S-arithmetic groups over function fields: Cohomology and finiteness properties.

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S-arithmetic subgroups Γ of reductive algebraic groups G over *number fields* are finitely presented and contain a torsion-free subgroup of finite index, which is of type FL (Ragunathan 1968, Borel-Serre 1976), therefore they are of type FP_∞, i.e. there exists a projective resolution

$$P_m \to P_{m-1} \to \cdots \to P_0 \to \mathbb{Z} \to 0$$

of finitely generated $\mathbb{Z}\Gamma$-modules P_i for all m, and also of type F_∞, i.e. there exists an Eilenberg-MacLane complex $K(\Gamma, 1)$ with finite m-skeleton for all m (cf. [Br2], VIII).

For *function fields* F $([F : \mathbb{F}_q(t)] < \infty,\ q = p^k,\ p = charF)$ however, many counter-examples are known: $SL_2(\mathbb{F}_q[t])$ is not even finitely generated, i.e. not of type F_1 (Nagao 1959, Serre 1968), $SL_2(\mathbb{F}_q[t, t^{-1}])$ and $SL_3(\mathbb{F}_q[t])$ are finitely generated, but not finitely presented, i.e. of type F_1, not F_2 (Stuhler 1976, Behr 1977); for the S-arithmetic ring O_S (S a finite, non-empty set of primes of F), $SL_2(O_S)$ is of type $F_{|S|-1}$, but not $F_{|S|}$ (Stuhler 1980), $SL_n(\mathbb{F}_q[t])$ is of type F_{n-2} but not F_{n-1} as long as $q \geq 2^{n-2}$ (Abels 1989) or $q \geq \binom{n-2}{[\frac{n-2}{2}]}$ (Abramenko 1987) and similar results hold for absolutely almost simple $\mathbb{F}_q$-groups G and $\Gamma = G(\mathbb{F}_q[t])$ (Abramenko 1994), the positive part was proved without restriction on q for Chevalley groups G, $\Gamma = G(O_S), |S| = 1$ (Behr 2004).

All these results provided evidence for the following *conjecture*: If G is an absolutely almost simple algebraic group, defined over F with F-rank $r > 0$ ("isotropic group"), $r_v = rank_{F_v}\ G$ for the completion F_v of F ($v \in S, S$ finite, $S \neq \emptyset$), Γ S-arithmetic subgroup (discrete in $G_S = \Pi_{v\in S} G(F_v)$), then Γ is of type F_{d-1}, but not of type F_d for $d = \sum_{v\in S} r_v$.

This conjecture was proved for the classical properties finite generation (iff $d \geq 2$, Behr 1969, Keller 1980) and finite presentation (iff $d \geq 3$: Behr 1998) and at last the negative part for arbitrary d (Bux-Wortman 2007). Moreover for anisotropic G (i.e. $rk_F\ G = 0$) it was known (Serre 1971), that Γ is of type F_∞.

This paper will give a proof of the positive part, using two old results on cohomology. They have the advantage not to need such precise local informations, which are necessary in (almost) all proofs of the results mentioned above, using filtrations of buildings, defined in very clever ways.

In 1976 Borel-Serre also computed the cohomology of spherical and affine buildings over non-archimedean local fields. If X_v is the Bruhat-Tits-building of $G(F_v)$ (G a semi-simple F-group, $v \in S$), then the product $X = \Pi_{v\in S} X_v$ is a contractible polysimplicial complex. X gives rise to a chain-complex $C = (C_n)_{n\in\mathbb{N}}$ with $\mathbb{Z}\Gamma$-modules C_n, generated by polysimplices, but these are (in general) not projective nor finitely generated. Borel-Serre proved that the reduced cohomology with compact support $\tilde{H}^i_c(X; M)$ for a $\mathbb{Z}\Gamma-$ module M vanishes in all dimensions, except for the top-dimension $d = \sum_{v\in S} d_v,\ d_v = dim X_v = r_v = rank_{F_v}\ G$. For $H^d_c(X; M)$ they gave an explicit description by locally-constant functions on unipotent groups. In section 1 we present a short version of their results.

On the other hand, K. Brown found in 1975 a very interesting cohomological criterion for finiteness properties. In his proof he constructed for a Γ-complex C_n of projective modules another complex C'_n with the same homology, but finitely generated Γ-modules. His assumptions are not all valid in our case and so we cannot use his general arguments. In section 2 we use his construction, but we must be more explicit, on the other side our situation is more special: The crucial point is Borel-Serre's vanishing result for cohomology with compact supports; in some sense, this substitutes the notion of being essentially trivial for filtrations (see [Br3],2). We obtain a partial resolution of $\mathbb{Z}$ by free Γ-modules up to dimension $d-1$, so Γ is of type FP_{d-1}. By this method we cannot prove type F_{d-1}, but this is implied together with finite presentation for $d \geq 3$ ([B3]).

It seems plausible, that the information on the top-cohomology should provide a new proof for being not of type FP_∞, but I don't know how to use it.

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1 Borel-Serre: Cohomology of buildings and S-arithmetic groups (see [BS])

1.1 Spherical or Tits buildings

Let k be a non-archimedean local field, G a connected semi-simple k-group of rank $l \geq 1$ and Y the Tits-building of $G(k)$. It is well-known by the Solomon-

Tits-theorem that Y has the homotopy-type of a bouquet of $(l-1)$-spheres with respect to its simplicial topology. Borel-Serre provide Y with the analytic topology, induced by the valuation on k and prove an analogue for the Alexander-Spanier-cohomology (cf. [Sp1] or [Sp2]).

Denote by P a minimal k-parabolic subgroup of G and by C the closed chamber of Y fixed by $P(k)$. Then Y can be described as $G(k)\big/P(k)\times C$ with identifications. The homogeneous space $G(k)\big/P(k) = (G\big/P)(k)$ is compact for the k-analytic topology, C carries the simplicial topology and Y gets the quotient-topology: Y_t.

We have $P(k) = Z_G(T)(k)\cdot U(k)$, where T is a maximal k-split torus of G, Z_G its centralizer and U the unipotent radical of P, W the corresponding Weyl group. The Bruhat-decomposition $G(k) = P(k)WP(k) = U(k)WP(k)$ gives a more concrete description of Y. Especially for the longest element w_0 of W this product-decomposition is unique; therefore we have a 1-1-correspondence between $U(k)$ and the set $\mathcal{C}_P$ of all chambers opposite to C, defined by $u \mapsto u\cdot w_0C$. Since two chambers determine an apartment (cf. [BT], 4), the set $\mathcal{A}_P$ of all apartments containing C is also in 1-1-correspondence with $U(k)$ and thus inherits a k-analytic structure from that of $U(k)$. $\mathcal{A}_P = \{uA_o | u\in U(k)\}$, where A_o is the apartment, defined by the opposite pair (C, w_0C) and fixed by $Z_G(T)(k) = P(k)\cap w_0P(k)w_0^{-1}$. In this setting Borel-Serre can compute the (reduced) Alexander-Spanier cohomology $H^*(Y_t;M)$ for a $\mathbb{Z}$-module M, using the group $C_c^\infty(\mathcal{A}_P;M)$ of locally constant functions with compact support on $\mathcal{A}_P$ or $U(k)$:

Proposition 1. *(=[BS], thm. 2.6)*

(i) $H^i(Y_t;M) = 0$ *for* $i \neq l-1$

(ii) $\tilde{H}^{l-1}(Y_t;M) \simeq C_c^\infty(U(k);M)$

1.2 Affine or Bruhat-Tits-buildings

The affine (or euclidean) building X of $G(k)$ (for a non-archimedean local field k) is more conveniently defined for the simply-connected covering $\tilde{G}$ of G in order to obtain X as a product of the buildings X_j for the almost simple factors G_j of $\tilde{G}$ – so it is a polysimplicial complex (see [BS], 4). An important part of this paper ([BS], 5) consists of the construction of a compactification of X by adding Y as a boundary at infinity. Thereby the direct sum $Z = X \amalg Y$ becomes a contractible compact space Z_t, inducing the natural topology on X and Y_t on Y.

Using the long exact cohomology sequence for $Z_t \mod Y_t$

$$\cdots \to \tilde{H}^i(Z_t;M) \to \tilde{H}^i(Y_t;M) \to H_c^{i+1}(X;M) \to H^{i+1}(Z_t;M) \to \dots$$

where M is a module over a ring R and H_c^* denotes the cohomology with compact supports and moreover by the vanishing of $\tilde{H}^*(Z_t; M)$ we can transfer proposition 1 to

Proposition 2. *(=[BS], thm. 5.6)*

(i) $H_c^i(X; M) = 0$ *for* $i \neq l$

(ii) $H_c^l(X; M) \simeq \begin{cases} H^{l-1}(Y_t; M) & \text{if } l \geq 1 \\ M & \text{if } l = 0 \end{cases}$

in particular for $R = \mathbb{Z}$ *and* $l \geq 1$*:*

$$H_c^l(X; M) \simeq C_c^\infty(U(k); M)$$

Remark: A function $f \in C_c^\infty(U(k); M)$ has compact support and is locally constant, so there exists a finite union of open subsets of $U(k)$, such that f is constant on each of them. This union corresponds to a neighbourhood of Y_t in Z_t and f is determined on its compact complement on X.

1.3 S-arithmetic groups over function fields

Let F be a function field (i.e. $[F : \mathbb{F}_q(t)] < \infty,\ q = p^m,\ p = char\ F$) with a finite non-empty set S of places of F and F_v the completion of F with respect to $v \in S$.

G denotes a connected semi-simple algebraic F-group of rank r, $r_v := rank_{F_v}\ G$ $(v \in S)$, $G_S := \Pi_{v \in S} G(F_v);\ X = \Pi_{v \in S} X_v$ with Bruhat-Tits-buildings X_v of $G(F_v)$, $dim\ X_v = d_v = r_v$ and $d = dim\ X = \sum_{v \in S} d_v$. Finally Γ is a S-arithmetic subgroup, discrete in G_S.

G is called isotropic if $r > 0$ and anisotropic if $r = 0$. It is well known from reduction theory ("Godement's compactness criterion"; cf. [H] or [B1]) that X/Γ is compact iff G is anisotropic.

For this cocompact case Borel-Serre prove (thm. 6.2 in [BS])

Proposition 3. *A S-arithmetic subgroup Γ of an anisotropic connected semi-simple group G over a function field F is finitely presented and of type FP_∞ and also of type F_∞.*

Remarks:

a) More precisely they show that Γ has a torsion free subgroup Γ_0 of finite index, which is of type FL and so of type $F(P)_\infty$, thus Γ inherits this properties.

b) Moreover $H^i(\Gamma_0; \mathbb{Z}\Gamma_0) \simeq \begin{cases} 0 & \text{for } i \neq d \\ H_c^d(X; \mathbb{Z}), & \text{thus free} \end{cases}$

c) In the "number-field-case" (over a number field K instead of F) all these results are valid for arbitrary S-arithmetic groups, i.e. also for isotropic G.

In the function-field-case Bux and Wortman proved in [BW], that Γ can be of type F_∞ only for anisotropic G: They give a bound for the "finiteness length" ($\max\{n : \Gamma \text{ has type } F_n\}$) for isotropic groups. To obtain a sharp bound, one should restrict to absolutely almost simple groups: A simply connected semi-simple group G is the direct product of its almost simple factors G_i and so is $\Gamma = \prod \Gamma_i,\ \Gamma_i$ in G_i. For instance Γ can only be finitely generated if all Γ_i are so. In this situation they show, that Γ cannot be of type FP_d (so not F_d).

1.4 Isotropic groups

Now assume that G is an isotropic, absolutely almost simple F-group. So there exists a minimal F-parabolic subgroup P with unipotent radical U. For each $v \in S$ choose a minimal F_v-parabolic group Q_v, contained in P with unipotent radical U_v, such that $U_v(F_v) \supseteq U(F_v) \supset U(F)$.

Set $\overline{U}_S := \prod_{v\in S} U_v(F_v) \supseteq U_S := \prod_{v\in S} U(F_v) \supset U(F)$ (the last inclusion by diagonal embedding) and $\Gamma \subset G(F)$ is S-arithmetic. By propositions 1 and 2 we have the isomorphisms for an arbitrary $\mathbb{Z}-$module M

$$H_c^{d_v}(X_v; M) \simeq H_c^{d_v-1}(Y_v; M) \simeq C_c^\infty(U_v(F_v); M)$$

and $H_c^i(X_v; M) = 0$ for $i < d_v$, so for $X = \prod_{v\in S} X_v$ we obtain by Künneth's formula

Theorem 1. *(cf. [BS] , 6.6)*

a) $H_c^i(X; M) = 0$ *for* $i < d = \sum_{v\in S} dim\ X_v$*;*

b) $H_c^d(X; M) = \bigotimes_{v\in S} H_c^{d_v}(X_v; M) \simeq C_c^\infty(\overline{U}_S; M) \supset C_c^\infty(U_S; M)$

2 Construction of a FP_{d-1}-resolution for Γ

The Bruhat-Tits-building X provides an augmented chain-complex $C = (C_n)_{-1\leq n\leq d}$ with $\mathbb{Z}\Gamma-$ modules C_n, generated by the $n-$dimensional polysimplices of X for $n \geq 0$ and $C_{-1} = \mathbb{Z},\ C_n \overset{\partial n}{\to} C_{n-1}$. Since X is a contractible space, C has trivial reduced homology.

Following K. Brown (see [Br1]) we shall construct inductively a projective — or even free — resolution of $\mathbb{Z}$ by defining chain-complexes $C'(k) = \big(C'(k)_n\big)_{-1\leq n\leq d-1}$ with finitely generated $\mathbb{Z}\Gamma-$modules $C'(k)_n$, where $C'(k)_n = C'(k-1)_n$ for

$n \leq k-1$ and $C'(k)_n = 0$ for $n > k$, beginning with $C'_{-1} = \mathbb{Z}$ (derivation $\partial_n : C'_n \to C'_{n-1}$).

Moreover we define chain-maps $f_k : C'(k) \to C$, starting with $f_{-1} = id_{\mathbb{Z}}$ and f_k an extension of f_{k-1}. Thereby we obtain finally a subcomplex C' of C, whose support in X is compact modulo Γ.

We consider the mapping-cones $C''(k)$ for f_{k-1}, given by $C''(k)_n := C_n \oplus C'(k-1)_{n-1}$, $C''(k)_{-1} = \mathbb{Z}$ and $\partial''_n(c,c') = \big(\partial_n c - f_{n-1}(c'), -\partial'_{k-1}(c')\big)$, $\partial'_{-1} = 0$.

2.1 Homology and the beginning of induction

There is a short exact sequence

$$0 \to C \to C'' \to \Sigma C' \to 0, \ (\Sigma C')_n := C'_{n-1},$$

giving rise to a long exact sequence for homology

$$(1) \ldots \to H_n(C) \to H_n\big(C''(k)\big) \to H_{n-1}\big(C'(k-1)\big) \to H_{n-1}(C) \to \ldots$$

Denote by X_o the set of vertices of X, then we have

$$C''(0)_0 = C_0(X) \oplus \mathbb{Z} = \{(\Sigma z_i x_i, z') \mid x_i \in X_0; z_i, z' \in \mathbb{Z}\}$$

and with $\partial_0(\Sigma z_i x_i) = \Sigma z_i$ (augmentation-map) we obtain $\partial''_0\big((\Sigma z_i x_i, z')\big) = 0 \Leftrightarrow \Sigma z_i = z'$. Furthermore $C''(0)_1 = C_1(X) \oplus \{0\}$; $H_0(C) = 0$ implies

$$Z_0(C) = B_0(C) \simeq B_0\big(C''(0)\big) = \{\Sigma z_i x_i, 0) \mid \Sigma z_i = 0\}$$

Choose now a base point $x_0 \in X_0$, then $\big(\Sigma z_i(x_0 - x_i), 0\big) \in B_0\big(C''(0)\big)$ and we see that

$$H_0\big(C''(0)\big) \simeq \{\big((\Sigma z_i)\cdot x_0, (\Sigma z_i)\big)\} = \{(zx_0, z) \mid z \in \mathbb{Z}\} \simeq \mathbb{Z}$$

We can give evidence of its $\mathbb{Z}\Gamma-$module-structure:

$$H_0\big(C''(0)\big) \simeq \{\big(\Sigma z_\gamma(\gamma x_0), \Sigma z_\gamma\big) \mid z_\gamma \in \mathbb{Z}, \gamma \in \Gamma\} \Big/ _{\{\left(\Sigma z_\gamma(\gamma x_0), 0\right) \mid \Sigma z_\gamma = 0\}}$$

Using this description, we can lift the augmentation-map $\epsilon : \mathbb{Z}\Gamma \to \mathbb{Z}$ to a $\mathbb{Z}\Gamma-$homomorphism φ_0 of $\mathbb{Z}\Gamma$ into $Z_0\big(C''(0)\big)$, defined by $\varphi_0(\Sigma z_\gamma \gamma) := \big(\Sigma z_\gamma(\gamma x_0), \Sigma z_\gamma\big)$; so φ_0 surjects onto $H_0\big(C''(0)\big)$. $Z_0\big(C''(0)\big)$ can be viewed as a fiber-product $C_0 \times_{\mathbb{Z}} C'_{-1}$, given by the maps ∂_0 and f_{-1}:

$$\begin{tikzcd}
\mathbb{Z}\Gamma \arrow[dr, "\varphi_0"] \arrow[drr, dashed, "\partial'_0"] \arrow[ddr, dashed, "f_0"'] & & \\
& Z_0\big(C''(0)\big) \arrow[r] \arrow[d] & C'_{-1} = \mathbb{Z} \arrow[d, "f_{-1}"] \\
& C_0 \arrow[r, "\partial_0"'] & \mathbb{Z}
\end{tikzcd}$$

According to this diagram we define

$$
\begin{aligned}
C'_0 := \mathbb{Z}\Gamma, \quad f_0 : C'_0 \to C_0 \text{ by } f_0(\Sigma z_\gamma \gamma) : &= \Sigma z_\gamma(\gamma x_0)\\
\partial'_0 : C'_0 \to C'_{-1} \text{ by } \partial'_0(\Sigma z_\gamma \gamma) : &= \Sigma z_\gamma
\end{aligned}
$$

As a consequence we have to set $C''(1)_1 = C_1 \oplus C'_0$ and $C''(1)_0 = C_0 \oplus \mathbb{Z}$, $C''(1)_n = C_n$ for $n > 1$. We confirm that

$$
\begin{aligned}
\partial''_0 \circ \partial''_1(c_1, c'_0) &= \partial''_0\big(\partial_1(c_1) - f_0(c'_0), -\partial'_0(c'_0)\big)\\
&= \big(\partial_0 \circ \partial_1(c_1) - \partial_0 \circ f_0(c'_0) - f_{-1} \circ (-\partial'_0(c'_0)\big), \big(-\partial'_1 \circ (-\partial'_0)(c'_0)\big)\\
&= 0, (\text{since } \partial_0 \circ f_0 = f_{-1} \circ \partial'_0)
\end{aligned}
$$

Moreover $\partial''_1(0, -c'_0) = \big(-f_0(-c'_0)\big), -\partial'_0(-c'_0)\big) = \big(\Sigma z_\gamma(\gamma x_0), \Sigma z_\gamma\big)$ for $c'_0 = \Sigma z_\gamma \gamma$, which means that $\partial''_1 : C''(1)_1 \to C''(1)_0$ is surjective on $H_0\big(C''(0)\big)$, thus $H_0\big(C''(1)\big) = 0$.

Observe that $f_0(C_0) = f_0(\mathbb{Z}\Gamma) = \mathbb{Z} \cdot (\Gamma x_0)$, whose support is the subcomplex $\Gamma \cdot x_0 =: X'_0$ of X.

For the next step consider $H_0\big(C'(0)\big) = Z_0\big(C'(0)\big) = \{\Sigma z_\gamma \cdot \gamma \mid \Sigma z_\gamma = 0\} =: I\Gamma$, the augmentation ideal of $\mathbb{Z}\Gamma$. It is well known that $I\Gamma$ is a finitely generated $\mathbb{Z}\Gamma-$module iff Γ is a finitely generated group. Now Γ is a $S-$arithemtic subgroup of $G(F)$, G an absolutely almost simple algebraic $F-$group — what we assume from now on. Therefore Γ is finitely generated iff the sum of local ranks $d \geq 2$ (see [B1] or [B3]); so we find a free module $(\mathbb{Z}\Gamma)^{r_1} =: C'(1)_1$, which surjects on $I\Gamma = H_0\big(C'(0)\big)$. By sequence (1) $H_0\big(C'(0)\big)$ is isomorphic to $H_1\big(C''(1)\big)$, because $H_1(C) = H_0(C) = 0$. We can lift the surjection of $C'(1)_1$ onto $H_1\big(C''(1)\big)$ to $Z_1\big(C''(1)\big)$, which is by definition a fiberproduct $C_1 \times_{C_0} Z_0(C'_0)$with respect to the maps ∂_1 and f_0. As above in the diagram we get the maps $f_1 : C'(1)_1 \to C_1$ and $\partial'_1 : C'(1)_1 \to C'(1)_0 = C'(0)_0$, more concretely:

$$
\begin{tikzcd}
c'_1 \arrow[dr] \arrow[drr, dashed, "\partial'_1"] \arrow[ddr, dashed, "f_1"'] & & \\
& (c_1, c'_0) \arrow[r] \arrow[d] & c'_0 \arrow[d, "f_0"] \\
& c_1 \arrow[r, "\partial_1"'] & \partial_1(c_1) = f_0(c'_0)
\end{tikzcd}
\qquad \text{with } \partial'_0(c'_0) = 0
$$

Let us point out, that c_1 is a $1-$chain in $C_1(X)$, whose boundary is contained in $f_0(C'_0)$. Since $C'(1)_1$ is finitely generated, we get finitely many elements $c_1^1, \ldots, c_1^{r_1}$, whose supports are pathes $p_1, \ldots, p_{r_1}$ in X, which generate a $\Gamma-$subcomplex X'_1 of X with X'_1/Γ compact.

For each $c'_0 \in Z_0\big(C'(0)\big)$ there exists $c_1 \in C_1$ with $\partial_1 c_1 = f_0(c'_0)$, since $H_0(C) = 0$ and also $c'_1 \in C'(1)_1$ with $\partial'_1(c'_1) = c'_0 : H_0\big(C'(1)\big) = 0$. Alternatively we could use $C''(2)_2 := C_2 \oplus C'(1)_1$ and compute ∂_2'', proving that $H_1\big(C''(2)\big) = 0$, which implies by (1) $H_0\big(C'(1)\big) = 0$.

2.2 Some examples

1. $\Gamma = SL_2(\mathbb{F}_q[t])$ is not finitely generated, due to Nagao-Serre (see [S], II. 1.6); its Bruhat-Tits-building is a tree X, which has a half-line H as a fundamental domain mod Γ. H has vertices $x_i (i \in \mathbb{N}_0)$ with stabilizers $\Gamma_i = \mathrm{stab}_\Gamma x_i$, where $\Gamma_0 = SL_2(\mathbb{F}_q), \Gamma_i = \begin{pmatrix} \mathbb{F}_q^* & \mathbb{F}_q[t]_i \\ 0 & \mathbb{F}_q^* \end{pmatrix}$ with $\mathbb{F}_q[t]_i = \{p \in \mathbb{F}_q[t] \mid \deg p \leq i\}$ for $i > 0$.

   We have $X_0' = \Gamma \cdot x_0$, which contains $\gamma_i x_0$ for $\gamma_i \in \Gamma_i \diagdown \Gamma_{i-1}$: the shortest path p_i in X, connecting x_0 with $\gamma_i x_0$ must contain the vertex x_i. Thus the complex X_1', generated by the p_i is not compact $\mod \Gamma$. If $c_i' = \gamma_i x_0 - x_0 \in C(0)_0'$ and c_i is the chain corresponding to p_i in C_1' we have $(c_i, c_i') \in Z_1\big(C''(1)\big)$, which shows, that $H_1\big(C''(1)\big)$ cannot be a finitely generated $\mathbb{Z}\Gamma-$module — just as $H_0\big(C'(0)\big)$.

2. $\Gamma = SL_3(\mathbb{F}_q[t])$ is finitely generated: it is easy to see, that $E = SL_3(\mathbb{F}_q) \cup \{e_{12}(p), e_{23}(q) \mid p, q \in \mathbb{F}_q[t]_1\}$ is a set of generators. A standard apartment A of its Bruhat-Tits-building X is a triangulated plane and a fundamental domain for $X \mod \Gamma$ is given by a cone C in A with vertex x_0 and angle $\frac{\pi}{3}$. Let $\triangle_0$ be the triangle with vertex x_0 and contained in C, then for each $\gamma \in E$ we have $\gamma\triangle_0 \cap \triangle_0 \neq \phi$ (γ fixes at least one vertex of $\triangle_0$). This implies, that every vertex γx_0 is connected with x_0 by a path p, that projects into $\triangle_0$, so p is contained in $\Gamma\triangle_0 =: X_1'$, which means X_1'/Γ is compact. In the language of chains: For each $c' \in Z_0\big(C'(0)\big)$ with $f_0(c') \in C_0(X_0') \subset C_0(X), X_0' = \Gamma \cdot x_0$ we find $c \in C_1(X_1') \subset C_1(X)_1$ s. th. $(c, c') \in Z_1\big(C''(1)\big)$. But these pairs generate $H_1\big(C''(1)\big)$, since $(c, 0) \in Z_1\big(C''(1)\big)$ is a boundary by $H_1(X) = 0$ and for $(c_1, c'), (c_2, c') \in Z_1\big(C''(1)\big)$ we have $(c_1, c') - (c_2, c') = (c_1 - c_2, 0) \in B_1\big(C''(1)\big)$. Conclusion: $H_1\big(C''(1)\big)$ is a finitely generated $\mathbb{Z}\Gamma-$ module, because the elements $c' = \gamma x_0 - x_0$ with $\gamma \in E$ generate $Z_0\big(C'(0)\big)$.

   On the other side $SL_3(\mathbb{F}_q[t])$ is not finitely presented: this is shown in [B2] by constructing an infinite series of paths in X_1' $\big($or $1-$cycles $c_n' \in C_1'(1)_1\big)$, which cannot be contracted in ΓC_n where $C_n (n \in \mathbb{N})$ are compact subsets of C with $\bigcup\limits_n C_n = C$. In the same way as in example 1 we obtain elements (c_n, c_n') in $Z_2\big(C''(2)\big)$, which cannot be contained in a finitely generated $\mathbb{Z}\Gamma-$module and of course are inequivalent $\mod B_2\big(C''(2)\big)$ — s.th. $H_2\big(C''(2)\big)$ is not finitely generated as a $\mathbb{Z}\Gamma-$module and the supports of all $2-$chains c_n cannot be contained in a complex X_2' with X_2'/Γ compact.

3. $\Gamma = SL_2(\mathbb{F}_q[t,t^{-1}])$ is finitely generated by the set $E = SL_2(\mathbb{F}_q) \cup \{e_{12}(p) \mid p \in \mathbb{F}_q[t]_1\} \cup \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right\}$. An apartment A of $X = X_1 \times X_2$ $\big($the buildings X_1 of $SL_2(\mathbb{F}_q[t])$ and X_2 of $SL_2(\mathbb{F}_q[t^{-1}])\big)$ is a plane, divided into squares. Choose $x_0 \in A$ with $\mathrm{stab}_\Gamma x_0 = SL_2(\mathbb{F}_q)$, then there exists a finite union Q of squares, containing x_0, s.th. $\gamma Q \cap Q \neq \phi$ for all $\gamma \in E$. Then each $\gamma x_0 \in \Gamma x_0 =: X_0'$ is connected with x_0 by a path in $\Gamma Q =: X_1'$, so X_1'/Γ is compact and we conclude as in example 2, that $H_1\big(C''(1)\big)$ is generated by elements (c, c') with $c \in C_1(X_1')$, so it is a finitely generated $\mathbb{Z}\Gamma-$module.

   On the other hand, Γ is not finitely presented, which was shown in [St1] in a similar way as for example 2, s. th. $H_2\big(C''(2)\big)$ cannot be finitely generated.

4. $G = SL_2 \times SL_2$ is semi-simple, but not almost simple and $\Gamma = \Gamma_1 \times \Gamma_2 = SL_2(\mathbb{F}_q[t]) \times SL_2(\mathbb{F}_q[t])$ is not finitely generated, since the Γ_i are not and $H_0\big(C'(0)\big) = I\Gamma_1 \times I\Gamma_2$, isomorphic to $H_1\big(C''(1)\big)$, is not a finitely generated $\mathbb{Z}\Gamma-$module — although the sum d of local ranks is 2!

2.3 Cohomology

The examples show that in order to construct a resolution and a $\bmod \Gamma$ finite subcomplex of X one should prove that $H_k\big(C''(k)\big)$ is a finitely generated $\mathbb{Z}\Gamma-$module for $k \leq d-1$. Brown's construction in [Br1] uses cohomology for this purpose, but he assumes, that the given modules of chains are projective, which is not true in our case: So we need more concrete arguments. We shall apply Borel-Serre's theorem (thm. 1) on cohomology with compact supports. They refer to Alexander-Spanier-cohomology, but for the polysimplicial complex X it can also be described by homomorphisms on the $\mathbb{Z}\Gamma-$modules $C_n(X)$ of chains (of [Sp1], section 19 or [Sp2], 6.9).

Recall that an element of $H_c^n(X; M)$ with an arbitrary coefficient-module M is then given by an abelian group homomorphism φ on $C_n(X)$, that must be zero on the module $B_n(X)$ of boundaries (since $\delta_n\varphi := \varphi \circ \partial_{n+1}$) and can be modified by $\delta_{n-1}\psi = \psi \circ \partial_n$ with $\psi : C_{n-1}(X) \to M$; the index c means compact support, so it consists in X of finitely many polysimplices. The action of Γ on φ is defined by $(\varphi\gamma)(c) = \varphi(\gamma c)$.

Maybe it is more convenient to consider $\Gamma-$homomorphisms $\tilde{\varphi}$ instead of φ, whose support is compact $\bmod \Gamma$, i.e. represented by finitely many polysimplices. If a module of φ's is finitely generated, then the union of all supports of $\tilde{\varphi}$'s is also compact $\bmod \Gamma$. Moreover we shall regard two variants of complexes C', either it is a chain complex of finitely generated free $\Gamma-$modules or it is $f(C') = C(X')$, where X' is a subcomplex of X of lower dimension, X'/Γ finite: then f becomes inclusion and ∂' the restriction of ∂ to X'.

Now we shall continue the induction, beginning with $C'_0 = \mathbb{Z}\Gamma; f_0 : C'_0 \to C_0$ with $f_0(\Sigma z_\gamma \gamma) = \Sigma z_j(\gamma x_0)$, so f_0 has finite kernel, since $\mathrm{stab}_\Gamma x_0$ is finite; $X'_0 = \Gamma \cdot x_0$. We define the (abelian-group-) homomorphism $\varphi : C''(k)_k = C_k \times C(k-1)_{k-1} \to C_{k-1}$ by $\varphi(c, c') := \partial_k(c)$, assuming that C'_{k-1} is a finitely generated free $\mathbb{Z}\Gamma-$module and f_{k-1} has finite kernel.

Since $C'(k-1)_k = 0$ we have $B_k\big(C''(k)\big) = B_k(C)$ and φ vanishes on $B_k(C)$, so it is an element of $Z^k\big(C''(k), C_{k-1}\big)$. We modify φ in its cohomology class by $\psi := \pi_1 \circ \partial''_k : C''(k)_n \to C_{k-1}, \psi(c, c') = \pi_1(\partial_k c - f_{k-1}c', \partial'_{k-1}c') = (\partial_k c - f_{k-1}c')$; ψ vanishes on $Z^k\big(C''(k), C_{k-1}\big)$ and $\varphi - \psi$ induces the same element in $H^k\big(C''(k), C_{k-1}\big)$ as $\varphi : \overline{\varphi}$. We obtain $\overline{\varphi}(c, c') = f_{k-1}(c')$; its restriction to $H_k\big(C''(k)\big)$ is injective in the first component, since $H_k(C) = 0$ and has finite kernel in the second, since f_{k-1} has. Because then $\partial'_{k-1}(c') = 0, \overline{\varphi}$ induces a surjective homomorphism with finite kernel from $H_k\big(C''(k)\big)$ to $f_{k-1}[H_{k-1}\big(C'(k-1)\big]$ or even onto $f_{k-1}\Big(H_k\big(C''(k)\big)\Big)$, since $H_k\big(C''(k)\big) \simeq H_{k-1}\big(C'(k-1)\big)$, due to sequence (1), given by $(c, c') \mapsto c'$.

For isotropic groups G X/Γ is not compact, so we have to consider an ascending sequence $X_m (m \in \mathbb{N})$ of subcomplexes with $\bigcup\limits_m X_m = X$ and X_m/Γ compact ("filtration"). Denote by φ_m the restriction of φ to $C(X_m)_k \times C'(k-1)_{k-1}$ and observe by induction, that the support X'_{k-1} of $C'(k-1)$ is also compact: we may suppose that $X'_{k-1} \subseteq X_m$. Thus we have $\varphi_m \in H^k_c\big(C''(k); C_{k-1}\big)$.

For cohomology there exists also a long exact sequence

$$(2) \ldots \to H^{n-1}_c(C; M) \to H^{n-1}_c\big(C'(k-1); M\big) \to H^n_c\big(C''(k); M\big) \to H^n_c(C; M) \to \ldots$$

for an arbitrary $\mathbb{Z}\Gamma-$ module M.

According to Borel-Serre (see thm.1) we know, that $H^n_c(C; M) = 0$ for $n \leq d-1$, which implies $H^k_c\big(C''(k); M\big) \simeq H^{k-1}_c\big(C'(k); M\big)$ for $k \leq d-1$. The cohomology of the complex $C'(k)$ of finitely generated free $\mathbb{Z}\Gamma-$modules is also finitely generated and so is $H^k_c\big(C''(k); M\big)$.

We apply this result to the homomorphisms $\overline{\varphi}_m$ with $M = C_{k-1}$: All $\overline{\varphi}_m$ are $\mathbb{Z}\Gamma-$linear combinations of finitely many elements, say $\phi_1, \ldots, \phi_s \in H^k_c\big(C''(k); M\big)$ the union of the supports of all ϕ_i is contained in some $X_{m_0} (m_0 \in \mathbb{N})$, which means $\overline{\varphi} = \overline{\varphi}_m V m \geq m_0$ and $\overline{\varphi} = 0$ outside X_{m_0}. Consider now finitely generated submodules of $H_k\big(C''(k)\big)$: each of them is mapped by some $\overline{\varphi}_m$ to $f_{k-1}\Big(H_{k-1}\big(C'(k)\big)\Big)$ with finite kernel. Since $\lim\limits_{\overrightarrow{m}} \varphi_m = \varphi$, we conclude that $H_k\big(C''(k)\big)$ is also finitely generated; take $z_i = (c_i, c'_i) \in Z_k\big(C''(k)\big), o = 1, \ldots, r_k$ as generators and define

C'_k as the free module $(\mathbb{Z}\Gamma)^{r_k}$, projecting to $\bigoplus\limits_1^{r_k} \mathbb{Z}\Gamma \cdot z_i$ and ∂'_k and f_k as in 2.1. Since the stabilisators of the c_i are finite, $f_k : C'_k \to C_k$ has finite kernel and for the complex $C'(k)$ with $C'(k)_k = C'(k)$ we get $H_{k-1}\big(C'(k)\big) = 0$. Furthermore $X'_k = \bigcup\limits_1^{r_k} \mathrm{supp}(C_i) \cup X'_{k-1}$ is a subcomplex of X with X'_k/Γ compact and $H_{k-1}\big(C(X'_k)\big) = 0$.

Remarks

1. Under the assumptions of [Br1], one can even show, that $H^k(C;M) \simeq \mathrm{Hom}_{\mathbb{Z}\Gamma}\big(H_k(C);M\big)$ and if the cohomology of C preserves direct limits for the coefficient modules, Brown constructs the chain-complex C' with finitely generated $\mathbb{Z}\Gamma-$modules.

2. For the semi-simple group G of example 4 the construction above should not work, although for $d = 2$ the sequence (2) implies that $H^1_c\big(C''(1), C_0\big)$ is finitely generated.

   Here we have $X'_0 = \mathbb{Z}(\Gamma_1 \times \Gamma_2) \cdot (x^1_0, x^2_0)$ and the shortest path between (x^1_0, x^2_0) and $(\gamma^1_i x^1_0, x^2_0)$ must contain the vertex (x^1_i, x^2_0) (notations as in example 1 with upper index 1 or 2, s.th. the distance between x^1_0 and $\gamma^1_i x^1_0$ goes to infinity with $i-$analogous in the second component. There exists a closed path in $X = X_1 \times X_2$ with the following vertices:

   $(x^1_0, x^2_0) \to (x^1_i, x^2_0) \to (\gamma^1_i x^1_0, x^2_0) \to (\gamma^1_i x^1_0, x^2_j) \to (x^1_i, x^2_j) \to (x^1_0, x^2_j) \to (x^1_0, x^2_0)$. This path p (or chains, whose support is p) is a boundary in $X(d \geq 2)$, therefore a homomorphism from cohomology has to vanish on p, e.g. ∂_1, which should define the isomorphism on homology. Bur for fixed x^1_i we get x^i_2 arbitrary far from x^0_2, which means, that ∂_1 cannot be restricted to compact supports.

We summarize our results in the following

Proposition 4. *a) There exists a partial resolution of Z with free $\mathbb{Z}\Gamma-$modules of finite rank: $C'_{d-1} \to C'_{d-2} \to \ldots \to C'_0 \to Z$*

b) There exists a $(d-1)-$dimensional subcomplex $X' = X'(d-1)$ of X with $H^0_k(X') = 0$ for $k < d-1$.

Both properties imply the following finiteness theorem: Cf. for the first [Br2], VIII. 4.3 and for the second [Br3], 1.1 (observing that stabilizers in Γ of cells in X are finite).

Theorem 2. *A $S-$arithmetic subgroup Γ of an absolutely almost simple algebraic group G, defined over a function field F with $rank_F G > 0$ and $d = \sum_{r \in S} rank_{F_v} G$ is of type FP_{d-1}.*

Remarks:

1. For semi-simple groups we obtain type $FP_{d'-1}$, where d' is the minimal d for the simple factors.
2. Our (co)homological method cannot prove that Γ is also of type F_{d-1}; but this is true, since finite presentability was shown for $d = 3$ (see [B3]: unfortunately this proof is case-by-case and lengthy and part II of it was not published, but exists!): cf [Br2], VIII. 7.

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