SYMBOL LENGTH AND STABILITY INDEX

KARIM JOHANNES BECHER AND PAWEŁ GŁADKI

Abstract. We show that a pythagorean field (more generally, a reduced abstract Witt ring) has finite stability index if and only if it has finite 2-symbol length. We give explicit bounds for the two invariants in terms of one another. To approach the question whether those bounds are optimal we consider examples of pythagorean fields.

Classification (MSC 2010): 11E04, 11E10, 11E81

1. Introduction

The aim of this article is to study the relation between two field invariants appearing in the theory of quadratic forms over fields, with a special focus on real pythagorean fields. We shall first recall some facts from the theory of quadratic forms over fields, Milnor K-Theory, Galois cohomology, and real algebra, refering to [8], [11], and [12] for details, and shall then formulate our main results in the context of fields. From Section 2 on, we will mainly work in the abstract theory of quadratic forms, where the field is replaced by an abstract Witt ring, and prove the results in this more general setting.

Let F always denote a field of characteristic different from 2. Let WF denote the Witt ring of quadratic forms over F and IF its fundamental ideal. For $n \in \mathbb{N}$ let $I^nF = (IF)^n$, $\overline{I}^nF = I^nF/I^{n+1}F$, $H^n(F) = H^n(\Gamma_F, \mu_2)$, the nth cohomology group for the trivial action of the absolute Galois group Γ_F of F on $\mu_2 = \{+1, -1\}$, and k_nF the nth group of Milnor K-theory modulo 2 of F defined in [14]. The group $\overline{I}^nF$ is generated by the classes of n-fold Pfister forms $\langle\langle a_1, \dots, a_n\rangle\rangle = \langle 1, -a_1\rangle \otimes \cdots \otimes \langle 1, -a_n\rangle$, and similarly k_nF is generated by 'symbols' $\{a_1, \dots, a_n\}$, while $H^n(F)$ contains 'cup products' $(a_1) \cup \cdots \cup (a_n)$, where $a_1, \dots, a_n \in F^\times$. Milnor [14] asked whether for any $n \in \mathbb{N}$ there are natural isomorphisms between the groups $\overline{I}^nF$, H^nF and k_nF making those elements correspond with one another for fixed $a_1, \dots, a_n \in F^\times$. We have $k_0F = \overline{I}^0F = H^0F = \mathbb{Z}/2\mathbb{Z}$, by convention, and $k_1F \cong \overline{I}^1F \cong H^1F \cong F^\times/F^{\times 2}$, via $\{a\} \mapsto \langle\langle a\rangle\rangle + I^2F \mapsto (a) \mapsto aF^{\times 2}$. Moreover, H^2F can be identified with $\mathrm{Br}_2(F)$, the 2-torsion part of the Brauer group of F, by interpreting $(a_1) \cup (a_2)$ as the class of the quaternion algebra $(a_1, a_2)_F$. For any $n \in \mathbb{N}$, Milnor [14] defined a natural homomorphism $s_n : k_nF \longrightarrow \overline{I}^nF$ with $s_n(\{a_1, \dots, a_n\}) = \langle\langle a_1, \dots, a_n\rangle\rangle + I^{n+1}F$,

Date: December 9, 2010.

which is trivially surjective, and he showed that $s_2 : k_2F \longrightarrow \overline{I}^2F$ is an isomorphism. It was proven in [15] that Milnor's aforementioned question has a positive answer in general and, in particular, that s_n is an isomorphism for any $n \in \mathbb{N}$.

As in [1] we denote by $\lambda_n(F)$ the supremum in $\mathbb{N}\cup\{\infty\}$ over the numbers r such that there exists an element of k_nF that can not be expressed as a sum of less than r symbols, and we call $\lambda_n(F)$ the *n-symbol length of F*. While $\lambda_0(F) = \lambda_1(F) = 1$ independently of F, the 2-symbol length $\lambda_2(F)$ is of particular interest and was studied (with slightly different notation) in [9] and in [10] relative to quadratic forms and the u-invariant of F.

Let $\sum F^2$ denote the subgroup of $F^\times$ consisting of the nonzero sums of squares in F. We say that F is *pythagorean* if $\sum F^2 = F^{\times 2}$. We say that F is *real* if $-1 \notin \sum F^2$, and *nonreal* if $-1 \in \sum F^2$. A *preordering of F* is a subset $T \subseteq F$ that contains all squares in F and is closed under addition and multiplication and such that $-1 \notin T$; if in addition $T \cup -T = F$, then T is called an *ordering of F*. For a preordering T of F we write $T^\times = T \setminus \{0\}$, which is a subgroup of $F^\times$. We denote by X_F the set of all orderings of F and by X_T the set of all orderings containing the preordering T. For any preordering T of F we have $T = \bigcap X_T$. Any $P \in X_F$ yields a map $\mathrm{sign}_P : WF \longrightarrow \mathbb{Z}$ called the *signature at P*. If F is real, then $P \longmapsto \ker(\mathrm{sign}_P)$ gives a one-to-one correspondence between X_F and the set of non-maximal prime ideals of WF.

We recall the definition of fans introduced in [3]. Let T be a preordering of F and $n \in \mathbb{N}$ such that $[F^\times : T^\times] = 2^{n+1}$. Then by [3] we have $n \leq |X_T| \leq 2^n$, and equality $|X_T| = 2^n$ holds if and only if the image of the homomorphism $\mathrm{sign}_T : WF \longrightarrow \mathbb{Z}^{X_T}, \varphi \longmapsto (\mathrm{sign}_P(\varphi))_{P\in X_T}$ is isomorphic to $\mathbb{Z}[(\mathbb{Z}/2\mathbb{Z})^n]$; in this case T is called a *fan of degree n*. A fan of degree 0 is the same as an ordering. A fan of degree 1 is the same as the intersection of two different orderings.

The (*reduced*) *stability index* of a field was introduced in [5]. In [6] this field invariant was characterized in terms of fans. By this characterization the *stability index of F* is given as

$$st(F) \;=\; \sup\{\deg(T) \mid T \text{ fan of } F\} \;\in\; \mathbb{N}\cup\{\infty\},$$

with the convention that $\sup\emptyset = 0$. Hence, $st(F) = 0$ if and only if $|X_F| \leq 1$.

Our aim is to relate the stability index to the symbol lengths, in particular to the 2-symbol length. This will be done in the more general context of abstract Witt rings, introduced in Section 2. For fields (2.3) reads as follows:

> *If $\lambda_i(F) < \infty$, for some $i \geq 2$, then $st(F) < \infty$. In particular, for $i = 2$ we have $st(F) \leq 2\lambda_2(F) - 1$.*

In Section 3 we focus on reduced Witt rings. For fields (3.7) reads as follows:

> *If F is pythagorean, then $\lambda_2(F) < \infty$ if and only if $st(F) < \infty$.*

In order to prove the right-to-left implication, we actually show in (3.6):

> *Let F be pythagorean and $s = st(F)$. If $1 \leq s \leq 2$, then $\lambda_2(F) = s$. If $3 \leq s \leq \infty$, then $[\frac{s}{2}] + 1 \;\leq\; \lambda_2(F) \;\leq\; 2^{s-1}(2^{s-2} - 1)$.*

Here and the sequel, we use the notation $[x] = \max\{z \in \mathbb{Z} \mid z \leq x\}$ for $x \in \mathbb{R}$. We do not know whether the upper bound on λ_2 in the pythagorean case is optimal. To approach this question, we construct in (3.8) for any $r \in \mathbb{N}$ a pythagorean field F with $\lambda_2(F) = st(F) = r$. As for the lower bound, for any $s \in \mathbb{N}$ we have $st(F) = s$ and $\lambda_2(F) = [\frac{s}{2}] + 1$ for the field $F = \mathbb{R}(\!(t_1)\!)\ldots(\!(t_s)\!)$.

2. Abstract Witt rings

We recall the notion of (abstract) Witt rings from [13]. A *Witt ring* is a triple (W, G, I) where W is a commutative ring, I is the unique ideal of index 2 of W, called the *fundamental ideal*, and $G \subseteq W^\times$ is a group that additively generates W and such that $G \longrightarrow I/I^2, a \longmapsto (1-a) + I^2$ is a group isomorphism.

Let $n \in \mathbb{N}$. The nth power of the fundamental ideal I^n is additively generated by the products $(1+a_1)\cdots(1+a_n)$ where $a_1, \ldots, a_n \in G$. We set $\overline{I}^n = I^n/I^{n+1}$.

For $\varphi \in W$ the least number of summands needed to write φ as a sum of elements of G is called the *anisotropic dimension of* φ and denoted by $\operatorname{diman}(\varphi)$. For $\alpha \in \overline{I}^n$ let $l(\alpha)$ denote the least number of summands needed to write α as a sum of classes of elements of the shape $(1+a_1)\cdots(1+a_n)$ with $a_1, \ldots, a_n \in G$.

2.1. **Lemma.** *Let $\alpha \in \overline{I}^2$ and $m \geq 1$. Then $l(\alpha) \leq m$ if and only if $\alpha = \varphi + I^3$ for some $\varphi \in I^2$ with* $\operatorname{diman}(\varphi) \leq 2m+2$.

Proof: The proof is easy, and basically the same as in [1, (3.2)] □

We define the *nth symbol length of W* as

$$\lambda_n(W) \;=\; \sup\{l(\alpha) \mid \alpha \in \overline{I}^n\} \;\in\; \mathbb{N} \cup \{\infty\}\,.$$

It is easy to see that $\lambda_0(W) = \lambda_1(W) = 1$. For the Witt ring $W = WF$ of the field F we then have $\lambda_n(F) = \lambda_n(W)$ in view of the isomorphism $s_n : k_nF \longrightarrow \overline{I}^n F$.

For $i \in \mathbb{N}$ we define

$$\Lambda_i : \mathbb{N} \longrightarrow \mathbb{N},\; n \longmapsto \lambda_i(\mathbb{Z}\,[(\mathbb{Z}/2\mathbb{Z})^n])\,.$$

Note that $\mathbb{Z}\,[(\mathbb{Z}/2\mathbb{Z})^n]$ is the Witt ring of the pythagorean field $\mathbb{R}(\!(t_1)\!)\ldots(\!(t_n)\!)$. So Λ_i yields the values of the ith symbol length for a particular sequence of fields. By [1] we have $\Lambda_2(n) = [\frac{n}{2}] + 1$, but no formula is known for Λ_i when $i > 2$.

Let X_W be the set of non-maximal prime ideals in W. By [13, Corollary 4.18], elements of X_W are in one-to-one correspondence with ring homomorphisms $W \longrightarrow \mathbb{Z}$, called *signatures of W*. The signature corresponding to $P \in X_W$ is denoted by sign_P. We say that W is *real* if $X_W \neq \emptyset$, and *nonreal* otherwise.

For $d \in \mathbb{N}$, a subset $\mathcal{F} \subseteq X_W$ is called a *fan of degree d on W* if $|\mathcal{F}| = 2^d$ and $W/\bigcap\mathcal{F} \cong \mathbb{Z}\,[(\mathbb{Z}/2\mathbb{Z})^d]$. The *stability index of W* is then defined as

$$st(W) \;=\; \sup\{n \in \mathbb{N} \mid \text{there exists a fan of degree } n \text{ on } W\} \;\in\; \mathbb{N} \cup \{\infty\}\,.$$

Given the Witt ring $W = WF$ of a field F, associating to a preordering T of F the set of prime ideals $\{\ker(\operatorname{sign}_P) \mid P \in X_T\}$ gives a degree preserving one-to-one correspondence between the two concepts of fans, so that $st(F) = st(W)$.

2.2. **Theorem.** *For $n \le st(W)$ we have $\lambda_i(W) \ge \Lambda_i(n)$ for any $i \in \mathbb{N}$.*

Proof: As $\mathbb{Z}\left[(\mathbb{Z}/2\mathbb{Z})^n\right]$ is a quotient of W, this is obvious. $\square$

2.3. **Corollary.** *We have $st(W) \le 2\lambda_2(W) - 1$. Moreover, if $\lambda_i(W) < \infty$ for some $i \ge 2$, then $st(W) < \infty$.*

Proof: For any $n \le st(W)$ one has $\lambda_2(W) \ge \Lambda_2(n) = [\frac{n}{2}] + 1 \ge \frac{n+1}{2}$, which shows the first statement. For fixed i, one has $\Lambda_i(n) \longrightarrow \infty$ for $n \longrightarrow \infty$, and the second statement thus follows using (2.2). $\square$

3. Reduced Witt rings

A commutative ring is *reduced* if it contains no nonzero nilpotent elements. By [13, Corollary 4.22], if the Witt ring W is reduced, then $G = W^\times$.

3.1. **Question.** *Assume that W is reduced. If $st(W) < \infty$, does it follow that $\lambda_i(W) < \infty$ for every $i \in \mathbb{N}$?*

We are going to give a positive answer to this question for $i = 2$.

3.2. **Lemma.** *Let $r \ge st(W)$. For every $\varphi \in W$, there exists $\varphi' \in W$ such that $\varphi \equiv \varphi' \bmod I^r$ and $0 \le \mathrm{sign}_P(\varphi') < 2^r$ for all $P \in X_W$; if W is reduced, then φ' is uniquely determined by φ.*

Proof: The proof is essentially given in [2, (2.2)]. $\square$

Given $\varphi \in W$ we put

$$\Delta(\varphi) = \max\{|\,\mathrm{sign}_P(\varphi)| \mid P \in X_W\}$$

and call this number the *amplitude of φ*. In the reduced case, the anisotropic dimension is bounded in terms of the amplitude and the stability index.

3.3. **Theorem** (Bonnard). *Assume that W is reduced of stability index $s \ge 1$. Then $\mathrm{diman}(\varphi) \le 2^{s-1}\Delta(\varphi)$ for any $\varphi \in W$.*

Proof: See [4, Proposition 4] or [16, Theorem 1]. $\square$

3.4. **Theorem.** *Assume that W is reduced with $st(W) \ge 2$. Let $s = st(W)$ and $r = \max\{s, 3\}$. Any element of $\overline{I}^2$ is of the shape $(\psi + 2) + I^3$ with some $\psi \in I$ of discriminant -1 and with $\mathrm{diman}(\psi) \le 2^s(2^{r-2} - 1)$.*

Proof: Let $\alpha \in \overline{I}^2$. By (3.2), there exists $\varphi \in I^2$ such that $\alpha = \varphi + I^3$ and $0 \le \mathrm{sign}_P(\varphi) \le 2^r - 4$ for all $P \in X_W$. Put $\psi = \varphi - 2$ if $s \le 3$ and $\psi = 2(2^{r-2} - 1) - \varphi$ if $s > 3$. Then $\alpha = \psi + 2 + I^3$ and $\Delta(\psi) \le 2(2^{r-2} - 1)$. Hence, $\mathrm{diman}(\psi) \le 2^s(2^{r-2} - 1)$ by (3.3). $\square$

The Witt ring W is said to be *linked* if $\lambda_2(W) \le 1$; in this case $\lambda_n(W) \le 1$ for all $n \ge 1$. If W is real, then $\lambda_n(W) \ge 1$ for any $n \in \mathbb{N}$, so that W is linked if and only if $\lambda_n(W) = 1$ for all $n \ge 1$.

3.5. **Proposition.** *Assume that W is reduced. Then W is linked if and only if $I^2 = 2I$, if and only if $st(W) \le 1$.*

Proof: If $st(W) \ge 2$, then we have $\lambda_2(W) \ge \Lambda_2(2) = 2$ by (2.2). Assume now that $st(W) \le 1$. Then to any $a, b \in W^\times$ there exists $c \in W^\times$ such that $(1+a)\cdot(1+b) - 2(1+c) \in \bigcap_{\mathfrak{p}\in X_W} \mathfrak{p} = 0$. This shows that $I^2 = 2I$, which in turn implies that W is linked. □

3.6. **Theorem.** *Assume that W is reduced with $s = st(W) < \infty$.*

(*a*) *If $s \le 1$ then $\lambda_2(W) = 1$.*
(*b*) *If $s = 2$ then $\lambda_2(W) = 2$.*
(*c*) *If $s \ge 3$, then $[\frac{s}{2}] + 1 \le \lambda_2(W) \le 2^{s-1}(2^{s-2} - 1)$.*

Proof: Part (*a*) follows from (3.5). If $s \ge 2$, then (2.2) and (3.4) yield that $[\frac{s}{2}] + 1 \le \lambda_2(W) \le 2^{s-1}(2^{r-2} - 1)$ with $r = \max\{s, 3\}$. This shows (*b*) and (*c*). □

3.7. **Corollary.** *If W is reduced, then $s(W) < \infty$ if and only if $\lambda_2(W) < \infty$.*

Proof: This is clear from (2.3) and (3.6). □

3.8. **Example.** Let r be a positive integer and let K be a pythagorean SAP-field having exactly r different orderings. For example, such a field K is obtained as the intersection of any r different real closures of $\mathbb{Q}$. It follows from the assumption on K that $|K^\times/K^{\times 2}| = 2^r$. Let P be an ordering of K. There exist $a_1, \ldots, a_{r-1} \in P^\times$ such that the square classes $a_1K^{\times 2}, \ldots, a_{r-1}K^{\times 2}$ form an $\mathbb{F}_2$-basis of $P^\times/K^{\times 2}$. Let $F = K(\!(t_1)\!)\ldots(\!(t_{r-1})\!)$. Then F is pythagorean, $st(F) = r$, and $|F^\times/F^{\times 2}| = 2^{2r-1}$. By [1, (1.1)] the latter implies that $\lambda_2(F) \le r$. As the ordering P extends to $K(\sqrt{a_1}, \ldots, \sqrt{a_{r-1}})$, the quaternion algebra $(-1,-1)_{K(\sqrt{a_1},\ldots,\sqrt{a_{r-1}})}$ is not split. Using the results in [17, Sect. 2], it follows that the product of quaternion algebras

$$(-1,-1)_F \otimes_F (a_1, t_1)_F \otimes_F \ldots \otimes_F (a_{r-1}, t_{r-1})_F$$

is a division algebra and thus not Brauer equivalent to a product of less than r quaternion algebras, whence $\lambda_2(F) \ge r$. Therefore $\lambda_2(F) = r = st(F)$. Hence, $W = WF$ is a reduced Witt ring with $\lambda_2(W) = r = st(W)$.

If W is reduced with $st(W) = 3$, then we have $2 \le \lambda_2(W) \le 4$ by (3.4) and (3.5). The Witt ring of $\mathbb{R}(\!(t_1)\!)(\!(t_2)\!)(\!(t_3)\!)$ and the one obtained for $r = 4$ in (3.8) show that the values 2 and 3 are both possible for $\lambda_2(W)$ in this situation, but this is open for the value 4. More generally, we are left with the following question.

3.9. **Question.** *Is there a reduced Witt ring W with $\lambda_2(W) > st(W)$?*

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Universität Konstanz, Zukunftskolleg / FB Mathematik und Statistik, 78457 Konstanz, Germany.

E-mail address: becher@maths.ucd.ie

Instytut Matematyki, Uniwersytet Śląski, ul. Bankowa 14, 40-007 Katowice, Poland.

E-mail address: pgladki@ux1.mat.mfc.us.edu.pl