On the 3-Pfister number of quadratic forms

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Abstract

For a field F of characteristic different from 2, containing a square root of -1, endowed with an $F^{\times 2}$-compatible valuation v such that the residue field has at most two square classes, we use a combinatorial analogue of the Witt ring of F to prove that an anisotropic quadratic form over F with even dimension d, trivial discriminant and Hasse-Witt invariant can be written in the Witt ring as the sum of at most $(d^2)/8$ 3-fold Pfister forms.

Introduction

Let F be a field of characteristic different from 2 which contains a square root of -1. Let $W(F)$ be the Witt ring of F, $I(F)$ the fundamental ideal of $W(F)$ and $I^m(F)$ the m-th power of $I(F)$. Since -1 is a square in F, the Witt ring $W(F)$ is an algebra over the field $\mathbb{F}_2$ with two elements.

The quadratic form $a_1X_1^2 + \ldots + a_dX_d^2$ with $a_i \in F^\times$ is denoted by $\langle a_1, \ldots, a_d\rangle$. Each $[q] \in I^m(F)$ is the sum of Witt classes of m-fold Pfister forms, i.e. quadratic forms of the type

$$\langle\langle a_1, \ldots, a_m\rangle\rangle = \langle 1, a_1\rangle \otimes \ldots \otimes \langle 1, a_m\rangle.$$

Brosnan, Reichstein and Vistoli [2010] define the *m-Pfister number* $\mathsf{Pf}_m(q)$ of a quadratic form $[q] \in I^m(F)$ to be the least number of terms needed to write the Witt class of q as a sum of Witt classes of m-fold Pfister forms. They prove the existence of quadratic forms whose 3-Pfister number increases exponentially with their dimension. In general, no upper bound is known for the 3-Pfister number.

In this paper, we shall prove that for a valued field (F, v) such that $1 + \mathfrak{m}_v \subset F^{\times 2}$ and $(\overline{F}_v^\times : \overline{F}_v^{\times 2}) \leq 2$, the 3-Pfister number of a quadratic form in $I^3(F)$ of dimension d is less than or equal to $(d^2)/8$. This result on 3-Pfister numbers extends those on the 1- and 2-Pfister numbers in [Parimala et al., 2009]. Indeed, our proof technique is similar: we use the group algebra $\mathbb{F}_2[F^\times/F^{\times 2}]$ as a combinatorial analogue of the Witt ring of F, allowing us to make explicit computations.

However, our result does not hold for general fields: the combinatorial analogue – which works for the computation of 1- and 2-Pfister numbers over general fields, see [Parimala et al., 2009] – is not powerful enough to deal with 3-Pfister numbers in general. We shall indicate the obstruction.

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1. A combinatorial analogue of the Witt ring

For F a field of characteristic different from 2 which contains a square root of -1, we set $V_F := F^\times/F^{\times 2}$; it is a vector space over $\mathbb{F}_2$, so let $\mathbb{F}_2[V_F]$ denote its group algebra. This algebra can be seen as a combinatorial analogue of the Witt ring for the following reason: it is a discrete object which only depends on the square classes of F such that the map

$$\Psi\colon \mathbb{F}_2[V_F] \to W(F)\colon (a_1F^{\times 2}) + \ldots + (a_dF^{\times 2}) \mapsto [\langle a_1, \ldots, a_d\rangle]$$

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is a well-defined surjective $\mathbb{F}_2$-algebra homomorphism. Moreover, in the case where F is the field of iterated Laurent series $\mathbb{C}((x_1))\dots((x_n))$, this homomorphism is in fact an isomorphism [Parimala et al., 2009, p. 333].

It is natural to study a more general situation: the group algebra $\mathbb{F}_2[V]$ with V any vector space over $\mathbb{F}_2$. Since the addition in V induces the multiplication in $\mathbb{F}_2[V]$, we write X^v for the image of $v \in V$ in $\mathbb{F}_2[V]$, so that

$$\mathbb{F}_2[V] = \Big\{ \sum_{v\in V} \alpha_v X^v \mid \alpha_v \in \mathbb{F}_2 \text{ and } \{v \in V \mid \alpha_v \neq 0\} \text{ is finite} \Big\}$$

where $X^0 = 1$ and $X^u \cdot X^v = X^{u+v}$.

We now define an ideal of $\mathbb{F}_2[V]$, which is an analogue of the fundamental ideal of $W(F)$. Let $\epsilon_0\colon \mathbb{F}_2[V] \to \mathbb{F}_2$ be the $\mathbb{F}_2$-algebra homomorphism defined by

$$\epsilon_0(\sum_{v\in V} \alpha_v X^v) = \sum_{v\in V} \alpha_v$$

and let $I[V]$ be its kernel. It has the property that for all $\xi \in \mathbb{F}_2[V_F]$,

$$\xi \in I[V_F] \iff \Psi(\xi) \in I(F).$$

For $m \geq 1$, let $I^m[V]$ denote the m-th power of $I[V]$; it is generated as a group by elements of the type

$$[\![v_1, \dots, v_m]\!] := (1 + X^{v_1}) \cdots (1 + X^{v_m}),$$

which we call *m-fold Pfister elements*. The *m-Pfister number* $\mathsf{Pf}_m(\xi)$ of an element $\xi \in I^m[V]$ is the least number of terms needed to write ξ as a sum of m-fold Pfister elements. It is related to the m-Pfister number of a quadratic form in the following way: for all $\xi \in I^m[V_F]$,

$$\mathsf{Pf}_m(\Psi(\xi)) \leq \mathsf{Pf}_m(\xi).$$

This is so because Ψ carries m-fold Pfister elements in $\mathbb{F}_2[V_F]$ to m-fold Pfister forms in $W(F)$. Furthermore, we define the *m-scaled Pfister number* of an element $\xi \in I^m[V]$, and we write $\mathsf{GPf}_m(\xi)$, as the least number of terms needed to write ξ as a sum of *scaled m-fold Pfister elements*, i.e. elements of the form $X^a[\![b_1, \dots, b_m]\!]$. Clearly $\mathsf{GPf}_m(\xi) \leq \mathsf{Pf}_m(\xi)$ and since $X^a(1+X^b) = (1+X^a)+(1+X^{a+b})$, we have $\mathsf{Pf}_m(\xi) \leq 2\mathsf{GPf}_m(\xi)$.

In this section, we shall first study the structure of $I^3[V]$ and, by way of example, compute the 3-Pfister number of certain of its elements. Next we exhibit an algorithm which allows us to classify the elements in $I^3[V]$. Finally, we use the scaled Pfister numbers to find an upper bound for the 3-Pfister number of an element in $I^3[V]$.

1.1. Structure of $I^3[V]$

We shall prove that $I^3[V]$ is the kernel of some group homomorphism. By [Parimala et al., 2009, Corollary 1.2],

$$I^2[V] = \{\xi \in I[V] \mid \epsilon_1(\xi) = 0\},$$

where $\epsilon_1\colon \mathbb{F}_2[V] \to V$ is the group homomorphism defined by $\epsilon_1(\sum_{v\in V} \alpha_v X^v) = \sum_{v\in V} \alpha_v v$. We define $\epsilon_2\colon \mathbb{F}_2[V] \to V \wedge V$ by

$$\epsilon_2(X^{v_1} + \ldots + X^{v_r}) = \sum_{1\leq i<j\leq r} v_i \wedge v_j.$$

To prove that ϵ_2 is well-defined it is sufficient to check the obvious fact that if $v_r = v_{r-1}$, then

$$\sum_{1\leq i<j\leq r-2} v_i \wedge v_j = \sum_{1\leq i<j\leq r} v_i \wedge v_j.$$

Even though the map ϵ_2 is not a group homomorphism, the following property holds:

Lemma 1.1 *Let $\xi_1, \xi_2 \in \mathbb{F}_2[V]$ be such that either $\epsilon_1(\xi_1+\xi_2)=0$ or $\epsilon_1(\xi_1)=0$ or $\epsilon_1(\xi_2)=0$. Then $\epsilon_2(\xi_1+\xi_2)=\epsilon_2(\xi_1)+\epsilon_2(\xi_2)$. In particular, $\epsilon_2|_{I^2[V]}$ is a group homomorphism.*

Proof : Write $\xi_1 = X^{v_1}+\ldots+X^{v_r}$ and $\xi_2 = X^{w_1}+\ldots+X^{w_s}$. Then

$$\begin{aligned}\epsilon_2(\xi_1+\xi_2) &= \sum_{1\le i<j\le r} v_i\wedge v_j + \sum_{i=1}^{r}\sum_{j=1}^{s} v_i\wedge w_j + \sum_{1\le i<j\le s} w_i\wedge w_j \\ &= \epsilon_2(\xi_1)+\epsilon_1(\xi_1)\wedge\epsilon_1(\xi_2)+\epsilon_2(\xi_2).\end{aligned}$$

By assumption, $\epsilon_1(\xi_1)\wedge\epsilon_1(\xi_2)=0$, hence $\epsilon_2(\xi_1+\xi_2)=\epsilon_2(\xi_1)+\epsilon_2(\xi_2)$. □

Observe that the image of a 3-fold Pfister element by ϵ_2 is zero, hence

$$I^3[V]\subset\{\xi\in I^2[V]\mid \epsilon_2(\xi)=0\}.$$

We shall prove that the inclusion is in fact an equality. First we define the *support* of an element $\xi=\sum_{v\in V}\alpha_v X^v$ as the set

$$D(\xi):=\{v\in V\mid \alpha_v=1\}.$$

Let $\xi\in I^2[V]$ be a nonzero element such that $\epsilon_2(\xi)=0$. Let $a\in V\setminus\{0\}$ and W a subspace of V be such that $\mathsf{span}_{\mathbb{F}_2}\langle D(\xi)\rangle = W\oplus\mathbb{F}_2\cdot a$ (where $\mathsf{span}_{\mathbb{F}_2}\langle D(\xi)\rangle$ is the subspace of V spanned by $D(\xi)$). Then there exist unique $\eta_1,\eta_2\in\mathbb{F}_2[W]$ such that $\xi=\eta_1+X^a\cdot\eta_2$.

Lemma 1.2 *We have $\eta_2, \eta_1+\eta_2\in I^2[W]$ and $\epsilon_2(\eta_1+\eta_2)=0$.*

Proof : Since $\xi\in I^2[W]$, we have $\epsilon_0(\xi)=0$ and $\epsilon_1(\xi)=0$. The map ϵ_0 is a ring homomorphism, so

$$0=\epsilon_0(\xi)=\epsilon_0(\eta_1)+\epsilon_0(\eta_2)$$

and thus $\epsilon_0(\eta_1)=\epsilon_0(\eta_2)$. Because ϵ_1 is a group homomorphism, we have

$$0=\epsilon_1(\xi)=\epsilon_1(\eta_1)+\epsilon_1(X^a\cdot\eta_2).$$

Observe that

$$\epsilon_1(X^a\cdot\eta_2)=\sum_{v\in D(\eta_2)} a+v = |D(\eta_2)|a+\epsilon_1(\eta_2)=\epsilon_0(\eta_2)a+\epsilon_1(\eta_2);$$

hence $\epsilon_1(\eta_1)+\epsilon_1(\eta_2)+\epsilon_0(\eta_2)a=0$ where $\epsilon_1(\eta_1)+\epsilon_1(\eta_2)\in W$. It implies that $\epsilon_1(\eta_1)+\epsilon_1(\eta_2)=0$ and $\epsilon_0(\eta_2)=0$. Therefore $\eta_1+\eta_2\in I^2[W]$ and $\eta_2\in I[W]$. Write

$$\eta_1 = X^{v_1}+\ldots+X^{v_r} \quad\text{and}\quad \eta_2=X^{w_1}+\ldots+X^{w_s}.$$

By Lemma 1.1, $\epsilon_2(\xi)=\epsilon_2(\eta_1)+\epsilon_2(X^a\cdot\eta_2)$. But

$$\begin{aligned}\epsilon_2(X^a\cdot\eta_2) &= \sum_{1\le i<j\le s}(a+w_i)\wedge(a+w_j)\\ &= \sum_{j=2}^{s}(j-1)a\wedge w_j + \sum_{i=1}^{s-1} w_i\wedge(s-i)a + \sum_{1\le i<j\le s} w_i\wedge w_j\\ &= \sum_{i=1}^{s}(s-1)a\wedge w_i + \epsilon_2(\eta_2)\\ &= a\wedge\epsilon_1(\eta_2)+\epsilon_2(\eta_2).\end{aligned}$$

Thus $\epsilon_2(\xi)=0$ implies $\epsilon_2(\eta_1)+\epsilon_2(\eta_2)+a\wedge\epsilon_1(\eta_2)=0$ where $\epsilon_2(\eta_1)+\epsilon_2(\eta_2)\in W\wedge W$ and $\epsilon_1(\eta_2)\in W$. Hence $\epsilon_2(\eta_1)+\epsilon_2(\eta_2)=0$ and $a\wedge\epsilon_1(\eta_2)=0$. It implies that $\epsilon_2(\eta_1+\eta_2)=0$ and $\epsilon_1(\eta_2)=0$. Therefore $\eta_2, \eta_1+\eta_2\in I^2[W]$ and $\epsilon_2(\eta_1+\eta_2)=0$. □

Corollary 1.3 *We have* $I^3[V] = \{\xi \in I^2[V] \mid \epsilon_2(\xi) = 0\}$.

Proof : Let $\xi \in I^2[V]$ be such that $\epsilon_2(\xi) = 0$. We shall prove that $\xi \in I^3[V]$ by induction on $m := \dim \mathsf{span}_{\mathbb{F}_2}\langle D(\xi)\rangle$, the dimension of $\mathsf{span}_{\mathbb{F}_2}\langle D(\xi)\rangle$. If $m = 0$, then $\xi = 0$ (by [Parimala et al., 2009, Lemma 1.1], $\xi \in I^2[V] \setminus \{0\}$ implies $|D(\xi)| \geq 4$). Suppose that $m > 0$, then $m \geq 2$ because $|D(\xi)| \geq 4$. Let $a \in V \setminus \{0\}$ and W be a subspace of V such that $\mathsf{span}_{\mathbb{F}_2}\langle D(\xi)\rangle = W \oplus \mathbb{F}_2 \cdot a$. Let $\eta_1, \eta_2 \in \mathbb{F}_2[W]$ be such that $\xi = \eta_1 + X^a \cdot \eta_2$. Then $\xi = (\eta_1 + \eta_2) + (1 + X^a) \cdot \eta_2$ and by Lemma 1.2, $\eta_2, \eta_1 + \eta_2 \in I^2[W]$ and $\epsilon_2(\eta_1 + \eta_2) = 0$. Since $\dim\langle D(\eta_1 + \eta_2)\rangle \leq \dim W < m$, by induction assumption, $\eta_1 + \eta_2 \in I^3[W]$. But $(1 + X^a) \cdot \eta_2 \in I^3[V]$ because $\eta_2 \in I^2[W]$, so $\xi \in I^3[V]$. □

1.2. Computation of 3-Pfister numbers

Let $e = (e_1, \ldots, e_n)$ be a family of linearly independent vectors of V with $n \geq 3$. Set $e_0 := e_1 + \ldots + e_n$ and

$$\eta_e := n + X^{e_1} + \ldots + X^{e_n} + nX^{e_0} + X^{e_0+e_1} + \ldots + X^{e_0+e_n}.$$

Then $|D(\eta_e)| = 2n$ if n is even and $|D(\eta_e)| = 2(n + 1)$ otherwise. Set

$$\xi = n + X^{e_1} + \ldots + X^{e_{n-1}} + X^{e_1+\ldots+e_{n-1}},$$

then $\xi \in I^2[V]$ and $\eta_e = (1 + X^{e_0}) \cdot \xi$, hence $\eta_e \in I^3[V]$. By [Parimala et al., 2009, Proposition 1.4], $\mathsf{Pf}_2(\xi) = n - 2$, thus $\mathsf{Pf}_3(\eta_e) \leq n - 2$. The following proposition proves that we have in fact an equality.

Proposition 1.4 $\mathsf{Pf}_3(\eta_e) = n - 2$.

Proof : We prove the equality by induction on n. Suppose that $n = 3$, then $\eta_e = [\![e_1, e_2, e_3]\!]$ and $\mathsf{Pf}_3(\eta_e) = 1$. Suppose now that $n > 3$. Let W be a subspace of V containing $e_1, \ldots, e_{n-1}$ such that $V = W \oplus \mathbb{F}_2 \cdot e_n$. Let $\varphi\colon V \to V$ be the linear map defined by $\varphi(x) = x$ for all $x \in W$ and $\varphi(e_n) = 0$. Let $\varphi_*\colon \mathbb{F}_2[V] \to \mathbb{F}_2[V]$ be the $\mathbb{F}_2$-algebra homomorphism induced by φ. Then $\varphi_*(\eta_e) = \eta_{e'}$, where $e' = (e_1, \ldots, e_{n-1})$. Suppose that $\eta_e = \pi_1 + \ldots + \pi_p$ where the π_i's are 3-fold Pfister elements and $p = \mathsf{Pf}_3(\eta_e)$. Renumbering if necessary, we may assume that $e_n \in D(\pi_p)$ and thus $\pi_p = [\![e_n, a, b]\!]$ for some $a, b \in V$. Observe that $\varphi_*(\pi_i)$ is a 3-fold Pfister element, for all $i \in \{1, \ldots, p\}$ and $\varphi_*(\eta_e) = \varphi_*(\pi_1) + \ldots + \varphi_*(\pi_{p-1})$. Therefore $\mathsf{Pf}_3(\eta_{e'}) \leq p - 1$. By induction assumption, $\mathsf{Pf}_3(\eta_{e'}) = n - 3$, hence $p \geq n - 2$. Since the other inequality holds, the proof is complete. □

1.3. Algorithm and classification

We shall give an algorithm to classify elements in $I^3[V]$ which is based on the following lemma.

Lemma 1.5 *Let $\xi \in I^3[V]$ be nonzero and let $a \in D(\xi) \setminus \{0\}$ and W be a subspace of V such that $\mathsf{span}_{\mathbb{F}_2}\langle D(\xi)\rangle = W \oplus \mathbb{F}_2 \cdot a$. Then there exist unique $\xi_1 \in I^3[W]$ and $\xi_2 \in I^2[W]$ such that $\xi = \xi_1 + (1 + X^a)\xi_2$. Moreover, the elements ξ_1 and ξ_2 satisfy the following properties:*

1. $0 \in D(\xi_2)$;
2. $|D(\xi_1)| = |D(\xi)| - 2|D(\xi_1 + \xi_2) \cap D(\xi_2)|$;
3. $0 \in D(\xi) \iff 0 \in D(\xi_1 + \xi_2) \iff 0 \notin D(\xi_1)$.

In particular, $|D(\xi_1)| < |D(\xi)|$ if $0 \in D(\xi)$.

Proof: By Lemma 1.2, the elements $\eta_1, \eta_2 \in \mathbb{F}_2[W]$ such that $\xi = \eta_1 + X^a \cdot \eta_2$ satisfy $\eta_1 + \eta_2 \in I^3[W]$ and $\eta_2 \in I^2[W]$. Hence $\xi = \xi_1 + (1 + X^a)\xi_2$ with $\xi_1 = \eta_1 + \eta_2 \in I^3[W]$ and $\xi_2 = \eta_2 \in I^2[W]$. □

These properties are the key to the construction of an algorithm to classify the elements in $I^3[V]$, which goes as follows. Assuming we have already classified all the elements $\xi \in I^3[V]$ such that $|D(\xi)| < d$, where $d > 0$, we classify first all the $\xi \in I^3[V]$ such that $|D(\xi)| = d$ and $0 \in D(\xi)$. To do so, we fix a nonzero $a \in V$ and a vector space W such that $V = W \oplus \mathbb{F}_2 \cdot a$. Then we choose $\xi_1 \in I^3[W]$ such that $|D(\xi_1)| < d$ and $0 \notin D(\xi_1)$, and we set

$$r := \frac{d - |D(\xi_1)|}{2} - 1$$

($r \in \mathbb{N}$ because d and $|D(\xi_1)|$ are even). We construct now an element $\xi_2 \in I^2[W]$ such that $|D(\xi_1 + \xi_2) \cap D(\xi_2)| = r + 1$. Thereto, we pick distinct nonzero $a_1, \ldots, a_r \in W \setminus D(\xi_1)$ and distinct $b_1, \ldots, b_s \in D(\xi_1)$ such that $r + s$ is odd and $a_1 + \ldots + a_r + b_1 + \ldots + b_s = 0$. Set

$$\xi_2 := X^0 + X^{a_1} + \ldots + X^{a_r} + X^{b_1} + \ldots + X^{b_s}$$

and $\xi := \xi_1 + (1 + X^a)\xi_2$. Then $\xi_2 \in I^2[W]$, so $\xi \in I^3[V]$ and

$$|D(\xi)| = |D(\xi_1 + \xi_2)| + |D(\xi_2)| = \big(|D(\xi_1)| - s + r + 1\big) + (r + 1 + s) = |D(\xi_1)| + 2(r + 1) = d.$$

Moreover all the $\xi \in I^3[V]$ such that $|D(\xi)| = d$ and $0 \in D(\xi)$ are constructed in this way. The second step is to classify all the elements $\xi \in I^3[V]$ such that $|D(\xi)| = d$ and $0 \notin D(\xi)$. We fix a nonzero $a \in V$ and a vector space W such that $V = W \oplus \mathbb{F}_2 \cdot a$. Choose $\xi_1 \in I^3[W]$ such that $|D(\xi_1)| \leq d$ and $0 \in D(\xi_1)$, and set

$$r := \frac{d - |D(\xi_1)|}{2}.$$

To construct an element $\xi_2 \in I^2[W]$ such that $|D(\xi_1 + \xi_2) \cap D(\xi_2)| = r$, we pick distinct nonzero $a_1, \ldots, a_r \in W \setminus D(\xi_1)$ and distinct nonzero $b_1, \ldots, b_s \in D(\xi_1)$ such that $r + s$ is odd and $a_1 + \ldots + a_r + b_1 + \ldots + b_s = 0$. Set

$$\xi_2 := X^0 + X^{a_1} + \ldots + X^{a_r} + X^{b_1} + \ldots + X^{b_s}$$

and $\xi := \xi_1 + (1 + X^a)\xi_2$. Then $\xi_2 \in I^2[W]$, so $\xi \in I^3[V]$ and

$$|D(\xi)| = |D(\xi_1 + \xi_2)| + |D(\xi_2)| = \big(|D(\xi_1)| - (s + 1) + r\big) + (r + s + 1) = |D(\xi_1)| + 2r = d.$$

Again, all the $\xi \in I^3[V]$ such that $|D(\xi)| = d$ and $0 \notin D(\xi)$ are constructed in this way.

Below we shall classify the elements $\xi \in I^3[V]$ with $|D(\xi)| \leq 14$. (As $|D(\xi)|$ grows, the number of different types of elements $\xi \in I^3[V]$ becomes huge.) Observe first that $|D(\xi)|$ is even for all $\xi \in I^3[V]$ because $I^3[V] \subset I[V]$. In the next proposition, we start with the elements $\xi \in I^3[V]$ such that $|D(\xi)| \leq 12$. Here we do not need our algorithm to classify those elements (even if it works perfectly) because we can use well-known results on quadratic forms.

Proposition 1.6 *Let $\xi \in I^3[V]$.*

1. *If $|D(\xi)| < 8$, then $\xi = 0$.*
2. *If $|D(\xi)| = 8$, then $\xi = X^a[\![e_1, e_2, e_3]\!]$ for some vectors $e_1, e_2, e_3, a \in V$ where e_1, e_2, e_3 are linearly independent. In particular,*

$$\mathsf{GPf}_3(\xi) = 1 \quad \textit{and} \quad \mathsf{Pf}_3(\xi) = \begin{cases} 1 & \textit{if } 0 \in D(\xi), \\ 2 & \textit{if } 0 \notin D(\xi). \end{cases}$$

3. *There is no $\eta \in I^3[V]$ such that $|D(\eta)| = 10$.*
4. *If $|D(\xi)| = 12$, then $\xi = \xi_1 \cdot \xi_2$ for some $\xi_1, \xi_2 \in \mathbb{F}_2[V]$ such that $\xi_2 \in I^2[V]$, $|D(\xi_1)| = 2$ and $|D(\xi_2)| = 6$. In particular,*

$$\mathsf{GPf}_3(\xi) = 2 \textit{ and } \quad \mathsf{Pf}_3(\xi) = \begin{cases} 3 & \textit{if } 0 \in D(\xi), \\ 2 \textit{ or } 4 & \textit{if } 0 \notin D(\xi). \end{cases}$$

Proof : Replacing V by $\mathsf{span}_{\mathbb{F}_2}\langle D(\xi)\rangle$ if necessary, we may assume that V is finite dimensional. By the proof of Theorem 2.2 in [Parimala et al., 2009], the vector space V may be identified with $V_E = E^\times/E^{\times 2}$ for some field E of iterated Laurent series so that the map $\Psi\colon \mathbb{F}_2[V_E] \to W(E)$ is an isomorphism. Let q be a quadratic form over E such that $\Psi(\xi) = [q] \in I^3(E)$ and the dimension of q is equal to $|D(\xi)|$.
(1) If $|D(\xi)| < 8$, then q is isotropic by [Lam, 2005, Hauptsatz 5.1, page 352], hence $\xi = 0$.
(2) If $|D(\xi)| = 8$, then q is a scalar multiple of a 3-fold Pfister form (see [Lam, 2005, Theorem 5.6, page 355]), hence $\xi = X^a[\![e_1, e_2, e_3]\!]$ for some $e_1, e_2, e_3, a \in V$ and $\mathsf{GPf}_3(\xi) = 1$. The vectors $e_1, e_2, e_3 \in V$ are linearly independent because otherwise $\xi = 0$. If $0 \in D(\xi)$, then $\xi = [\![e_1, e_2, e_3]\!]$ and $\mathsf{Pf}_3(\xi) = 1$; otherwise $\xi = \eta_e$ where $e = (a, a+e_1, a+e_2, a+e_3)$ is a family of 4 linearly independent vectors and in particular $\mathsf{Pf}_3(\xi) = 2$.
(3) There is no $\eta \in I^3[V]$ such that $|D(\eta)| = 10$, because if $|D(\xi)| = 10$, then q is isotropic (see [Pfister, 1966, page 123]); hence $\xi = 0$ which leads to a contradiction.
(4) Suppose that $|D(\xi)| = 12$, then by [Pfister, 1966, page 123-124], q can be written as $\phi_1 \perp \phi_2 \perp \phi_3$ where the ϕ_i's are 4-dimensional, $[\phi_i] \in I^2(E)$ and the product of the Clifford invariants $C(\phi_1)C(\phi_2)C(\phi_3)$ is trivial in the Brauer group of E. Hence there exists a quadratic extension $E(\sqrt{\delta})$ of E which splits all $C(\phi_i)$. Then $q = q_1 \otimes q_2$ where $q_1 = \langle 1, -\delta\rangle$ and for some 6-dimensional form q_2 such that $[q_2] \in I^2(E)$. Hence $\xi = \xi_1 \cdot \xi_2$ for some $\xi_1, \xi_2 \in \mathbb{F}_2[V]$ with $\xi_2 \in I^2[V]$, $|D(\xi_1)| = 2$ and $|D(\xi_2)| = 6$. We may assume that $\xi_1 = X^0 + X^a$ for some $a \in V$. Let $e_1, \dots, e_5 \in V$ be such that

$$\xi_2 = X^{e_1} + \ldots + X^{e_5} + X^{e_1+\ldots+e_5}.$$

Then $\xi = X^{e_1}[\![a, e_1+e_2, e_1+e_3]\!] + X^{e_4}[\![a, e_4+e_5, e_1+e_2+e_3+e_4]\!]$ and $\mathsf{GPf}_3(\xi) \neq 1$ because $|D(\xi)| \neq 8$; hence $\mathsf{GPf}_3(\xi) = 2$. If $0 \in D(\xi)$, then we may assume that $e_5 = 0$. Since $|D(\xi)| = 12$, the vectors $e_1, \dots, e_4, a$ are linearly independent. Then $\xi = \eta_e$ where $e = (e_1, e_2, e_3, e_4, e_1 + e_2 + e_3 + e_4 + a)$ is a family of 5 linearly independent vectors and in particular $\mathsf{Pf}_3(\xi) = 3$. If $0 \notin D(\xi)$ and $e_1, \dots, e_5, a$ are linearly independent, then $\xi = \eta_e$ where $e = (e_1, \dots, e_5, e_1 + \ldots + e_5 + a)$ is a family of 6 linearly independent vectors, so $\mathsf{Pf}_3(\xi) = 4$. Suppose that $0 \notin D(\xi)$ and $e_1, \dots, e_5, a$ are linearly dependent. If $e_1, \dots, e_5$ are linearly independent, then there exists a permutation σ of $\{1, \dots, 5\}$ such that $e_{\sigma(5)} = e_{\sigma(1)} + e_{\sigma(2)}$ because $|D(\xi)| = 12$. Renumbering the $e_1, \dots, e_5$ if necessary, we may assume that $e_5 = e_1 + e_2$. Then

$$\xi = (X^0 + X^a)(X^{e_1} + X^{e_2} + X^{e_1+e_2} + X^{e_3} + X^{e_4} + X^{e_3+e_4}) = [\![a, e_1, e_2]\!] + [\![a, e_3, e_4]\!]$$

where $e_1, \dots, e_4, a$ are linearly independent. If $a \in \mathsf{span}_{\mathbb{F}_2}\langle e_1, \dots, e_5\rangle$, then there exists a permutation σ of $\{1, \dots, 5\}$ such that $a = e_{\sigma(1)} + e_{\sigma(2)} + e_{\sigma(3)}$. Renumbering the $e_1, \dots, e_5$ if necessary, we may assume that $a = e_1 + e_2 + e_3$. Then

$$\xi = (X^0 + X^{e_1+e_2+e_3})(X^{e_1} + X^{e_2} + X^{e_1+e_2} + X^{e_4} + X^{e_5} + X^{e_4+e_5})$$

and we are in the situation of the previous case. Hence, if $0 \in D(\xi)$ and $e_1, \dots, e_5, a$ are linearly dependent, then $\mathsf{Pf}_3(\xi) = 2$. □

In the next proposition, we use the above algorithm to classify the elements $\xi \in I^3[V]$ such that $|D(\xi)| = 14$.

Proposition 1.7 *Suppose that $\xi \in I^3[V]$ is such that $|D(\xi)| = 14$. Then there exist $e_1, \dots, e_7 \in V$ such that*

$$\xi = [\![e_1, e_2, e_3]\!] + [\![e_4, e_5, e_6]\!] + [\![e_1, e_2, e_6]\!] + [\![e_4, e_5, e_7]\!] = X^{e_6}\Big([\![e_1, e_2, e_3 + e_6]\!] + [\![e_4, e_5, e_6 + e_7]\!]\Big)$$

and in particular, $\mathsf{GPf}_3(\xi) = 2$. More precisely:

1. *If $0 \in D(\xi)$, then $\xi = [\![e_1, e_2, e_3]\!] + [\![e_4, e_5, e_6]\!] + [\![e_1, e_2, e_6]\!]$ for some linearly independent vectors $e_1, \dots, e_6 \in V$ and $\mathsf{Pf}_3(\xi) = 3$.*

2. *If $0 \notin D(\xi)$, then there exist linearly independent vectors $e_1, \ldots, e_6 \in V$ such that*

- *either* $\xi = [\![e_1, e_2, e_3]\!] + [\![e_4, e_5, e_6]\!]$ *and* $\mathsf{Pf}_3(\xi) = 2$*;*
- *or* $\xi = [\![e_1, e_2, e_3]\!] + [\![e_4, e_5, e_6]\!] + [\![e_1, e_2, e_4]\!] + [\![e_1, e_4, e_5]\!]$ *and* $\mathsf{Pf}_3(\xi) = 4$*;*
- *or* $\xi = [\![e_1, e_2, e_3]\!] + [\![e_4, e_5, e_6]\!] + [\![e_1, e_2, e_6]\!] + [\![e_4, e_5, e_7]\!]$ *for some* $e_7 \in V \setminus \mathsf{span}_{F_2}\langle e_1, \ldots, e_6\rangle$*, and* $\mathsf{Pf}_3(\xi) = 4$*.*

Proof: Let a, W, ξ_1, ξ_2 be as in Lemma 1.5.

(1) Suppose that $0 \in D(\xi)$. Then $|D(\xi_1)| < 14$ and $0 \notin D(\xi_1)$. Observe that $\xi_1 \neq 0$ because otherwise $|D(\xi_2)| = 7$ which is impossible. Hence $|D(\xi_1)|$ is equal to 8 or 12.

a) If $|D(\xi_1)| = 12$, then $D(\xi_1 + \xi_2) \cap D(\xi_2) = \{0\}$. Suppose that $\xi = \eta_e$ where e is a family of 6 linearly independent vectors, then $\xi_2 = X^0 + X^{v_1} + \ldots + X^{v_r}$ where r is odd, $v_1, \ldots, v_r \in D(\eta_e)$ are distinct pairwise and $v_1 + \ldots + v_r = 0$; but such vectors $v_1, \ldots, v_r$ do not exist in $D(\eta_e)$. Hence, by the proof of Proposition 1.6, there exist linearly independent vectors $e_1, \ldots, e_5 \in V$ such that $\xi_1 = [\![e_1, e_2, e_3]\!] + [\![e_3, e_4, e_5]\!]$. Then

$$\begin{aligned} \xi_2 &= X^0 + X^{v_1} + \ldots + X^{v_r}, \\ \xi_1 &= X^{v_1} + \ldots + X^{v_{12}}, \\ \xi &= X^0 + X^{v_{r+1}} + \ldots + X^{v_{12}} + X^a(X^0 + X^{v_1} + \ldots + X^{v_r}) \end{aligned}$$

for some vectors $v_1, \ldots, v_r \in D(\xi_1)$. Replacing W by $\mathsf{span}_{\mathbb{F}_2}\langle e_1, e_2, e_3, e_4, a\rangle$ if necessary, we may assume that $|D(\xi_2)| = 4$. Choosing another basis of $\mathsf{span}_{\mathbb{F}_2}\langle D(\xi_1)\rangle$ if necessary, we may assume that $\xi_2 = X^0 + X^{e_1} + X^{e_2} + X^{e_1+e_2}$ and then

$$\xi = [\![e_1, e_2, a]\!] + [\![e_4, e_5, e_3]\!] + [\![e_1, e_2, e_3]\!].$$

b) If $|D(\xi_1)| = 8$, then $D(\xi_1 + \xi_2) \cap D(\xi_2) = \{0, b, c\}$ and $\xi_1 = \eta_e$ where e is a family of 4 linearly independent vectors. Therefore

$$\xi_2 = X^0 + X^b + X^c + X^{v_1} + \ldots + X^{v_r}$$

for some pairwise distinct vectors $v_1, \ldots, v_r \in D(\xi_1)$ such that r is odd and $v_1 + \ldots + v_r = b + c$. Observe that $b + c \in D(\eta_e)$. Since $b, c \notin D(\eta_e)$ and there do not exist vectors $u, v \in \mathsf{span}_{\mathbb{F}_2}\langle D(\eta_e)\rangle \setminus D(\eta_e)$ such that $u + v \in D(\eta_e)$, we have $b, c \notin \mathsf{span}_{\mathbb{F}_2}\langle D(\eta_e)\rangle$. Replacing W by $\mathsf{span}_{\mathbb{F}_2}\langle e_1, e_2, e_3, e_4, a\rangle$ if necessary, we may assume that $|D(\xi_2)| = 4$ and choosing another basis of $\mathsf{span}_{\mathbb{F}_2}\langle D(\eta_e)\rangle$ we may assume that $\xi_2 = X^0 + X^{e_1} + X^b + X^{e_1+b}$. Then

$$\xi = [\![e_0, e_1 + e_2, e_3]\!] + [\![a, b, e_1]\!] + [\![e_0, e_1 + e_2, e_1]\!]$$

where $e_0 := e_1 + \ldots + e_4$.

In both the cases,

$$\xi = [\![f_1, f_2, f_3]\!] + [\![f_4, f_5, f_6]\!] + [\![f_1, f_2, f_6]\!] + [\![f_4, f_5, f_7]\!]$$

for some linearly independent vectors $f_1, \ldots, f_6 \in V$ and with $f_7 = 0$. Since $0 \in D(\xi)$, the 3-Pfister number of ξ is odd. Hence $\mathsf{Pf}_3(\xi) = 3$ because if $\mathsf{Pf}_3(\xi) = 1$, then $|D(\xi)| = 8$.

(2) Suppose that $0 \notin D(\xi)$. Then $0 \in D(\xi_1)$ and $|D(\xi_1)| \in \{8, 12, 14\}$.

a) If $|D(\xi_1)| = 14$, then $\xi_1 = [\![e_1, e_2, e_3]\!] + [\![e_4, e_5, e_6]\!] + [\![e_1, e_2, e_6]\!]$ for some linearly independent vectors $e_1, \ldots, e_6 \in V$ and $D(\xi_2) \subset D(\xi_1)$. Replacing W by $\mathsf{span}_{\mathbb{F}_2}\langle e_2, \ldots, e_6, a\rangle$ and choosing another basis of $\mathsf{span}_{\mathbb{F}_2}\langle D(\xi_1)\rangle$, we may assume that $\xi_2 = X^0 + X^{e_4} + X^{e_5} + X^{e_4+e_5}$. Hence

$$\xi = [\![e_1, e_2, e_3]\!] + [\![e_4, e_5, e_6]\!] + [\![e_1, e_2, e_6]\!] + [\![e_4, e_5, a]\!]$$

where $e_1, \ldots, e_6, a$ are linearly independent. Since $0 \notin D(\xi)$, the 3-Pfister number of ξ is even. If $\mathsf{Pf}_3(\xi) = 2$, then the dimension of $\mathsf{span}_{\mathbb{F}_2}\langle D(\xi)\rangle$ is less than or equal to 6 and we get a contradiction; so $\mathsf{Pf}_3(\xi) = 4$.

b) If $|D(\xi_1)| = 12$, then $D(\xi_1 + \xi_2) \cap D(\xi_2) = \{b\}$ and $\xi_1 = \eta_e$ for some family e of 5 linearly independent vectors. Since $b \in \mathsf{span}_{\mathbb{F}_2}\langle D(\xi_1)\rangle \setminus D(\xi_1)$, there exists a permutation σ of $\{1, \dots, 5\}$ such that either $b = e_{\sigma(1)} + e_{\sigma(2)}$ or $b = e_{\sigma(1)} + e_{\sigma(2)} + e_{\sigma(3)}$. Renumbering the e_i's, we may assume that $b = e_1 + e_2$ or $b = e_1 + e_2 + e_3$. Then replacing W by $\mathsf{span}_{\mathbb{F}_2}\langle e_1 + e_2, e_3, e_4, e_5, a\rangle$ and choosing another basis of $\mathsf{span}_{\mathbb{F}_2}\langle D(\xi_1)\rangle$, we may assume that $\xi_2 = X^0 + X^{e_1} + X^{e_2} + X^{e_1+e_2}$. Thus

$$\xi = [\![e_0, e_1 + e_2 + e_3, e_4]\!] + [\![e_1 + e_2, e_1, e_0]\!] + [\![e_0, e_1 + e_2 + e_3, e_1 + e_2]\!] + [\![e_1 + e_2, e_1, a]\!]$$

where $e_0 := e_1 + \ldots + e_5$. So ξ can be written as

$$[\![f_1, f_2, f_3]\!] + [\![f_4, f_5, f_6]\!] + [\![f_1, f_2, f_6]\!] + [\![f_4, f_5, f_7]\!]$$

for some linearly independent vectors $f_1, \dots, f_5, f_7 \in V$ with $f_6 = f_1 + f_4$. We know that $\mathsf{Pf}_3(\xi)$ is even and $\mathsf{Pf}_3(\xi) \leq 4$. To prove that $\mathsf{Pf}_3(\xi) = 4$, we consider a linear map $\varphi \colon V \to V$ such that $\varphi|_W$ is the identity and $\varphi(a) = 0$. The map φ induces an algebra homomorphism $\varphi_* \colon \mathbb{F}_2[V] \to \mathbb{F}_2[V]$ such that $\varphi_*(\xi) = \eta_e$. Then $3 = \mathsf{Pf}_3(\eta_e) \leq \mathsf{Pf}_3(\xi)$, so $\mathsf{Pf}_3(\xi) = 4$.

c) If $|D(\xi_1)| = 8$, then $D(\xi_1 + \xi_2) \cap D(\xi_2) = \{b, c, d\}$ and $\xi_1 = [\![e_1, e_2, e_3]\!]$ for some linearly independent vectors $e_1, e_2, e_3 \in W$. Therefore

$$\xi_2 = X^{v_1} + \ldots + X^{v_r} + X^b + X^c + X^d$$

for some vectors $v_1, \dots, v_r \in D(\xi_1)$ such that $v_1 + \ldots + v_r = b + c + d$. The vectors e_1, e_2, e_3, b are linearly independent because $b \notin D(\xi_1)$. Observe that e_1, e_2, e_3, b, c are linearly independent because otherwise either $c \in D(\xi_1)$ or $d \in D(\xi_1)$ and we get a contradiction. Replacing W by $\mathsf{span}_{\mathbb{F}_2}\langle e_1, e_2, e_3, a, b\rangle$ if necessary, we may assume that $|D(\xi_2)| = 4$. Then $\xi_2 = X^0 + X^b + X^c + X^{b+c}$,

$$\xi = [\![e_1, e_2, e_3]\!] + [\![a, b, c]\!]$$

and $\mathsf{Pf}_3(\xi) = 2$. □

Remark 1.8 Proposition 1.7 shows that, already for 14-dimensional quadratic forms, the combinatorial analogue $\mathbb{F}_2[V]$ diverges from the Witt ring of a general field. Indeed, Hoffmann and Tignol [1998, Example 6.3] constructed a field over which there exist 14-dimensional quadratic forms in the third power of the fundamental ideal which are not a scalar multiple of a sum of two 3-fold Pfister forms. However, Proposition 1.7 shows that, in the combinatorial analogue, all the elements $\xi \in I^3[V]$ such that $|D(\xi)| = 14$ can be written as a scaled sum of two 3-fold Pfister elements.

1.4. Bounds for the 3-Pfister number

We shall compute an upper bound for the 3-Pfister number of $\xi \in I^3[V]$, which only depends on $|D(\xi)|$. We first prove some preliminary results.

Lemma 1.9 *Let $\xi \in I^2[V] \setminus \{0\}$, then $\mathsf{GPf}_2(\xi) \leq \big(|D(\xi)| - 2\big)/2$.*

Proof: We use an induction on $d := |D(\xi)|$. If $d = 4$, then

$$\xi = X^a + X^b + X^c + X^{a+b+c} = X^a[\![a + b, a + c]\!]$$

for some $a, b, c \in V$, so $\mathsf{GPf}_2(\xi) = 1$. Now suppose that $d > 4$, then let $a, b, c \in D(\xi)$ be distinct pairwise. Set $\xi' := X^a + X^b + X^c + X^{a+b+c} + \xi$. Then $\xi' \in I^2[V]$ and $|D(\xi')| \leq d - 2$. By induction, $\mathsf{GPf}_2(\xi') \leq (d - 4)/2$. It implies that

$$\mathsf{GPf}_2(\xi) = \mathsf{GPf}_2(X^a[\![a + b, a + c]\!] + \xi') \leq 1 + \mathsf{GPf}_2(\xi') \leq 1 + (d - 4)/2 \leq (d - 2)/2$$

as wanted. □

Lemma 1.10 *Let $e = (e_1, \ldots, e_n)$ be a family of linearly independent vectors of V where $n \geq 3$. Then*

$$\mathsf{GPf}_3(\eta_e) = \begin{cases} (n-2)/2 & \text{if } n \text{ is even;} \\ (n-1)/2 & \text{otherwise.} \end{cases}$$

Proof: Set $e_0 := e_1 + \ldots + e_n$ and $\xi := n + X^{e_1} + \ldots + X^{e_{n-1}} + X^{e_0+e_n}$. Then $\xi \in I^2[V]$, $|D(\xi)| \leq n+1$ and $\eta_e = (1 + X^{e_0})\xi$. Hence $\mathsf{GPf}_3(\eta_e) \leq \mathsf{GPf}_2(\xi) \leq \big(|D(\xi)| - 2\big)/2 \leq (n-1)/2$. Since $\mathsf{Pf}_3(\eta_e) = n-2$, we have $n - 2 \leq 2\mathsf{GPf}_3(\eta_e)$. We obtain that

$$(n-2)/2 \leq \mathsf{GPf}_3(\eta_e) \leq (n-1)/2.$$

Therefore $\mathsf{GPf}_3(\eta_e) = (n-2)/2$ if n is even and $\mathsf{GPf}_3(\eta_e) = (n-1)/2$ otherwise. □

Lemma 1.11 *Let $\xi \in I^3[V] \setminus \{0\}$. Then there exist linearly independent $e_1, \ldots, e_n \in D(\xi)$ such that $e_1 + \ldots + e_{n-1} \in D(\xi)$ and $n \geq 3$.*

Proof: Let m be the dimension of $\mathsf{span}_{\mathbb{F}_2}\langle D(\xi)\rangle$. Then $m \geq 3$ because $|D(\xi)| \geq 8$. Let $e_1, \ldots, e_m \in D(\xi)$ be linearly independent vectors, then there exist $v_1, \ldots, v_r \in \mathsf{span}_{\mathbb{F}_2}\langle e_1, \ldots, e_m\rangle$ such that

$$\xi = X^{e_1} + \ldots + X^{e_m} + X^{v_1} + \ldots + X^{v_r}.$$

Since $\epsilon_1(\xi) = 0$, we have $r \geq 1$. If $r = 1$, then $v_1 = e_1 + \ldots + e_m$ because $\epsilon_1(\xi) = 0$; it implies that $\epsilon_2(\xi) = \sum_{1 \leq i < j \leq m} e_i \wedge e_j \neq 0$ (because $m \geq 3$) which is impossible. Hence $r \geq 2$. Renumbering the vectors $e_1, \ldots, e_m$ and $v_1, \ldots, v_r$ if necessary, we may assume that $v_r = e_1 + \ldots + e_{n-1}$ for some $n \in \mathbb{N}$, $3 \leq n \leq m$. So $e_1, \ldots, e_n, e_1 + \ldots + e_{n-1} \in D(\xi)$ with $n \geq 3$ and $e_1, \ldots, e_n$ linearly independent. □

In the next proposition we establish an upper bound for the dimension of the space spanned by the support of a $\xi \in I^3[V]$. (Originally, our proof of this proposition was given by induction on $|D(\xi)|$; the shorter proof included here was pointed out to us by Karim Becher.)

Proposition 1.12 *Let $\xi \in I^3[V]$. Set $d := |D(\xi)|$ and $m := \dim \mathsf{span}_{\mathbb{F}_2}\langle D(\xi)\rangle$. Then $m \leq d/2$; if moreover $0 \in D(\xi)$, then $m \leq (d/2) - 1$.*

Proof: Let $\epsilon\colon \mathbb{F}_2[V] \to \bigwedge^* V$ be the map defined by

$$\epsilon(X^{v_1} + \ldots + X^{v_d}) = (1 + v_1) \cdot \ldots \cdot (1 + v_d),$$

where $\cdot$ is the multiplication in the exterior algebra $\bigwedge^* V$ (Compare with the definition of the Stiefel-Whitney invariant in [Milnor, 1970]). It is easy to check that $\epsilon(\xi_1 + \xi_2) = \epsilon(\xi_1) \cdot \epsilon(\xi_2)$ for all $\xi_1, \xi_2 \in \mathbb{F}_2[V]$ and $\epsilon(\xi) = 1$ for all $\xi \in I^3[V]$. Let $\xi \in I^3[V]$, set $d := |D(\xi)|$ and $m := \dim \mathsf{span}_{\mathbb{F}_2}\langle D(\xi)\rangle$. Suppose that $m \geq d/2$. Write

$$\xi = X^{v_1} + \ldots + X^{v_n} + X^{w_1} + \ldots + X^{w_n}$$

where $v_1, \ldots, v_n \in V$ are linearly independent and $d = 2n$. Since $\xi \in I^3[V]$, we have

$$\epsilon(X^{v_1} + \ldots + X^{v_n}) = \epsilon(\xi)\epsilon(X^{w_1} + \ldots + X^{w_n}) = \epsilon(X^{w_1} + \ldots + X^{w_n}).$$

We obtain in particular that,

$$w_1 \wedge \ldots \wedge w_n = v_1 \wedge \ldots \wedge v_n \neq 0 \in \textstyle\bigwedge^n V.$$

This implies that $\mathsf{span}_{\mathbb{F}_2}\langle v_1, \ldots, v_n\rangle = \mathsf{span}_{\mathbb{F}_2}\langle w_1, \ldots, w_n\rangle$ and $m = d/2$. Now assume that $0 \in D(\xi)$, then one of the w_i's is equal to zero and $w_1 \wedge \ldots \wedge w_n = 0$ which leads to a contradiction. Hence $m \leq d/2$ and $m \leq d/2 - 1$ if $0 \in D(\xi)$. □

From the previous results we can now deduce our upper bound for the 3-Pfister number of elements in $I^3[V]$.

Theorem 1.13 *Let $\xi \in I^3[V]$, then $\mathsf{GPf}_3(\xi) \leq |D(\xi)|^2/16$, hence $\mathsf{Pf}_3(\xi) \leq |D(\xi)|^2/8$.*

Proof : We use an induction on d. If $d = 0$, then $\xi = 0$ and $\mathsf{GPf}_3(\xi) = 0$. Suppose that $d > 0$. Observe that $\mathsf{GPf}_3(\xi) = \mathsf{GPf}_3(X^b\xi)$ for all $b \in V$. Hence, replacing ξ by $X^b\xi$ for some $b \in D(\xi)$, we may assume that $0 \in D(\xi)$. Let $e_1, \dots, e_n \in D(\xi)$ be linearly independent such that $n \geq 3$ and $e_1 + \dots + e_{n-1} \in D(\xi)$. Set $e := (e_1, \dots, e_n)$ and $e_0 := e_1 + \dots + e_n$. Then

$$\xi = X^0 + X^{e_1} + \dots + X^{e_n} + X^{e_0+e_n} + X^{v_1} + \dots + X^{v_r}$$

for some $v_1, \dots, v_r \in V$ and we have

$$\xi + \eta_e = \begin{cases} X^0 + X^{e_0+e_1} + \dots + X^{e_0+e_{n-1}} + X^{v_1} + \dots + X^{v_r} & \text{if } n \text{ is even,} \\ X^{e_0} + X^{e_0+e_1} + \dots + X^{e_0+e_{n-1}} + X^{v_1} + \dots + X^{v_r} & \text{if } n \text{ is odd.} \end{cases}$$

So $\xi + \eta_e \in I^3[V]$ and $|D(\xi + \eta_e)| < |D(\xi)|$. By induction, $\mathsf{GPf}_3(\xi + \eta_e) \leq (d-2)^2/16$. Since $n \leq \dim \mathsf{span}_{\mathbb{F}_2}\langle D(\xi)\rangle$ and $0 \in D(\xi)$, by Proposition 1.12, we have $n \leq d/2 - 1$. Therefore

$$\mathsf{GPf}_3(\xi) \leq \mathsf{GPf}_3(\eta_e) + \mathsf{GPf}_3(\xi + \eta_e) \leq (n-1)/2 + (d-2)^2/16 \leq d/4 - 1 + (d-2)^2/16 \leq d^2/16$$

which ends the proof. □

2. The 3-Pfister number of quadratic forms

In this section, we shall apply the results in the previous section to quadratic forms over a field F of characteristic different from 2 which contains a square root of -1.

Recall that $V_F := F^\times/F^{\times 2}$ and that

$$\Psi\colon \mathbb{F}_2[V_F] \to W(F)\colon (a_1F^{\times 2}) + \dots + (a_dF^{\times 2}) \mapsto [\langle a_1, \dots, a_d\rangle]$$

is a surjective $\mathbb{F}_2$-algebra homomorphism. For all $m \in \mathbb{N}$, $m \geq 1$, the map Ψ carries m-fold Pfister elements in $\mathbb{F}_2[V_F]$ to m-fold Pfister forms in $W(F)$. Hence $\Psi\big(I^m[V_F]\big) = I^m(F)$ for all $m \geq 1$ but, for $[q] \in I^m(F)$, the number of elements in $D(\xi)$ may be strictly greater than the dimension of q for all $\xi \in I^m[V_F]$ such that $\Psi(\xi) = [q]$.

For $m \in \{1, 2\}$, every field F satisfies the property

$$\forall a_1, \dots, a_d \in F^\times, \quad [\langle a_1, \dots, a_d\rangle] \in I^m(F) \quad \Longrightarrow \quad (a_1F^{\times 2}) + \dots + (a_dF^{\times 2}) \in I^m[V_F], \qquad (1)$$

meaning in particular that for $m \in \{1, 2\}$ and for all $[q] \in I^m(F)$, there exists $\xi \in I^m[V_F]$ such that $\Psi(\xi) = [q]$ and $|D(\xi)|$ is less than or equal to the dimension of q. However, for $m = 3$ this is no longer true for every field F. Theorem 1.1 in [Brosnan et al., 2010] gives counterexamples: for any field k of characteristic different from 2 and for all even integer $d \geq 2$, there exist a field extension F/k and $[q] \in I^3(F)$ with q of dimension d, such that

$$\mathsf{Pf}_3(q) \geq \frac{2^{(d+4)/4} - d - 2}{7}.$$

If we take in particular a field k which contains a square root of -1 and a $d \in \mathbb{N}$ such that

$$\frac{2^{(d+4)/4} - d - 2}{7} > (d^2)/8,$$

then we have an example of a $[q] \in I^3(F)$ with q of dimension d such that $\mathsf{Pf}_3(q) > (d^2)/8$. Write $q = \langle a_1, \dots, a_d\rangle$ and set $\xi = (a_1F^{\times 2}) + \dots + (a_dF^{\times 2})$ then $\xi \notin I^3[V_F]$ because otherwise

$$\mathsf{Pf}_3(q) = \mathsf{Pf}_3\big(\Psi(\xi)\big) \leq \mathsf{Pf}_3(\xi) \leq (d^2)/8$$

which leads to a contradiction. Thus, the combinatorial analogue of the Witt ring is less powerful to compute 3-Pfister numbers than it is to compute the 1- and 2-Pfister numbers.

Below we shall first characterize those fields for which property (1) holds for $m = 3$; for these particular fields, we thus find an upper bound for the 3-Pfister number using the combinatorial analogue. Thereafter we shall discuss the general case.

2.1. Case where we deduce an upper bound from the combinatorial analogue

The homomorphism Ψ induces a map

$$\overline{\Psi}\colon I^2[V_F]/I^3[V_F] \to I^2(F)/I^3(F)\colon \xi + I^3[V_F] \mapsto \Psi(\xi) + I^3(F)$$

which is a surjective group homomorphism. Observe that the property (1) holds for $m = 3$ if and only if $\overline{\Psi}$ is an isomorphism. In fact, we shall prove that this is further equivalent to Ψ being an isomorphism. Before we give other equivalent conditions to the assertion that $\overline{\Psi}$ is an isomorphism, we need some definitions. In [1981], Ware says that $a \in F^\times$ is $F^{\times 2}$-*rigid* if $F^{\times 2} + aF^{\times 2} \subset F^{\times 2} \cup aF^{\times 2}$ and that F is $F^{\times 2}$-*rigid* (or *rigid*) if a is $F^{\times 2}$-rigid for all $a \in F^\times \setminus F^{\times 2}$ (here Ware's original definition of rigidity for a field F is simplified by the assumption that -1 is a square in F). Following [Ware, 1981], we say that a valuation v on F is $F^{\times 2}$-*compatible* if $1 + \mathfrak{m}_v \subset F^{\times 2}$ where $\mathfrak{m}_v$ is the maximal ideal of the valuation ring of F.

Proposition 2.1 *The following conditions are equivalent:*

1. *$\overline{\Psi}$ is an isomorphism;*
2. *for all $a \in F^\times \setminus F^{\times 2}$, $F^{\times 2} + aF^{\times 2} \subset F^{\times 2} \cup aF^{\times 2}$ (i.e., F is $F^{\times 2}$-rigid);*
3. *There exists an $F^{\times 2}$-compatible valuation v on F such that $(\overline{F}_v^\times : \overline{F}_v^{\times 2}) \le 2$, where $\overline{F}_v$ is the residue field of F.*
4. *Ψ is an isomorphism;*

Proof: (4)⇒(1) Trivial.
(1)⇒(2) Let $a \in F^\times \setminus F^{\times 2}$ and $x, y \in F^\times$. Set $b := x^2 + ay^2$, then $b \neq 0$ because $a \notin F^{\times 2}$ and the quadratic form $\langle 1, a, b, ab\rangle$ is isotropic. Since it is a Pfister form, we have $[\langle 1, a, b, ab\rangle] = 0 \in W(F)$. Hence $(F^{\times 2}) + (aF^{\times 2}) + (bF^{\times 2}) + (abF^{\times 2}) \in I^3[V_F]$ because $\overline{\Psi}$ is an isomorphism. By Proposition 1.6, the element $(F^{\times 2}) + (aF^{\times 2}) + (bF^{\times 2}) + (abF^{\times 2})$ is equal to zero. Since $a \notin F^{\times 2}$, we obtain that $b \in F^{\times 2} \cup aF^{\times 2}$.
(2)⇒(3) By [Ware, 1981, Theorem 3.3], there exists an $F^{\times 2}$-compatible valuation v on F such that $(\overline{F}_v^\times : \overline{F}_v^{\times 2}) \le 2$.
(3)⇒(2) Let v be an $F^{\times 2}$-compatible valuation on F such that $(\overline{F}_v^\times : \overline{F}_v^{\times 2}) \le 2$. Let $\mathcal{O}_v$ denote the valuation ring of F and for $a \in \mathcal{O}_v$, let $\overline{a}$ denote the residue class of a in $\overline{F}_v$. By [Arason et al., 1987, Proposition 1.5 (1)], all elements $a \in F^\times$ such that $a \notin O_v^\times \cdot F^{\times 2}$ are $F^{\times 2}$-rigid. Now let $a \in O_v^\times \cdot F^{\times 2}$ be such that $a \notin F^{\times 2}$ and let $x, y \in F^\times$. Then there exist $b \in O_v^\times$ and $z \in F^\times$ such that $a = bz^2$ and $b \notin F^{\times 2}$. Since $1 + \mathfrak{m}_v \subset F^{\times 2}$ and $b \notin F^{\times 2}$, we have $\overline{b} \notin \overline{F}_v^{\times 2}$. Observe that $\overline{F}_v$ is $\overline{F}_v^{\times 2}$-rigid because $(\overline{F}_v^\times : \overline{F}_v^{\times 2}) \le 2$, so $\overline{b}$ is $\overline{F}_v^{\times 2}$-rigid. By [Arason et al., 1987, Proposition 1.5 (2)], we deduce that b is $F^{\times 2}$-rigid. Hence

$$x^2 + ay^2 = x^2 + b(zy)^2 \in F^{\times 2} \cup bF^{\times 2} = F^{\times 2} \cup aF^{\times 2}.$$

Thus F is $F^{\times 2}$-rigid.
(2)⇒(4) It is easy to prove that for all $a_1F^{\times 2}, \ldots, a_dF^{\times 2} \in F^\times/F^{\times 2}$ distinct pairwise,

$$\{a_1x_1^2 + \ldots + a_dx_d^2 \in F^\times \mid x_1, \ldots, x_d \in F\} = a_1F^{\times 2} \cup \ldots \cup a_dF^{\times 2}.$$

Let ξ be in the kernel of Ψ. Let $a_1F^{\times 2}, \ldots, a_dF^{\times 2} \in F^\times/F^{\times 2}$ be pairwise distinct elements such that $\xi = (a_1F^{\times 2}) + \ldots + (a_dF^{\times 2})$ and assume that $d \ge 1$. Since $\Psi(\xi) = 0$, the quadratic form $\langle a_1, \ldots, a_d\rangle$ is isotropic. So $a_d \in a_1F^{\times 2} \cup \ldots \cup a_{d-1}F^{\times 2}$, leading to a contradiction. Hence $\xi = 0$ and Ψ is an isomorphism. □

By [Wadsworth, 1983, Proposition 1.2]), if v is an $F^{\times 2}$-compatible valuation on F and the residue characteristic is different from 2, then (F, v) is a 2-Henselian valued field. Any valued field with residue characteristic different from 2 is contained in a Henselian (hence 2-Henselian) valued field (F, v) such that $(\overline{F}_v^\times : \overline{F}_v^{\times 2}) \le 2$. Indeed, let (k, w) be a valued field with residue characteristic different from 2. Let (K, v) be a Henselization of (k, w) (see [Efrat, 2006, Theorem 15.3.5]). Assume that $(\overline{K}_v^\times : \overline{K}_v^{\times 2}) > 2$ and choose $\alpha \in \overline{K}_v^\times \setminus \overline{K}_v^{\times 2}$. Now we take a maximal field extension M of $\overline{K}_v$ in a separable closure of $\overline{K}_v$ such that $\alpha \notin M^{\times 2}$, then $M^\times = M^{\times 2} \cup \alpha M^{\times 2}$, so $(M^\times : M^{\times 2}) = 2$. By [Efrat, 2006, Corollary 16.1.4], there exists an algebraic field extension F over K such that $\overline{F}_v = M$. Then $(\overline{F}_v^\times : \overline{F}_v^{\times 2}) = 2$ and (F, v) is Henselian since (K, v) is Henselian and F is an algebraic extension of K. Similarly, taking for M a quadratic closure of $\overline{K}_v$ we can also find an F such that $(\overline{F}_v^\times : \overline{F}_v^{\times 2}) = 1$.

Now we can give an upper bound for the 3-Pfister number:

Theorem 2.2 *Let (F, v) be a valued field such that $1 + \mathfrak{m}_v \subset F^{\times 2}$ and $(\overline{F}_v^\times : \overline{F}_v^{\times 2}) \leq 2$, and let $[q] \in I^3(F)$ be such that q is of dimension d. Then $\mathsf{Pf}_3(q) \leq (d^2)/8$.*

Proof : Write $q = \langle a_1, \ldots, a_d \rangle$. By Proposition 2.1, Ψ is an isomorphism. Hence the element $\xi := (a_1 F^{\times 2}) + \ldots + (a_d F^{\times 2})$ is in $I^3[V_F]$. Since $|D(\xi)| \leq d$, one has

$$\mathsf{Pf}_3(q) = \mathsf{Pf}_3\big(\Psi(\xi)\big) \leq \mathsf{Pf}_3(\xi) \leq (d^2)/8$$

and we are done. □

2.2. General case

Now we study the case where Ψ may not be an isomorphism. Let $[q] \in I^3(F)$ with $q = \langle a_1, \ldots, a_d \rangle$. Set $\xi := (a_1 F^{\times 2}) + \ldots + (a_d F^{\times 2})$. Then $\overline{\Psi}(\xi + I^3[V_F]) = [q] + I^3(F) = 0$, hence $\xi + I^3(F)$ is in the kernel of $\overline{\Psi}$. That is why we shall investigate the structure of the kernel of $\overline{\Psi}$.

First we introduce some notations. For $e = (x_1 F^{\times 2}, \ldots, x_r F^{\times 2})$ a family of linearly independent vectors of V_F, we set $\xi_e := (r+1)(F^{\times 2}) + (x_1 F^{\times 2}) + \ldots + (x_r F^{\times 2}) + (x_1 \ldots x_r F^{\times 2}) \in I^2[V_F]$.

Lemma 2.3 *Let $\xi \in I^2[V_F]$ and set $d := |D(\xi)|$. Suppose that $d > 0$, then we have one of the following situations:*

1. *$\xi = \xi_e$ for some family e of linearly independent vectors of V_F;*
2. *there exist an integer n, $3 \leq n \leq d-2$, and $a_1 F^{\times 2}, \ldots, a_n F^{\times 2} \in D(\xi)$ linearly independent such that $a_1 \ldots a_{n-1} F^{\times 2} \in D(\xi)$.*

Proof : Let $a_1 F^{\times 2}, \ldots, a_d F^{\times 2} \in D(\xi)$ be distinct pairwise such that $(a_1 F^{\times 2}, \ldots, a_m F^{\times 2})$ is a basis of $\mathsf{span}_{\mathbb{F}_2}\langle D(\xi) \rangle$, where $m \leq d$. Because $d > 0$ and $\xi \in I^2[V_F]$, we have $d \geq 4$. Suppose that $\xi \neq \xi_e$ for all e. Then $m \leq d-2$ and $m \geq 3$ because otherwise $\xi = (a_1 F^{\times 2}) + (a_2 F^{\times 2}) + (F^{\times 2}) + (a_1 a_2 F^{\times 2})$ with $a_1 F^{\times 2}, a_2 F^{\times 2}$ linearly independent. Since the elements $a_1 F^{\times 2}, \ldots, a_d F^{\times 2}$ are distinct pairwise, there exists $i \in \{m+1, \ldots, d\}$ such that $a_i F^{\times 2} = a_{\sigma(1)} \ldots a_{\sigma(n-1)} F^{\times 2}$ for some permutation σ of $\{1, \ldots, m\}$ and for some integer n, $3 \leq n \leq m$. Hence the situation (2) occurs. □

We recall a notation introduced in Subsection 1.2: for $e := (x_1 F^{\times 2}, \ldots, x_r F^{\times 2})$ a family of linearly independent vectors of V_F with $r \geq 3$, we write η_e for the element

$$r(F^{\times 2}) + (x_1 F^{\times 2}) + \ldots + (x_r F^{\times 2}) + r(x_0 F^{\times 2}) + (x_0 x_1 F^{\times 2}) \ldots + (x_0 x_r F^{\times 2})$$

of $\mathbb{F}_2[V_F]$, where $x_0 F^{\times 2} = x_1 \ldots x_r F^{\times 2}$. We have $\eta_e \in I^3[V_F]$ and $\mathsf{Pf}_3(\eta_e) = r - 2$.

Lemma 2.4 *Let r be an odd integer, $r \geq 3$ and let $e = (x_1 F^{\times 2}, \ldots, x_r F^{\times 2})$ be a family of linearly independent vectors of V_F. Then $\xi_e = \xi_{e'} + \eta_e$, where $e' := (x_0 x_1 F^{\times 2}, \ldots, x_0 x_{r-1} F^{\times 2})$ and $x_0 = x_1 \ldots x_r$.*

Proof : It is an easy computation. □

The next proposition describes the elements in the kernel of $\overline{\Psi}$.

Proposition 2.5 *Let $\xi + I^3[V_F]$ be in the kernel of $\overline{\Psi}$ and set $d := |D(\xi)|$. Then either $\xi \in I^3[V_F]$ or there exist $\eta \in I^3[V_F]$ and a family e of r linearly independent vectors of V_F such that $\mathsf{Pf}_3(\eta) \leq d^2$, r is even, $2 \leq r \leq d-2$ and $\xi = \xi_e + \eta$.*

Proof : We use an induction on d. Suppose that $d = 0$, then $\xi = 0 \in I^3[V_F]$.
From now on, we assume that $d > 0$ and $\xi \notin I^3[V_F]$. Since $\xi \in I^2[V_F]$ and $\xi \neq 0$, we have $d \geq 4$. By Lemma 2.3, either $\xi = \xi_e$ for some family $e = (x_1F^{\times 2}, \ldots, x_rF^{\times 2})$ of linearly independent vectors of V_F, where $r \in \{d-2, d-1\}$, or $\xi = (a_1F^{\times 2}) + \ldots + (a_dF^{\times 2})$, with $a_{n+1} = a_1 \ldots a_{n-1}$, where the vectors $a_1F^{\times 2}, \ldots, a_nF^{\times 2}$ are linearly independent and $3 \leq n \leq d-2$. If $\xi = \xi_e$ and $r = d-2$, then we choose $\eta := 0$. If $\xi = \xi_e$ and $r = d-1$, then by Lemma 2.4, $\xi = \xi_{e'} + \eta_e$, where $e' := (x_0x_1F^{\times 2}, \ldots, x_0x_{d-2}F^{\times 2})$ and $x_0 := x_1 \ldots x_{d-1}$. One can choose $\eta := \eta_e$ since $\eta_e \in I^3[V_F]$ and $\mathsf{Pf}_3(\eta_e) = d-3 \leq d^2$. Hence, we may assume that $\xi = (a_1F^{\times 2}) + \ldots + (a_dF^{\times 2})$, with $a_{n+1} = a_1 \ldots a_{n-1}$, where the vectors $a_1F^{\times 2}, \ldots, a_nF^{\times 2}$ are linearly independent and $3 \leq n \leq d-2$. Set $e := (a_1F^{\times 2}, \ldots, a_nF^{\times 2})$ and $a_0 := a_1 \ldots a_n$.

1. Suppose that n is even. Then

$$\xi + \eta_e = (a_0a_1F^{\times 2}) + \ldots + (a_0a_{n-1}F^{\times 2}) + (a_{n+2}F^{\times 2}) + \ldots + (a_dF^{\times 2}),$$

   hence $\xi + \eta_e + I^3[V_F] = \xi + I^3[V_F] \neq 0$ with $|D(\xi + \eta_e)| \leq d-2$. By induction, there exist a family f of r linearly independent vectors of V_F and $\eta \in I^3[V_F]$ such that r is even, $2 \leq r \leq d-4$, $\mathsf{Pf}_3(\eta) \leq (d-2)^2$ and $\xi + \eta_e = \xi_f + \eta$. Then $\xi = \xi_f + \eta + \eta_e$ with $\eta + \eta_e \in I^3[V_F]$ and

$$\mathsf{Pf}_3(\eta + \eta_e) \leq (d-2)^2 + n - 2 \leq (d-2)^2 + d - 2 \leq d^2$$

   as wanted.

2. Suppose that n is odd, then

$$\xi + \eta_e = (F^{\times 2}) + (a_0a_1F^{\times 2}) + \ldots + (a_0a_{n-1}F^{\times 2}) + (a_0F^{\times 2}) + (a_{n+2}F^{\times 2}) + \ldots + (a_dF^{\times 2}).$$

   If $|D(\xi + \eta_e)| < d$ then we can proceed as in the case where n is even. Hence we may assume that $|D(\xi + \eta_e)| = d$. Suppose that $\xi + \eta_e = \xi_f$ where $f = (y_1F^{\times 2}, \ldots, y_rF^{\times 2})$ is a family of linearly independent vectors of V_F; if $r = d-2$ then $\xi = \xi_f + \eta_e$ with $\mathsf{Pf}_3(\eta_e) \leq n-2 \leq d^2$ and if $r = d-1$, then $\xi = \xi_{f'} + \eta_f + \eta_e$ where $f' := (y_0y_1F^{\times 2}, \ldots, y_0y_{d-2}F^{\times 2})$, $y_0 = y_1 \ldots y_{d-1}$ and $\mathsf{Pf}_3(\eta_f + \eta_e) \leq d - 3 + n - 2 \leq 2(d-2) \leq d^2$. Now suppose that $\xi + \eta_e \neq \xi_f$ for all family f, then by Lemma 2.3,

$$\xi + \eta_e = (b_1F^{\times 2}) + \ldots + (b_{d-1}F^{\times 2}) + (F^{\times 2})$$

   with $b_{m+1} = b_1 \ldots b_{m-1}$, $3 \leq m \leq d-2$ and $b_1F^{\times 2}, \ldots, b_mF^{\times 2}$ linearly independent, because $F^{\times 2} \in D(\xi + \eta_e)$. Set $f := (b_1F^{\times 2}, \ldots, b_mF^{\times 2})$ and $b_0 := b_1 \ldots b_m$. If m is even, then

$$\xi + \eta_e + \eta_f = (b_0b_1F^{\times 2}) + \ldots + (b_0b_{m-1}F^{\times 2}) + (b_{m+2}F^{\times 2}) + \ldots + (b_{d-1}F^{\times 2}) + (F^{\times 2})$$

   and if m is odd then

$$\xi + \eta_e + \eta_f = (b_0b_1F^{\times 2}) + \ldots + (b_0b_{m-1}F^{\times 2}) + (b_0F^{\times 2}) + (b_{m+2}F^{\times 2}) + \ldots + (b_{d-1}F^{\times 2}).$$

   In both cases, $\xi + \eta_e + \eta_f + I^3[V_F] = \xi + I^3[V_F] \neq 0$ with $|D(\xi + \eta_e + \eta_f)| \leq d-2$. By induction, $\xi + \eta_e + \eta_f = \xi_g + \eta$ where g is a family of r linearly independent vectors of V_F, r is even, $2 \leq r \leq d-4$ and $\mathsf{Pf}_3(\eta) \leq (d-2)^2$. Then $\xi = \xi_g + \eta + \eta_e + \eta_f$ with

$$\mathsf{Pf}_3(\eta + \eta_e + \eta_f) \leq (d-2)^2 + n - 2 + m - 2 \leq (d-2)^2 + 2(d-2) \leq d^2$$

   and we are done. □

We deduce two corollaries. First the existence of quadratic forms $q = \langle 1, x_1, \ldots, x_r, x_1 \ldots x_r \rangle$ such that $[q] \in I^3(F)$ and $x_1F^{\times 2}, \ldots, x_rF^{\times 2}$ are linearly independent is precisely the obstruction for $\overline{\Psi}$ to be an isomorphism.

Corollary 2.6 *The equivalent conditions in Proposition 2.1 are also equivalent to the following condition: there do not exist linearly independent vectors $x_1F^{\times 2}, \ldots, x_rF^{\times 2} \in V_F$ such that $r \geq 2$ is even and $[\langle 1, x_1, \ldots, x_r, x_1 \ldots x_r\rangle] \in I^3(F)$.*

Proof: If there exist linearly independent vectors $x_1F^{\times 2}, \ldots, x_rF^{\times 2} \in V_F$ such that $r \geq 2$ is even and $[\langle 1, x_1, \ldots, x_r, x_1 \ldots x_r\rangle] \in I^3(F)$, then $(F^{\times 2}) + (x_1F^{\times 2}) + \ldots + (x_rF^{\times 2}) + (x_1 \ldots x_rF^{\times 2}) + I^3[V_F]$ is a non-trivial element in the kernel of $\overline{\Psi}$, hence $\overline{\Psi}$ is not an isomorphism. Conversely, assume that $\overline{\Psi}$ is not an isomorphism, then there exists a non-trivial $\xi + I^3[V_F]$ in the kernel of $\overline{\Psi}$. By Proposition 2.5, there exists a family $e = (x_1F^{\times 2}, \ldots, x_rF^{\times 2})$ of linearly independent vectors of V_F such that $r \geq 2$ is even and $\xi + I^3[V_F] = \xi_e + I^3[V_F]$. Then $[\langle 1, x_1, \ldots, x_r, x_1 \ldots x_r\rangle] \in I^3(F)$. □

Next we relate the 3-Pfister number of a given quadratic form to the one of some particular quadratic form.

Corollary 2.7 *Let $[q] \in I^3(F)$ with $q = \langle a_1, \ldots, a_d\rangle$. Then we have one of the following situations:*

1. *$(a_1F^{\times 2}) + \ldots + (a_dF^{\times 2}) \in I^3[V_F]$ and $\mathsf{Pf}_3(q) \leq (d^2)/8$;*
2. *there exist an even integer $2 \leq r \leq d-2$ and linearly independent vectors $x_1F^{\times 2}, \ldots, x_rF^{\times 2} \in V_F$ such that $[\langle 1, x_1, \ldots, x_r, x_1 \ldots x_r\rangle] \in I^3(F)$ and*

$$\mathsf{Pf}_3(q) \leq d^2 + \mathsf{Pf}_3(\langle 1, x_1, \ldots, x_r, x_1 \ldots x_r\rangle).$$

Proof: Set $\xi := (a_1F^{\times 2}) + \ldots + (a_dF^{\times 2})$, then $\overline{\Psi}(\xi + I^3[V_F]) = [q] + I^3(F) = 0$, hence $\xi + I^3[V_F]$ is in the kernel of $\overline{\Psi}$. Suppose that $\xi \in I^3[V_F]$, then $|D(\xi)| \leq d$ and $\mathsf{Pf}_3(q) = \mathsf{Pf}_3\big(\Psi(\xi)\big) \leq \mathsf{Pf}_3(\xi) \leq (d^2)/8$. Suppose that $\xi \notin I^3[V_F]$, then by Proposition 2.5, there exist a family $e = (x_1F^{\times 2}, \ldots, x_rF^{\times 2})$ of linearly independent vectors of V_F and $\eta \in I^3[V_F]$ such that $\xi = \xi_e + \eta$, r is even, $2 \leq r \leq d-2$ and $\mathsf{Pf}(\eta) \leq d^2$. Since $\xi_e + I^3[V_F] = \xi + I^3[V_F]$ is in the kernel of $\overline{\Psi}$ we have $[\langle 1, x_1, \ldots, x_r, x_1 \ldots x_r\rangle] \in I^3(F)$. Moreover $[q] = \Psi(\xi) = \Psi(\xi_e) + \Psi(\eta)$, so

$$\mathsf{Pf}_3(q) \leq \mathsf{Pf}_3\big(\Psi(\xi_e)\big) + \mathsf{Pf}_3\big(\Psi(\eta)\big) \leq \mathsf{Pf}_3\big(\Psi(\xi_e)\big) + \mathsf{Pf}_3(\eta) \leq \mathsf{Pf}_3(\langle 1, x_1, \ldots, x_r, x_1 \ldots x_r\rangle) + d^2$$

and the proof is complete. □

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