonal involutions. □

**Proposition 6.11.** *Let $(A,\sigma)$ be a central simple algebra with unitary involution over a field K. Let k be the fixed field of $\sigma|_K$. The following assertions are equivalent:*

(i) *There exist quaternion algebras with unitary involution $(A_1,\sigma_1),\cdots,(A_n,\sigma_n)$ over K such that for all i and j we have $\sigma_i|_K=\sigma_j|_K$ and*
$$(A,\sigma)\simeq(A_1,\sigma_1)\otimes_K\cdots\otimes_K(A_n,\sigma_n).$$

(ii) *There exist quaternion algebras with canonical involution $(Q_1,\gamma_1),\cdots,(Q_n,\gamma_n)$ over k such that*
$$(A,\sigma)\simeq(Q_1,\gamma_1)\otimes_k\cdots\otimes_k(Q_n,\gamma_n)\otimes_k(K,\sigma|_K). \tag{20}$$

(iii) *There exists a quadratic space $(V,q)$ of dimension $2n+1$ over k with nontrivial discriminant such that*
$$(A,\sigma)\simeq(C(V,q),J_q^{\mathrm{id}}).$$

*If n is even, the assertions (i)-(iii) are equivalent to the following:*

(iv) *There exists a quadratic space $(V',q')$ of dimension $2n+2$ over k such that*
$$(A,\sigma)\simeq(C_0(V',q'),J_{q'}^{\mathrm{id}}).$$

*Moreover, if n is odd the assertions (i)-(iii) are equivalent to the following:*

(v) *There exists a quadratic space $(V'',q'')$ of dimension $2n+2$ over k and an orthogonal symmetry $\tau:V''\to V''$ such that*
$$(A,\sigma)\simeq(C_0(V'',q''),J_{q''}^{\tau}).$$

**Proof.** The equivalence of (i) and (ii) is obvious. According to 6.10 (ii), in (20), one may replace $(Q_1,\gamma_1)$ by $(Q_1',\rho_1)$, where $(Q_1',\rho_1)$ is a suitable quaternion algebra with orthogonal involution over $k$. This implies that one can write
$$(A,\sigma)\simeq(A_0,\sigma_0)\otimes_k(K,\sigma|_K),$$
where $(A_0,\sigma_0)$ is a tensor product of $n$ quaternion algebras with involution over $k$ and $\sigma_0$ can be chosen to be orthogonal or symplectic. Using 6.8 and 6.9, there exists a quadratic space $(V,q)$ of dimension $2n$ over $k$ with $q\simeq\langle a_1,\cdots,a_{2n}\rangle$ such that
$$(A_0,\sigma_0)\simeq(C(V,q),J_q^{-\mathrm{id}}). \tag{21}$$
There also exists an element $c\in k^\times\backslash k^{\times2}$ such that $(K,\sigma)\simeq(C(\langle c\rangle),J^{-\mathrm{id}})$. Take $q'=\langle a_1,\cdots,a_{2n-1}\rangle$. We thus obtain
$$\begin{array}{rcl}(A_0,\sigma_0)\otimes(K,\sigma|_K) & \simeq & (C(V,q),J_q^{-\mathrm{id}})\otimes(C(\langle c\rangle),J^{-\mathrm{id}})\\ \text{using 5.10 (a)} & \simeq & (C(q\perp\langle \mathrm{d}_\pm q\cdot c\rangle),J^{-\mathrm{id}\oplus\mathrm{id}})\\ \text{using 5.9 (a)} & \simeq & (C(q'),J^{-\mathrm{id}})\otimes(C(\langle a_{2n},\mathrm{d}_\pm q\cdot c\rangle),J^{-+})\\ \text{using 6.1, }\exists d,\ e\in k^\times & \simeq & (C(q'),J^{-\mathrm{id}})\otimes(C(\langle d,e\rangle),J^{++})\\ \text{using 5.10 (a)} & \simeq & (C(-de\cdot q'\perp\langle d,e\rangle),J^{\mathrm{id}\oplus\mathrm{id}})\end{array}$$
This implies the equivalence of (i) and (iii). Using 5.5 and the relation (21), we obtain
$$(A_0,\sigma_0)\simeq(C_0(\langle-1\rangle\perp q),J^{\mathrm{id}}).$$
Suppose that $n$ is even. Take $q_1=\langle-1\rangle\perp q$. We thus obtain
$$\begin{array}{rcl}(A_0,\sigma_0)\otimes(K,\sigma|_K) & \simeq & (C_0(q_1),J^{\mathrm{id}})\otimes(C(\langle c\rangle),J^{-\mathrm{id}})\\ \text{using 5.4 (a)} & \simeq & (C_0(q_1\perp\langle-\mathrm{d}_\pm q_1\cdot c\rangle),J^{\mathrm{id}\oplus\mathrm{id}}).\end{array}$$
Thus it suffices to put $q'=q_1\perp\langle-\mathrm{d}_\pm q_1\cdot\rangle$. This implies the equivalence of (i) and (iv).

Suppose that $n$ is odd. We can write

$$(A,\sigma) = (Q_1,\rho_1)\otimes_k \cdots \otimes_k (Q_n,\rho_n)\otimes_k (K,\sigma|_K),$$

where $(Q_i,\rho_i)$, $i = 1,\cdots,n$, are quaternion algebras with orthogonal involution. According to (iii), there exists a quadratic space $(V,q)$ of dimension $2n-1$ over $k$ such that

$$(C(V,q),J_q^{\mathrm{id}}) \simeq (Q_2,\rho_2)\otimes_k \cdots \otimes_k (Q_n,\rho_n)\otimes_k (K,\sigma|_K).$$

According to 6.1, there exists a quadratic space $(V_1,q_1)$ of dimension 2 over $k$ such that $(C(V_1,q_1),J_{q_1}^{\mathrm{id}}) \simeq (Q_1,\rho_1)$. According to 5.6, there exists a reflection $\tau_1$ of the underlying vector space of $\langle -1\rangle \perp q_1$ such that $(C(q_1),J^{\mathrm{id}}) \simeq (C_0(\langle -1\rangle \perp q_1),J^{\tau_1})$.

Take $q' = \langle -1\rangle \perp q_1$. We thus obtain

$$\begin{array}{rcl} (A,\sigma) & \simeq & (C_0(q'),J^{\tau_1})\otimes_k (C(V,q),J_q^{\mathrm{id}}) \\ \text{using 5.4 (a)} & \simeq & (C_0(q' \perp -\mathrm{d}_\pm q'\cdot q),J^{\tau_1\oplus \mathrm{id}}). \end{array}$$

Therefore it suffices to put $q'' = q' \perp -\mathrm{d}_\pm q' \cdot q$ and $\tau = \tau_1 \oplus \mathrm{id}$. □

## 7. Type of involutions induced by orthogonal symmetries

**Proposition 7.1.** (Compare with [19, 7.4]) Let $(V,q)$ be a nondegenerate quadratic space of even dimension $n$. Let $\sigma$ is an orthogonal symmetry of $V$. Then the involution $J_q^\sigma$ is orthogonal if and only if $tr(\sigma) = n-2s \equiv 0$ or $2 \mod 8$, where $s$ is the dimension of the subspace $V^-$ of the anti-symmetric elements of $V$ (cf. 4.2).

**Proof.** If $n = 2$, the equivalence of the conditions is obvious (cf. 4.5). So let assume that $n \geqslant 4$. At least one of the subspaces $V^+$ or $V^-$ is of dimension greater or equal to 2. As $V^+$ and $V^-$ are orthogonal, relative to the bilinear form associated to $q$, we conclude that $q|_{V^+}$ and $q|_{V^-}$ are nondegenerate. Consider a subspace $W$ of dimension 2 of $V$ with $q|_W$ nondegenerate, such that either $W \subset V^+$ or $W \subset V^-$. The orthogonal subspace $W^\perp$ is stable under $\sigma$. We thus have a decomposition $\sigma = \sigma|_W \oplus \sigma_{W^\perp}$. In order to simplify the notation, let us set $\sigma_W = \sigma|_W$, $\sigma_{W^\perp} = \sigma|_{W^\perp}$, $q_W = q|_W$ and $q_{W^\perp} = q|_{W^\perp}$. According to 5.9, we have an isomorphism of algebras with involution:

$$(C(V,q),J_q^\sigma) \simeq (C(W,q_W)\otimes C(W^\perp,q_{W^\perp}),J_{q_W}^{\sigma_W}\otimes J^{-\sigma_{W^\perp}}_{\mathrm{d}_\pm q_W\cdot q_{W^\perp}})$$

If $W \subset V^+$, then we have $\sigma_W = \mathrm{id}$ and $J_{q_W}^{\sigma_W}$ is consequently orthogonal (cf. 4.5). In order to prove that $J_q^\sigma$ is orthogonal, it is necessary and sufficient to show that $J' = J^{-\sigma_{W^\perp}}_{\mathrm{d}_\pm q_W\cdot q_{W^\perp}}$ is orthogonal. By induction, $J'$ is orthogonal if and only if $n'-2s' \equiv 0$ or $2 \mod 8$, where $n' = \dim W^\perp$ and $s'$ is the dimension of the subspace of the anti-symmetric elements of $-\sigma_{W^\perp}$. As $n' = n-2$ and $s' = n-2-s$, we obtain $n-2s = -(n'-2s')+2$ and the proof is achieved.

If $W \subset V^-$, we have $\sigma_W = -\mathrm{id}$ and $J_{q_W}^\sigma$ is symplectic. In order to prove that $J_q^\sigma$ is orthogonal, it is necessary and sufficient to show that $J'$ is symplectic. By induction, $J'$ is symplectic if $n'-2s' \equiv 4$ or $6 \mod 8$. In this case, we have $n' = n-2$ and $s' = n-s$. Therefore we have $n-2s = -(n'-2s')-2$ and the proof is achieved. □

**Corollary 7.2.** ([13, Prop. 3]) Let $(V,q)$ be a nondegenerate quadratic space of even dimension $n$. Consider the involutions $J_q^{\mathrm{id}}$ and $J_q^{-\mathrm{id}}$ of $C(V,q)$. Then we have:

(i) If $n \equiv 0 \mod 8$ then $J_q^{\mathrm{id}}$ and $J_q^{-\mathrm{id}}$ are of orthogonal type.
(ii) If $n \equiv 2 \mod 8$ then $J_q^{\mathrm{id}}$ is of orthogonal type and $J_q^{-\mathrm{id}}$ is of symplectic type.
(iii) If $n \equiv 4 \mod 8$ then $J_q^{\mathrm{id}}$ and $J_q^{-\mathrm{id}}$ are of symplectic type.
(iv) If $n \equiv 6 \mod 8$ then $J_q^{\mathrm{id}}$ is of symplectic type and $J_q^{-\mathrm{id}}$ is if orthogonal type.

**Corollary 7.3.** ([9, 8.4]) Let $(V,q)$ be a nondegenerate quadratic space of odd dimension $n$. Consider the involution $J_q^{\mathrm{id}}$ of $C_0(V,q)$. Then we have:

(i) If $n \equiv 1,7 \mod 8$ then $J_q^{\mathrm{id}}$ is orthogonal.

(ii) If $n \equiv 3,5 \mod 8$, then $J_q^{\mathrm{id}}$ is symplectic.

**Proof.** Use 7.2 and 5.7. □

**Corollary 7.4.** *Let $(V,q)$ be a nondegenerate quadratic space of even dimension $n$ and let $\tau$ be a reflection of $V$. Consider the involution $J_q^{\tau}$ of $C(V,q)$. We have:*

(i) *If $n \equiv 2,4 \mod 8$, then $J_q^{\tau}$ is of orthogonal type.*

(ii) *If $n \equiv 0,6 \mod 8$, then $J_q^{\tau}$ is of symplectic type.*

**Proof.** It suffices to note that for a reflection $\tau$, the dimension $s$, of the subspace of the anti-symmetric elements is equal to 1 and to use 7.1. □

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