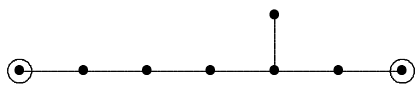
THE KNESER-TITS CONJECTURE FOR GROUPS WITH TITS-INDEX $E_{8,2}^{66}$ OVER AN ARBITRARY FIELD

R. PARIMALA, J.-P. TIGNOL, AND R. M. WEISS

Abstract. We prove: (1) The group of multipliers of similitudes of a 12-dimensional anisotropic quadratic form over a field K with trivial discriminant and split Clifford invariant is generated by norms from quadratic extensions E/K such that q_E is hyperbolic. (2) If G is the group of K-rational points of an absolutely simple algebraic group whose Tits index is $E_{8,2}^{66}$, then G is generated by its root groups, as predicted by the Kneser-Tits conjecture.

1. Introduction

The Kneser-Tits conjecture—first formulated in [21]—predicts that the group of K-rational points (for some field K of arbitrary characteristic) of an absolutely simple algebraic group with Tits index



is generated by its root groups. This Tits-index is denoted by $E_{8,2}^{66}$ in [19]. Groups with this Tits index are classified by similarity classes of anisotropic 12-dimensional quadratic forms over K with trivial discriminant and split Clifford invariant. By [22, 42.6], they are also the groups whose corresponding spherical building is a Moufang quadrangle of type E_8 as defined in [22, 16.6].

Given a quadratic form q defined over a field K, we denote by $\mathrm{clif}(q)$ the Clifford invariant of q, by $G(q)$ the group of multipliers of similitudes of q, by $\mathrm{Hyp}(q)$ the subgroup of $K^\times$ generated by $K^{\times 2}$ and the norms from finite extensions E/K such that q_E is hyperbolic and by $\mathrm{Hyp}_2(q)$ the subgroup of $\mathrm{Hyp}(q)$ generated by $K^{\times 2}$ and the norms from *quadratic* extensions E/K such that q_E is hyperbolic (including inseparable ones).

Our goal is to prove the following closely related statements.

Theorem 1.1. *If q is an anisotropic quadratic form with trivial discriminant, then $G(q) = \mathrm{Hyp}_2(q)$ in the following cases:*

(i) $\dim q = 8$ *and the index of* $\mathrm{clif}(q)$ *is* 2*;*
(ii) $\dim q = 12$ *and* $\mathrm{clif}(q)$ *is split.*

Theorem 1.2. *If G is the group of K-rational points of an absolutely simple algebraic group whose Tits index is $E_{8,2}^{66}$, then G is generated by its root groups.*

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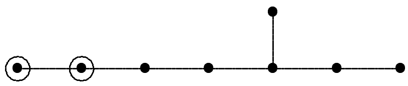
We give two very different proofs of these theorems. In §2 we lay the groundwork that is common to the two proofs, and show that the equality $G(q) = \mathrm{Hyp}(q)$ holds for quadratic forms as in Theorem 1.1. As a consequence, the connected component of the identity $\mathrm{PGO}_+(q)$ in the group of projective similitudes is R-trivial if $\mathrm{char}(K) \neq 2$: see Corollary 2.19. In §3, we give proofs of Theorems 1.1 and 1.2 based on results in [23] and [24]. In particular, the notion of a *quadrangular algebra* introduced in Chapters 12–13 of [22] and in [23] plays a central role in these proofs. In §4 we show how the R-triviality of $\mathrm{PGO}_+(q)$ for q as in Theorem 1.1(ii) in characteristic 0 yields another proof of Theorem 1.2 in arbitrary characteristic. In §5 we give an entirely different proof of Theorem 1.1 under the assumption that $\mathrm{char}(K) \neq 2$, using the triality-defined correspondence between 8-dimensional quadratic forms of trivial discriminant and hermitian forms over the simple components of their even Clifford algebra.

For a survey of what is known about the Kneser-Tits conjecture; see [9]. We call attention especially to §6 of that paper, where the Kneser-Tits conjecture over an arbitrary field is discussed. By [17], [9, 6.1] and Theorem 1.2, the only exceptional groups of relative rank at least 2 for which the Kneser-Tits conjecture remains to be verified over arbitrary fields are those whose Tits index is



(called $E_{8,2}^{78}$ in [19]). Groups with this Tits index are classified by isotopy classes of Albert division algebras, and the corresponding spherical buildings are the Moufang hexagons defined in [22, 16.8] for "hexagonal systems" of dimension 27. See also [9, 8.6] and [22, 37.41].

Acknowledgement. The proof in §4 that the R-triviality in characteristic 0 of $\mathrm{PGO}_+(q)$ for q as in Theorem 1.1(ii) implies Theorem 1.2 in arbitrary characteristic is due to Skip Garibaldi. We would like to thank him for allowing us to reproduce his proof here.

2. Similitudes of quadratic forms

Our main background reference for quadratic forms is [6], although we mostly use the notation of [23]. Let (K, L, q) be a quadratic space. Thus K is a field, L is a K-vector space and $q\colon L \to K$ is a quadratic form on L. We let $f = \partial q$ denote the polar bilinear form of q. Thus

$$f(x, y) = q(x + y) - q(x) - q(y) \qquad \text{for } x,\, y \in L.$$

The quadratic space (K, L, q) is *nondegenerate* if $\dim_K \mathrm{rad}\, f \leq 1$; see [6, 7.17]. If $\dim_K L$ is even and (K, L, q) is nondegenerate, then f is nondegenerate, the discriminant $\mathrm{disc}(q)$ is the isomorphism class of the center of the even Clifford algebra $C_0(q)$ and the Clifford invariant $\mathrm{clif}(q)$ is the Brauer class of the full Clifford algebra $C(q)$; see [6, §§13, 14]. As in [6, §§8, 9], we let I_qK denote the quadratic Witt group of K and let $I_q^nK = I^{n-1}K \cdot I_qK$ for all $n > 0$, where $I^{n-1}K$ is the $(n-1)$st power of the fundamental ideal IK of even-dimensional forms in the bilinear Witt ring WK.

The following definitions are taken from [22, 21.31].

Definition 2.1. A quadratic space (K, L, q) is *of type* E_7 if it is anisotropic and there exists a separable quadratic extension E/K with norm N and scalars $\alpha_1, \ldots, \alpha_4$ such that

$$(K, L, q) \cong (K, E^4, \alpha_1 N \perp \alpha_2 N \perp \alpha_3 N \perp \alpha_4 N)$$

and

$$\alpha_1\alpha_2\alpha_3\alpha_4 \notin N(E).$$

In other words, q is anisotropic,

$$q \cong \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle \cdot N,$$

and the quaternion algebra $(E/K, \alpha_1\alpha_2\alpha_3\alpha_4)$, which represents $\mathrm{clif}(q)$, is not split.

Definition 2.2. A quadratic space (K, L, q) is *of type* E_8 if it is anisotropic and there exists a separable quadratic extension E/K with norm N and scalars $\alpha_1, \ldots, \alpha_6$ such that

$$(K, L, q) \cong (K, E^6, \alpha_1 N \perp \alpha_2 N \perp \cdots \perp \alpha_6 N)$$

and

$$-\alpha_1\alpha_2 \cdots \alpha_6 \in N(E).$$

In other words, q is anisotropic,

$$q \cong \langle \alpha_1, \ldots, \alpha_6 \rangle \cdot N,$$

and $\mathrm{clif}(q)$ is split.

Proposition 2.3. *Suppose that (K, L, q) is an anisotropic quadratic space. Then the following hold:*

(i) *(K, L, q) is of type E_7 if and only if $\dim q = 8$, $\mathrm{disc}(q)$ is trivial and $\mathrm{clif}(q)$ is of index 2.*
(ii) *(K, L, q) is of type E_8 if and only if $\dim q = 12$, $\mathrm{disc}(q)$ is trivial and $\mathrm{clif}(q)$ is split. These conditions are also equivalent to $\dim q = 12$ and $q \in I_q^3 K$.*

Proof. If $\mathrm{char}(K) \neq 2$, (i) is in [11, Ex. 9.12] and (ii) in [16, p. 123]. In arbitrary characteristic, see [5, 4.12] and (for the second part of (ii)) [6, Thm. 16.3] if $\mathrm{char}(K) = 2$. □

Remark 2.4. Suppose that (K, L, q) is a quadratic space of type E_7 and that $q(1) = 1$ for a distinguished element 1 of L. Let E and $\alpha_1, \ldots, \alpha_4$ be as in 2.1. By [23, 2.24],

$$C(q, 1) \cong M(4, D) \oplus M(4, D), \tag{2.5}$$

where $C(q, 1)$ is the Clifford algebra with base point as defined in [22, 12.47] and D is the quaternion division algebra $(E/K, \alpha_1\alpha_2\alpha_3\alpha_4)$. By [22, 12.51], $C(q, 1)$ is isomorphic to the even Clifford algebra $C_0(q)$. Since D represents $\mathrm{clif}(q)$, it is independent of the choice of the orthogonal decomposition of q in 2.1.

Our goal in this section is to prove the equality $G(q) = \mathrm{Hyp}(q)$ for q of type E_7 or E_8. We start with some general observations. The following is essentially [23, 2.18].

Proposition 2.6. *Suppose that the polynomial $p(x) = x^2 - \alpha x + \beta \in K[x]$ is separable and irreducible over K. Let E be the splitting field of $p(x)$ over K and let N denote the norm of the extension E/K, so that (K, E, N) is a nondegenerate anisotropic 2-dimensional quadratic space. Let (K, L, q) be a finite-dimensional quadratic space. Then the following assertions are equivalent:*

(i) *The K-vector space structure on L extends to an E-vector space structure such that $q(u \cdot v) = N(u)q(v)$ for all $u \in E$, $v \in L$;*
(ii) *There exists a similitude T of q such that $q(T(v)) = \beta q(v)$ and $f(v, T(v)) = \alpha$ for all non-zero $v \in L$ and $p(T) = 0$;*
(iii) *For each $v_1 \in L$ there exists a decomposition $L = V_1 \oplus \cdots \oplus V_d$ for some $d \in \mathbb{N}$ such that $v_1 \in V_1$, the restriction q_i of q to V_i is similar to N for each $i \in [1, d]$ and $q = q_1 \perp \cdots \perp q_d$;*
(iv) *for some $d \in \mathbb{N}$ and some $\alpha_1, \alpha_2, \ldots, \alpha_d \in K^\times$,*

$$(K, L, q) \cong (K, E^d, \alpha_1 N \perp \alpha_2 N \perp \cdots \perp \alpha_d N);$$

(v) *q_E is hyperbolic.*

Proof. Suppose that (i) holds, choose a root $\gamma \in E$ of $p(x)$ and let $T(v) = \gamma \cdot v$ for all $v \in L$. Then T is a similitude of q as in (ii). If T is a similitude of q as in (ii), then for each nonzero $v \in L$ the restriction of q to $\langle v, T(v)\rangle$ is similar to N (and, in particular, is nondegenerate). Therefore (iii) holds, and (iii) of course implies (iv). Fixing an isomorphism as in (iv), we may transfer to L the natural E-vector space structure on E^d to obtain (i). The equivalence of (iv) and (v) follows readily from [6, Prop. 34.8]. □

Definition 2.7. A similitude φ of a quadratic space (K, L, q) is called *inseparable* if $\mathrm{char}(K) = 2$, the multiplier of φ is not in $K^{\times 2}$ and

$$f(v, \varphi(v)) = 0 \qquad \text{for all } v \in L,$$

where $f = \partial q$. We call a similitude of q *separable* if it is not inseparable. Thus, if $\mathrm{char}(K) \neq 2$ all similitudes are separable.

Proposition 2.8. *Let (K, L, q) be a finite-dimensional quadratic space such that $f = \partial q$ is nondegenerate. If (K, L, q) admits an inseparable similitude with multiplier γ, then*

$$q \simeq \langle 1, \gamma\rangle \cdot q_0$$

for some non-degenerate quadratic form q_0. In particular, $\dim L \equiv 0 \bmod 4$ and $q_{K(\sqrt{\gamma})}$ is hyperbolic.

Proof. Let $E = K(\sqrt{\gamma})$ be a purely inseparable quadratic extension of K, and let φ be an inseparable similitude of (K, L, q) with multiplier γ. Linearizing the condition $f\big(v, \varphi(v)\big) = 0$, we obtain

$$f\big(v, \varphi(w)\big) = f(\varphi(v), w) \qquad \text{for all } v,\ w \in L.$$

Since $f\big(\varphi(v), \varphi(w)\big) = \gamma f(v, w)$ for all v, $w \in L$, it follows that

$$f\big(v, \varphi^2(w)\big) = f\big(\varphi(v), \varphi(w)\big) = \gamma f(v, w) \qquad \text{for all } v,\ w \in L,$$

hence $\varphi^2(w) = \gamma w$ for all $w \in L$. We then define on L an E-vector space structure by

$$(\lambda + \mu\sqrt{\gamma}) \cdot v = \lambda v + \mu\varphi(v) \qquad \text{for } \lambda,\ \mu \in K \text{ and } v \in L,$$

and we define a map $f' \colon L \times L \to E$ by

$$f'(v,w) = f(v,w) + \sqrt{\gamma^{-1}} f\big(v, \varphi(w)\big) \qquad \text{for } v,\, w \in L.$$

A straightforward computation shows that f' is a bilinear alternating form on L. It is nondegenerate since f is nondegenerate. Therefore, the dimension of L over E is even, hence its dimension over K is a multiple of 4. Let $(e_i, e_i')_{i=1}^d$ be a symplectic E-base of L for f'. If $L_0 \subset L$ is the K-span of $(e_i, e_i')_{i=1}^d$ and q_0 is the restriction of q to L_0, we have $L = L_0 \perp \varphi(L_0)$ and $q = q_0 \perp \langle\gamma\rangle q_0$. □

Corollary 2.9. *Let (K, L, q) be a finite-dimensional quadratic space such that ∂q is non-degenerate. Then the multiplier of every inseparable similitude of q is in* $\mathrm{Hyp}_2(q)$.

Proof. Let γ be the multiplier of an inseparable similitude of q. Clearly, $\gamma \in N\big(K(\sqrt{\gamma})\big)$, and Proposition 2.8 shows that q is hyperbolic over $K(\sqrt{\gamma})$. □

We now consider quadratic forms of low dimension. The following result is presumably well-known:

Proposition 2.10. *Every 10-dimensional quadratic form in $I_q^3(K)$ is isotropic.*

Proof. This was proved by Pfister [16, p. 123] under the hypothesis that $\mathrm{char}(K) \neq 2$. The arguments also apply when $\mathrm{char}(K) = 2$; see [5, Thm. 4.10]. □

We now consider quadratic spaces of type E_7. For the next statement, we do not require the form to be anisotropic.

Lemma 2.11. *Let (K, L, q) be a nondegenerate quadratic space of dimension 8. If $\mathrm{disc}(q)$ is trivial and $\mathrm{clif}(q)$ is represented by a quaternion algebra Q with norm form N_Q, then q is Witt-equivalent to the sum of a multiple of N_Q and a multiple of some 3-fold Pfister quadratic form π: there exist α, $\beta \in K^\times$ such that*

$$q = \langle\alpha\rangle \cdot N_Q + \langle\beta\rangle \cdot \pi \qquad \text{in } I_qK. \tag{2.12}$$

Moreover, $G(q) = G(N_Q) \cap G(\pi)$.

Proof. Let $\alpha \in K^\times$ be a value represented by q. Consider the form

$$q' = q \perp \langle -\alpha\rangle \cdot N_Q.$$

This 12-dimensional form is isotropic and has trivial discriminant and Clifford invariant, hence it is in I_q^3K and is Witt-equivalent to a 10-dimensional form. By Proposition 2.10, it is actually equivalent to an 8-dimensional form. By the Arason–Pfister Hauptsatz [6, Thm. 23.7], this 8-dimensional quadratic form becomes hyperbolic over the function field of the corresponding quadric, hence it is a multiple of some 3-fold Pfister quadratic form π by [6, Cor. 23.4]. Letting $q' = \langle\beta\rangle \cdot \pi$ in I_qK, we have (2.12).

Now, for $\gamma \in G(q)$ we have $\langle 1, -\gamma\rangle \cdot q = 0$ in I_qK, hence

$$\langle 1, -\gamma\rangle \cdot \langle\alpha\rangle \cdot N_Q = -\langle 1, -\gamma\rangle \cdot \langle\beta\rangle \cdot \pi \qquad \text{in } I_qK.$$

Since the left side is a form of dimension 8 and the right side is a form of dimension 16, the right side must be isotropic. It is then hyperbolic by [6, Cor. 9.10], since it is a multiple of a Pfister form. The left side is then also hyperbolic, which means that γ is in $G(\pi)$ and in $G(N_Q)$. We have thus proved $G(q) \subset G(N_Q) \cap G(\pi)$. Since the reverse inclusion is clear, the proof is complete. □

Lemmas 2.13 and 2.14 are well-known when $\operatorname{char}(K) \neq 2$; see [7, 2.13] for Lemma 2.13. The proofs we give below do not require any separability hypothesis.

Lemma 2.13. *Let E_1, E_2 be linearly disjoint quadratic extensions of a field K, and let $M = E_1 \otimes_K E_2$. The norm groups of E_1, E_2, and M are related as follows:*

$$N(E_1/K) \cap N(E_2/K) = K^{\times 2} \cdot N(M/K).$$

Proof. Since $K^{\times 2} \subset N(E_i/K)$ and $N(M/K) \subset N(E_i/K)$ for $i = 1, 2$, the inclusion $N(E_1/K) \cap N(E_2/K) \supset K^{\times 2} \cdot N(M/K)$ is clear, and it suffices to prove the reverse inclusion. We identify E_1 and E_2 with subfields of M and consider $\alpha \in N(E_1/K) \cap N(E_2/K)$. Let $x_1 \in E_1^\times$, $x_2 \in E_2^\times$ be such that

$$\alpha = N_{E_1/K}(x_1) = N_{E_2/K}(x_2).$$

Let $T_{E_i/K} \colon E_i \to K$ be the trace map, for $i = 1, 2$. Computation shows that

$$x_1 N_{M/E_1}(1 + x_1^{-1} x_2) = T_{E_1/K}(x_1) + T_{E_2/K}(x_2).$$

If $x_1 \neq -x_2$, the left side is nonzero. Taking the norm from E_1 to K of each side yields

$$\alpha N_{M/K}(1 + x_1^{-1} x_2) = \big(T_{E_1/K}(x_1) + T_{E_2/K}(x_2)\big)^2 \in K^{\times 2},$$

hence $\alpha \in K^{\times 2} \cdot N(M/K)$. If $x_1 = -x_2$, then $x_1 \in E_1 \cap E_2 = K$, hence $\alpha \in K^{\times 2}$. □

Lemma 2.14. *Any multiplier of similitude of an anisotropic quadratic Pfister space (K, L, π) is a square in K or is the norm of a quadratic extension over which π is hyperbolic.*

Proof. By [6, Cor. 9.9], the multipliers of π are the represented values of π, so any $\gamma \in G(\pi)$ has the form $\gamma = \pi(v)$ for some $v \in L$. Let $e \in L$ be such that $\pi(e) = 1$. If e and v are not linearly independent, then $\gamma \in K^{\times 2}$. Otherwise, let V be the K-span of e and v. The restriction of π to V is the norm form of a quadratic extension of K over which π is isotropic, hence hyperbolic by [6, Cor. 9.10]. By construction, this norm form represents γ. □

Proposition 2.15. *For any nondegenerate quadratic space (K, L, q) of dimension 8 such that $\operatorname{disc}(q)$ is trivial and $\operatorname{clif}(q)$ has index 1 or 2, we have $G(q) = \operatorname{Hyp}(q)$.*

Proof. It suffices to show $G(q) \subset \operatorname{Hyp}(q)$, since the reverse inclusion follows from the similarity norm principle [6, Thm. 20.14]. Let $\gamma \in G(q)$, and consider a decomposition of q as in (2.12). We may assume q is not hyperbolic, otherwise $\operatorname{Hyp}(q) = K^\times = G(q)$ and there is nothing to prove. If N_Q or π is isotropic, hence hyperbolic, the proposition readily follows from Lemma 2.14. For the rest of the proof, we may thus assume N_Q and π are anisotropic. By Lemma 2.11 we have $\gamma \in G(N_Q) \cap G(\pi)$, hence Lemma 2.14 yields quadratic extensions E_1, E_2 of K that split N_Q and π respectively, such that $\gamma \in N(E_1/K) \cap N(E_2/K)$. If $E_1 \cong E_2$, then E_1 splits N_Q and π, hence also q. Since $\gamma \in N(E_1/K)$, it follows that $\gamma \in \operatorname{Hyp}(q)$. If $E_1 \not\cong E_2$, then E_1 and E_2 are linearly disjoint over K, and the tensor product $M = E_1 \otimes_K E_2$ is a field that splits N_Q and π, hence also q. Since γ is a norm from E_1 and from E_2, Lemma 2.13 shows that $\gamma \in K^{\times 2} \cdot N(M/K)$, hence $\gamma \in \operatorname{Hyp}(q)$. □

Proposition 2.15 applies in particular to quadratic spaces of type E_7. We now turn to spaces of type E_8.

Proposition 2.16. *Suppose that (K, L, q) is of type E_8, that γ is the multiplier of a separable similitude of q and that $\gamma \notin K^{\times 2}$. Then there exists a decomposition*

$$(K, L, q) = (K, L_1, q_1) \perp (K, L_2, q_2)$$

such that q_2 is of type E_7, q_1 is similar to the reduced norm of the quaternion division algebra representing $\operatorname{clif}(q_2)$ *and γ is a multiplier of similitudes of both q_1 and q_2.*

Proof. Let φ be a separable similitude of q with multiplier γ. If $\operatorname{char}(K) = 2$, we choose $v \in L$ such that $f(v, \varphi(v)) \neq 0$; if $\operatorname{char}(K) \neq 2$, we let v be an arbitrary non-zero vector in L. Next we set $W = \langle v, \varphi(v)\rangle$. Since $\gamma \notin K^{\times 2}$, we have $\dim_K W = 2$. Let $\hat{q}_1$ denote the restriction of q to W and let $\hat{q}_2$ denote the restriction of q to $W^\perp$. The form $\hat{q}_1$ is similar to the norm N of a quadratic extension E/K such that $\gamma \in N(E)$. Since $\hat{q}_1$ is nondegenerate, the extension E/K is separable and $q = \hat{q}_1 \perp \hat{q}_2$. Since q_E and $(\hat{q}_1)_E$ have trivial discriminant and split Clifford invariant, also $(\hat{q}_2)_E$ has trivial discriminant and split Clifford invariant. By Proposition 2.10, it follows that $(\hat{q}_2)_E$ is isotropic. Hence we can choose a 2-dimensional subspace U of $W^\perp$ such that the restriction of $\hat{q}_2$ to U is hyperbolic over E. By Proposition 2.6, the restriction of $\hat{q}_2$ to U is similar to N. Let $L_1 = W \oplus U$, let $L_2 = L_1^\perp$ and let q_i denote the restriction of q to L_i for $i = 1$ and 2. Then q_1 is similar to the reduced norm of a quaternion division algebra D, γ is a multiplier of q_1 and

$$(K, L, q) = (K, L_1, q_1) \perp (K, L_2, q_2).$$

Since γ is a multiplier of both q and q_1, both $\langle 1, -\gamma\rangle \cdot q$ and $\langle 1, -\gamma\rangle \cdot q_1$ are hyperbolic. Since

$$\langle 1, -\gamma\rangle \cdot q = \langle 1, -\gamma\rangle \cdot q_1 \perp \langle 1, -\gamma\rangle \cdot q_2,$$

it follows by Witt's Cancellation Theorem that the product $\langle 1, -\gamma\rangle \cdot q_2$ is also hyperbolic. Hence $q_2 = \langle \gamma\rangle \cdot q_2$ in $I_q K$. By [6, 8.17], therefore, there is a similitude of q_2 with multiplier γ. Since $\operatorname{disc}(q)$ and $\operatorname{disc}(q_1)$ are both trivial, so is $\operatorname{disc}(q_2)$. Furthermore, $\operatorname{clif}(q_2) = \operatorname{clif}(q_1)$ since $\operatorname{clif}(q)$ is split. Since the quaternion division algebra D represents $\operatorname{clif}(q_1)$, it also represents $\operatorname{clif}(q_2)$. We conclude, in particular, that q_2 is of type E_7. □

Corollary 2.17. *For any nondegenerate quadratic space (K, L, q) of dimension* 12 *such that* $\operatorname{disc}(q)$ *and* $\operatorname{clif}(q)$ *are trivial, we have* $G(q) = \operatorname{Hyp}(q)$.

Proof. As in Proposition 2.15, it suffices to prove $G(q) \subset \operatorname{Hyp}(q)$. If q is isotropic, then Proposition 2.10 shows that q is Witt-equivalent to a multiple of a 3-fold Pfister form, hence the inclusion follows from Lemma 2.14. For the rest of the proof, we may thus assume q is anisotropic, i.e., q is of type E_8.

Let $\gamma \in G(q)$. If γ is the multiplier of an inseparable similitude, then we have $\gamma \in \operatorname{Hyp}(q)$ by Corollary 2.9. If γ is the multiplier of a separable similitude, we fix a decomposition $q = q_1 \perp q_2$ as in Proposition 2.16, so $\gamma \in G(q_1) \cap G(q_2)$. By Proposition 2.15 we have $G(q_2) = \operatorname{Hyp}(q_2)$. Now, if E/K is a finite extension such that $(q_2)_E$ is hyperbolic, then E splits $\operatorname{clif}(q_2)$. Hence $(q_1)_E$ is hyperbolic, and therefore q_E is hyperbolic. This shows $\operatorname{Hyp}(q_2) \subset \operatorname{Hyp}(q)$. Since $\gamma \in \operatorname{Hyp}(q_2)$, it follows that $\gamma \in \operatorname{Hyp}(q)$. □

Remark 2.18. Restricting to *quadratic* extensions that split q_2 in the last part of the proof above, we see that $\mathrm{Hyp}_2(q_2) \subset \mathrm{Hyp}_2(q)$. This observation will be used in §5.

When $\mathrm{char}(K) \neq 2$, Proposition 2.15 and Corollary 2.17 yield information on the connected component of the identity $\mathrm{PGO}_+(q)$ in the group of projective similitudes of (K, L, q), which is the group of algebra automorphisms of $\mathrm{End}(L)$ that commute with the adjoint involution of q. The property of R-triviality used in the following statement refers to Manin's R-equivalence; see [14, §1] for details.

Corollary 2.19. *Assume* $\mathrm{char}(K) \neq 2$. *For q a quadratic form of type E_7 or E_8 over K, the group* $\mathrm{PGO}_+(q)$ *is R-trivial.*

Proof. By [14, Thm. 1], it suffices to prove that $G(q_E) = \mathrm{Hyp}(q_E)$ for every field E containing K. If q is of type E_7, this property readily follows from Proposition 2.15. If it is of type E_8, it follows from Corollary 2.17. □

Remark 2.20. Skip Garibaldi has observed that if q is of type E_7, then it follows from Lemma 2.11 and [8, Prop. 6.1] that the group $\mathrm{PGO}_+(q)$ is actually stably rational.

3. Quadrangular algebras and proofs of Theorems 1.1 and 1.2

Most of this section is devoted to results about quadrangular algebras. At the very end of this section, we use these results to prove Theorems 1.1 and 1.2.

The notion of a quadrangular algebra arose in the course of the classification of Moufang polygons; see, in particular, Chapters 12–13 and 27 in [22]. For the definition, see [23, 1.17].

Proposition 3.1. *Let (K, L, q) be a quadratic space of type E_7 or E_8 as defined in 2.1 and 2.2 and suppose that $q(1) = 1$ for a distinguished element 1 of L. Then there exists a unique quadrangular algebra*

$$\Xi = (K, L, q, 1, X, \cdot, h, \theta)$$

as defined in [23, 1.17].

Proof. Existence holds by [23, Thm. 10.1] and uniqueness (up to equivalence as defined in [23, Thm. 1.22]) holds by [23, 6.42]. □

Notation 3.2. For the rest of this section, we let

$$\Xi = (K, L, q, 1, X, \cdot, h, \theta)$$

be as in 3.1 and $f = \partial q$. By [23, Prop. 4.2], we can assume that Ξ is δ-standard for some $\delta \in L$ as defined in [23, 4.1]. (This allows us to use the identities in Chapter 4 of [23].) In addition, we let σ be as [23, 1.2], we let u^{-1} for all non-zero $u \in L$ be as in [23, 1.3] and we let π be as in [23, 1.17(D1)].

Remark 3.3. Suppose that (K, L, q) is of type E_7 and let $C(q, 1)$ and D be as in 2.4. By (2.5) and [23, 1.17(A1)-(A3) and Prop. 2.22], there exists a unique map $*$ from $D \times X$ to X with respect to which X is a left vector space over D, $ta = t * a$ for all $(a, t) \in X \times K$ and $w * (a \cdot v) = (w * a) \cdot v$ for all $w \in D$, $a \in X$ and $v \in L$. This map is given explicitly in [24, 3.6].

Proposition 3.4. *Suppose that (K, L, q) is of type E_7, let D and $*$ be as in 3.3, let φ_1 be a similitude of q, let $u = \varphi_1(1)$ and let*

$$a \mathbin{\hat{\cdot}} v = (a \cdot v) \cdot u^{-1}$$

for all $(a, v) \in X \times L$, where u^{-1} is as in 3.2. Then there exists a similitude φ with the same multiplier as φ_1 such that $u = \varphi(1)$, an element $\omega \in K^\times$ and a D-linear automorphism ψ of X such that the following hold:

(i) *$\psi(a \cdot v) = \psi(a) \mathbin{\hat{\cdot}} \varphi(v)$ for all $a \in X$ and all $v \in L$.*
(ii) *$\varphi(h(a, b)) = \omega h(\psi(a), \psi(b)u)$ for all $a, b \in X$.*
(iii) *$\varphi(\theta(a, v)) \equiv \omega\theta(\psi(a), \varphi(v)) \pmod{\langle \varphi(v) \rangle}$ for all $a \in X$ and all $v \in L$.*

Proof. The map φ_1 is an isomorphism of pointed quadratic spaces from $(K, L, q, 1)$ to $(K, L, q/q(u), u)$. It therefore induces an isomorphism of Clifford algebras with base point from $C(q, 1)$ to $C(q/q(u), u)$. Let $\hat{\Xi}$, $\hat{h}$ and $\hat{\theta}$ be as in [23, Prop. 8.1]; thus, $\hat{\Xi} = (K, L, q/q(u), u, X, \hat{\cdot}, \hat{h}, \hat{\theta})$ is the isotope of Ξ at u as defined in [23, 8.7]. By [23, 1.17(A1)–(A3) and Prop. 2.22] applied to both Ξ and to $\hat{\Xi}$, X is a right $C(q, 1)$-module with respect to $\cdot$ and a right $C(q/q(u), u)$-module with respect to $\hat{\cdot}$. By [22, 12.55] (where the base point 1 is called ϵ), exactly one of the two direct summands in (2.5) acts nontrivially on X, and by [22, 12.54], there exists an isometry ρ of q fixing 1 that extends to an automorphism of $C(q, 1)$ interchanging the two direct summands. Thus for $j = 0$ or 1, the composition $\varphi_1 \circ \rho^j$ maps the direct summand A of $C(q, 1)$ acting nontrivially on X to the direct summand A_u of $C(q/q(u), u)$ acting nontrivially on X. Let $\varphi = \varphi_1 \circ \rho^j$. Choosing a basis for X as a left vector space over D, we can identify both A and A_u with $\mathrm{End}_D(X)$. It follows that there exists a D-linear automorphism ψ of X such that (i) holds. By [23, 1.25 and Prop. 6.38], there exists $\omega \in K^\times$ such that also (ii) and (iii) hold. □

Notation 3.5. Let g and ϕ be the maps that appear in [23, 1.17(C3)-(C4)], let

$$(U_+, U_1, U_2, U_3, U_4)$$

be the root group sequence and x_4 the isomorphism from L to U_4 obtained by applying the recipe in [22, 16.6] to Ξ, g and ϕ, let Γ be the corresponding Moufang quadrangle (see [22, 8.11]), let $G^\dagger$ be the subgroup of $\mathrm{Aut}(\Gamma)$ generated by the root groups of Γ, let H_0 be the subgroup of $\mathrm{Aut}(\Gamma)$ defined in [24, 1.4] (or [23, 11.20]), let $G = H_0 \cdot G^\dagger$ and let $H^\dagger = H_0 \cap G^\dagger$. By [22, 35.11], the similarity class of (K, L, q) is an invariant of Γ.

Proposition 3.6. *$G/G^\dagger \cong H_0/H^\dagger$ and if (K, L, q) is of type E_8, then G is the group of K-rational points of an absolutely simple algebraic group with Tits index $E_{8,2}^{66}$, and every such group arises in this way starting with some quadratic space of type E_8 defined over K.*

Proof. For the isomorphism $G/G^\dagger \cong H_0/H^\dagger$, see the top of page 193 of [24] (where G is called G_0), The remaining assertions hold by [22, 42.6]. □

Proposition 3.7. *Suppose that (K, L, q) is of type E_7 and that φ_1 is a similitude of q. Let H_0 and x_4 be as in 3.5. Then there exist an element h of H_0 and a similitude φ of q with the same multiplier as φ_1 such that*

$$x_4(v)^h = x_4(\varphi(v))$$

for all $v \in L$.

Proof. Let φ and ψ be the maps obtained by applying Proposition 3.4 to φ_1. By Proposition 3.4 and [23, Prop. 12.5], the pair (φ, ψ) is contained in the structure group of Ξ (as defined in [23, 12.4]). The claim follows now by [23, Thm. 12.11] and the first few lines of its proof (as well as [23, 11.22]). □

From now on we identify K with its image under the map $t \mapsto t \cdot 1$ from K to L. Thus when we write $\pi(a) + t$ for $(a, t) \in X \times K$, for example, we mean $\pi(a) + t \cdot 1$ (where π is as in 3.2).

Proposition 3.8. *Let a be a non-zero element of X, let*

$$p(x) = x^2 - f(1, \pi(a))x + q(\pi(a)) \in K[x],$$

let E be the splitting field of $p(x)$ over K and let N be the norm of the extension E/K. Let $T(v) = \theta(a, v)$ for all $v \in L$ and let I be the identity automorphism of L. Then $N(E^\times) = K^{\times 2} \cdot \{q(\pi(a) + t) \mid t \in K\}$, q_E is hyperbolic and for each $t \in K$, $T + tI$ is a similitude of q with multiplier $q(\pi(a) + t)$.

Proof. By [23, 1.17(D2)], $p(t) = q(\pi(a) - t) \neq 0$ for each $t \in K$. Thus $p(x)$ is irreducible over K. It follows that $N(E^\times) = K^{\times 2} \cdot \{q(\pi(a) + t) \mid t \in K\}$. By [23, Props. 4.9(i) and 4.22], $T + tI$ is a similitude of q with multiplier $q(\pi(a)+t)$ for each $t \in K$ and $f(T(v), v) = f(\pi(a), 1)q(v)$ for each non-zero $v \in L$. By [23, Prop. 4.21], $p(T) = 0$. Thus if $p(x)$ is separable, then q_E is hyperbolic by Propositions 2.6. If $p(x)$ is inseparable, then $f(\pi(a), 1) = 0$, hence T is inseparable and again q_E is hyperbolic, this time by Proposition 2.8. □

Definition 3.9. For each non-zero $a \in X$, the map $v \mapsto \theta(a, v)$ is a similitude of q by Proposition 3.8. We call an element $a \in X$ *separable* if $a \neq 0$ and the similitude $v \mapsto \theta(a, v)$ of q is separable as defined in 2.7. We let $X_{\rm sep}$ denote the set of separable elements of X. Thus if $\mathrm{char}(K) \neq 2$, then $X_{\rm sep} = X \backslash \{0\}$, but if $\mathrm{char}(K) = 2$, then by [23, Prop. 4.9(i)],

$$X_{\rm sep} = \{a \in X \mid f(\pi(a), 1) \neq 0\}.$$

If $\mathrm{char}(K) = 2$, then by [22, 13.42–13.43], $a \mapsto f(\pi(a), 1)$ is a nondegenerate quadratic form on X. In particular, the set $X_{\rm sep}$ is non-empty also if $\mathrm{char}(K) = 2$.

Proposition 3.10. *Every inseparable similitude of q (as defined in 2.7) is the product of two separable similitudes.*

Proof. Let φ be an inseparable similitude of q, so $\mathrm{char}(K) = 2$. It suffices to show that

$$f\big(\theta(a, \varphi(v)), v\big) \neq 0$$

for some $v \in L$ and some $a \in X_{\rm sep}$, where $X_{\rm sep}$ is as in 3.9. Suppose this is false and let $w = \varphi(1)$. Then

$$f\big(\theta(a, w), 1\big) = 0 \tag{3.11}$$

for all $a \in X_{\rm sep}$. Furthermore,

$$f(w, 1) = 0 \tag{3.12}$$

but $w \notin \langle 1 \rangle$ since φ is inseparable. Choose $a \in X_{\rm sep}$. Since f is nondegenerate, we can choose $v \in \langle w \rangle^\perp \backslash \langle 1 \rangle^\perp$. Replacing v by $v + w$ if necessary, we can assume in addition (by [23, Prop. 4.9(i)] again) that

$$f(\theta(a, w), v) \neq 0. \tag{3.13}$$

By [23, Prop. 3.21], $av \in X_{\rm sep}$, so $f\big(\theta(av,w),1)\big) = 0$ by (3.11). By [23, 1.17(C4)] and (3.12), it follows that

$$f(\theta(a,w^\sigma)^\sigma,1)q(v) = f(w,v^\sigma)f(\theta(a,v)^\sigma,1) + f(\theta(a,v),w^\sigma)f(v^\sigma,1),$$

where σ is as in 3.2. By [23, 1.4] and (3.12), we have $x^\sigma = x$ for $x = 1$ and $x = w$ and $f(x^\sigma,y) = f(x,y^\sigma)$ for all $x,y \in L$. Therefore

$$f(\theta(a,w^\sigma)^\sigma,1) = f(\theta(a,w),1) = 0$$

by (3.11) and

$$f(w,v^\sigma) = f(w^\sigma,v) = f(w,v) = 0$$

by the choice of v and hence

$$f(\theta(a,v),w)f(v,1) = f(\theta(a,v),w^\sigma)f(v^\sigma,1) = 0.$$

Since $f(v,1) \neq 0$ by the choice of v, we conclude that $f(\theta(a,v),w) = 0$. By [23, Prop. 4.22], therefore, $f\big(\theta(a,\theta(a,v)),\theta(a,w)\big) = 0$. By [23, Prop. 4.21], it follows that

$$f(\pi(a),1)f\big(\theta(a,v),\theta(a,w)\big) = q(\pi(a))f\big(v,\theta(a,w)\big).$$

By (3.13), therefore, $f\big(\theta(a,v),\theta(a,w)\big) \neq 0$. By one more application of [23, Prop. 4.22], however, $f\big(\theta(a,v),\theta(a,w)\big) = q(\pi(a))f(v,w) = 0$. □

Proposition 3.14. *Let E/K be a separable quadratic extension such that q_E is hyperbolic and let V_i and q_i for $i \in [1,d]$ be as in Proposition 2.6(iii) with $v_1 = 1$. Then there exists $e \in X_{\rm sep}$ such that $\theta(e,V_i) = V_i$ for each $i \in [1,d]$.*

Proof. Let $p(x) = x^2 - \alpha x + \beta \in K[x]$ be an irreducible polynomial that splits over E. We can choose $p(x)$ so that $\alpha = 0$ if and only if $\mathrm{char}(K) \neq 2$. Let $\gamma, \gamma_1 \in E$ be the two roots of $p(x)$. There exists an E-vector space structure on L as in Proposition 2.6(i) such that V_i is a 1-dimensional subspace for each $i \in [1,d]$. Let $T(v) = \gamma \cdot v$ for each $v \in L$ and let T^ϵ be the unique automorphism of L such that $T^\epsilon(v) = \gamma_1 \cdot v$ for all $v \in V_1$ and $T^\epsilon(v) = T(v)$ for all $v \in V_1^\perp$. Both T and T^ϵ are norm splitting maps of q as defined in [22, 12.14] and both map V_i to itself for each $i \in [1,d]$. By [22, 12.20 and 13.13(ii)], therefore, we can choose $R \in \{T, T^\epsilon\}$ such that R is linked to the map $(a,v) \mapsto a \cdot v$ at some point $e \in X$ as defined in [22, 13.2]. By [22, 13.61], $e \in X_{\rm sep}$ and there exists $r \in K^\times$ and $s \in K$ such that $R(v) = r\theta(e,v) + sv$ for all $v \in L$. □

Proposition 3.15. *Suppose that (K,L,q) is of type E_8 and that*

$$(K,L,q) = (K,L_1,q_1) \perp (K,L_2,q_2)$$

with q_2 of type E_7, q_1 similar to the reduced norm of the quaternion division algebra representing $\mathrm{clif}(q_2)$ *and $1 \in L_2$. Then the following hold:*

(i) *There exists $e \in X_{\rm sep}$ such that $\theta(e,L_i) = L_i$ for $i = 1$ and 2.*

(ii) *Let $e \in X$ be as in (i), let X_e be the subspace of X generated by elements of the form $ev_1v_2\cdots v_j$, where $v_i \in L_2$ for $i \in [1,j]$ and $j \geq 1$ is arbitrary, let $\cdot_e$, h_e, respectively, θ_e denote the restriction of $\cdot$, h, respectively, θ to $X_e \times L_2$, $X_e \times X_e$, respectively, $X_e \times L_2$ and let*

$$\Xi_e = (K,L_2,q_2,1,X_e,\cdot_e,h_e,\theta_e).$$

Then Ξ_e is a quadrangular algebra.

Proof. By 2.2, 2.4 and Proposition 2.6, both $(q_1)_E$ and $(q_2)_E$ are hyperbolic. We can thus choose V_i and q_i for $i \in [1, d]$ as in Proposition 2.6(iii) with $v_1 = 1$, $V_i \subset L_2$ for $i \in [1, 4]$ and $V_i \subset L_1$ for $i \in [5, 6]$. By Proposition 3.14, therefore, there exists $e \in X_{\text{sep}}$ such that $\theta(e, L_i) = L_i$ for $i = 1$ and 2. Thus (i) holds.

Let Ξ_e be as described in (ii). To show that Ξ_e is a quadrangular algebra, it therefore suffices to show that $X_e \cdot L_2 \subset X_e$, $\theta(X_e, L_2) \subset L_2$ and $h(X_e, X_e) \subset L_2$. The first of these inclusions holds by the definition of X_e. To show the other two inclusions, we first choose non-zero elements $v_i \in V_i$ for $i \in [2, 5]$. We can assume that e is the element of X chosen in [23, 6.4]. Thus the set $1, v_2, \ldots, v_5$ is e-orthogonal as defined in [23, 6.6]. By [23, 1.17(A3) and Prop. 6.16], there exists a non-zero $v_6 \in L$ such that $1, v_2, \ldots, v_5, v_6$ is e-orthogonal and $ev_2v_3v_4v_5v_6 = e$. (We are not claiming that $v_6 \in V_6$ or even $v_6 \in L_1$.) Let I_2 be as in [23, 6.32], let J denote subset of I_2 containing all the elements of I_2 that are subsets of $\{v_2, v_3, v_4\}$ together with the element $\{v_5, v_6\} \in I_2$ (so $|J| = 8$), let J_2 be the elements of J of cardinality 2 (so $|J_2| = 4$), let X_x for each $x \in J$ be as in [23, 6.35], let M be the subspace of X spanned by $\{X_m \mid m \in J\}$ and let N be the subspace of M spanned by $\{X_m \mid m \in J_2\}$. By [23, Prop. 6.34], $\dim_K M = 16$ and $M = eL_2 \oplus N$. By [23, 6.37], we have $M = X_e$, by [23, Prop. 6.13], we have $h(e, N) = 0$ and by [23, Props. 3.15 and 4.5(i)], $h(e, eL_2) \subset L_2$ (since $\theta(e, L_2) \subset L_2$). Hence $h(e, X_e) \subset L_2$. By repeated application of [23, 1.17(B1)–(B2)], it follows that $h(X_e, X_e) \subset L_2$. Since $1 \in L_2$, we have $L_2^\sigma \subset L_2$, where σ is as in 3.2. By repeated application of [23, 1.17(C3)–(C4)], it follows from $\theta(e, L_2) \subset L_2$ first that $\theta(eL_2, L_2) \subset L_2$ and then that $\theta(X_e, L_2) \subset L_2$. Thus (ii) holds. □

Definition 3.16. For each non-zero u in L, let π_u be the reflection of q given by

$$\pi_u(v) = f(u, v)u/q(u) - v$$

for all $v \in L$. Thus $\pi_1 = \sigma$, where σ is as in 3.2.

Proposition 3.17. *Let $H^\dagger$ and x_4 be as in 3.5. Suppose that φ is a product of an even number of reflections of q as defined in 3.16. Then there exists an element $h \in H^\dagger$ such that*

$$x_4(v)^h = x_4(\varphi(v))$$

for all v.

Proof. Let u be a non-zero element of L. By [24, eq. (6)–(14)], there are elements $w_1(0, q(u))$ and $w_4(u)$ in $H^\dagger$ such that

$$x_4(v)^{w_1(0,q(u))w_4(u)} = x_4(v/q(u))^{w_4(u)} = x_4(\pi_u\pi_1(v))$$

for each $v \in L$. □

Notation 3.18. Let M denote the subgroup of $K^\times$ generated by the non-zero elements in the set $\{q(\pi(a) + t) \mid (a, t) \in X \times K\}$.

Thus

$$M \subset G(q) \cap \text{Hyp}_2(q) \tag{3.19}$$

by Proposition 3.8 and $K^{\times 2} = \{q(\pi(a) + t) \mid (a, t) \in \{0\} \times K^\times\} \subset M$.

Proposition 3.20. *Let $H^\dagger$ and x_4 be as in 3.5. For each $h \in H^\dagger$, there exists a unique similitude φ_h of q such that*

$$x_4(v)^h = x_4(\varphi_h(v))$$

for all $v \in L$. Furthermore, the map $h \mapsto \gamma_h$ is a surjective homomorphism from $H^\dagger$ to M, where γ_h is the multiplier of φ_h.

Proof. Let $w_1(a,t)$ for non-zero $(a,t) \in X \times K$ and $w_4(u)$ for non-zero $u \in L$ be as in [24, eqs. (6)–(7)]. Then

$$x_4(v)^{w_4(u)} = x_4\big(uf(u, v^\sigma) - q(u)v^\sigma\big)$$

and

$$x_4(v)^{w_4(a,t)} = x_4\big((\theta(a,v) + tv)/q(\pi(a)+t)\big)$$

for all $v \in L$ and all non-zero $(a,t) \in X \times K$ by [24, eqs. (13)–(14)]. We have

$$q\big((\theta(a,v) + tv)/q(\pi(a)+t)\big) = q(v)/q(\pi(a)+t)$$

for all $v \in L$ and all non-zero $(a,t) \in X \times K$ (by 3.8) and

$$q\big(uf(u,v^\sigma) - q(u)v^\sigma\big) = q(v)q(u)^2$$

for all $u, v \in L$ since $q(v^\sigma) = q(v)$. The claim holds, therefore, by [24, Thm. 2.1]. □

Proposition 3.21. *If (K, L, q) is of type E_7, then $G(q) = M$.*

Proof. By (3.19), it suffices to show that $G(q) \subset M$. Let φ_1 be a similitude of q, let x_4, U_4, H_0 and $H^\dagger \subset H_0$ be as in 3.5 and let h and φ be as in Proposition 3.7. Thus

$$x_4(v)^h = x_4(\varphi(v))$$

for each $v \in L$ and φ is a similitude of q with the same multiplier as φ_1. Let H_1 and H_2 be the subgroups of H_0 defined in [24, 3.12 and 3.14]. By [24, Thm. 3.15(ii)], $H_0 = H_1H_2$ and by [24, Thm. 5.19], $H_2 \subset H_1H^\dagger$. We conclude that $H_0 = H_1H^\dagger$. By [24, Prop. 3.11], H_1 centralizes U_4. There thus exists $g \in H^\dagger$ such $x_4(v)^g = x_4(\varphi(v))$ for each $v \in L$. The claim holds, therefore, by Proposition 3.20. □

Proposition 3.22. *If (K, L, q) is of type E_8, then $G(q) = M$.*

Proof. By (3.19), it suffices to show that $G(q) \subset M$. Let φ be a similitude of q whose multiplier is not in $K^{\times 2}$. By Proposition 3.10, it suffices to assume that φ is separable. Let

$$(K, L, q) = (K, L_1, q_1) \perp (K, L_2, q_2)$$

be the decomposition of q obtained by applying Proposition 2.16 to φ. Replacing Ξ by an isotope as defined in [23, 8.7], we can assume that the base point 1 lies in L_2 (without changing the subgroup generated by the set of non-zero elements in $\{q(\pi(a)+t) \mid (a,t) \in X \times K\}$). We can thus let e and

$$\Xi_e = (K, L_2, q_2, 1, X_e, \cdot_e, h_e, \theta_e)$$

with $X_e \subset X$ be as in Proposition 3.15. By Proposition 3.21 (and the uniqueness assertion in Proposition 3.1), we conclude that γ is the product of elements in $\{q(\pi(a)+t) \mid (a,t) \in X_e \times K\}$. □

We can now prove Theorems 1.1 and 1.2. By Proposition 2.3, a quadratic form satisfying the hypotheses of Theorem 1.1 is of type E_7 or E_8. By the existence assertion in Proposition 3.1, we can apply all the results in this section. Hence $G(q) \subset \mathrm{Hyp}_2(q)$ by (3.19) and Propositions 3.21 and 3.22. By [6, Thm. 20.14], we have $\mathrm{Hyp}_2(q) \subset G(q)$. This concludes the proof of Theorem 1.1.

Suppose that (K, L, q) is of type E_8 and that x_4, H_0 and $H^\dagger$ are as in 3.5. To prove Theorem 1.2, it suffices by Proposition 3.6 (and the existence assertion in Proposition 3.1) to show that every element in H_0 lies in $H^\dagger$. Let $h \in H_0$. By [24, eq. (19)], there is a similitude φ of q such that

$$x_4(v)^h = x_4(\varphi(v))$$

for all $v \in L$. Replacing h by a suitable element in $hH^\dagger$, we can assume, by Propositions 3.20 and 3.22, that φ is an isometry of q and hence a product of reflections of q. Again replacing h by a suitable element of $hH^\dagger$, we can assume, by Proposition 3.17 and [24, Prop. 3.16], that φ is the identity. By [24, Thm. 3.12], $h = \alpha_u$ for some $u \in C^\times$, where $C = K$ by [24, 3.6] and α_u is as defined in [24, Prop. 3.11]. By [24, Prop. 3.13], it follows that $h \in H^\dagger$. This concludes the proof of Theorem 1.2.

4. R-equivalence and an alternative proof of Theorem 1.2

In this section, we give an alternative proof of Theorem 1.2 based on Corollary 2.19 and various other results about R-equivalence. This proof is due to Skip Garibaldi. The methods employed in this section are completely different from those employed in the previous section; in particular, we make no further reference to the Moufang quadrangle Γ of §3.

Let G denote a reductive algebraic group of absolute type E_8 whose Tits index over a field K is $E_{8,2}^{66}$. Our goal is to show that the group of K-rational points of G is generated by its root groups. By [9, 7.3], it suffices to assume that $\mathrm{char}(K) = 0$. This will allow us to apply Corollary 2.19. By [9, 7.2], it suffices to show that G is R-trivial.

Now fix a maximal K-torus T containing a maximal K-split torus S in G and fix a pinning for G with respect to T over an algebraic closure of K. Number the simple roots α_j as in [3, Chapter 6, Plate VII] and let $\omega_i^\vee$ be the corresponding fundamental dominant co-weights, so $\langle \alpha_j, \omega_i^\vee \rangle = \delta_{ij}$. The fundamental co-weights $\omega_1^\vee$ and $\omega_8^\vee$ belong to the co-root lattice and so define cocharacters, in other words, homomorphisms from $\mathbf{G}_m$ to T. Their images generate a subtorus S in T which is the connected component of the intersection (in T) of the kernels of the roots $\alpha_2, \ldots, \alpha_7$. There is a canonical isomorphism

$$\Phi : \bar{K}^\times \otimes_{\mathbb{Z}} T_* \xrightarrow{\sim} T(\bar{K}), \tag{4.1}$$

where T_* is the lattice of cocharacters of T and where $\bar{K}$ is an algebraic closure of K; see [18, 3.2.11]. Since the group G is of adjoint type, the $\omega_i^\vee$ form a $\mathbb{Z}$-basis for T_*. Thus we may view Φ as an isomorphism

$$\prod_{i=1}^{8} \bar{K}^\times \otimes_{\mathbb{Z}} \mathbb{Z}\omega_i^\vee \xrightarrow{\sim} T(\bar{K}).$$

This shows that the intersection of the kernels of the roots $\alpha_2 \ldots, \alpha_7$ is connected and that Φ restricts to an isomorphism

$$\left(\bar{K}^\times \otimes_{\mathbb{Z}} \mathbb{Z}\omega_1^\vee\right) \times \left(\bar{K}^\times \otimes_{\mathbb{Z}} \mathbb{Z}\omega_8^\vee\right) \xrightarrow{\sim} S(\bar{K}). \tag{4.2}$$

The cocharacters $\omega_1^\vee$ and $\omega_8^\vee$ are defined over K by [2, Cor. 6.9], so (4.2) implies that S is K-isomorphic to the direct product of the images of the cocharacters $\omega_1^\vee$ and $\omega_8^\vee$.

We next fix a parabolic P of G whose Levi subgroup is the connected reductive group $Z_G(S)$; see [18, §13.4 and Lemma 15.1.2]. Let U be the unipotent radical of P and let U^- be the unipotent radical of the opposite parabolic. The product $U^- \times U$ is isomorphic as a variety to an affine space. By [1, Proof of Thm. 21.20], the natural map from G to G/P restricts to an isomorphism from U^- to an open subset of G/P. Hence the product map from $U^- \times P$ to G defines an isomorphism from $U^- \times P$ to an open subset of G. It follows that G is birationally equivalent to

$$U^- \times Z_G(S) \times U.$$

(This subvariety is the analog of the big cell for the Bruhat decomposition of G over K; see [2, Prop. 4.10(d)].) We conclude that G is birationally equivalent to the product of $Z_G(S)$ and an affine space.

Let H denote the derived subgroup of $Z_G(S)$. The sequence

$$1 \to S \to Z_G(S) \to H/(H \cap S) \to 1$$

is exact on L-points for every extension L/K because S is split. Hence $Z_G(S)$ is birationally equivalent to the product of S with $H/(H \cap S)$.

The absolute Dynkin diagram of H is of type D_6. By [20, p. 211], the group H is $\mathrm{Spin}(q)$ for q a quadratic form over K with $\dim q = 12$, $\operatorname{disc} q = 1$ and $\operatorname{clif}(q)$ split. As S centralizes H, the intersection $H \cap S$ is contained in the center $\mu_2 \times \mu_2$ of $\mathrm{Spin}(q)$. We show that $H \cap S$ is equal to the center of $\mathrm{Spin}(q)$.

Since G is simply connected as well as adjoint, the co-roots $\alpha_j^\vee$ provide also a $\mathbb{Z}$-basis for the cocharacter lattice T_*. Thus we may view the isomorphism Φ in (4.1) as an isomorphism

$$\bar{K}^\times \otimes_{\mathbb{Z}} T_* = \prod_{i=1}^{8} \bar{K}^\times \otimes_{\mathbb{Z}} \mathbb{Z}\alpha_i^\vee \xrightarrow{\sim} T(\bar{K}).$$

Then it follows from [18, 8.1.8] that Φ restricts to an isomorphism

$$\prod_{i=2}^{7} \bar{K}^\times \otimes_{\mathbb{Z}} \mathbb{Z}\alpha_i^\vee \xrightarrow{\sim} (H \cap T)^0(\bar{K}). \tag{4.3}$$

The expressions for the fundamental dominant weights ω_i in terms of the roots α_j in [3, Chapter 6, Plate VII] imply expressions for the fundamental dominant co-weights $\omega_i^\vee$ in terms of the co-roots $\alpha_j^\vee$. These expressions yield

$$\omega_1^\vee(-1) = \alpha_2^\vee(-1)\alpha_3^\vee(-1) \quad \text{and} \quad \omega_8^\vee(-1) = \alpha_3^\vee(-1)\alpha_5^\vee(-1)\alpha_7^\vee(-1).$$

From (4.2) and (4.3), we see that these two elements both lie in $S(\bar{K})$ and in $(H \cap T)^\circ(\bar{K})$ and are nontrivial and distinct. We have thus produced two distinct nontrivial elements in $(H \cap S)(\bar{K})$. Hence $H \cap S$ is, in fact, the entire center of $\mathrm{Spin}(q)$.

Therefore $H/(H \cap S)$ is $\mathrm{PGO}_+(q)$. It follows by 2.19 that $H/(H \cap S)$ is R-trivial. Therefore G is birationally equivalent to the product of $\mathrm{PGO}_+(q)$ times an affine space. Thus G itself is R-trivial by [4, p. 197, Cor.]. This concludes our second proof of Theorem 1.2.

We observe that this proof goes through verbatim for every group G of absolute type E_8 in whose Tits index the roots α_1 and α_8 are circled. We conclude that for such groups, G is R-trivial and the group of K-rational points of G is generated by its root groups. With only minor modifications, the proof also shows that if G is adjoint of absolute type E_7 with trivial Tits algebras and the root α_1 is circled in the Tits index of G, then G is R-trivial.

5. Theorem 1.1 and triality

In this section, we assume that $\mathrm{char}(K) \neq 2$. We give an alternative proof of Theorem 1.1, based on completely different methods. We actually show:

Proposition 5.1. *Suppose the characteristic of the base field K is different from 2. If q is a quadratic form with trivial discriminant, then $\mathrm{Hyp}_2(q) = \mathrm{Hyp}(q)$ in the following cases:*

(i) $\dim q = 8$ *and the index of* $\mathrm{clif}(q)$ *is* 1 *or* 2*;*
(ii) $\dim q = 12$ *and* $\mathrm{clif}(q)$ *is split.*

Since in each case $G(q) = \mathrm{Hyp}(q)$ by Proposition 2.15 and Corollary 2.17, Theorem 1.1 follows from Proposition 5.1.

We start with the case of 8-dimensional quadratic forms. If $\mathrm{clif}(q)$ is split, then q is a multiple of a 3-fold Pfister form, and the result follows from Lemma 2.14. Similarly, if q is isotropic, then q is Witt-equivalent to a multiple of a 2-fold Pfister form, and the result follows from Lemma 2.14. We may thus assume that (K, L, q) is of type E_7 and let D be the quaternion division algebra over K that represents $\mathrm{clif}(q)$. We show next that the Clifford algebra construction associates to q a skew-hermitian form h of rank 4 over D, and we shall complete the proof of Proposition 5.1(i) by proving that

$$\mathrm{Hyp}(q) = \mathrm{Sn}(h) = \mathrm{Hyp}_2(q);$$

see Proposition 5.9.

Let (A, σ) be a central simple K-algebra of degree 8 with an orthogonal involution of trivial discriminant. The Clifford algebra $C(A, \sigma)$ decomposes into a direct product of two central simple K-algebras of degree 8:

$$C(A, \sigma) = C_+(A, \sigma) \times C_-(A, \sigma).$$

Recall that $C(A, \sigma)$ carries a canonical involution $\underline{\sigma}$, which induces orthogonal involutions σ_+ and σ_- on $C_+(A, \sigma)$ and $C_-(A, \sigma)$ respectively. By triality (see [12, (42.3)]), the Clifford algebras of $(C_+(A, \sigma), \sigma_+)$ and $(C_-(A, \sigma), \sigma_-)$ satisfy

$$\begin{aligned}(C(C_+(A, \sigma), \sigma_+), \underline{\sigma_+}) &= (C_-(A, \sigma), \sigma_-) \times (A, \sigma),\\ (C(C_-(A, \sigma), \sigma_-), \underline{\sigma_-}) &= (A, \sigma) \times (C_+(A, \sigma), \sigma_+).\end{aligned}$$

Proposition 5.2. *The following hold:*

(1) *If A is split, then $(C_+(A, \sigma), \sigma_+)$ and $(C_-(A, \sigma), \sigma_-)$ are isomorphic.*
(2) *If (A, σ) is split and isotropic, then $(C_+(A, \sigma), \sigma_+)$ and $(C_-(A, \sigma), \sigma_-)$ are hyperbolic.*
(3) *If (A, σ) is split and hyperbolic, then $(C_+(A, \sigma), \sigma_+)$ and $(C_-(A, \sigma), \sigma_-)$ are split and hyperbolic.*

Proof. (1) is well-known, (2) is in [12, (8.5)], and (3) follows from (2) and the fact that the Clifford invariant of a hyperbolic quadratic form is trivial. □

We apply this proposition in the following context: let (K, L, q) be an 8-dimensional quadratic space with $\mathrm{disc}\, q = 1$, and assume $\mathrm{clif}(q)$ is represented by a quaternion division algebra D. Let $\mathrm{ad}_q\colon \mathrm{End}_K L \to \mathrm{End}_K L$ be the adjoint involution of

q. We apply the discussion above with $(A,\sigma) = (\operatorname{End}_K L, \operatorname{ad}_q)$. Then $C(A,\sigma) = C_0(L,q)$ and $(C_+(A,\sigma),\sigma_+)$, $(C_-(A,\sigma),\sigma_-)$ are isomorphic to $(\operatorname{End}_D W, \operatorname{ad}_h)$ for some 4-dimensional skew-hermitian space (W,h) over D (with its conjugation involution).

Proposition 5.3. *For an arbitrary extension E/K, the following statements are equivalent:*

(a) *q_E is hyperbolic;*
(b) *D_E is split and $(\operatorname{End}_D W, \operatorname{ad}_h)_E$ is hyperbolic;*
(c) *D_E is split and $(\operatorname{End}_D W, \operatorname{ad}_h)_E$ is isotropic.*

Proof. (a) $\Rightarrow$ (b): This readily follows from Proposition 5.2(3).

(b) $\Rightarrow$ (c): Clear.

(c) $\Rightarrow$ (a): This follows from Proposition 5.2(2) with $(\operatorname{End}_D W, \operatorname{ad}_h)_E$ for (A,σ); then by triality $(C_+(A,\sigma),\sigma_+)$ or $(C_-(A,\sigma),\sigma_-)$ is isomorphic to $(\operatorname{End}_K L, \operatorname{ad}_q)_E)$. □

The next results 5.4–5.6 hold for skew-hermitian forms of arbitrary dimension.

Lemma 5.4. *Let (W,h) be a skew-hermitian space over a quaternion division algebra D over K and let E be a quadratic extension of K. If h is anisotropic, the following conditions are equivalent:*

(i) *$E \cong K\big(h(v,v)\big)$ for some $v \in W$;*
(ii) *D_E is split and h_E is isotropic.*

Proof. If (i) holds, then E is isomorphic to a maximal subfield of D, hence D_E is split. Let $h(v,v)^2 = a \in K^\times$, so $E \cong K(\sqrt{a})$. Then $v \cdot (h(v,v) + \sqrt{a}) \in W_E$ is isotropic for h_E. Thus, (ii) holds.

Conversely, if (ii) holds, then E is isomorphic to a maximal subfield of D. Let $E = K(\sqrt{a})$ for some $a \in K$, and let $\lambda \in D$ be a pure quaternion such that $\lambda^2 = a$. Suppose $x + y\sqrt{a} \in W_E$ is h_E-isotropic for some x, $y \in W$. The condition $h_E(x + y\sqrt{a}, x + y\sqrt{a}) = 0$ yields

$$h(x,x) + h(y,y)a = 0 \quad \text{and} \quad h(x,y) + h(y,x) = 0.$$

Since h is skew-hermitian, the second equation shows that $h(x,y) \in K$. Then

$$h(x + y\lambda, x + y\lambda) = 2h(x,y)\lambda - h(y,y)a - \lambda h(y,y)\lambda$$

and the right side commutes with λ. Therefore, $h(x + y\lambda, x + y\lambda) = \lambda b$ for some $b \in K^\times$, and we have $E \cong K\big(h(v,v)\big)$ with $v = x + y\lambda$. □

For any skew-hermitian space (W,h) over a quaternion division algebra D over K, we let $\operatorname{Sn}(h)$ denote the group of spinor norms of h, which is the image of the Clifford group $\Gamma(\operatorname{End}_D W, \operatorname{ad}_h) = \Gamma(W,h)$ under the multiplier map; see [12, (13.30)].

Proposition 5.5. *If h is anisotropic, then $\operatorname{Sn}(h) = \prod_E N(E/K)$, where E runs over the quadratic extensions of K satisfying the equivalent conditions (i) and (ii) of Lemma 5.4.*

Proof. The multiplier map $\Gamma(W,h) \to K^\times$ factors through the vector representation $\Gamma(W,h) \to \mathrm{O}_+(W,h)$, where $\mathrm{O}_+(W,h)$ is the group of direct isometries of the space

(W,h). By [10, Thm. 6.2.17], this group is generated by transformations of the form

$$\tau_{v,r}\colon W\to W,\quad x\mapsto x-vh(vr,x)$$

where $v\in W$ is an anisotropic vector and $r\in D^\times$ satisfies $r-\overline{r}=rh(v,v)\overline{r}$. To compute the spinor norm of that transformation, observe that $\tau_{v,r}$ is the identity on $v^\perp$, hence the spinor norm of $\tau_{v,r}$ is the spinor norm of its restriction to the 1-dimensional subspace vD. Let $\nu=h(v,v)\in D^\times$ and let h_v denote the restriction of h to vD, so

$$h_v(v\lambda,v\mu)=\overline{\lambda}\nu\mu\qquad\text{for }\lambda,\ \mu\in D.$$

We have

$$\mathrm{O}_+(vD,h_v)=\{\theta\in K(\nu)^\times\mid\theta\overline{\theta}=1\}\quad\text{and}\quad\Gamma(vD,h_v)=K(\nu)^\times$$

(where $\theta\in K(\nu)^\times$ is identified with the map $v\lambda\mapsto v\theta\lambda$ for $\lambda\in D$). The vector representation $\Gamma(vD,h_v)\to\mathrm{O}_+(vD,h_v)$ carries $u\in K(\nu)^\times$ to $u\overline{u}^{-1}$, hence the spinor norm of that isometry is $u\overline{u}K^{\times2}$; see [12, (13.17)]. This shows that $\mathrm{Sn}(h_v)$ consists of norms from the quadratic extension $K(\nu)/K$. Since $\mathrm{Sn}(h)$ is generated by the groups $\mathrm{Sn}(h_v)$ for the anisotropic vectors $v\in W$, the proposition follows. □

Corollary 5.6. *Let (W,h) be a skew-hermitian space over a quaternion division algebra D over K, and let $p=\mathrm{char}(K)>2$. For $\widetilde{K}=K^{-p^{-\infty}}$ the perfect closure of K, we have $\mathrm{Sn}(h_{\widetilde{K}})\cap K=\mathrm{Sn}(h)$.*

Proof. The inclusion $\mathrm{Sn}(h)\subset\mathrm{Sn}(h_{\widetilde{K}})\cap K$ is clear, so it suffices to prove the reverse inclusion. Let $x\in\mathrm{Sn}(h_{\widetilde{K}})\cap K$. If $x\in\widetilde{K}^{\times2}$, then $x\in K^{\times2}\subset\mathrm{Sn}(h)$. We may thus assume $x\notin\widetilde{K}^{\times2}$. By Proposition 5.5, there exist quadratic extensions $\widetilde{E}_1/\widetilde{K}$, …, $\widetilde{E}_r/\widetilde{K}$ such that $D_{\widetilde{E}_i}$ is split and $h_{\widetilde{E}_i}$ is isotropic for each $i\in[1,r]$, and elements $y_i\in\widetilde{E}_i\setminus\widetilde{K}$ for $i\in[1,r]$ such that

$$x=N_{\widetilde{E}_1/\widetilde{K}}(y_1)\cdot\ldots\cdot N_{\widetilde{E}_r/\widetilde{K}}(y_r).\tag{5.7}$$

Let $K'\subset\widetilde{K}$ be the subfield generated by $N_{\widetilde{E}_1/\widetilde{K}}(y_1)$, …, $N_{\widetilde{E}_r/\widetilde{K}}(y_r)$ and, for $i\in[1,r]$, let $E_i'=K'(y_i)$. Thus, K'/K is a purely inseparable extension of finite degree, (5.7) yields

$$x=N_{E_1'/K'}(y_1)\cdot\ldots\cdot N_{E_r'/K'}(y_r),\tag{5.8}$$

and each E_i'/K' is a quadratic extension. For $i\in[1,r]$, let E_i'' be the separable closure of K in E_i'; it is a quadratic extension of K and we have

$$E_i'\cong E_i''\otimes_K K'\qquad\text{and}\qquad\widetilde{E}_i\cong E_i''\otimes_K\widetilde{K}.$$

Since quaternion division algebras do not split over extensions of odd degree, the condition that $D_{\widetilde{E}_i}$ is split shows that $D_{E_i''}$ is split. Likewise, anisotropic skew-hermitian forms do not become isotropic over odd-degree extensions by [15, Thm. 3.5], hence $h_{E_i''}$ is isotropic. Now, let $[K':K]=p^d$; taking the norm from K' to K of each side of (5.8), we obtain

$$x^{p^d}=N_{E_1'/K}(y_1)\cdot\ldots\cdot N_{E_r'/K}(y_r)=N_{E_1''/K}\big(N_{E_1'/E_1''}(y_1)\big)\cdot\ldots\cdot N_{E_r''/K}\big(N_{E_r'/E_r''}(y_r)\big).$$

Since $x^{p^d}\equiv x\bmod K^{\times2}$, this equation shows that x is a product of norms from quadratic extensions over which D is split and h is isotropic, hence $x\in\mathrm{Sn}(h)$ by Proposition 5.5. □

We now return to the context of Proposition 5.3. The following proposition completes the proof of Proposition 5.1(i):

Proposition 5.9. *For q an anisotropic 8-dimensional quadratic form and h the corresponding 4-dimensional skew-hermitian form as in Proposition 5.3, we have*

$$\mathrm{Hyp}(q) = \mathrm{Sn}(h) = \mathrm{Hyp}_2(q).$$

Proof. Proposition 5.3 shows that the quadratic extensions E/K such that D_E is split and h_E is isotropic are exactly those such that q_E is hyperbolic, hence by Proposition 5.5 we have

$$\mathrm{Sn}(h) = \mathrm{Hyp}_2(q) \subset \mathrm{Hyp}(q).$$

To complete the proof, we show $\mathrm{Hyp}(q) \subset \mathrm{Sn}(h)$. Let E/K be a finite-degree extension such that q_E is hyperbolic, let $\widetilde{K}$ be the perfect closure of K in some algebraic closure of E, and let K_1 be the purely inseparable closure of K in E. The compositum $E \cdot \widetilde{K}$ of E and $\widetilde{K}$ satisfies $E \cdot \widetilde{K} \cong E \otimes_{K_1} \widetilde{K}$. Since q_E is hyperbolic, q is also hyperbolic over $E \cdot \widetilde{K}$, hence $D_{E\cdot\widetilde{K}}$ is split and $h_{E\cdot\widetilde{K}}$ is isotropic, by Proposition 5.3. Therefore, $\mathrm{Sn}(h_{E\cdot\widetilde{K}}) = (E \cdot \widetilde{K})^\times$. Since $\widetilde{K}$ is perfect, we may apply the norm principle for spinor norms (see [13, (6.2)]), which is a twisted analogue of Knebusch's norm theorem, to see that $N(E \cdot \widetilde{K}/\widetilde{K}) \subset \mathrm{Sn}(h_{\widetilde{K}})$. Since $N(E/K_1) \subset N(E \cdot \widetilde{K}/\widetilde{K})$, it follows that $N(E/K_1) \subset \mathrm{Sn}(h_{\widetilde{K}})$. Let $p = \mathrm{char}(K)$ if $\mathrm{char}(K) > 2$ and $p = 1$ if $\mathrm{char}(K) = 0$, so $[K_1 : K] = p^d$ for some $d \geq 0$. For all $x \in K_1$, we have $N_{K_1/K}(x) = x^{p^d}$, hence

$$N(E/K) = N_{K_1/K}\big(N(E/K_1)\big) = N(E/K_1)^{p^d} \subset \mathrm{Sn}(h_{\widetilde{K}}).$$

But $N(E/K) \subset K$, hence Corollary 5.6 shows that $N(E/K) \subset \mathrm{Sn}(h)$. Of course, we also have $K^{\times 2} \subset \mathrm{Sn}(h)$, hence $\mathrm{Hyp}(q) \subset \mathrm{Sn}(h)$. □

Part (ii) of Proposition 5.1 follows from part (i) by the same arguments as in the proof of Corollary 2.17: let q be a 12-dimensional nondegenerate form with trivial discriminant and Clifford invariant. If q is isotropic, then it is Witt-equivalent to a 3-fold Pfister form and $G(q) = \mathrm{Hyp}_2(q)$ by Lemma 2.14. For the rest of the proof, suppose q is anisotropic, i.e., q is of type E_8. Let $\gamma \in G(q)$. Since $\mathrm{char}(K) \neq 2$, all similitudes of q are separable. We can thus fix a decomposition $q = q_1 \perp q_2$ as in Proposition 2.16. Since $\gamma \in G(q_2)$, part (i) of Proposition 5.1 shows that $\gamma \in \mathrm{Hyp}_2(q_2)$. Since $\mathrm{Hyp}_2(q_2) \subset \mathrm{Hyp}_2(q)$ by Remark 2.18, it follows that $\gamma \in \mathrm{Hyp}_2(q)$. This proves Proposition 5.1(ii).

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Department of Mathematics and Computer Science, Emory University, MSC W401 400 Dowman Drive, Atlanta, GA 30322, USA
E-mail address: parimala@mathcs.emory.edu

Institute for Information and Communication Technologies, Electronics and Applied Mathematics, Université Catholique de Louvain, Avenue Georges Lemaître, 4, 1348 Louvain-la-Neuve, Belgium
E-mail address: Jean-Pierre.Tignol@uclouvain.be

Department of Mathematics, Tufts University, 503 Boston Avenue, Medford, MA 02155, USA
E-mail address: rweiss@tufts.edu