SUMS OF SQUARES IN CERTAIN QUATERNION AND OCTONION ALGEBRAS

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Abstract. Formulae for the levels and sublevels of certain quaternion and octonion algebras are established. Corollaries concerning the equality of levels and sublevels of quaternion algebras with those of associated octonion algebras are presented.

Let R be a not necessarily associative ring with unity. The *level* and *sublevel of* R, respectively denoted by $s(R)$ and $\underline{s}(R)$, are defined as follows:

$$s(R) = \inf\{n \in \mathbb{N} \mid \text{there exist } \ x_1, \ldots, x_n \in R \ \text{ such that } \textstyle\sum_{i=1}^{n} {x_i}^2 = -1\},$$

$$\underline{s}(R) = \inf\{n \in \mathbb{N} \mid \text{there exist } \ x_1, \ldots, x_{n+1} \in R \setminus \{0\} \ \text{ such that } \ \textstyle\sum_{i=1}^{n+1} {x_i}^2 = 0\}.$$

Let F be a field of characteristic different from 2. For $a, b \in F^{\times}$, the *quaternion algebra* $(a, b)_F$ is a 4-dimensional F-vector space with basis $\{1, i, j, k\}$ satisfying $i^2 = a, j^2 = b$ and $ij = -ji = k$. For $a, b, c \in F^{\times}$, the *octonion algebra* $(a, b, c)_F$ is isomorphic to $(a, b)_F \oplus (a, b)_F e$, where $e^2 = c$, with its multiplication being determined by $(u_1, v_1)(u_2, v_2) = (u_1 u_2 + c\overline{v_2} v_1, v_2 u_1 + v_1 \overline{u_2})$, where $u_1, u_2, v_1, v_2 \in (a, b)_F$ (here, $\overline{}$ denotes conjugation).

The related problems of determining the numbers attainable as the levels and sublevels of quaternion and octonion algebras remain open, and motivate our investigations. Given $a, b \in F^{\times}$, we study whether the level (respectively, the sublevel) of $(a, b)_F$ equals that of $(a, b, x)_{F((x))}$.

We will provide a partial answer to these questions, by showing that the respective equalities hold whenever the level or sublevel of $(a, b)_F$ belongs to an associated family of intervals. Moreover, we will show that these equalities always hold for a particular class of quaternion algebras, conjectured to contain members of level and sublevel n for all $n \in \mathbb{N}$ (see [4]).

Throughout, we will employ standard concepts and notation regarding quadratic forms. Our notation coincides with that employed in [3], aside from our usage of $n \times \varphi$ to denote the orthogonal sum of $n \in \mathbb{N}$ copies of a quadratic form φ. Moreover, for $a, b \in F^{\times}$, we will let $k(a)$ (respectively $k(a, b)$) denote the least $n \in \mathbb{N}$ such that $n \times \langle 1, -a \rangle$ (respectively $n \times \langle 1, -a, -b, ab \rangle$) is isotropic (over F, unless stated otherwise). If such an n exists, then $n = 2^k + 1$ for some $k \in \mathbb{Z}$. Otherwise, the quantity is said to be infinite.

In order to obtain the aforementioned results, we will establish characterisations of the levels and sublevels of certain quaternion and octonion algebras, namely those with "transcendental parameters". These characterisations provide analogues of a theorem of Tignol and Vast (see [5]), the statement of which is included in the following result.

Theorem 1. *Let* $a, b \in F^{\times}$*,* $Q = (a, x)_{F((x))}$ *and* $O = (a, b, x)_{F((x))}$*. Then*

(a) $s(Q) = \min\{s(F(\sqrt{a})), k(a)\}$ *and* $\underline{s}(Q) = \min\{s(F(\sqrt{a})), k(a) - 1\}$,

(b) $s(O) = \min\{s((a, b)_F), k(a, b)\}$ *and* $\underline{s}(O) = \min\{\underline{s}((a, b)_F), k(a, b) - 1\}$.

Proof. (a) The level equality is the aforementioned result from [5]. We will prove the sublevel equality.

As $F(\sqrt{a}) \subseteq Q$, we have that $\underline{s}(Q) \leqslant \underline{s}(F(\sqrt{a}))$, which equals $s(F(\sqrt{a}))$. Letting $k(a) = n$, there exist $\gamma_1, \dots, \gamma_n, \delta_1, \dots, \delta_n \in F$, not all zero, such that $\sum_{i=1}^n {\gamma_i}^2 - a \sum_{i=1}^n {\delta_i}^2 = 0$. Hence $\sum_{i=1}^n (\gamma_i j + \delta_i k)^2 = 0$ in Q, whereby $\underline{s}(Q) \leqslant n-1$. Hence, we have that $\underline{s}(Q) \leqslant \min\{s(F(\sqrt{a})), k(a)-1\}$.

Suppose that $\underline{s}(Q) = n$. Hence, there exist $\alpha_i, \beta_i, \gamma_i, \delta_i \in F(\!(x)\!)$, not all zero, such that

$$\sum_{i=1}^{n+1} \alpha_i^2 + a \sum_{i=1}^{n+1} \beta_i^2 + x\left(\sum_{i=1}^{n+1} \gamma_i^2 - a \sum_{i=1}^{n+1} \delta_i^2\right) = 0$$

and

$$\sum_{i=1}^{n+1} \alpha_i \beta_i = \sum_{i=1}^{n+1} \alpha_i \gamma_i = \sum_{i=1}^{n+1} \alpha_i \delta_i = 0.$$

Multiplying across these equations by x^{2d} for a suitable choice of $d \in \mathbb{Z}$, we have that

$$\sum_{i=1}^{n+1} (x^d\alpha_i)^2 + a \sum_{i=1}^{n+1} (x^d\beta_i)^2 + x\left(\sum_{i=1}^{n+1} (x^d\gamma_i)^2 - a \sum_{i=1}^{n+1} (x^d\delta_i)^2\right) = 0$$

and

$$\sum_{i=1}^{n+1} (x^d\alpha_i)(x^d\beta_i) = \sum_{i=1}^{n+1} (x^d\alpha_i)(x^d\gamma_i) = \sum_{i=1}^{n+1} (x^d\alpha_i)(x^d\delta_i) = 0,$$

where at least one of $x^d\alpha_i, x^d\beta_i, x^d\gamma_i, x^d\delta_i \in F[\![x]\!]$ is not divisible by x for some i.

If $x^d\alpha_i$ or $x^d\beta_i$ is not divisible by x for some i, then taking residues modulo x gives that

$$\sum_{i=1}^{n+1} \left(\overline{x^d\alpha_i}\right)^2 + a \sum_{i=1}^{n+1} \left(\overline{x^d\beta_i}\right)^2 = 0 \quad \text{and} \quad \sum_{i=1}^{n+1} \left(\overline{x^d\alpha_i}\right)\left(\overline{x^d\beta_i}\right) = 0,$$

whereby $\underline{s}(F(\sqrt{a})) = s(F(\sqrt{a})) \leqslant n$.

If $x^d\alpha_i$ and $x^d\beta_i$ are divisible by x for all i, then dividing by x and taking residues modulo x gives that

$$\sum_{i=1}^{n+1} \left(\overline{x^d\gamma_i}\right)^2 - a \sum_{i=1}^{n+1} \left(\overline{x^d\delta_i}\right)^2 = 0,$$

whereby $k(a) - 1 \leqslant n$, completing the proof of (a).

(b) The sublevel equality can be proven by arguing as above. We will prove the level equality.

As $(a,b)_F \subset O$, we clearly have that $s(O) \leqslant s\left((a,b)_F\right)$. For $k(a,b) = n$, we have that $n \times \langle 1, -a, -b, ab\rangle$ is isotropic over F, whereby it represents -1. Thus, there exist $\varepsilon_i, \zeta_i, \eta_i, \vartheta_i \in F$, not all zero, such that $\sum_{i=1}^n {\varepsilon_i}^2 - a\sum_{i=1}^n {\zeta_i}^2 - b\sum_{i=1}^n {\eta_i}^2 + ab\sum_{i=1}^n {\vartheta_i}^2 = -1$. Hence, we have that $\sum_{i=1}^n (\varepsilon_i e + \zeta_i ie + \eta_i je + \vartheta_i ke)^2 = -1$ in O, whereby $s(O) \leqslant n$. Hence $s(O) \leqslant \min\{s\left((a,b)_F\right), k(a,b)\}$.

Suppose that $s(O) = n$. Hence, there exist $\alpha_i, \beta_i, \gamma_i, \delta_i, \varepsilon_i, \zeta_i, \eta_i, \vartheta_i \in F(\!(x)\!)$, not all zero, such that

$$\sum_{i=1}^{n} \alpha_i^2 + a \sum_{i=1}^{n} \beta_i^2 + b \sum_{i=1}^{n} \gamma_i^2 - ab \sum_{i=1}^{n} \delta_i^2 + x\left(\sum_{i=1}^{n} \varepsilon_i^2 - a \sum_{i=1}^{n} \zeta_i^2 - b \sum_{i=1}^{n} \eta_i^2 + ab \sum_{i=1}^{n} \vartheta_i^2\right) = -1$$

and

$$\sum_{i=1}^{n} \alpha_i\beta_i = \sum_{i=1}^{n} \alpha_i\gamma_i = \sum_{i=1}^{n} \alpha_i\delta_i = \sum_{i=1}^{n} \alpha_i\varepsilon_i = \sum_{i=1}^{n} \alpha_i\zeta_i = \sum_{i=1}^{n} \alpha_i\eta_i = \sum_{i=1}^{n} \alpha_i\vartheta_i = 0.$$

If $\alpha_i, \ldots, \vartheta_i \in F[\![x]\!]$ for all i, then, taking residues modulo x, we obtain that $s\left((a,b)_F\right) \leqslant n$.

Alternatively, multiplying across these equations by x^{2d} for a suitable choice of $d \in \mathbb{N}$, we have that

$$\sum_{i=1}^{n}(x^d\alpha_i)^2 + \ \ldots \ -ab\sum_{i=1}^{n}(x^d\delta_i)^2 + x\left(\sum_{i=1}^{n}(x^d\varepsilon_i)^2 - \ \ldots \ + ab\sum_{i=1}^{n}(x^d\vartheta_i)^2\right) = -x^{2d},$$

and

$$\sum_{i=1}^{n}(x^d\alpha_i)(x^d\beta_i) = \ldots = \sum_{i=1}^{n}(x^d\alpha_i)(x^d\vartheta_i) = 0,$$

where at least one of $x^d\alpha_i, x^d\beta_i, x^d\gamma_i, x^d\delta_i, x^d\varepsilon_i, x^d\zeta_i, x^d\eta_i, x^d\vartheta_i \in F[\![x]\!]$ is not divisible by x for some i.

If $x^d\alpha_i, x^d\beta_i, x^d\gamma_i$ or $x^d\delta_i$ is not divisible by x for some i, then taking residues modulo x gives that

$$\sum_{i=1}^{n}\left(\overline{x^d\alpha_i}\right)^2 + \ \ldots \ -ab\sum_{i=1}^{n}\left(\overline{x^d\delta_i}\right)^2 = 0$$

and

$$\sum_{i=1}^{n}\left(\overline{x^d\alpha_i}\right)\left(\overline{x^d\beta_i}\right) = \ \ldots \ = \sum_{i=1}^{n}\left(\overline{x^d\alpha_i}\right)\left(\overline{x^d\delta_i}\right) = 0.$$

Hence $\underline{s}\left((a,b)_F\right) \leqslant n-1$, whereby $s\left((a,b)_F\right) \leqslant n$ by [2, Theorem].

If $x^d\alpha_i, x^d\beta_i, x^d\gamma_i$ and $x^d\delta_i$ are divisible by x for all i, then dividing by x and taking residues modulo x gives that

$$\sum_{i=1}^{n}\left(\overline{x^d\varepsilon_i}\right)^2 - a\sum_{i=1}^{n}\left(\overline{x^d\zeta_i}\right)^2 - b\sum_{i=1}^{n}\left(\overline{x^d\eta_i}\right)^2 + ab\sum_{i=1}^{n}\left(\overline{x^d\vartheta_i}\right)^2 = 0,$$

whereby $k(a,b) \leqslant n$. □

Corollary 1. *Let $a \in F^\times$.*

(a) For $Q = (x,y)_{F(\!(x)\!)(\!(y)\!)}$ and $O = (x,y,z)_{F(\!(x)\!)(\!(y)\!)(\!(z)\!)}$, we have that

$$s(Q) = \underline{s}(Q) = s(O) = \underline{s}(O) = s(F).$$

(b) For $O = (a,x,y)_{F(\!(x)\!)(\!(y)\!)}$, we have that

$$s(O) = \min\left\{s(F(\sqrt{a})), k(a)\right\} \quad \text{and} \quad \underline{s}(O) = \min\left\{s(F(\sqrt{a})), k(a)-1\right\}.$$

Proof. (a) It is clear that $\underline{s}(O) \leqslant s(O) \leqslant \underline{s}(Q) \leqslant s(Q) \leqslant s(F)$. If $\underline{s}(O) = n$, then $(n+1)\times\langle 1, x, y, -xy, z, -xz, -yz, xyz\rangle$ is isotropic over $F(\!(x)\!)(\!(y)\!)(\!(z)\!)$. By iteratively invoking Springer's Theorem (see [3, Ch.VI, Theorem 1.4]), one obtains that $(n+1)\times\langle 1\rangle$ is isotropic over F, whereby $s(F) \leqslant n$, establishing the result.

(b) Since $k(a,x)$ over $F(\!(x)\!)$ equals $k(a)$ over F, by [3, Ch.VI, Theorem 1.4], an application of Theorem 1(b), followed by one of Theorem 1(a), establishes these statements. □

For F a formally real field and $a \in F^\times$ a sum of squares, one sees that $k(a)$ is finite, whereas $s(F(\sqrt{a}))$ is infinite (see [3, Ch.VIII, Lemma 1.4]). In contrast, the finiteness of $k(a,b)$ encodes an upper bound on $s\left((a,b)_F\right)$, allowing us to establish the following corollary.

Corollary 2. *Let $k \geqslant 0$ be an integer and $a, b \in F^\times$.*

(a) If $1 + \left\lfloor \frac{2}{3}\cdot 2^k \right\rfloor < s\left((a,b)_F\right) \leqslant 2^k + 1$, then $s\left((a,b,x)_{F(\!(x)\!)}\right) = s\left((a,b)_F\right)$.

(b) If $\left\lfloor \frac{2}{3}\cdot 2^k \right\rfloor < \underline{s}\left((a,b)_F\right) \leqslant 2^k$, then $\underline{s}\left((a,b,x)_{F(\!(x)\!)}\right) = \underline{s}\left((a,b)_F\right)$.

Proof. If $k(a,b) < 2^k + 1$, then the Pfister form $2^k \times \langle 1, -a, -b, ab\rangle$ is hyperbolic, whereby its neighbour $\left(1 + \left\lfloor \frac{2}{3} \cdot 2^k \right\rfloor\right) \times \langle a, b, -ab\rangle$ is isotropic, implying that $\underline{s}\left((a,b)_F\right) \leqslant \left\lfloor \frac{2}{3} \cdot 2^k \right\rfloor$ and $s\left((a,b)_F\right) \leqslant 1 + \left\lfloor \frac{2}{3} \cdot 2^k \right\rfloor$. Hence, we must have that $k(a,b) \geqslant 2^k + 1$, whereby Theorem 1(b) gives the result. □

The existence of quaternion algebras whose levels and sublevels lie outside of the above intervals was established in [1] and [4], through the consideration of algebras of the form $(x,y)_{F_0(x,y)(\varphi)}$, where F_0 is formally real and the quadratic form φ over $F_0(x,y)$ is such that $\langle 1, x, y, -xy\rangle \subset \varphi \subset n \times \langle 1, x, y, -xy\rangle$ for some $n \in \mathbb{N}$. Without placing any restrictions on the quaternion algebra $(a,b)_F$, we cannot say whether Corollary 2 holds when $s((a,b)_F)$ or $\underline{s}((a,b)_F)$ take such values.

At present, the existence of quaternion algebras of level 6 (respectively, sublevel 5) remains unknown, prompting us to ask the following question.

Question 1. Let F_0 be a formally real field and $\psi = 8 \times \langle 1, -x, -y, xy\rangle$ be a quadratic form over $F_0(x,y)$. Is $(x,y)_{F_0(x,y)(\psi)}$ of level 6 (and thus sublevel 5)?

Since $6 \times \langle x, y, -xy\rangle$ is a Pfister neighbour of ψ, it is isotropic over $F_0(x,y)(\psi)$, whereby the sublevel and level of $(x,y)_{F_0(x,y)(\psi)}$ are at most 5 and 6 respectively. It seems reasonable to suggest that these upper bounds are attained. Should this be the case, Theorem 1(b) would imply that the associated octonion algebra $(x,y,z)_{F_0(x,y)(\psi)((z))}$ has strictly smaller level and sublevel, since $k(x,y) = 5$ over $F_0(x,y)(\psi)$. Thus, we suspect that Corollary 2 does not hold for all possible level and sublevel values of $(a,b)_F$.

However, restricting ourselves to the aforementioned class of quaternion algebras, conjectured to contain members of level and sublevel n for all $n \in \mathbb{N}$, we can prove the level and sublevel equalities.

Corollary 3. *Let F_0 be a formally real field and φ a quadratic form over $F_0(x,y)$ such that $\langle 1, x, y, -xy\rangle \subset \varphi \subset n \times \langle 1, x, y, -xy\rangle$ for some $n \in \mathbb{N}$. Let $Q = (x,y)_{F_0(x,y)(\varphi)}$ and $O = (x,y,z)_{F_0(x,y)(\varphi)((z))}$. Then $s(O) = s(Q)$ and $\underline{s}(O) = \underline{s}(Q)$.*

Proof. For S an ordering of F_0, let T denote an extension of S to $F_0(x,y)$ such that x and y are negative with respect to T. By [3, Ch.XIII, Theorem 3.1], T extends to an ordering of $F_0(x,y)(\varphi)$. However, $\langle 1, -x, -y, xy\rangle$ is positive definite with respect to T, whereby $k(x,y) = \infty$ over $F_0(x,y)(\varphi)$. Hence, the result follows from invoking Theorem 1(b). □

Remark The above results hold if the respective Laurent series fields are replaced by rational function fields.

Acknowledgements

I would like to thank Jean-Pierre Tignol, Thomas Unger and Karim Johannes Becher for their guidance and help. I am beholden to David Grimm for some helpful comments. I gratefully acknowledge the support I received through an IRCSET-Marie Curie International Mobility Fellowship from the Irish Research Council for Science, Engineering and Technology, co-funded by Marie Curie Actions under FP7. I greatly appreciate the hospitality and support of the Zukunftskolleg.

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