ON THE RELATION OF SYMPLECTIC ALGEBRAIC COBORDISM TO HERMITIAN K-THEORY

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Abstract. We reconstruct hermitian K-theory via algebraic symplectic cobordism. In the motivic stable homotopy category $SH(S)$ there is a unique morphism $\varphi\colon \mathbf{MSp} \to \mathbf{BO}$ of commutative ring T-spectra which sends the Thom class $th^{\mathbf{MSp}}$ to the Thom class $th^{\mathbf{BO}}$. Using φ we construct an isomorphism of bigraded ring cohomology theories on the category $\mathcal{S}m\mathcal{O}p/S$

$$\bar{\varphi}\colon \mathbf{MSp}^{*,*}(X,U) \otimes_{\mathbf{MSp}^{4*,2*}(pt)} \mathbf{BO}^{4*,2*}(pt) \cong \mathbf{BO}^{*,*}(X,U).$$

The result is an algebraic version of the theorem of Conner and Floyd reconstructing real K-theory using symplectic cobordism. Rewriting the bigrading as $\mathbf{MSp}^{p,q} = \mathbf{MSp}^{[q]}_{2q-p}$, we have an isomorphism

$$\bar{\varphi}\colon \mathbf{MSp}^{[*]}_{*}(X,U) \otimes_{\mathbf{MSp}^{[2*]}_{0}(pt)} KO^{[2*]}_{0}(pt) \cong KO^{[*]}_{*}(X,U),$$

where the $KO^{[n]}_{i}(X,U)$ are Schlichting's hermitian K-theory groups.

1. A motivic version of a theorem by Conner and Floyd

Our main result relates symplectic algebraic cobordism to hermitian K-theory. It is an algebraic version of the theorem of Conner and Floyd [2, Theorem 10.2] reconstructing real K-theory using symplectic cobordism. The algebraic version of the reconstruction of complex K-theory using unitary cobordism was done in [5].

In [7] the current authors constructed a commutative ring T-spectrum $\mathbf{BO}$ representing hermitian K-theory in the stable homotopy category $SH(S)$ for any regular noetherian separated base scheme S of finite Krull dimension without residue fields of characteristic 2. (These restrictions allowed us to use particularly strong results of Marco Schlichting [9]. We leave it to the expert(s) in negative hermitian K-theory to weaken them.) It has a standard family of Thom classes for special linear vector bundles and hence for symplectic bundles. The symplectic Thom classes can all be derived from a single class $th^{\mathbf{BO}} \in \mathbf{BO}^{4,2}(\operatorname{Th}\mathcal{U}_{HP^\infty}) = \mathbf{BO}^{4,2}(\mathbf{MSp}_2)$, the symplectic Thom orientation.

In [6] we constructed the commutative ring T-spectrum $\mathbf{MSp}$ of algebraic symplectic cobordism. It is a commutative monoid in the model category of symmetric $T^{\wedge 2}$-spectra, just as $\mathbf{MSL}$ and Voevodsky's $\mathbf{MGL}$ are commutative monoids in the model category of symmetric T-spectra. The canonical map $\Sigma^\infty_T \mathbf{MSp}_2(-2) \to \mathbf{MSp}$ gives the symplectic Thom orientation $th^{\mathbf{MSp}} \in \mathbf{MSp}^{4,2}(\mathbf{MSp}_2)$. It is the universal symplectically oriented commutative ring T-spectrum.

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Therefore there is a unique morphism $\varphi\colon \mathbf{MSp} \to \mathbf{BO}$ of commutative monoids in $SH(S)$ with $\varphi(th^{\mathbf{MSp}}) = th^{\mathbf{BO}}$. Our main result is the following theorem. Our notation is that for a motivic space Y and a bigraded cohomology theory we write $A^{*,*}(Y) = \bigoplus_{p,q\in\mathbb{Z}} A^{p,q}(Y)$ and $A^{4*,2*}(Y) = \bigoplus_{i\in\mathbb{Z}} A^{4i,2i}(Y)$. A motivic space Y is *small* if $Hom_{SH(S)}(\Sigma_T^\infty Y, -)$ commutes with arbitrary coproducts.

Theorem 1.1. *Let S be a regular noetherian separated scheme of finite Krull dimension with $\frac{1}{2} \in \Gamma(S, \mathcal{O}_S)$. For all small pointed motivic spaces Y over S the map*

$$\bar{\varphi}\colon \mathbf{MSp}^{*,*}(Y) \otimes_{\mathbf{MSp}^{4*,2*}(pt)} \mathbf{BO}^{4*,2*}(pt) \to \mathbf{BO}^{*,*}(Y).$$

induced by φ is an isomorphism.

This has as a consequence the result mentioned in the abstract. For a pair (X,U) consisting of a smooth S-scheme of finite type X and an open subscheme U, there is a quotient pointed motivic space X_+/U_+. We define $\mathbf{MSp}^{*,*}(X,U) = \mathbf{MSp}^{*,*}(X_+/U_+)$ and $\mathbf{BO}^{*,*}(X,U) = \mathbf{BO}^{*,*}(X_+/U_+)$. There are natural isomorphisms $\mathbf{BO}^{p,q}(X,U) = KO^{[q]}_{2q-p}(X,U)$ with the hermitian K-theory of X with supports in $X - U$ as defined by Schlichting [11]. The weight q is the degree of the shift in the duality used for the symmetric bilinear forms on the chain complexes of vector bundles.

For a field k of characteristic not 2 the ring $\mathbf{BO}^{4*,2*}(k)$ is not large. For all i one has $\mathbf{BO}^{8i,4i}(k) \cong GW(k)$ and $\mathbf{BO}^{8i+4,4i+2}(k) \cong \mathbb{Z}$. All members of $\mathbf{BO}^{0,0}(k)$ therefore come from composing endomorphisms in $SH(k)$ of the sphere T-spectrum $\mathbf{1} = \Sigma_T^\infty pt_+$ with the unit $e\colon \mathbf{1} \to \mathbf{BO}$ of the monoid. (See Morel [3, Theorem 4.36] and Cazanave [1] for calculations of the endomorphisms of the sphere T-spectrum.) Consequently $\varphi^{0,0}\colon \mathbf{MSp}^{0,0}(k) \to \mathbf{BO}^{0,0}(k)$ is surjective. We do not know what happens in other bidegrees.

This is the fourth in a series of papers about symplectically oriented motivic cohomology theories. All depend on the quaternionic projective bundle theorem proven in the first paper [8].

2. Preliminaries

Let S be a Noetherian separated scheme of finite Krull dimension. We will be dealing with hermitian K-theory, and we prefer avoiding the subtleties of negative K-theory, so we will assume as we did in [7] that S is regular and that $\frac{1}{2} \in \Gamma(S, \mathcal{O}_S)$. Let $\mathcal{S}m/S$ be the category of smooth S-schemes of finite type. Let $\mathcal{S}m\mathcal{O}p/S$ be the category whose objects are pairs (X,U) with $X \in \mathcal{S}m/S$ and $U \subset X$ an open subscheme and whose arrows $f\colon (X,U) \to (X',U')$ are morphisms $f\colon X \to X'$ of S-schemes with $f(U) \subset U'$. Note that all X in $\mathcal{S}m/S$ have an ample family of line bundles.

A *motivic space over S* is a simplicial presheaf on $\mathcal{S}m/S$. We will often write *pt* for the base scheme regarded as a motivic space over itself. Inverting the motivic weak equivalences in the category of pointed motivic spaces gives the pointed motivic unstable homotopy category $H_\bullet(S)$.

Let $T = \mathbf{A}^1/(\mathbf{A}^1 - 0)$ be the Morel-Voevodsky object. A *T-spectrum* M is a sequence of pointed motivic spaces $(M_0, M_1, M_2, \dots)$ equipped with structural maps $\sigma_n\colon M_n \wedge T \to M_{n+1}$. Inverting the stable motivic weak equivalences gives the motivic stable homotopy category $SH(S)$. A pointed motivic space X has a T-suspension spectrum $\Sigma_T^\infty X$. For any T-spectrum M there are canonical maps of spectra

$$u_n\colon \Sigma_T^\infty M_n(-n) \to M. \tag{1}$$

Both $H_\bullet(S)$ and $SH(S)$ are equipped with closed symmetric monoidal structures, and $\Sigma_T^\infty\colon H_\bullet(S) \to SH(S)$ is a strict symmetric monoidal functor. The symmetric monoidal structure $(\wedge, \mathbf{1}_S = \Sigma_T^\infty pt_+)$ on the homotopy category $SH(S)$ can be constructed on the model category level using symmetric T-spectra.

Any T-spectrum A defines a cohomology theory on the category of pointed motivic spaces. Namely, for a pointed space (X,x) one sets $A^{p,q}(X,x) = Hom_{H_\bullet(S)}(\Sigma_T^\infty(X,x), \Sigma^{p,q}(A))$ and $A^{*,*}(X,x) = \bigoplus_{p,q\in\mathbb{Z}} A^{p,q}(X,x)$. We write (somewhat inconsistently)

$$A^{4*,2*}(X,x) = \bigoplus_{i\in\mathbb{Z}} A^{4i,2i}(X,x).$$

For an unpointed space X we set $A^{p,q}(X) = A^{p,q}(X_+,+)$, with $A^{*,*}(X)$ and $A^{4*,2*}(X)$ defined accordingly. We will not always write the pointings explicitly.

Each $Y \in \mathcal{S}m/S$ defines an unpointed motivic space which is constant in the simplicial direction $Hom_{\mathcal{S}m/S}(-,Y)$. So we regard smooth S-schemes as motivic spaces and set $A^{p,q}(Y) = A^{p,q}(Y_+,+)$. Given a monomorphism $U \hookrightarrow Y$ of smooth S-schemes, we write $A^{p,q}(Y,U) = A^{p,q}(Y_+/U_+, U_+/U_+)$.

A *commutative ring T-spectrum* is a commutative monoid (A,μ,e) in $(SH(S),\wedge,1)$.

The cohomology theory $A^{*,*}$ defined by a commutative ring T-spectrum is a ring cohomology theory satisfying a certain bigraded commutativity condition described by Morel. Namely, let $\varepsilon \in A^{0,0}(pt)$ be the element such that $\Sigma_T^2\varepsilon \in Hom_{SH(S)}(T\wedge T, T\wedge T)$ is the map exchanging the two factors T. Then for $\alpha \in A^{p,q}(X,x)$ and $\beta \in A^{p',q'}(X,x)$ we have $\alpha\cup\beta = (-1)^{pp'}\varepsilon^{qq'}\beta\cup\alpha$. In particular, $A^{4*,2*}(X,x)$ is contained in the center of $A^{*,*}(X,x)$.

We work in this text with the algebraic cobordism T-spectrum $\mathbf{MSp}$ of [6, §6] and the hermitian K-theory T-spectrum $\mathbf{BO}$ of [7, §8]. The spectrum $\mathbf{MSp}$ is a commutative ring T-spectrum because it is naturally a commutative monoid in the category of symmetric $T^{\wedge 2}$-spectra. The T-spectrum $\mathbf{BO}$ has a commutative monoid structure as shown in [7, Theorem 1.3].

3. The first Pontryagin class $p_1(E,\phi)$

Let V be a vector bundle over a smooth S-scheme X with zero section $z\colon X \hookrightarrow V$. The *Thom space* of V is the quotient motivic space $\operatorname{Th} V = V/(V - z(X))$. It is pointed by the image of $V - z(X)$. It comes with a canonical structure map $z\colon X_+ \to \operatorname{Th} V$ induced by the zero section. For the trivial bundle $\mathbf{A}^n \to pt$ one has $\operatorname{Th}\mathbf{A}^n = T^{\wedge n}$.

We write H for the trivial rank 2 symplectic bundle $\left(\mathcal{O}^{\oplus 2}, \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0\end{smallmatrix}\right)\right)$. The orthogonal direct sum $\mathsf{H}^{\oplus n}$ is the trivial symplectic bundle of rank $2n$.

The most basic form a symplectic orientation is a symplectic Thom structure [8, Definition 7.1]. We will use the following version of the definition.

Definition 3.1. Let (A,μ,e) be a symmetric ring T-spectrum. A *symplectic Thom structure* on the cohomology theory $A^{*,*}$ is a rule which assigns to each rank 2 symplectic bundle (E,ϕ) over an X in $\mathcal{S}m/S$ an element $th(E,\phi) \in A^{4,2}(\operatorname{Th} E) = A^{4,2}(E, E-X)$ with the following properties:

(1) For an isomorphism $u\colon (E,\phi) \cong (E_1,\phi_1)$ one has $th(E,\phi) = u^*\, th(E_1,\phi_1)$.
(2) For a morphism $f\colon Y \to X$ with pullback map $f_E\colon f^*E \to E$ one has $f_E^*\, th(E,\phi) = th(f^*E, f^*\phi)$.

(3) For the rank 2 trivial symplectic bundle H over pt the map

$$-\times th(\mathsf{H})\colon A^{*,*}(X) \to A^{*+4,*+2}(X \times \mathbf{A}^2, X \times (\mathbf{A}^2 - 0))$$

is an isomorphism for all X.

The *Pontryagin class* of (E,ϕ) is $p_1(E,\phi) = -z^* \, th(E,\phi) \in A^{4,2}(X)$ where $z\colon X \to E$ is the zero section.

The sign in the Pontryagin class is simply conventional. It is chosen so that if $A^{*,*}$ is an oriented cohomology theory with an additive formal group law, then the Chern and Pontryagin classes satisfy the traditional formula $p_i(E,\phi) = (-1)^i c_{2i}(E)$.

From Mayer-Vietoris one sees that for any rank 2 symplectic bundle

$$\cup\, th(E,\phi)\colon A^{*,*}(X) \xrightarrow{\cong} A^{*,*}(E, E - X)$$

is an isomorphism.

The *quaternionic Grassmannian* $HGr(r,n) = HGr(r,\mathsf{H}^{\oplus n})$ is defined as the open subscheme of $Gr(2r,2n) = Gr(2r,\mathsf{H}^{\oplus n})$ parametrizing subspaces of dimension $2r$ of the fibers of $\mathsf{H}^{\oplus n}$ on which the symplectic form of $\mathsf{H}^{\oplus n}$ is nondegenerate. We write $\mathcal{U}_{HGr(r,n)}$ for the restriction to $HGr(r,n)$ of the tautological subbundle of $Gr(2r,2n)$. The symplectic form of $\mathsf{H}^{\oplus n}$ restricts to a symplectic form on $\mathcal{U}_{HGr(r,n)}$ which we denote by $\phi_{HGr(r,n)}$. The pair $(\mathcal{U}_{HGr(r,n)}, \phi_{HGr(r,n)})$ is the *tautological symplectic subbundle* of rank $2r$ on $HGr(r,n)$.

More generally, given a symplectic bundle (E,ϕ) of rank $2n$ over X, the *quaternionic Grassmannian bundle* $HGr(r,E,\phi)$ is the open subscheme of the Grassmannian bundle $Gr(2r,E)$ parametrizing subspaces of dimension $2r$ of the fibers of E on which ϕ is nondegenerate.

For $r = 1$ we have *quaternionic projective spaces* and *bundles* $HP^n = HGr(1,n+1)$ and $HP(E,\phi) = HGr(1,E,\phi)$.

The quaternionic projective bundle theorem is proven in [8] using the symplectic Thom structure and not any other version of a symplectic orientation. It is proven first for trivial bundles.

Theorem 3.2 ([8, Theorem 8.1])**.** *Let (A,μ,e) be a commutative ring T-spectrum with a symplectic Thom structure on $A^{*,*}$. Let $(\mathcal{U}_{HP^n}, \phi_{HP^n})$ be the tautological rank 2 symplectic subbundle over HP^n and $t = p_1(\mathcal{U}_{HP^n}, \phi_{HP^n}) \in A^{4,2}(HP^n)$ its Pontryagin class. Then for any X in $\mathcal{S}m/S$ we have an isomorphism of bigraded rings*

$$A^{*,*}(HP^n \times X) \cong A^{*,*}(X)[t]/(t^{n+1}).$$

A Mayer-Vietoris argument gives the more general theorem [8, Theorem 8.2].

Theorem 3.3 (Quaternionic projective bundle theorem)**.** *Let (A,μ,e) be a commutative ring T-spectrum with a symplectic Thom structure on $A^{*,*}$. Let (E,ϕ) be a symplectic bundle of rank $2n$ over X, let $(\mathcal{U},\phi|_{\mathcal{U}})$ be the tautological rank 2 symplectic subbundle over the quaternionic projective bundle $HP(E,\phi)$, and let $t = p_1(\mathcal{U},\phi|_{\mathcal{U}})$ be its Pontryagin class. Then we have an isomorphism of bigraded $A^{*,*}(X)$-modules*

$$(1,t,\dots,t^{n-1})\colon A^{*,*}(X) \oplus A^{*,*}(X) \oplus \cdots \oplus A^{*,*}(X) \to A^{*,*}(HP(E,\phi)).$$

Definition 3.4. Under the hypotheses of Theorem 3.3 there are unique elements $p_i(E,\phi) \in A^{4i,2i}(X)$ for $i = 1,2,\dots,n$ such that

$$t^n - p_1(E,\phi) \cup t^{n-1} + p_2(E,\phi) \cup t^{n-2} - \cdots + (-1)^n p_n(E,\phi) = 0.$$

The classes $p_i(E,\phi)$ are called the *Pontryagin classes* of (E,ϕ) with respect to the symplectic Thom structure of the cohomology theory (A,∂). For $i>n$ one sets $p_i(E,\phi)=0$, and one sets $p_0(E,\phi)=1$.

Corollary 3.5. *The Pontryagin classes of a trivial symplectic bundle vanish*: $p_i(\mathsf{H}^{\oplus n})=0$.

The Cartan sum formula holds for Pontryagin classes [8, Theorem 10.5]. In particular:

Theorem 3.6. *Let (A,μ,e) be a commutative ring T-spectrum with a symplectic Thom structure on $A^{*,*}$. Let (E,ϕ) and (F,ψ) be symplectic bundles over X. Then we have*

$$p_1\big((E,\phi)\oplus(F,\psi)\big)=p_1(E,\phi)+p_1(F,\psi). \tag{2}$$

We also have the following result [8, Proposition 8.5].

Proposition 3.7. *Suppose that (E,ϕ) is a symplectic bundle over X with a totally isotropic subbundle $L\subset E$. Then for all i we have*

$$p_i(E,\phi)=p_i\left((L^\perp/L,\overline{\phi})\oplus\left(L\oplus L^\vee,\left(\begin{smallmatrix}0 & 1_{L^\vee}\\ -1_L & 0\end{smallmatrix}\right)\right)\right).$$

This is because there is an $\mathbf{A}^1$-deformation between the two symplectic bundles.

Definition 3.8. The *Grothendieck-Witt group of symplectic bundles* $GW^-(X)$ is the abelian group of formal differences $[E,\phi]-[F,\psi]$ of symplectic vector bundles over X modulo three relations:

(1) For an isomorphism $u\colon(E,\phi)\cong(E_1,\phi_1)$ one has $[E,\phi]=[E_1,\phi_1]$.
(2) For an orthogonal direct sum one has $[(E,\phi)\oplus(E_1,\phi_1)]=[E,\phi]+[E_1,\phi_1]$.
(3) If (E,ϕ) is a symplectic bundle over X with a totally isotropic subbundle $L\subset E$, then we have $[E,\phi]=[L^\perp/L,\overline{\phi}]+\big[L\oplus L^\vee,\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)\big]$.

The *Grothendieck-Witt group of orthogonal bundles* $GW^+(X)$ is defined analogously.

Theorem 3.9. *Let (A,μ,e) be a commutative ring T-spectrum with a symplectic Thom structure on $A^{*,*}$. Then the associated first Pontryagin class induces a well-defined additive map*

$$p_1\colon GW^-(X)\to A^{4,2}(X)$$

which is functorial in X.

In [10] Schlichting constructed hermitian K-theory spaces for exact categories. This gives hermitian K-theory spaces $KO(X)$ and $KSp(X)$ for orthogonal and symplectic bundles on schemes. Their π_0 are $GW^+(X)$ and $GW^-(X)$ respectively. In [11] he constructed Hermitian K-theory spaces $KO^{[m]}(X,U)$ for complexes of vector bundles on X acyclic on the open subscheme U equipped with a nondegenerate symmetric bilinear form for the duality shifted by m. For an even integer $2n$ an orthogonal bundle (U,ψ) gives a chain complex $U[2n]$ equipped with a nondegenerate symmetric bilinear form $\psi[4n]\colon U[2n]\otimes_{\mathcal{O}_X}U[2n]\to\mathcal{O}_X[4n]$ in the symmetric monoidal category $D^b(VB_X)$. For an odd integer $2n+1$ a symplectic bundle (E,ϕ) gives a chain complex $E[2n{+}1]$ equipped with a nondegenerate symmetric bilinear form $\phi[4n{+}2]\colon E[2n{+}1]\otimes_{\mathcal{O}_X}E[2n{+}1]\to\mathcal{O}_X[4n{+}2]$. These functors induce homotopy equivalences of spaces $KO(X)\to KO^{[4n]}(X)$ and $KSp(X)\to KO^{[4n+2]}(X)$ [11, Proposition 6].

The simplicial presheaves $X\mapsto KO^{[n]}(X)$ are pointed motivic spaces. Dévissage gives schemewise weak equivalences $KO^{[n]}(X)\to KO^{[n+1]}(X\times\mathbf{A}^1,X\times(\mathbf{A}^1-0))$ which are adjoint to maps $KO^{[n]}\times T\to KO^{[n+1]}$. These are the structural maps of a T-spectrum $(KO^{[0]},KO^{[1]},KO^{[2]},\dots)$ of which our $\mathbf{BO}$ is a fibrant replacement [7, §§7–8]. One has

$KO_i^{[n]}(X,U) = \mathbf{BO}^{4n-i,2n}(X_+/U_+)$ for all $i \geq 0$ and n. Hence $\mathbf{BO}^{4n,2n}(X_+/U_+)$ is the Grothendieck-Witt group for the usual duality shifted by n of symmetric chain complexes of vector bundles on X which are acyclic on U.

Definition 3.10. The *right isomorphisms* are

$$\begin{array}{rrcl} \mathsf{unsign.trans}_{4n}\colon & GW^+(X) & \xrightarrow{\cong} & KO_0^{[4n]}(X) = \mathbf{BO}^{8n,4n}(X) \\ & [U,\psi] & \longmapsto & \big[U[2n],\psi[4n]\big] \end{array}$$

and

$$\begin{array}{rrcl} \mathsf{sign.trans}_{4n+2}\colon & GW^-(X) & \xrightarrow{\cong} & KO_0^{[4n+2]}(X) = \mathbf{BO}^{8n+4,4n+2}(X) \\ & [E,\phi] & \longmapsto & -\big[E[2n+1],\phi[4n+2]\big] \end{array}$$

The sign in $\mathsf{sign.trans}_{4n+2}$ is chosen so that it commutes with the forgetful maps to $K_0(X)$, where we have $[E] = -\big[E[2n+1]\big]$. Most authors of papers on Witt groups do not use this sign because Witt groups do not have forgetful maps to $K_0(X)$.

Definition 3.11. The *periodicity elements* $\beta_8 \in \mathbf{BO}^{8,4}(pt)$ and $\beta_8^{-1} \in \mathbf{BO}^{-8,-4}(pt)$ correspond to the unit $1 = [\mathcal{O}_X, 1] \in GW^+(X)$ under the isomorphisms $\mathbf{BO}^{8,4}(pt) \cong GW^+(pt) \cong \mathbf{BO}^{-8,-4}(pt)$ of Definition 3.10.

We have the composition

$$\widetilde{p}_1^A\colon \mathbf{BO}^{4,2}(X) \underset{\cong}{\xleftarrow{\mathsf{sign.trans}_2}} GW^-(X) \xrightarrow{p_1} A^{4,2}(X) \tag{3}$$

The Thom classes for hermitian K-theory are constructed by the same method that Nenashev used for Witt groups [4, §2]. Suppose we have an SL_n-bundle (E,λ) consisting of a vector bundle $\pi\colon E \to X$ of rank n and $\lambda\colon \mathcal{O}_X \cong \det E$ an isomorphism of line bundles. The pullback $\pi^*E = E \oplus E \to E$ has a canonical section Δ_E, the diagonal. There is a Koszul complex

$$K(E) = \big(0 \to \Lambda^n\pi^*E^\vee \to \Lambda^{n-1}\pi^*E^\vee \to \cdots \to \Lambda^2\pi^*E^\vee \to E^\vee \to \mathcal{O}_E \to 0\big)$$

in which each boundary map the contraction with Δ_E. It is a locally free resolution of the coherent sheaf $z_*\mathcal{O}_X$ on E. There is a canonical isomorphism $\Theta(E,\lambda)\colon K(E) \to K(E)^\vee[n]$ induced by λ which is symmetric for the shifted duality.

Definition 3.12. In the *standard special linear Thom structure* on $\mathbf{BO}$ the Thom class of the special linear bundle (E,λ) of rank n is

$$th^{\mathbf{BO}}(E,\lambda) = [K(E),\Theta(E,\lambda)] \in KO_0^{[n]}(E,E-X) = \mathbf{BO}^{2n,n}(E,E-X)$$

In the *standard symplectic Thom structure* on $\mathbf{BO}$ the Thom class of the symplectic bundle (E,ϕ) of rank $2r$ is

$$th^{\mathbf{BO}}(E,\phi) = th^{\mathbf{BO}}(E,\lambda_\phi) \in \mathbf{BO}^{4r,2r}(E,E-X)$$

for $\lambda_\phi = (\operatorname{Pf}\phi)^{-1}$ where $\operatorname{Pf}\phi \in \Gamma(X,\det E^\vee)$ denotes the Pfaffian of $\phi \in \Gamma(X,\Lambda^2E^\vee)$.

The corresponding first Pontryagin class of a rank 2 symplectic bundle is therefore

$$p_1^{\mathbf{BO}}(E,\phi) = -[K(E),\Theta(E,\lambda_\phi)]|_X \in \mathbf{BO}^{4,2}(X).$$

A short calculation shows that this is the class which corresponds to $[E,\phi] - [\mathsf{H}] \in GW^-(X)$ under the isomorphism $\mathsf{sign.trans}_2$. The symplectic splitting principle [8, Theorem 10.2] and Theorem 3.6 now give the next proposition.

Proposition 3.13. *Let (E,ϕ) be a symplectic bundle of rank $2r$ on X. Then $p_1^{\mathbf{BO}}(E,\phi) \in \mathbf{BO}^{4,2}(X)$ is the class which corresponds to $[E,\phi] - r[\mathsf{H}] \in GW^-(X)$ under the isomorphism* $\mathsf{sign.trans}_2$.

Let $X = \bigsqcup X_i$ be the connected components of X. We consider the elements and functions

$$1_{X_i} \in \mathbf{BO}^{0,0}(X), \qquad \mathrm{rk}_{X_i}\colon \mathbf{BO}^{4,2}(X) \to \mathbb{Z}, \qquad \mathsf{h} \in \mathbf{BO}^{4,2}(pt). \tag{4}$$

The first is the central idempotent which is the image of the unit $1_{X_i} \in \mathbf{BO}^{0,0}(X_i)$. The second is the rank function on the Grothendieck-Witt group $KO_0^{[2]}(X)$ of bounded chain complexes of vector bundles. The third is the class corresponding to $[\mathsf{H}] \in GW^-(pt)$ under the right isomorphism $\mathsf{sign.trans}_2\colon GW^-(pt) \cong \mathbf{BO}^{4,2}(pt)$.

Let $\widetilde{p}_1^{\mathbf{BO}}\colon \mathbf{BO}^{4,2}(X) \to \mathbf{BO}^{4,2}(X)$ be the map of (3).

Corollary 3.14. *For all $\alpha \in \mathbf{BO}^{4,2}(X)$ we have $\alpha = \widetilde{p}_1^{\mathbf{BO}}(\alpha) + \mathsf{h}\prod_i \frac{1}{2}(\mathrm{rk}_{X_i}\alpha)1_{X_i}$.*

4. Symplectically oriented commutative ring T-spectra

Embed $\mathsf{H}^{\oplus n} \subset \mathsf{H}^{\oplus\infty}$ as the direct sum of the first n summands. The ensuing filtration $\mathsf{H} \subset \mathsf{H}^{\oplus 2} \subset \mathsf{H}^{\oplus 3} \subset \cdots$ for each r a direct system of schemes

$$pt = HGr(r,r) \hookrightarrow HGr(r,r+1) \hookrightarrow HGr(r,r+2) \hookrightarrow \cdots .$$

The ind-scheme and motivic space

$$BSp_{2r} = HGr(r,\infty) = \mathrm{colim}_{n\geq r} HGr(r,n)$$

is pointed by $h_r\colon pt = HGr(r,r) \hookrightarrow BSp_{2r}$. Each $HGr(r,n)$ has a tautological symplectic subbundle $(\mathcal{U}_{HGr(r,n)}, \phi_{HGr(r,n)})$, and their colimit is an ind-scheme $\mathcal{U}_{BSp_{2r}}$ which is a vector bundle over the ind-scheme BSp_{2r}. It has a Thom space $\mathrm{Th}\,\mathcal{U}_{BSp_{2r}}$ just like for ordinary schemes. We write

$$\mathbf{MSp}_{2r} = \mathrm{Th}\,\mathcal{U}_{BSp_{2r}} = \mathrm{Th}\,\mathcal{U}_{HGr(r,\infty)} = \mathrm{colim}_{n\geq r}\,\mathrm{Th}\,\mathcal{U}_{HGr(r,n)}.$$

We refer the reader to [6, §6] for the complete construction of $\mathbf{MSp}$ as a commutative monoid in the category of symmetric $T^{\wedge 2}$-spectra. The unit comes from the pointings $h_r\colon pt \hookrightarrow BSp_{2r}$, which induce canonical inclusions of Thom spaces

$$e_r\colon T^{\wedge 2r} \hookrightarrow \mathbf{MSp}_{2r}.$$

Let (A,μ,e) be a commutative ring T-spectrum. The unit of the monoid defines the unit element $1_A \in A^{0,0}(pt_+)$. Applying the T-suspension isomorphism twice gives an element $\Sigma_T^2 1_A \in A^{4,2}(T^{\wedge 2}) = A^{4,2}(\mathrm{Th}\,\mathbf{A}^2)$.

Definition 4.1. A *symplectic Thom orientation* on a commutative ring T-spectrum (A,μ,e) is an element $th \in A^{4,2}(\mathbf{MSp}_2) = A^{4,2}(\mathrm{Th}\,\mathcal{U}_{HP^\infty})$ with $th|_{T^{\wedge 2}} = \Sigma_T^2 1_A \in A^{4,2}(T^{\wedge 2})$.

The element th should be regarded as the symplectic Thom class of the tautological quaternionic line bundle $\mathcal{U}_{HP^\infty}$ over HP^∞.

Example 4.2. The *standard symplectic Thom orientation* on algebraic symplectic cobordism is the element $th^{\mathbf{MSp}} = u_2 \in \mathbf{MSp}^{4,2}(\mathbf{MSp}_2)$ corresponding to the canonical map $u_2\colon \Sigma_T^\infty \mathbf{MSp}_2(-2) \to \mathbf{MSp}$ described in (1).

The main theorem of [6] gives seven other structures containing the same information as a symplectic Thom orientation. First:

Theorem 4.3 ([6, Theorem 10.2])**.** *Let (A,μ,e) be a commutative monoid in $SH(S)$. There is a canonical bijection between the sets of*

(a) *symplectic Thom structures on the ring cohomology theory $A^{*,*}$ such that for the trivial rank 2 symplectic bundle H over pt we have $th(\mathsf{H}) = \Sigma_T^2 1_A$ in $A^{4,2}(T^{\wedge 2})$, and*

(α) *symplectic Thom orientations on (A,μ,e).*

Thus a symplectic Thom orientation determines Thom and Pontryagin classes for all symplectic bundles.

Lemma 4.4. *In the standard special linear and symplectic Thom structures on* $\mathbf{BO}$ *we have $th(\mathbf{A}^1,1) = \Sigma_T 1_{\mathbf{BO}}$ and $th(\mathsf{H}) = \Sigma_T^2 1_{\mathbf{BO}}$.*

Proof. The structural maps $\mathbf{KO}^{[n]} \wedge T \to \mathbf{KO}^{[n+1]}$ of the spectrum are by definition [7, §8] adjoint to maps $\mathbf{KO}^{[n]} \to \mathbf{Hom}_\bullet(T, \mathbf{KO}^{[n+1]})$ which are fibrant replacements of maps of simplicial presheaves

$$(-\boxtimes (K(\mathcal{O}), \Theta(\mathcal{O},1)))_* \colon KO^{[n]}(-) \to KO^{[n+1]}(-\wedge T)$$

which act on the homotopy groups as $-\cup [K(\mathcal{O}), \Theta(\mathcal{O},1)] = -\cup th(\mathbf{A}^1,1)$. So we have $\Sigma_T 1_{\mathbf{BO}} = th(\mathbf{A}^1,1)$. It then follows that we have $th(\mathsf{H}) = th(\mathbf{A}^1,1)^{\cup 2} = \Sigma_T^2 1_{\mathbf{BO}}$. □

The standard symplectic Thom structure on $\mathbf{BO}$ thus satisfies the normalization condition of Theorem 4.3. It corresponds to the *standard symplectic Thom orientation* on hermitian K-theory $th^{\mathbf{BO}} \in \mathbf{BO}^{4,2}(\mathbf{MSp}_2)$. It is given by the formulas of Definition 3.12 for $(E,\phi) = (\mathcal{U}_{HP^\infty}, \phi_{HP^\infty})$ tautological subbundle on $HP^\infty = BSp_2$.

A *symplectically oriented commutative T-ring spectrum* is a pair (A,ϑ) with A a commutative monoid in $SH(S)$ and ϑ a symplectic Thom orientation on A. We could write the associated Thom and Pontryagin classes as $th^\vartheta(E,\phi)$ and $p_i^\vartheta(E,\phi)$.

A *morphism of symplectically oriented commutative T-ring spectra* $\varphi\colon (A,\vartheta) \to (B,\varpi)$ is a morphism of commutative monoids with $\varphi(\vartheta) = \varpi$. For such a φ one has $\varphi(th^\vartheta(E,\phi)) = th^\varpi(E,\phi)$ and $\varphi(p_i^\vartheta(E,\phi)) = p_i^\varpi(E,\phi)$ for all symplectic bundles.

Theorem 4.5 (Universality of $\mathbf{MSp}$)**.** *Let (A,μ,e) be a commutative monoid in $SH(S)$. The assignments $\varphi \mapsto \varphi(th^{\mathbf{MSp}})$ gives a bijection between the sets of*

(ε) *morphisms $\varphi\colon (\mathbf{MSp}, \mu_{\mathbf{MSp}}, e_{\mathbf{MSp}}) \to (A,\mu,e)$ of commutative monoids in $SH(S)$, and*

(α) *symplectic Thom orientations on (A,μ,e).*

This is [6, Theorems 12.3, 13.2]. Thus $(\mathbf{MSp}, th_{\mathbf{MSp}})$ is the universal symplectically oriented commutative T-ring spectrum.

Let $\varphi\colon (A,\vartheta) \to (B,\varpi)$ be a morphism of symplectically oriented commutative T-ring spectra. For a space X the isomorphisms $X \wedge pt_+ \cong X \cong pt_+ \wedge X$ make $A^{*,*}(X)$ into a two-sided module over the ring $A^{*,*}(pt)$ and into a bigraded-commutative algebra over the commutative ring $A^{4*,2*}(pt)$. The morphism φ induces morphisms of graded rings

$$\begin{aligned} \bar\varphi_X \colon A^{*,*}(X) \otimes_{A^{4*,2*}(pt)} B^{4*,2*}(pt) &\to B^{*,*}(X) \\ \bar\varphi_X \colon A^{4*,2*}(X) \otimes_{A^{4*,2*}(pt)} B^{4*,2*}(pt) &\to B^{4*,2*}(X) \end{aligned} \tag{5}$$

which are natural in X, with the pullbacks acting on the left side of the $\otimes$.

Theorem 4.6 (Weak quaternionic cellularity of $\mathbf{MSp}_{2r}$)**.** *Let $\varphi\colon (A,\vartheta) \to (B,\varpi)$ be a morphism of symplectically oriented commutative T-ring spectra. Then for all r the natural morphism of graded rings*

$$\bar\varphi_{\mathbf{MSp}_{2r}} \colon A^{4*,2*}(\mathbf{MSp}_{2r}) \otimes_{A^{4*,2*}(pt)} B^{4*,2*}(pt) \to B^{4*,2*}(\mathbf{MSp}_{2r})$$

is an isomorphism.

Proof. Let $t_1, \dots, t_r$ be independent indeterminates with t_i of bidegree $(4i, 2i)$. By [6, Theorems 9.1, 9.2, 9.3] there is a commutative diagram of isomorphisms

$$\begin{array}{ccc} A^{*,*}(pt)[[t_1,\dots,t_r]]^{hom} & \xrightarrow[\cong]{t_i \mapsto p_i^{\vartheta}(\mathcal{U}_{BSp_{2r}}, \phi_{BSp_{2r}})} & A^{*,*}(BSp_{2r}) \\ \times t_r \downarrow \cong & & \cong \downarrow \cup th^{\vartheta}(\mathcal{U}_{BSp_{2r}}, \phi_{BSp_{2r}}) \\ t_r A^{*,*}(pt)[[t_1,\dots,t_r]]^{hom} & \xrightarrow[\cong]{} & A^{*+4r,*+2r}(\mathbf{MSp_{2r}}) \end{array}$$

The notation on the left refers to homogeneous formal power series. There is a similar diagram for (B, ϖ). The maps $\varphi\colon A^{*,*} \to B^{*,*}$ commute with the maps of the two diagrams because φ sends the Thom and Pontryagin classes of (A, ϑ) onto the Thom and Pontryagin classes of (B, ϖ). The morphism $\bar{\varphi}^{4*,2*}_{\mathbf{MSp}_{2r}}$ is an isomorphism because

$$t_r A^{4*,2*}(pt)[[t_1,\dots,t_r]]^{hom} \otimes_{A^{4*,2*}(pt)} B^{4*,2*}(pt) \to t_r B^{4*,2*}(pt)[[t_1,\dots,t_r]]^{hom}$$

is an isomorphism. □

5. Where the class p_1 takes the place of honour

We suppose that $(\mathrm{U}, \mathrm{u}) \to (\mathbf{BO}, th^{\mathbf{BO}})$ is a morphism of symplectically oriented commutative ring T-spectra. We set

$$\bar{\mathrm{U}}^{*,*}(X) = \mathrm{U}^{*,*}(X) \otimes_{\mathrm{U}^{4*,2*}(pt)} \mathbf{BO}^{4*,2*}(pt),$$
$$\bar{\mathrm{U}}^{4*,2*}(X) = \mathrm{U}^{4*,2*}(X) \otimes_{\mathrm{U}^{4*,2*}(pt)} \mathbf{BO}^{4*,2*}(pt),$$

and we write $\bar{\varphi}_X$ for the morphisms of (5).

Theorem 5.1. *Let $(\mathrm{U}, \mathrm{u}) \to (\mathbf{BO}, th^{\mathbf{BO}})$ be a morphism of symplectically oriented commutative ring T-spectra. Suppose there exists an N such that for all $n \geq N$ the maps $\bar{\varphi}_{\mathrm{U}_{2n}}\colon \bar{\mathrm{U}}^{4i,2i}(\mathrm{U}_{2n}) \to \mathbf{BO}^{4i,2i}(\mathrm{U}_{2n})$ are isomorphisms for all i. Then for all small pointed motivic spaces X and all (p,q) the homomorphism $\bar{\varphi}_X\colon \bar{\mathrm{U}}^{p,q}(X) \to \mathbf{BO}^{p,q}(X)$ is an isomorphism.*

Before turning to the theorem itself, we prove a series of lemmas. The first three demonstrate the significance of the first Pontryagin class for this problem.

Lemma 5.2. *The functorial map $\bar{\varphi}_X\colon \bar{\mathrm{U}}^{4,2}(X) \to \mathbf{BO}^{4,2}(X)$ has a section s_X which is functorial in X.*

Proof. Write $HGr = \operatorname{colim}_r HGr(r, \infty)$. According to Theorem [7, Theorem 10.1, (11.1)] there is an isomorphism *à la* Morel-Voevodsky $\bar{\tau}\colon (\mathbb{Z} \times HGr, (0, x_0)) \cong \mathbf{KSp}$ in $H_\bullet(S)$ such that the restrictions are

$$\bar{\tau}|_{\{i\} \times HGr(n,2n)} = [\mathcal{U}_{HGr(n,2n)}, \phi_{HGr(n,2n)}] + (i-n)[\mathsf{H}]$$

in $KSp_0(HGr(n,2n)) = GW^-(HGr(n,2n))$. Composing with the isomorphisms in $H_\bullet(S)$

$$(\mathbb{Z} \times HGr, (0, x_0)) \xrightarrow{\bar{\tau}} \mathbf{KSp} \xrightarrow{\mathsf{trans}_1} \mathbf{KO}^{[2]} \xrightarrow{-1} \mathbf{KO}^{[2]}.$$

where the trans_1 comes from the translation functor $(\mathcal{F}, \phi) \mapsto (\mathcal{F}[1], \phi[2])$, and the -1 is the inverse operation of the H-space structure. It gives us an element

$$\tau_2 \in \mathbf{KO}_0^{[2]}(\mathbb{Z} \times HGr, (0, x_0)) = \mathbf{BO}^{4,2}(\mathbb{Z} \times HGr, (0, x_0))$$

corresponding to the composition. By Corollary 3.14 we have

$$\tau_2|_{\{i\}\times HGr(n,2n)} = p_1(\mathcal{U}_{HGr(n,2n)}, \phi_{HGr(n,2n)}) + i\mathsf{h}.$$

For any symplectically oriented cohomology theory $A^{*,*}$ we have [7, (9.3)]

$$A^{*,*}(\mathbb{Z}\times HGr) = \big(A^{*,*}(pt)[[p_1,p_2,p_3,\dots]]^{hom}\big)^{\times\mathbb{Z}}.$$

For such a theory let

$$\tfrac{1}{2}\mathrm{rk}^A = (i1_{HGr})_{i\in\mathbb{Z}} \in A^{0,0}(\mathbb{Z}\times HGr), \qquad p_1^A = (p_1)_{i\in\mathbb{Z}} \in A^{4,2}(\mathbb{Z}\times HGr)$$

Then $\tau_2 = p_1^{\mathbf{BO}} + \frac{1}{2}\mathrm{rk}^{\mathbf{BO}}\mathsf{h}$. Consider the element

$$s = p_1^{\mathrm{U}}\otimes 1_{\mathbf{BO}} + \tfrac{1}{2}\mathrm{rk}^{\mathrm{U}}\otimes\mathsf{h} \in \bar{\mathrm{U}}^{4,2}(\mathbb{Z}\times HGr).$$

Clearly one has $\bar{\varphi}(s) = \tau_2$. The element s may be regarded as a morphism of functors $Hom_{H_\bullet(S)}(-,\mathbb{Z}\times HGr)\to\bar{\mathrm{U}}^{4,2}(-)$ by the Yoneda lemma. The composite map

$$Hom_{H_\bullet(S)}(-,\mathbb{Z}\times HGr)\xrightarrow{s}\bar{\mathrm{U}}^{4,2}(-)\xrightarrow{\bar{\varphi}}\mathbf{BO}^{4,2}(-)$$

coincides with a functor transformation given by the adjoint $\Sigma_T^\infty(\mathbb{Z}\times HGr)(-2)\to\mathbf{BO}$ of the motivic weak equivalence $\tau_2\colon\mathbb{Z}\times HGr\to\mathbf{KO}^{[2]}$. Thus for every pointed motivic space X the map

$$s_X\colon\mathbf{BO}^{4,2}(X) = Hom_{H_\bullet(S)}(X,\mathbf{KO}^{[2]}) = Hom_{H_\bullet(S)}(X,\mathbb{Z}\times HGr)\xrightarrow{s}\bar{\mathrm{U}}^{4,2}(X)$$

is a section of the map $\bar{\varphi}_X\colon\bar{\mathrm{U}}^{4,2}(X)\to\mathbf{BO}^{4,2}(X)$ which is natural in X. □

Lemma 5.3. *For any integer i the functorial map $\bar{\varphi}_X\colon\bar{\mathrm{U}}^{8i+4,4i+2}(X)\to\mathbf{BO}^{8i+4,4i+2}(X)$ has a section t_X which is functorial in X.*

Proof. We have $\mathbf{BO}^{8*+4,4*+2} = \mathbf{BO}^{4,2}[\beta_8,\beta_8^{-1}]$ for the periodicity element $\beta_8\in\mathbf{BO}^{8,4}(pt)$ of Definition 3.11. So any element of $\mathbf{BO}^{8*+4,4*+2}(X)$ may be written uniquely in the form $a\cup\beta_8^i$ with $a\in\mathbf{BO}^{4,2}(X)$ and $i\in\mathbb{Z}$. We define

$$t_X(a\cup\beta_8^i) = s_X(a)\cup(1_{\mathrm{U}}\otimes\beta_8^i)\in\bar{\mathrm{U}}^{8*+4,4*+2}(X).$$

Then t_X is a section of $\bar{\varphi}_X$ which is natural in X. □

Lemma 5.4. *If X is a small pointed motivic space and i is an integer, then for any $\alpha\in\bar{\mathrm{U}}^{4i,2i}(X)$ there exists an $n\geq 0$ with $t_{X\wedge T^{\wedge 2n}}\circ\bar{\varphi}_{X\wedge T^{\wedge 2n}}(\Sigma_T^{2n}\alpha) = \Sigma_T^{2n}\alpha$.*

Proof. We may assume that $\alpha = a\otimes b$ with $a\in\mathrm{U}^{4d,2d}(X)$ and $b\in\mathbf{BO}^{4i-4d,2i-2d}(pt)$. For a small motivic space X there is a canonical isomorphism [12, Theorem 5.2]

$$\mathrm{U}^{4d,2d}(X) = \mathrm{colim}_m\, Hom_{H_\bullet(S)}(X\wedge T^{\wedge m},\mathrm{U}_{2d+m}).$$

This isomorphism implies that there exists an integer $n\geq 0$ such that $\Sigma_T^{2n}a = f^*[u_{2d+2n}]$ for an appropriate map $f\colon X\wedge T^{\wedge 2n}\to\mathrm{U}_{2d+2n}$ in $H_\bullet(S)$. We may assume that $d+n\geq N$ and that $n+i$ is odd.

We have $[u_{2d+2n}]\otimes b\in\bar{\mathrm{U}}^{4n+4i,2n+2i}(\mathrm{U}_{2d+2n})$. By hypothesis

$$\bar{\varphi}_{\mathrm{U}_{2d+2n}}\colon\bar{\mathrm{U}}^{4n+4i,2n+2i}(\mathrm{U}_{2d+2n})\to\mathbf{BO}^{4n+4i,2n+2i}(\mathrm{U}_{2d+2n})$$

is an isomorphism. So its section $t_{\mathrm{U}_{2d+2n}}$ is the inverse isomorphism. Hence we have

$$(t_{\mathrm{U}_{2d+2n}}\circ\bar{\varphi}_{\mathrm{U}_{2d+2n}})([u_{2d+2n}]\otimes b) = [u_{2d+2n}]\otimes b.$$

Then by the functoriality of $\bar{\mathrm{U}}$, t and $\bar{\varphi}$ we have

$$\Sigma_T^{2n}\alpha = f^*([u_{2d+2n}]\otimes b) = f^*\circ t_{\mathrm{U}_{2d+2n}}\circ\bar{\varphi}_{\mathrm{U}_{2d+2n}}([u_{2d+2n}]\otimes b) = t_{X\wedge T^{\wedge 2n}}\circ\bar{\varphi}_{X\wedge T^{\wedge 2n}}(\Sigma_T^{2n}\alpha). \qquad \square$$

Lemma 5.5. *Suppose for some (p,q) that the homomorphism $\bar{\varphi}_X\colon \bar{\mathrm{U}}^{p,q}(X)\to \mathbf{BO}^{p,q}(X)$ is an isomorphism for all small pointed motivic spaces X. Then the same holds for $(p-1,q)$ and $(p-1,q-1)$.*

Proof. For $(p-1,q)$ this is because the suspension Σ_{S^1} induces isomorphisms $\mathrm{U}^{p-1,q}(X)\cong \mathrm{U}^{p,q}(X\wedge S^1)$ and similar isomorphisms for $\bar{\mathrm{U}}$ and $\mathbf{BO}$, and these are compatible with φ and $\bar{\varphi}$. For $(p-1,q-1)$ use the suspension $\Sigma_{\mathbb{G}_m}$. $\square$

Proof of Theorem 5.1. First suppose $(p,q)=(8i+4,4i+2)$ for some i. Then for any small motivic space X the map $\varphi_X\colon \bar{\mathrm{U}}^{8i+4,4i+2}(X)\to \mathbf{BO}^{8i+4,4i+2}(X)$ is surjective because it has the section t_X of Lemma 5.3. To show it injective, we suppose α is in its kernel. The suspension Σ_T is compatible with φ and $\bar{\varphi}$, so we have $\bar{\varphi}_{X\wedge T^{\wedge 2n}}(\Sigma_T^{2n}\alpha)=\Sigma_T^{2n}\varphi_X(\alpha)=0$. By Lemma 5.4 we therefore also have $\Sigma_T^{2n}\alpha=0$. But Σ_T^{2n} induces an isomorphism of cohomology groups. So we have $\alpha=0$. Thus $\bar{\varphi}_X\colon \bar{\mathrm{U}}^{p,q}(X)\to \mathbf{BO}^{p,q}(X)$ is an isomorphism for all small motivic spaces X for $(p,q)=(8i+4,4i+2)$.

The result for other values of (p,q) follows from Lemma 5.5 and a numerical argument. $\square$

6. Last details

Proof of Theorem 1.1. By the universality of the symplectically oriented commutative ring T-spectrum $(\mathbf{MSp}, th^{\mathbf{MSp}})$ (Theorem 4.5) there is a unique morphism $\varphi\colon \mathbf{MSp}\to\mathbf{BO}$ of commutative ring T-spectra with $\varphi(th^{\mathbf{MSp}})=th^{\mathbf{BO}}$. It induces the morphisms of (5):

$$\bar{\varphi}_X\colon \mathbf{MSp}^{*,*}(X)\otimes_{\mathbf{MSp}^{4*,2*}(pt)}\mathbf{BO}^{4*,2*}(pt)\to\mathbf{BO}^{*,*}(X),$$
$$\bar{\varphi}_X\colon \mathbf{MSp}^{4*,2*}(X)\otimes_{\mathbf{MSp}^{4*,2*}(pt)}\mathbf{BO}^{4*,2*}(pt)\to\mathbf{BO}^{4*,2*}(X).$$

The second morphism, with the bidegrees $(4i,2i)$ only, is an isomorphism for $X=\mathbf{MSp}_{2r}$ for all r by Theorem 4.6. So all the hypotheses of Theorem 5.1 hold with $(\mathrm{U},\mathrm{u})=(\mathbf{MSp}, th^{\mathbf{MSp}})$. The conclusions of Theorem 5.1 imply Theorem 1.1. $\square$

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