ISOTROPY OF UNITARY INVOLUTIONS

NIKITA KARPENKO AND MAKSIM ZHYKHOVICH

Abstract. We prove the so-called *Unitary Isotropy Theorem*, a result on isotropy of a unitary involution. The analogous previously known results on isotropy of orthogonal and symplectic involutions as well as on hyperbolicity of orthogonal, symplectic, and unitary involutions are formal consequences of this theorem. A component of the proof is a detailed study of the quasi-split unitary grassmannians.

Contents

1. Introduction

Let K be an arbitrary field of characteristic different from 2, A a central simple K-algebra, τ an involution on A (i.e., a self-inverse ring anti-automorphism), $F \subset K$ the subfield of τ-invariant elements of K. In this paper we finish the proof of *Isotropy Theorem* saying that if τ becomes isotropic over any field extension of F splitting A, then τ becomes isotropic over some finite odd degree field extension of F (note that by the example of [13], τ over F does not need to be isotropic).

We refer to [12] for generalities on central simple algebras with involutions. The involutions τ is *isotropic*, if the algebra A contains a non-zero right ideal I satisfying $\tau(I)\cdot I = 0$. The algebra A is *split*, if it represents $0 \in \mathrm{Br}(K)$ in the Brauer group of K, that is, if it is isomorphic to a full matrix algebra over K.

Isotropy Theorem has been proved in [5] for algebras A of exponent 2 (for the symplectic case see also [10]). More precisely, it has been reduced to the case of orthogonal τ by J.-P. Tignol and proved in the orthogonal case by the first named author. In the remaining case, considered in the present paper (see Theorem 6.1), the involution τ is of unitary type (and, in particular, K is of degree 2 over F).

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The proof in the unitary case, made in Section 6, goes along the line of the proof of the orthogonal case, but there are (at least) two important differences. First of all, the information on orthogonal grassmannians needed in the orthogonal case was already available: partially from topology (in the split case), partially from more recent works of A. Vishik [16], [14] (partially remaking in algebraic terms the available topological material). In contrast with this, almost no information (even in the quasi-split case) on unitary grassmannians was available. It seems to be, almost completely, a terra incognito in the literature. Sections 3 and 4 cover this gap. (More study of unitary grassmannian is undertaken in [2], a successor paper).

To explain the second difference, we have to sketch the proof. It is easily reduced to the case of A of even 2-primary index. Let Y be the F-variety of isotropic right ideals in A of reduced dimension $\operatorname{ind} A$. Let X be the F-variety of all right ideals in D of reduced dimension $(\operatorname{ind} A)/2$, where D is a central *division* K-algebra Brauer-equivalent to A. Considering Chow motives with coefficients in $\mathbb{F}_2 := \mathbb{Z}/2\mathbb{Z}$ of smooth projective F-varieties, we manage to show, that certain indecomposable direct summand of the motive of X (the so-called *upper* motive of X introduced in [6]) is isomorphic to a direct summand of the motive of Y. This produces a cycle class π in the modulo 2 Chow group $\operatorname{Ch}_{\dim Y}(Y \times Y)$ (the corresponding projector namely). With some more effort, we come to the case where π is *symmetric* (i.e., invariant under the factor exchange automorphism of the Chow group). We finish by applying to π a certain operation which transforms it to a 0-cycle class in $\operatorname{Ch}_0(Y \times Y)$ of degree 1 (modulo 2) and therefore terminates the proof.

The shortest way to explain where the operation comes from is as follows. By [17], the projector π can be lifted to the algebraic cobordism (modulo 2). Then, applying an appropriate (modulo 2) symmetric operation of [15] and projecting back from cobordism to the Chow group, we get the required 0-cycle class.

Fortunately, the symmetric operations and algebraic cobordism theory are not really needed here (and thus we are not restricted to the characteristic 0). Actually, we succeed to compute the above symmetric operation *because* it can be described (on symmetric projectors) in terms of the Steenrod operations on the modulo 2 Chow groups. (The need of Steenrod operations explains our characteristic assumption.) This is done in Section 2.

The needed operation is related with the difference of two other operations: sq, given by the squaring, and st, given by a Steenrod operation. The proof succeeds if the value of one operation turns out to be trivial and the value of the other one – non-trivial (see Lemma 2.10). The value $\operatorname{sq}(\pi)$ is computed due to its relation with the *rank* of the motive (see Lemmas 2.5 and 2.2); the needed ranks are calculated in Section 5. To compute the value $\operatorname{st}(\pi)$ we use the information on the Steenrod operations on quasi-split unitary grassmannians obtained in Section 4.

The second difference between the orthogonal and the unitary cases is as follows: $\operatorname{st}(\pi)$ is the trivial value in the orthogonal case while $\operatorname{sq}(\pi)$ is the trivial value in the unitary case. In particular, we have to check the non-triviality of the more sophisticated $\operatorname{st}(\pi)$ here, which is certainly more difficult than to show its triviality (in the orthogonal case).

To finish the introduction, let us mention *Hyperbolicity Theorem* – a corollary of Isotropy Theorem: if τ becomes *hyperbolic* over any field extension of F splitting A, then τ is hyperbolic (over F). This corollary has been already proved (in a different way) in [9] (the exponent 2 case) and in [3] (the unitary case). We also note that the orthogonal and symplectic cases of Isotropy Theorem are formal consequences of its unitary case – Theorem 6.1 (cf. [3, §5] and [10]).

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2. Operations sq and st

Let F be a field of characteristic $\neq 2$. Let X be a connected smooth projective variety over F.

We write CH for the integral Chow group and we write Ch for $\mathrm{CH}/2\,\mathrm{CH}$ (the Chow group with coefficients in $\mathbb{F}_2$).

We are going to use the following statement due to A. Merkurjev:

Lemma 2.1. *Let $\delta : X \to X \times X$ be the diagonal morphism. For any $\mathfrak{a}, \mathfrak{b} \in \mathrm{CH}(X \times X)$ one has $\deg(\mathfrak{b}^t \cdot \mathfrak{a}) = \deg(\delta^*(\mathfrak{b} \circ \mathfrak{a}))$, where $\cdot$ stands for the product in the Chow ring, $\circ$ stands for the composition of correspondences, and ^{t} stands for the transposition of correspondences.*

Proof. The commutative diagram

$$\begin{array}{ccccc} X \times X & \xrightarrow{e} & X \times X \times X & \xrightarrow{s} & X \times X \times X \times X \\ {\scriptstyle pr_1}\downarrow & & {\scriptstyle pr_{13}}\downarrow & & \\ X & \xrightarrow{\delta} & X \times X & & \end{array}$$

where $e(x, y) = (x, y, x)$ and $s(x, y, z) = (x, y, y, z)$, yields the commutative diagram

$$\begin{array}{ccccc} \mathrm{CH}(X \times X) & \xleftarrow{e^*} & \mathrm{CH}(X \times X \times X) & \xleftarrow{s^*} & \mathrm{CH}(X \times X \times X \times X) \\ {\scriptstyle pr_{1*}}\downarrow & & {\scriptstyle pr_{13*}}\downarrow & & \\ \mathrm{CH}(X) & \xleftarrow{\delta^*} & \mathrm{CH}(X \times X). & & \end{array}$$

We have $pr_{1*}(\mathfrak{b}^t \cdot \mathfrak{a}) = pr_{1*}e^*s^*(\mathfrak{a} \times \mathfrak{b}) = \delta^* pr_{13*}s^*(\mathfrak{a} \times \mathfrak{b}) = \delta^*(\beta \circ \alpha)$, hence $\deg(\mathfrak{b}^t \cdot \mathfrak{a}) = \deg pr_{1*}(\mathfrak{b}^t \cdot \mathfrak{a}) = \deg \delta^*(\mathfrak{b} \circ \mathfrak{a})$. □

Our first basic operation is a map $\mathrm{sq} : \mathrm{Ch}(X \times X) \to \mathbb{Z}/4\mathbb{Z}$ defined as follows. For any $\alpha \in \mathrm{Ch}(X \times X)$ we take its integral representative $\mathfrak{a} \in \mathrm{CH}(X \times X)$ and set

$$\mathrm{sq}(\alpha) := \deg(\mathfrak{a}^2) \mod 4,$$

where by $\mathfrak{a}^2$ we mean the product of cycles $\mathfrak{a} \cdot \mathfrak{a}$ and not the composition of correspondences $\mathfrak{a} \circ \mathfrak{a}$. Since any other integral representative of the same α is of the form $\mathfrak{a} + 2\mathfrak{b}$ with some $\mathfrak{b} \in \mathrm{CH}(X \times X)$ and $\deg((\mathfrak{a} + 2\mathfrak{b})^2) \equiv \deg(\mathfrak{a}^2) \pmod 4$, the definition is correct.

We also define an auxiliary operation $\mathrm{sq}' : \mathrm{Ch}(X \times X) \to \mathbb{Z}/4\mathbb{Z}$ as follows. For any $\alpha \in \mathrm{Ch}(X \times X)$ we take its integral representative $\mathfrak{a} \in \mathrm{CH}(X \times X)$ and set $\mathrm{sq}'(\alpha) := \deg(\mathfrak{a}^t \cdot \mathfrak{a})$

mod 4. Since any other integral representative of the same α is of the form $\mathfrak{a}+2\mathfrak{b}$ with some $\mathfrak{b}\in \mathrm{CH}(X\times X)$ and $\deg((\mathfrak{a}+2\mathfrak{b})^t\cdot(\mathfrak{a}+2\mathfrak{b}))\equiv\deg(\mathfrak{a}^2) \pmod 4$, because

$$\deg(\mathfrak{b}^t\cdot\mathfrak{a})=\deg\big((\mathfrak{b}^t\cdot\mathfrak{a})^t\big)=\deg(\mathfrak{a}^t\cdot\mathfrak{b}),$$

the definition is correct.

Lemma 2.2. *One has* $\mathrm{sq}'(\alpha)=\mathrm{sq}(\alpha)$ *for any symmetric projector* $\alpha\in\mathrm{Ch}(X\times X)$.

Proof. Indeed, such α has a symmetric integral representative: if $\mathfrak{a}$ is any integral representative, then $\mathfrak{a}^t\circ\mathfrak{a}$ is a symmetric integral representative of α. Computing $\mathrm{sq}(\alpha)$ and $\mathrm{sq}'(\alpha)$ with the help of a symmetric integral representative of α, we get the same. □

The operations sq and sq′ are natural at least in the following sense:

Lemma 2.3. *Each of the operations* sq *and* sq′ *commutes with the change of field homomorphism with respect to any field extension of* F. □

The operation sq′ also enjoys the additivity with respect to sums of orthogonal correspondences:

Lemma 2.4. *For any orthogonal correspondences* $\alpha,\beta\in\mathrm{Ch}(X\times X)$ *one has*

$$\mathrm{sq}'(\alpha+\beta)=\mathrm{sq}'(\alpha)+\mathrm{sq}'(\beta).$$

Proof. Let $\mathfrak{a},\mathfrak{b}\in\mathrm{CH}(X\times X)$ be integral; representatives of α,β. It suffices to show that $\deg(\mathfrak{b}^t\cdot\mathfrak{a})\equiv 0 \pmod 2$. By Lemma 2.1, $\deg(\mathfrak{b}^t\cdot\mathfrak{a})=\deg(\delta^*(\mathfrak{b}\circ\mathfrak{a}))$. Since the correspondences β and α are orthogonal, $\mathfrak{b}\circ\mathfrak{a}\in 2\,\mathrm{CH}(X\times X)$. □

We are working with the Chow motives over F with coefficients in $\mathbb{F}_2$, [1, Chapter XII]. A motive is *split*, if it is isomorphic to a (finite) direct sum of Tate motives. A motive is *geometrically split*, if it splits over a field extension of F. The *rank* $\mathrm{rk}\,M$ of a geometrically split motive M is the number of Tate summands in the decomposition of M_E for a field extension E/F such that M_E is split (this number does not depend on the choice of E). If α is a projector on a smooth projective variety X such that the motive (X,α) is geometrically split, we set $\mathrm{rk}\,\alpha:=\mathrm{rk}(X,\alpha)$.

Lemma 2.5. *Let* $\alpha\in\mathrm{Ch}(X\times X)$ *be a projector and assume that the motive* (X,α) *is geometrically split (so that the rank* $\mathrm{rk}(\alpha)\in\mathbb{Z}$ *of* α *is defined). Then* $\mathrm{sq}'(\alpha)=\mathrm{rk}(\alpha)$ mod 4.

Proof. By the naturality of sq′ (Lemma 2.3), we may assume that the motive (X,α) is split, that is, that (X,α) is isomorphic to a finite sum of Tate motives. The number of the summands is the rank. By additivity of sq′ (Lemma 2.4), we may assume that the rank is 1. In this case α has an integral representative of the form $a\times b$ with some homogeneous $a,b\in\mathrm{CH}(X)$ having odd $\deg(a\cdot b)$. It follows that $\mathrm{sq}'(\alpha)=1 \mod 4$. □

Now we are going to define our second basic operation. We write $S^\bullet$ for the modulo 2 total cohomological Steenrod operation, [1, Chapter XI]. Let $pr: X\times X\to X$ be the projection onto the first factor.

Lemma 2.6. *For any $\alpha, \beta \in \mathrm{Ch}(X \times X)$ one has*

$$\deg S^\bullet(\beta^t \circ \alpha) = \deg\big(pr_*(S(\alpha)) \cdot pr_*(S(\beta)) \cdot c_\bullet(-T_X)\big),$$

where T_X is the tangent bundle of X, and $c_\bullet$ is the total (modulo 2) Chern class.

Note that deg in Lemma 2.6 refers to the degree homomorphism on the modulo 2 Chow group taking its values in $\mathbb{Z}/2\mathbb{Z}$.

Here is the definition of the operation $\mathrm{st} : \mathrm{Ch}(X \times X) \to \mathbb{Z}/4\mathbb{Z}$. For any $\alpha \in \mathrm{Ch}(X \times X)$ we choose an integral representative $\mathfrak{a} \in \mathrm{CH}(X \times X)$ of $S^\bullet(\alpha) \in \mathrm{Ch}(X \times X)$ and set

$$\mathrm{st}(\alpha) := \deg\big(pr_*(\mathfrak{a})^2 \cdot c_\bullet(-T_X)\big) \pmod 4,$$

where $c_\bullet$ refers now to the *integral* total Chern class. Clearly, the definition is correct because the choice of $\mathfrak{a}$ does not affect the value of the operation.

The operation st is a sort of refinement modulo 4 of a modulo 2 operation related with the Steenrod operation:

Lemma 2.7. *We have* $\mathrm{st}(\alpha) \mod 2 = \deg S^\bullet(\alpha^t \circ \alpha)$. *In particular,*

$$\mathrm{st}(\alpha) \mod 2 = \deg S^\bullet(\alpha)$$

if the correspondence α is a symmetric projector.

Proof. This is the particular case $\beta = \alpha$ of Lemma 2.6. □

The operation st is natural:

Lemma 2.8. *The operation* st *commutes with the change of field homomorphism with respect to any field extension of F.* □

The operation st also enjoys a sort of additivity:

Lemma 2.9. *If* $\deg S(\beta^t \circ \alpha) = 0$, *then*

$$\mathrm{st}(\alpha + \beta) = \mathrm{st}(\alpha) + \mathrm{st}(\beta).$$

In particular, the additivity holds for orthogonal symmetric correspondences α, β.

Proof. Let $\mathfrak{a}, \mathfrak{b} \in \mathrm{CH}(X \times X)$ be integral; representatives of $S^\bullet(\alpha), S^\bullet(\beta)$. Then $\mathfrak{a} + \mathfrak{b}$ is an integral representative of $S^\bullet(\alpha + \beta)$ and it suffices to show that

$$\deg\big(pr_*(\mathfrak{a}) \cdot pr_*(\mathfrak{b}) \cdot c_\bullet(-T_X)\big) \equiv 0 \pmod 2.$$

This is indeed so by Lemma 2.6 and the condition on α, β. □

The two operations sq and st are related as follows:

Lemma 2.10. *Let $d = \dim X$. For any symmetric projector $\alpha \in \mathrm{Ch}^d(X \times X)$ one has* $\mathrm{sq}(\alpha) \equiv \mathrm{st}(\alpha) \pmod 2$. *If moreover X has no closed points of odd degree, then* $\mathrm{sq}(\alpha) = \mathrm{st}(\alpha)$.

Proof. The value $\mathrm{sq}(\alpha)$ is given by the degree of certain integral representative $\mathfrak{a}$ of α^2. By Lemma 2.7, the value $\mathrm{st}(\alpha)$ is given by the degree of certain integral representative $\mathfrak{b}$ of $S^d\alpha$. Since $S^d\alpha = \alpha^2$, it follows that $\mathfrak{a} - \mathfrak{b} \in 2\,\mathrm{CH}(X \times X)$. Since $\deg \mathrm{CH}(X \times X) = \deg \mathrm{CH}(X)$, we get that $\deg \mathfrak{a} - \deg \mathfrak{b} \in 2 \deg \mathrm{CH}(X)$. In particular, $\deg \mathfrak{a} - \deg \mathfrak{b} \in 4\mathbb{Z} + 2 \deg \mathrm{CH}(X)$ as claimed. □

Lemma 2.10 shows that the difference $\mathrm{st} - \mathrm{sq}$ is a sort of weak replacement for a certain symmetric operation. This replacement suffices for our further purposes. One advantage of this replacement is that it is available over any F with $\operatorname{char} F \neq 2$, not only over F with $\operatorname{char} F = 0$ like the symmetric operation itself.

3. Chow ring of quasi-split unitary grassmannians

Let V be a finite-dimensional vector space over a field K (of arbitrary characteristic). We set $n := \dim V$. (Although in relation with our main purpose we are only interested in the case of even n, we treat the case of odd n because it differs from the even one only in a few places.) For any subset $I \subset \{1, 2 \dots, [n/2]\}$ (where $[n/2] = n/2$ for even n and $[n/2] = (n-1)/2$ for odd n), we write $G_I(V)$ for the variety of flags of subspaces in V of dimensions given by I. In particular, for any integer $k \in [1,\ [n/2]]$, the variety $G_{\{k\}}(V)$ (which we simply denote as $G_k(V)$) is the grassmannian of k-planes.

Let us consider the closed subvariety $H_I = H_I(V)$ of the product $G_I(V) \times G_I(V^\#)$, where $V^\#$ is the dual vector space of V, defined by the orthogonality condition: H_I is the variety of pairs of flags such that each space of the first flag is orthogonal with respect to the corresponding space of the second flag (or, equivalently, the biggest space of the first flag is orthogonal with the biggest space of the second flag).

Example 3.1. The variety H_k is the variety of pairs of k-planes $U \subset V$, $U' \subset V^\#$ such that $U \cdot U' = 0$. It is canonically isomorphic to the variety of flags in V consisting of a k-plane contained in a $(n-k)$-plane.

Any isomorphism of the vector spaces V and $V^\#$ provides a (non-canonical) isomorphism of the varieties $G_I(V)$ and $G_I(V^\#)$. Since the automorphism group of V acts trivially on $\mathrm{CH}(G_I(V))$, the ring $\mathrm{CH}(G_I(V))$ is canonically identified with $\mathrm{CH}(G_I(V^\#))$. In particular, we obtain a canonical involution (i.e., a self-inverse automorphism) of the ring

$$\mathrm{CH}(G_I(V) \times G_I(V^\#)) = \mathrm{CH}(G_I(V)) \otimes \mathrm{CH}(G_I(V^\#)) = \mathrm{CH}(G_I(V)) \otimes \mathrm{CH}(G_I(V))$$

given by the exchange of the factors in the last tensor product.

Lemma 3.2. *There exists one and only one involution of* $\mathrm{CH}(H_I)$ *such that the following square of the involutions and a pull-back commutes:*

$$\begin{array}{ccc} \mathrm{CH}(G_I(V) \times G_I(V^\#)) & \longrightarrow & \mathrm{CH}(G_I(V) \times G_I(V^\#)) \\ \downarrow & & \downarrow \\ \mathrm{CH}(H_I) & \longrightarrow & \mathrm{CH}(H_I). \end{array}$$

Proof. The uniqueness is a consequence of the surjectivity of the pull-back which we have because the ring $\mathrm{CH}(H_I)$ is generated by the Chern classes of the tautological bundles on H_I which are pull-backs of the tautological bundles on $G_I(V) \times G_I(V^\#)$.

To show the existence, we fix a non-degenerate symmetric bilinear form on V (giving a self-dual isomorphism $V \simeq V^\#$). This provides the variety $G_I(V) \times G_I(V^\#)$ with the switch involution (inducing our involution on the Chow group), and the subvariety H_I is stable under it. □

The canonical involution on $\mathrm{CH}(H_I)$ just constructed will be denoted by σ.

Example 3.3. Assume that the field K is separable quadratic over some subfield $F \subset K$ and let h be a K/F-hermitian form on V. Let Y_I be the flag variety of totally isotropic subspaces in V. The K-variety $(Y_I)_K$ is canonically isomorphic to H_I. The non-trivial automorphism of K/F induces an automorphism of $\mathrm{CH}(Y_I)_K$ identified with σ. The image of the change of field homomorphism $\mathrm{CH}(Y_I) \to \mathrm{CH}(H_I)$ is contained in the subring $\mathrm{CH}(H_I)^\sigma \subset \mathrm{CH}(H_I)$ of the σ-invariant elements. Moreover, if h is hyperbolic, the change of field homomorphism $\mathrm{CH}(Y_I) \to \mathrm{CH}(H_I)$ is injective and its image coincides with $\mathrm{CH}(H_I)^\sigma$ so that we have a canonical identification $\mathrm{CH}(Y_I) = \mathrm{CH}(H_I)^\sigma$; the ideal $(1+\sigma)\,\mathrm{CH}(H_I) \subset \mathrm{CH}(H_I)^\sigma$ coincides with the image of the norm homomorphism $\mathrm{CH}(Y_I)_K \to \mathrm{CH}(Y_I)$.

We are going to study the subring $\mathrm{CH}(H_I)^\sigma \subset \mathrm{CH}(H_I)$ of the σ-invariant elements. More precisely, we will study the quotient of this subring by its "elementary part" – the norm ideal $(1+\sigma)\,\mathrm{CH}(H_I)$. We are basically interested in the case of $\#I = 1$.

We start with the case $I = \{1\}$. Let h be the hyperplane class in $\mathrm{CH}^1(G_1(V))$ or in $\mathrm{CH}^1(G_1(V^\#))$ and let us define the elements $a, b \in \mathrm{CH}^1(H_1)$ as the pull-backs of $h \times 1$ and $1 \times h$. For any $i \geq 0$, one has $a^i = c_i(-\mathcal{A})$ and $b^i = c_i(-\mathcal{B})$, where $\mathcal{A}$ and $\mathcal{B}$ are the corresponding tautological vector bundles on H_1.

The ring $\mathrm{CH}(H_1)$ is generated by the two elements a, b subject to the relations $a^n = 0$ and $a^{n-1} + a^{n-2}(-b) + \cdots + (-b)^{n-1} = 0$ (implying $b^n = 0$). The involution σ exchanges the generators a and b.

Let $\mathcal{T}$ be the vector bundle on H_1 whose fiber over a point (U, U') is $U \oplus U'$ (i.e., $\mathcal{T} = \mathcal{A} \oplus \mathcal{B}$). (Note that the isomorphism $(Y_1)_K = H_1$ of Example 3.3 transforms $\mathcal{T}$ to the tautological vector bundle on $(Y_1)_K$ (defined over F).) We write c_i for $c_i(-\mathcal{T})$. We have $c_i = a^i + a^{i-1}b + \cdots + b^i \in \mathrm{CH}(H_1)^\sigma$.

The elements $c_{n-1}, c_n, \ldots$ are divisible by 2. Indeed,

$$c_{n-1}/2 = a^{n-1} + a^{n-3}b^2 + \cdots + ab^{n-2} = a^{n-2}b + \cdots + b^{n-1}$$

and $c_{n-1+i}/2 = (c_{n-1}/2) \cdot a^i = (c_{n-1}/2) \cdot b^i$ for any $i \geq 0$.

Lemma 3.4. *The ring* $\mathrm{CH}(H_1)^\sigma/(1+\sigma)\,\mathrm{CH}(H_1)$ *is* additively *generated by the classes of the following elements:*

$$c_0, c_1, \ldots, c_{n-2} \quad \textit{and} \quad c_{n-1}/2, c_n/2, \ldots.$$

Moreover, for any odd $i \leq n-2$ the class of c_i is 0, for any even $i \geq n-1$ the class of $c_i/2$ is 0, and for any $i > 2n-3$ the class of $c_i/2$ is 0.

Proof. The group $\mathrm{CH}^{<n-1}(H_1)$ is freely generated by $a^i b^j$ with $i + j < n-1$. Therefore the quotient $\mathrm{CH}(H_1)^\sigma/(1+\sigma)\,\mathrm{CH}(H_1)$ in codimensions $< n-1$ is (additively) generated by the classes of $a^i b^i$ ($2i < n-1$) which are also represented by c_{2i}.

For any $i = n-1, n, \ldots, 2n-3$, the group $\mathrm{CH}^i(H_1)$ is generated by the elements

$$a^{n-1}b^{i-(n-1)}, a^{n-2}b^{i-(n-2)}, \ldots, a^{i-(n-1)}b^{n-1}$$

whose alternating sum is 0, and this is the only relation on the generators. The quotient of the subgroup of σ-invariant elements by the norms is therefore trivial for even i and generated by the class of $c_i/2$ for odd i.

Finally, for $i > 2n-3 = \dim H_1$, the group $\mathrm{CH}^i(H_1)$ is trivial. □

Remark 3.5. Here is a complete analysis of the graded ring

$$R := \mathrm{CH}(H_1)^\sigma/(1+\sigma)\,\mathrm{CH}(H_1),$$

which is now easily done. Similarities as well as differences with the Chow ring of a split projective quadric are striking.

In the case of even n, the ring R is generated (as a ring) by two elements: (the classes of) $ab \in R^2$ and $c := c_{n-1}/2 \in R^{n-1}$ (R^2 and R^{n-1} are the graded components of R). The relations are: $(ab)^{n/2} = 0$ and $c^2 = 0$. The non-zero homogeneous elements of R are as follows:

$$(ab)^i = c_{2i},\ \ c(ab)^i = c_{n-1+2i}/2,\ \text{ with } i = 0, 1, \dots, (n-2)/2.$$

If n is odd, the ring R is generated by two elements: (the classes of) $ab \in R^2$ and $c := c_n/2 \in R^n$. The relations are: $(ab)^{(n-1)/2} = 0$ and $c^2 = 0$. The non-zero homogeneous elements of R are as follows:

$$(ab)^i = c_{2i},\ \ c(ab)^i = c_{n+2i}/2,\ \text{ with } i = 0, 1, \dots, (n-3)/2.$$

The geometric description of the generators (for arbitrary parity of n) is as follows. The element $(ab)^i$ is the pullback of $h^i \times h^i \in \mathrm{CH}^{2i}\big(G_1(V) \times G_1(V^\#)\big)$. To describe $c(ab)^i$, we take some orthogonal subspaces $U \subset V$ and $U' \subset V^\#$ of dimension $[n/2] - i$. Then $c(ab)^i$ is the class of the (closed) subvariety $L_i \subset H_1$ of pairs of lines: one line in U, the other in U'.

Now we start to study the case of $I = \{k\}$ where k satisfies $1 \le k \le [n/2]$. We write $\mathcal{T}_k$ for the vector bundle on H_k whose fiber over a point (U, U') is $U \oplus U'$ (in particular, $\mathcal{T}_1 = \mathcal{T}$). Note that the isomorphism $(Y_k)_K = H_k$ of Example 3.3 transforms $\mathcal{T}_k$ to the tautological vector bundle on $(Y_k)_K$ (defined over F).

We consider the natural projections $\pi_1 : H_{\{1,k\}} \to H_1$ and $\pi_k : H_{\{1,k\}} \to H_k$.

Lemma 3.6 (cf. [16, Proposition 2.1])**.** *For any integer i one has*

$$c_i(-\mathcal{T}_k) = (\pi_k)_*\pi_1^* c_{i+2(k-1)}(-\mathcal{T}_1).$$

Proof. For any smooth scheme X with a rank k vector bundle $\mathcal{E}$ one has

$$c_i(-\mathcal{E}) = \pi_* c_{i+k-1}(-\mathcal{O}(-1)),$$

where π is the morphism $\mathbb{P}(\mathcal{E}) \to X$ and $\mathcal{O}(-1)$ is the tautological (line) bundle on $\mathbb{P}(\mathcal{E})$. If now $\mathcal{E}_1, \mathcal{E}_2$ are two rank k vector bundles on X and π is the morphism $\mathbb{P}(\mathcal{E}_1)\times_X \mathbb{P}(\mathcal{E}_2) \to X$, we get that

$$c_i(-(\mathcal{E}_1 \oplus \mathcal{E}_2)) = \pi_* c_{i+2(k-1)}\big(-(\mathcal{O}_1(-1) \oplus \mathcal{O}_2(-1))\big).$$

In particular, taking $\pi = \pi_k$, we see that

$$c_i(-\mathcal{T}_k) = (\pi_k)_* c_{i+2(k-1)}\big(-(\mathcal{O}_1(-1) \oplus \mathcal{O}_2(-1))\big).$$

Since $\mathcal{O}_1(-1) \simeq \pi_1^*\mathcal{A}$ and $\mathcal{O}_2(-1) \simeq \pi_1^*\mathcal{B}$, we are done. □

Corollary 3.7. *The (σ-invariant) elements*

$$c_{n-2k+1}(-\mathcal{T}_k), c_{n-2k+2}(-\mathcal{T}_k), \dots, c_{2n-2k-1}(-\mathcal{T}_k) \in \mathrm{CH}(H_k)^\sigma$$

are divisible by 2. □

We consider the projections $\pi_k : H_{\{k,k+1\}} \to H_k$ and $\pi_{k+1} : H_{\{k,k+1\}} \to H_{k+1}$. The vector bundle $\pi_k^*\mathcal{T}_k$ is a subbundle of the vector bundle $\pi_{k+1}^*\mathcal{T}_{k+1}$ (the quotient is a direct sum of two line bundles), and we write $\alpha \in \operatorname{CH}^2(H_{\{k,k+1\}})$ for $c_2(\pi_{k+1}^*\mathcal{T}_{k+1}/\pi_k^*\mathcal{T}_k)$.

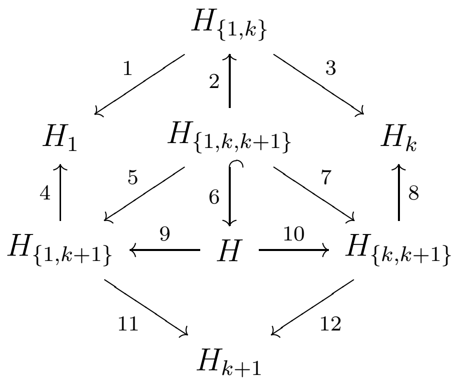
Lemma 3.8 (cf. [16, Lemma 2.5]). *For $i \in \{0, 1, \dots, n-2k\}$ one has*

$$\pi_k^* c_i(-\mathcal{T}_k) \equiv \pi_{k+1}^* c_i(-\mathcal{T}_{k+1}) + \alpha \cdot c_{i-2}(-\mathcal{T}_{k+1}) \pmod{1+\sigma}.$$

For $i \geq n-2k+1$ one has

$$\pi_k^* c_i(-\mathcal{T}_k)/2 \equiv \pi_{k+1}^* c_i(-\mathcal{T}_{k+1})/2 + \alpha \cdot c_{i-2}(-\mathcal{T}_{k+1})/2 \pmod{1+\sigma}.$$

Proof. We play with the following commutative diagram



where H is defined as the fiber product of $H_{\{1,k+1\}}$ and $H_{\{k,k+1\}}$ over H_{k+1}. The variety $H_{\{1,k,k+1\}}$ is naturally a closed subvariety (of codimension 2) in H and 6 is the closed imbedding. Note that $\pi_k = 8$ and $\pi_{k+1} = 12$. By Lemma 3.6, the elements $c_i(-\mathcal{T}_k)$ and $c_i(-\mathcal{T}_k)/2$ are $3_*1^*(x)$ for certain $x \in \operatorname{CH}(H_1)^\sigma$. Let us compute $y := 8^*3_*1^*(x)$ for an arbitrary $x \in \operatorname{CH}(H_1)^\sigma$.

The square 3-8-7-2 is transversal cartesian. Therefore $8^*3_* = 7_*2^*$. By commutativity of the square 1-2-5-4, $2^*1^* = 5^*4^*$ so that $y = 7_*5^*4^*(x)$. By commutativity of the triangles 5-6-9 and 6-7-10, $y = 10_*6_*6^*9^*4^*(x) = 10_*[H_{\{1,k,k+1\}}] \cdot 9^*4^*(x)$. The class $[H_{\{1,k,k+1\}}] \in \operatorname{CH}^2(H)$ is computed modulo $1+\sigma$ as $9^*4^*(ab) + 10^*(\alpha)$. It follows that $y \equiv 10_*9^*4^*(abx) + \alpha \cdot 10_*9^*4^*(x)$. Since the square 9-10-12-11 is transversal cartesian, $10_*9^* = 12^*11_*$, so that we finally get $y \equiv 12^*11_*4^*(abx) + \alpha \cdot 12^*11_*4^*(x) \pmod{1+\sigma}$.

We get the first (resp. second) desired congruence taking $x = c_{i+2(k-1)}(-\mathcal{T}_1)$ (resp. $x = c_{i+2(k-1)}(-\mathcal{T}_1)/2$) by Lemma 3.6, because $abc_{i+2(k-1)}(-\mathcal{T}_1) \equiv c_{i+2(k+1)}(-\mathcal{T}_1) \pmod{1+\sigma}$ (resp. $abc_{i+2(k-1)}(-\mathcal{T}_1)/2 \equiv c_{i+2(k+1)}(-\mathcal{T}_1)/2 \pmod{1+\sigma}$) for the corresponding values of i (cf. Remark 3.5). □

Proposition 3.9 (cf. [16, Proposition 2.9]). *The ring $\operatorname{CH}(H_k)^\sigma$ is generated (as a ring) modulo the ideal $(1+\sigma)\operatorname{CH}(H_k)$ by the elements $c_i(-\mathcal{T}_k)$ with even i satisfying $0 \leq i \leq n-2k$ and the elements $c_i(-\mathcal{T}_k)/2$ with odd i satisfying $n-2k+1 \leq i \leq 2n-2k-1$.*

Proof. For each integer l with $1 \leq l \leq k$, we consider the projection $\pi_l : H_{\{1,\dots,k\}} \to H_l$ and the elements

(∗) $\pi_l^* c_i(-\mathcal{T}_l)$ with even i satisfying $0 \leq i \leq n-2l$ and

$\pi_l^* c_i(-\mathcal{T}_l)/2$ with odd i satisfying $n-2l+1 \leq i \leq 2n-2l-1$.

Lemma 3.10 (cf. [16, Lemma 2.10])**.** *The ring* $\mathrm{CH}(H_{\{1,\dots,k\}})^\sigma$ *is generated modulo the ideal* $(1+\sigma)\,\mathrm{CH}(H_k)$ *by the elements* (∗) *(with* l *running over* $1,\dots,k$*).*

Proof. Since the statement does not depend on the base field K, we may assume that K is quadratic separable over some subfield F. Then we fix a hyperbolic K/F-hermitian form on V and replace $\mathrm{CH}(H_k)^\sigma$ by $\mathrm{CH}(Y_k)$ (see Example 3.3).

We do induction on k. The case $k=1$ is Lemma 3.4. To pass from $k-1$ to k, we apply a variant of [16, Statement 2.11] taking as $Y \to X$ the projection $Y_{\{1,2,\dots,k\}} \to Y_{\{1,2,\dots,k-1\}}$ and taking as B the subgroup generated by the norms and the elements $(*)$ with $l=k$. Each fiber of this projection is a hermitian quadric given by a hyperbolic hermitian space of dimension $n-2(k-1)$. Let us check that condition (a) of [16, Statement 2.11] is verified. The restriction of $\pi_k^*\mathcal{T}_k$ to the generic fiber of the projection is isomorphic to the direct sum of $\mathcal{T}_1$ and a trivial vector bundle (of rank $2(k-1)$). Therefore the pull-backs of the elements $(*)$ to the generic fiber give the elements

$$c_i(-\mathcal{T}_k) \text{ with even } i \text{ satisfying } 0 \le i \le n-2k \text{ and}$$
$$c_i(-\mathcal{T}_k)/2 \text{ with odd } i \text{ satisfying } n-2k+1 \le i \le 2n-2k-1.$$

which generate the group $\mathrm{CH}(Y)$ modulo the norms by Lemma 3.4 (note that $2(n-2(k-1))-3 \le 2n-2k-1$).

However condition (b) of [16, Statement 2.11] is not satisfied: the specialization homomorphism from the Chow group of the generic fiber to the Chow group of the fiber over a point x is not surjective in general. It is surjective if the residue field of x does not contain a subfield isomorphic to K. We finish the proof by showing that in the opposite case the image of $\mathrm{CH}(Y_x)$ in the associated graded group of the filtration on $\mathrm{CH}(Y)$ is in the image of $1+\sigma$. We are speaking about the filtration on $\mathrm{CH}(Y)$ whose term $\mathcal{F}^r\,\mathrm{CH}(Y)$ is the subgroup generated by the classes of cycles on Y whose image in X has codimension $\ge r$.

Let T be the closure of x in X. Let $Y_T = Y \times_X T \hookrightarrow Y$ be the preimage of T under $Y \to X$. The image of the homomorphism $\mathrm{CH}(Y_x) \to \mathcal{F}^r\,\mathrm{CH}(Y)/\mathcal{F}^{r+1}\,\mathrm{CH}(Y)$, where $r = \mathrm{codim}_X\, x$, is in the image of the push-forward $\mathrm{CH}(Y_T) \to \mathcal{F}^r\,\mathrm{CH}(Y)/\mathcal{F}^{r+1}\,\mathrm{CH}(Y)$. Since x is the generic point of T and $F(x) = F(T) \supset K$, a non-empty open subset $U \subset T$ possesses a morphism to $\mathrm{Spec}\, K$. Its preimage $Y_U \subset Y_T$ is open and also possesses a morphism to $\mathrm{Spec}\, K$. Therefore $(Y_U)_K \simeq Y_U \coprod Y_U$ and, in particular, the push-forward $\mathrm{CH}((Y_U)_K) \to \mathrm{CH}(Y_U)$ is surjective.

We play with the following commutative diagram:

$$\begin{array}{ccccc}
Y_K & \longrightarrow & Y & \longrightarrow & X \\
\uparrow & & \uparrow & & \uparrow \\
(Y_T)_K & \longrightarrow & Y_T & \longrightarrow & T \\
\uparrow & & \uparrow & & \uparrow \\
(Y_U)_K & \longrightarrow & Y_U & \longrightarrow & U
\end{array}$$

It follows that the image of the push-forward $\mathrm{CH}((Y_T)_K) \to \mathrm{CH}(Y_T)$ generates $\mathrm{CH}(Y_T)$ modulo the image of $\mathrm{CH}(Y_T \setminus Y_U)$. Since the image of $\mathrm{CH}(Y_T \setminus Y_U) \to \mathrm{CH}(Y)$ is in $\mathcal{F}^{r+1}\,\mathrm{CH}(Y)$, it follows that the image of $\mathrm{CH}(Y_T)$ in the quotient of the filtration on $\mathrm{CH}(Y)$ is contained in the image of $\mathcal{F}^r\,\mathrm{CH}(Y_K)$, that is, in the image of $1+\sigma$. □

Let $I = [1,\ k] = \{1, 2, \dots, k\}$. For every $i \in I$, let $\mathcal{A}_i$ and $\mathcal{B}_i$ the the tautological vector bundles on H_i (so that $\mathcal{T}_i = \mathcal{A}_i \oplus \mathcal{B}_i$). We define $a_i, b_i \in \mathrm{CH}^1(H_I)$ as the first Chern classes of the linear bundles $(H_I \to H_i)^*\mathcal{A}_i/(H_I \to H_{i-1})^*\mathcal{A}_{i-1}$ and $(H_I \to H_i)^*\mathcal{B}_i/(H_I \to H_{i-1})^*\mathcal{B}_{i-1}$.

For any $l \in I$, we identify $\mathrm{CH}(H_{[l\ ,k]})$ with a subring in $\mathrm{CH}(H_I)$ via the pull-back. Note that $a_i, b_i \in \mathrm{CH}(H_{[l\ ,k]})$ for $i \in [l+1\ , k]$.

By induction on $l \in I$, we prove the following statement: the ring $\mathrm{CH}(H_{[l,\ k]})^\sigma$ is generated modulo $(1+\sigma)$ by the elements of Proposition 3.9 and the elements $\{a_ib_i\}_{i\in[l+1,\ k]}$. Note that this statement for $l = k$ is the statement of Proposition 3.9.

The induction base $l = 1$ follows from Lemma 3.10 and Lemma 3.8 (the latter showing that the missing generators of Lemma 3.10 are expressible in terms of the kept generators and the added generators). Let us do the passage from $l-1$ to l.

The projection $H_{[l-1,\ k]} \to H_{[l,\ k]}$ is (canonically isomorphic to) a product of two rank $l-1$ projective bundles (given by the dual of the rank l tautological vector bundles $\mathcal{A}_l$ and $\mathcal{B}_l$ on $H_{[l,\ k]}$). The $\mathrm{CH}(H_{[l,\ k]})$-algebra $\mathrm{CH}(H_{[l-1,\ k]})$ is therefore generated by the two elements a_l, b_l subject to the two relations

$$\sum_{i=0}^{l} c_i(\mathcal{A}_l)a_l^{l-i} = 0,\ \ \sum_{i=0}^{l} c_i(\mathcal{B}_l)b_l^{l-i} = 0.$$

In particular, the $\mathrm{CH}(H_{[l,\ k]})$-module $\mathrm{CH}(H_{[l-1,\ k]})$ is free, a basis is given by the products $a_l^ib_l^j$ with $i, j \in [0,\ l-1]$.

The involution σ exchanges a and b. Therefore the module $\mathrm{CH}(H_{[l-1,\ k]})^\sigma/(1+\sigma)$ over the ring $\mathrm{CH}(H_{[l,\ k]})^\sigma/(1+\sigma)$ is free of rank l, a basis is given by the (classes of the) products $a_l^ib_l^i$ with $i \in [0,\ l-1]$. In particular, the $\mathrm{CH}(H_{[l,\ k]})^\sigma/(1+\sigma)$-algebra $\mathrm{CH}(H_{[l-1,\ k]})^\sigma/(1+\sigma)$ is generated by a_lb_l. This generator satisfy the relation

$$\sum_{i=0}^{l} c_{2i}(\mathcal{T}_l)(a_lb_l)^{l-2i} = 0$$

(this is an equality in the quotient $\mathrm{CH}(H_{[l-1,\ k]})^\sigma/(1+\sigma)$!). This is the only relation on the generator because its powers up to $l-1$ form a basis.

Now let $C \subset \mathrm{CH}(H_{[l,\ k]})^\sigma/(1+\sigma)$ be the subring generated by the elements of Proposition 3.9 and the elements $\{a_ib_i\}_{i\in[l+1,\ k]}$. Note that the coefficients of the above relation are in C: they are expressible in terms of $c_i(-\mathcal{T}_l)$ (which are non-zero modulo $1+\sigma$ only for $i = 0, 2, \dots, n-2l$ by Lemmas 3.6 and 3.4. Therefore the subring of $\mathrm{CH}(H_{[l-1,\ k]})^\sigma/(1+\sigma)$ generated by C and a_lb_l is also a free C-module of rank l. On the other hand, this subring coincides with the total ring by the induction hypothesis and it follows that $C = \mathrm{CH}(H_{[l,\ k]})^\sigma/(1+\sigma)$. □

Remark 3.11 (Geometric description of the generators). Proposition 3.9 provides us with generators of the ring $\mathrm{CH}(Y_k)$ modulo the $K/F-$norms via the identification $\mathrm{CH}(Y_k) = \mathrm{CH}(H_k)^\sigma$ of Example 3.3. These generators have precisely the same geometric description as the standard generators of the Chow ring of an orthogonal grassmannian. Namely, they are obtained as via the composition $(Y_{\{1,k\}} \to Y_k)_* \circ (Y_{\{1,k\}} \to Y_1)^*$ out of the additive generators of $\mathrm{CH}(Y_1)$ modulo the norms. Moreover, for any odd i satisfying $n-2k+1 \le i \le 2n-2k-1$, the generator $c_i(-\mathcal{T}_k)/2$ is the class of the Schubert subvariety of the subspaces intersecting non-trivially a fixed totally isotropic subspace in V of certain K-dimension. This is a consequence of Remark 3.5 and Lemma 3.6.

4. Steenrod operations for split unitary grassmannians

In this section, dimension n of the K-vector space V is supposed to be even.

Let $H = H_k$. One more tool for study of $\mathrm{CH}(H)$ is given by the morphism $in : H \to X$, where X is the variety of totally isotropic $2k$-planes of the hyperbolic quadratic form $\mathbb{H}(V) = V \oplus V^{\#}$. The morphism associates to a point (U, U') of H the point $U \oplus U'$ of X. This is a closed imbedding by [7, Corollary 10.4].

Note that the image of the pull-back $in^* : \mathrm{CH}(X) \to \mathrm{CH}(H)$ is contained in $\mathrm{CH}(H)^\sigma$. Indeed, fixing a non-degenerated symmetric bilinear form on V giving an identification of V with $V^{\#}$, we get the exchange involution on H (inducing σ on $\mathrm{CH}(H)$) and an involution on X given by the automorphism $V \oplus V^{\#} = V^{\#} \oplus V$. The imbedding $H \hookrightarrow X$ commute with these involutions, and the involution induced on $\mathrm{CH}(X)$ is the identity because V is of even dimension.

The power of this tool of studying $\mathrm{CH}(H)$ is explained by the fact that $\mathrm{CH}(X)$ (in contrast to $\mathrm{CH}(H)$) is very well studied. One more advantage of the variety X is that (in contrast to H) it has twisted forms with "high" degrees of the closed points.

The meaning of the imbedding $H \hookrightarrow X$ is as follows. Assume that V is endowed with a K/F-hermitian form h. We consider the variety Y_k. Let X_{2k} be the variety of $2k$-planes in the vector F-space V totally isotropic with respect to the quadratic form on V given by h. We have a natural closed imbedding $in : Y_k \hookrightarrow X_{2k}$ which becomes the above imbedding over K. Choosing a hyperbolic h, we get another proof of the fact that the image of $\mathrm{CH}(X) \to \mathrm{CH}(H)$ is in $\mathrm{CH}(H)^\sigma$: this is so because $\mathrm{CH}(X) = \mathrm{CH}(X_{2k})$ and $\mathrm{CH}(Y_k) = \mathrm{CH}(H)^\sigma$.

Recall (see [16]) that the ring $\mathrm{CH}(X)$ is generated by certain elements $w_i \in \mathrm{CH}^i(X)$, $i = 0, 1, \ldots, n-2k$ and $z_i \in \mathrm{CH}^i(X)$, $i = n-2k, n-2k+1, \ldots, 2n-2k-1$. They satisfy $w_i = c_i(-\mathcal{T}_X)$ for all i and $z_i = c_i(-\mathcal{T}_X)/2$ for $i \neq n-2k$, where $\mathcal{T}_X$ is the tautological vector bundle on X.

Lemma 4.1. *The pull-back* $\mathrm{CH}(X) \to \mathrm{CH}(H)^\sigma/(1+\sigma)$ *is surjective. The image of each* z_i *with even* $i \neq n-2k$ *is* 0.

Remark 4.2. One may show (see [2]) that the pull-back $in^* : \mathrm{CH}(X_{2k}) \to \mathrm{CH}(Y_k)/(1+\sigma)$ is surjective (for any h). Moreover, the push-forward in_* induces an injection $in_* : \mathrm{CH}(Y_k)/(1+\sigma) \to \mathrm{Ch}(X_{2k})$, where $\mathrm{Ch} := \mathrm{CH}/2$. It follows that the ring $\mathrm{CH}(Y_k)/(1+\sigma)$ is naturally identified with $\mathrm{Ch}(X_{2k})$ modulo the kernel of the multiplication by $[Y_k] \in \mathrm{Ch}(X_{2k})$. In the case of $k = n/2$ and hyperbolic h, the computation of the class $[Y_k]$ given below together with the computation of $\mathrm{Ch}(X_{2k})$ given [1], provides the following presentation of $\mathrm{CH}(Y_{n/2})/(1+\sigma)$ by generators and relations: generators are $e_i \in \mathrm{CH}^i$, $i = 1, 3, \ldots, n-1$; relations are $e_i^2 = 0$ for each i.

Proof of Lemma 4.1. The generators of the ring on the right-hand side given in Proposition 3.9 come from $\mathrm{CH}(X)$ because the pull-back of the tautological vector bundle $\mathcal{T}_X$ on X to H is $\mathcal{T}_k$ and the Chern classes $c_i(-\mathcal{T}_X)$ are divisible by 2 for $i > n-2k$. This gives the surjectivity.

The image of z_i with even $i \neq n-2k$ is 0 by Lemmas 3.6 and 3.4. $\square$

Lemma 4.3. *The element* $[H] \in \mathrm{CH}(X)$ *is a square.*

Proof. Let $x \in \mathrm{CH}(X)$ be the class of the Schubert subvariety $S \subset X$ of the subspaces $U \subset V \oplus V^{\#}$ satisfying $\dim U \cap V \geq k$. We claim that $[H] = x^2$. Indeed, x can be also represented by the Schubert subvariety $S' \subset X$ of the subspaces $U \subset V \oplus V^{\#}$ satisfying $\dim U \cap V^{\#} \geq k$. Since $S \cap S' = H$ and $\operatorname{codim}_X H = \operatorname{codim}_X S + \operatorname{codim}_X S'$, $[H] = [S] \cdot [S']$ by [1, Corollary 57.22]. $\square$

We now pass to the modulo 2 Chow group $\mathrm{Ch}(X) = \mathrm{CH}(X)/2$ and we use the notion of *level* for elements of $\mathrm{Ch}(X)$ introduced in [5]. Namely, an element of $\mathrm{Ch}(X)$ is of level l if it can be written as a polynomial in the generators of the z-degree $\leq l$ (we use the same notation w_i, z_i for the classes of the integral generators). We recall that (see [5, proof of Proposition 12]) that by the formula of [16, Proposition 2.8] the cohomological Steenrod operation preserves the level. In particular, the squaring preserves the level.

We also recall that the generators satisfy the relation

$$z_i^2 = z_i c_i(-\mathcal{T}_X) - z_{i+1} c_{i-1}(-\mathcal{T}_X) + z_{i+2} c_{i-2}(-\mathcal{T}_X) - \dots$$

which shows that any element of $\mathrm{Ch}(X)$ can be written as a polynomial in the generators of z_i-degree ≤ 1 for each i. A polynomial satisfying this restriction is called *standard* below.

Corollary 4.4. *The element* $[H] \in \mathrm{Ch}(X)$ *is of level* k.

Proof. Since squaring does not affect the level, it suffices to show that the level of a homogeneous element x with $x^2 = [H]$ is k. The codimension of x is equal to

$$(\dim X - \dim H)/2 = \big(k(4n-6k-1) - k(2n-3k)\big)/2 = (k/2)(2n-3k-1) =$$
$$(n-2k) + (n-2k+1) + \dots + (n-k-1),$$

and the minimal codimension of an element which is not of level k is this number plus $n-k$. $\square$

Theorem 4.5 (cf. [5, Proposition 12])**.** *Let F be a field of characteristic $\neq 2$, K/F a quadratic field extension, V a vector space over K of even positive dimension n, h a K/F-hermitian form on V, k an integer satisfying* $1 \leq k \leq n/2$, *Y the variety of totally isotropic k-planes in V. Then for any $i > k(n-2k)$, one has* $\deg S\,\mathrm{Ch}_i(Y_k) = 0$, *where S is the cohomological Steenrod operation and* $\deg$ *is the degree homomorphism on the modulo* 2 *Chow groups.*

Proof. Assume that $\deg S\,\mathrm{Ch}^j(Y) \neq 0$ for some j. Then $\deg S\,\mathrm{Ch}^j(H)^\sigma \neq 0$. Since S commutes with σ, S is trivial on $(1+\sigma)$. Therefore $\deg S\,\mathrm{Ch}^j(H)^\sigma/(1+\sigma) \neq 0$. It follows by Lemma 4.1 that $\deg in^* S\,\mathrm{Ch}^j(X) \neq 0$, or, equivalently, $\deg S\, in^*\,\mathrm{Ch}^j(X) \neq 0$. Let $y \in \mathrm{Ch}^j(X)$ be a standard monomial in the generators with $\deg S\, in^*(y) \neq 0$. Since $in^*(y) \neq 0$, the monomial y does not contain any z_i with even $i \neq n = 2k$ by the second half of Lemma 4.1. We may also assume that y does not contain z_{n-2k}. Indeed, $in^*(z_{n-2k})$ is a polynomial in the generators of $\mathrm{Ch}(H)^\sigma/(1+\sigma)$ of codimension $\leq n-2k$. In particular, $in^*(z_{n-2k})$ is a polynomial in $c_i(-\mathcal{T}_k)$ with $i \leq n-2k$. Let $P \in \mathrm{Ch}^{n-2k}(X)$ be the same polynomial in $c_i(-\mathcal{T}_X) = w_i$. Then $in^*(P) = in^*(z_{n-2k})$ and we may replace z_{n-2k} by P in y without changing $in^*(y)$.

We have

$$0 \neq \deg in^* S(y) = \deg in_* in^* S(y) = \deg S([H] \cdot y).$$

Since degree of any level $2k-1$ element is 0, [5, proof of Proposition 12], the element $S([H]y)$ is not of level $2k-1$. Since the Steenrod operation preserves the level, the product $[H] \cdot y$ is not of level $2k-1$. Since $[H]$ is of level k by Corollary 4.4, y is not of level $k-1$. The smallest possible codimension of a monomial of level not $k-1$ without z-generators of even codimension is the sum of k summands

$$(n-2k+1) + (n-2k+3) + \cdots + (n-1) = k(n-2k+1) + k(k-1) = k(n-k).$$

It follows that $j < k(n-k)$. Since $\dim Y - k(n-k) = k(n-2k)$, we are done. □

5. Some ranks of some motives

Let K/F be a separable quadratic field extension. Let M be a motive over F with coefficients in $\mathbb{F}_2$. We assume that there exists a field extension F'/F such that F is algebraically closed in F' and the motive $M_{F'}$ decomposes in a sum of shifts of the motives of $\operatorname{Spec} F'$ and $\operatorname{Spec} K'$, where K' is the field $K \otimes_F F'$. Note that the number of F' and the number of K' appearing in the decomposition do not depend on the choice of F': if F'' is another field like that, the Krull-Schmidt principle over the field $F' \otimes_F F''$ gives the equalities. The number of F' in the decomposition is the *F-rank* rk_F of M, the number of K' is the *K-rank* rk_K of M. The usual rank $\operatorname{rk} M$ is also defined for such M and is equal to $\operatorname{rk}_F M + 2\operatorname{rk}_K M$.

Recall that there are functors

$$\operatorname{tr}, \operatorname{cor} : \operatorname{CM}(K, \mathbb{F}_2) \to \operatorname{CM}(F, \mathbb{F}_2).$$

The first one (non-additive and not commuting with the shift, see [8]) is induced by the Weil transfer. The second one (additive and commuting with the shift, see [11]) is induced by the functor associating to a K-variety the same variety considered as a variety over F via the composition with $\operatorname{Spec} K \to \operatorname{Spec} F$.

Here is an example of computation of ranks.

Lemma 5.1. *Let M be a motive over K isomorphic to a sum of n shifts of the Tate motive. Then $\operatorname{rk}_F \operatorname{tr} M = \operatorname{rk} M = n$, $\operatorname{rk}_K \operatorname{tr} M = n(n-1)/2$, $\operatorname{rk}_F \operatorname{cor} M = 0$, and $\operatorname{rk}_K \operatorname{cor} M = n$.*

Proof. Since $\operatorname{cor} M(\operatorname{Spec} K) = M(\operatorname{Spec} K)$, the formulas for cor follow. The formulas for tr follow from [4, Lemma 2.1]. □

Let D be a central division K algebra admitting a K/F-unitary involution, and assume that $\deg D = 2^n$ for some $n \geq 0$. For an integer $k \in [0,\ n-1]$, let X_k be the Weil transfer with respect to K/F of the generalized Severi-Brauer variety $X(2^k, D)$. The motive $M(X_k)$ satisfy the above conditions (one may take as F' the function field of the variety X_0) so that the ranks $\operatorname{rk}_F M$ and $\operatorname{rk}_K M$ are defined for any summand M of $M(X_k)$. In particular, the ranks $\operatorname{rk}_F U(X_k)$ and $\operatorname{rk}_K U(X_k)$ are defined for the *upper* (indecomposable) motive $U(X_k)$.

Proposition 5.2. $v_2(\operatorname{rk}_F U(X_k)) = n - k$, $v_2(\operatorname{rk}_K U(X_k)) = n - k - 1$.

Proof. We induct on k. Let us do the induction base $k = 0$. According to [4, Theorem 1.2], $U(X_0) = M(X_0)$. Since $\operatorname{rk} M(X(1, D)) = 2^n$, it follows from Lemma 5.1 that $\operatorname{rk}_F U(X_0) = 2^n$, $\operatorname{rk}_K U(X_0) = 2^{n-1}(2^n - 1)$.

Now we assume that $k > 0$. Since $\operatorname{rk} M(X(2^k, D)) = b := \binom{2^n}{2^k}$, it follows from Lemma 5.1 that $\operatorname{rk}_F M(X_k) = b$ and $\operatorname{rk}_K M(X_k) = b(b-1)/2$. In particular, $v_2(\operatorname{rk}_F M(X_k)) = n - k > 0$ and $v_2(\operatorname{rk}_K M(X_k)) = n - k - 1$. Therefore, it suffices to show that for each summand M of the complete motivic decomposition of X_k different from $U(X_k)$ we have $v_2(\operatorname{rk}_F M) > n - k$ and $v_2(\operatorname{rk}_K M) > n - k - 1$.

By [11] and [4], M is a shift of the motive $U(X_l)$ with some $l \in [0,\ k-1]$ or a shift of the motive $\operatorname{cor}_{K/F} U(X(2^l, D))$ with some $l \in [0,\ k]$. In the first case we are done by the induction hypothesis. In the second case we have $\operatorname{rk}_F M = 0$ and $\operatorname{rk}_K M = \binom{2^n}{2^l}$. □

6. Unitary isotropy theorem

Let K be a field of characteristic $\neq 2$, A a central simple K-algebra, τ a unitary involution on A, F the subfield of the elements of K fixed under τ. We say that τ is *isotropic*, if $\tau(I) \cdot I = 0$ for some non-zero right ideal $I \subset A$; otherwise we say that τ is *anisotropic.*

Theorem 6.1 (**Unitary Isotropy Theorem).** *If τ becomes isotropic over any field extension F'/F such that $K' := K \otimes_F F'$ is a field and the central simple K'-algebra $A' := A \otimes_F F'$ is split, then τ becomes isotropic over some finite odd degree field extension of F.*

Proof. We can easily reduce this theorem to the case of 2-primary $\operatorname{ind} A$. Indeed, it suffices to find a finite odd degree field extension L/F, such that A becomes 2-primary over L. For such L/F we can take the field extension of F corresponding to a Sylow 2-subgroup of the Galois groups of the normal closure of E/F, where E is a separable finite odd degree field extension of K such that $\operatorname{ind}(A \otimes_F E)$ is 2-primary.

Because of the above redaction, we assume that the index of A is a power of 2.

We follow the lines of the proof of [5, Theorem 1]. We prove Theorem 6.1 over all fields simultaneously using an induction on $\operatorname{ind} A$. The case of $\operatorname{ind} A = 1$ is trivial. From now we are assuming that $\operatorname{ind} A = 2^r$ for some integer $r \geq 1$, and we fix the following notations:

F is a field of characteristic different from 2;

K/F is a quadratic field extension;

A is a central simple K-algebra of the index 2^r (with $r \geq 1$);

τ is an F-linear unitary involution on A;

D is a central division F-algebra (of degree 2^r) Brauer-equivalent to A;

V is a right D-module of D-dimension v with an isomorphism $\operatorname{End}_D(V) \simeq A$ (in particular, $\operatorname{rdim} V = \deg A = 2^r \cdot v$, where $\operatorname{rdim} V := \dim_F V / \deg D$ is the reduced dimension);

we fix an arbitrary F-linear unitary involution ε on D;

h is a hermitian (with respect to ε) form on V such that the involution τ is adjoint to h;

$Y = X(2^r; (V, h)) \simeq X(2^r; (A, \tau))$ is the variety of totally isotropic submodules in V of reduced dimension $\operatorname{rdim} = 2^r$ which is isomorphic (via Morita equivalence) to the variety of right totally isotropic ideals in A of the same reduced dimension;

X is the Weil transfer (with respect to K/F) of the generalized Severi-Brauer K-variety $X(2^{r-1}; D)$.

Let F' be the function field of the Weil transfer of the Severi-Brauer variety $X(1; D)$ of D. Clearly, K' is a field and A' is split for such F'. We assume that the involution τ' (and therefore, the hermitian form $h_{F'}$) is isotropic and we want to show that h (and τ) becomes isotropic over a finite odd degree extension of F. According to [3, Theorem 1.4], the Witt index of $h_{F'}$ is a multiple of $2^r = \operatorname{ind} A$. In particular, $v \geq 2$. If the Witt index is greater than 2^r, we replace V by a submodule in V of D-codimension 1 and we replace h by its restriction on this new V. The Witt index of $h_{F'}$ drops by 2^r or stays unchanged. We repeat the procedure until the Witt index becomes equal to 2^r. In particular, v is still ≥ 2.

The variety Y has an F'-point and the index of the central simple $K \otimes_F F(X)$-algebra $A \otimes_F F(X)$ is equal to 2^{r-1} (note that $K \otimes_F F(X)$ is a field). Consequently, by the induction hypothesis, the variety $Y_{F(X)}$ has an odd degree closed point. We prove Theorem 6.1 by showing that the variety Y has an odd degree closed point.

We will use and we recall the following statement from [5].

Proposition 6.2. *Let $\mathfrak{X}$ be a geometrically split, geometrically irreducible F-variety satisfying the nilpotence principle and let $\mathcal{Y}$ be a smooth complete F-variety. Assume that there exists a field extension E/F such that*

(1) *for some field extension $\overline{E(\mathfrak{X})}/E(\mathfrak{X})$, the image of the change of field homomorphism $\operatorname{Ch}(\mathcal{Y}_{E(\mathfrak{X})}) \to \operatorname{Ch}(\mathcal{Y}_{\overline{E(\mathfrak{X})}})$ coincides with the image of the change of field homomorphism $\operatorname{Ch}(\mathcal{Y}_{F(\mathfrak{X})}) \to \operatorname{Ch}(\mathcal{Y}_{\overline{E(\mathfrak{X})}})$;*
(2) *the E-variety $\mathfrak{X}_E$ is p-incompressible;*
(3) *a shift of the upper indecomposable summand of $M(\mathfrak{X})_E$ is a summand of $M(\mathcal{Y})_E$.*

Then the same shift of the upper indecomposable summand of $M(\mathfrak{X})$ is a summand of $M(\mathcal{Y})$.

We are going to apply Proposition 6.2 (with $p = 2$) $\mathfrak{X} = X$, $\mathcal{Y} = Y$, and $E = F(Y)$. We need to check that conditions (1) - (3) are satisfied for these X, Y, E. First of all, we need a motivic decomposition of Y over a field extension $\tilde{F}/F$, such that $Y(\tilde{F}) \neq \emptyset$ and $\tilde{K} = K \otimes_F \tilde{F}$ is a field. Over such $\tilde{F}$, the hermitian form h decomposes in the orthogonal sum of the hyperbolic $\tilde{D}$-plane and a hermitian form h' on a right $\tilde{D}$-module V' with $\operatorname{rdim} V' = 2^r(v-2)$, where $\tilde{D}$ is central simple $\tilde{K}$-algebra $D \otimes_F \tilde{F}$. Let $L/F(X)$ be a finite odd degree extension such that $Y(L) \neq \emptyset$. Recall that a smooth projective variety is *anisotropic*, if it has no odd degree closed points (by [9, lemma 6.3], the motive of an anisotropic variety does not contain a Tate summand).

Lemma 6.3. *The shift of the motive of $X_{\tilde{F}}$ and two Tate motives are the motivic summands of $Y_{\tilde{F}}$. In the case $\tilde{F} = L$, any other motivic summand of Y_L is a shift of some anisotropic L-variety.*

Proof. According to [7, Theorem 15.8], the variety $Y_{\tilde{F}}$ is a relative cellular space (as defined in [1, §66]) over the (non-connected) variety Z of triples (I, J, N), where I and J are right ideals in D, and where N is a submodule in V' such that the submodule $I \oplus J \oplus N \subset V$ is a point of $Y_{\tilde{F}}$ (that is, $\varepsilon_{\tilde{F}}(I) \cdot J = 0$, N is totally isotropic, and the reduced dimension of the $\tilde{D}$-module $I \oplus J \oplus N \subset V$ is equal to $\deg \tilde{D}$). Therefore, by [1, Corollary 66.4], the motive of $Y_{\tilde{F}}$ is the sum of shifts of the motives of the components of Z.

The shift of the motive of $X_{\tilde{F}}$ is given by the motive of the component of the triples $\{(I, J, 0) | \operatorname{rdim} I = \operatorname{rdim} J = (\deg \tilde{D})/2\}$. The rational points $(0, \tilde{D}, 0)$ and $(\tilde{D}, 0, 0)$ of Z are components of Z which produce the two promised Tate summands. In the case $\tilde{F} = L$ we have $\operatorname{ind} \tilde{D} = (\deg \tilde{D})/2 = 2^{r-1}$. Therefore to prove the second statement of this lemma, we only need to check that the component of Z of triples $(0, 0, N)$ is anisotropic. It is true, because this component is naturally identified with anisotropic L-variety $Y' = X(2^r; (V', h'))$. □

Remark 6.4. Two Tate motives mentioned in Lemma 6.3 are clearly $\mathbb{F}_2$ and $\mathbb{F}_2(\dim Y)$. In the case $\tilde{F} = L$, by duality, the motivic summand $M(X_L)$ of Y_L has as the shifting number the integer

$$n := (\dim Y - \dim X)/2 .$$

Since $Y(F(Y)) \neq \emptyset$, the condition (3) of Proposition 6.2 is checked by Lemma 6.3. Let us check now the condition (2). By [4, Theorem 1.1], the variety $X_{F(Y)}$ is 2-incompressible if (and only if) the $K \otimes_F F(Y)$-algebra $D \otimes_F F(Y)$ is division. This is indeed the case:

Lemma 6.5. *The algebra $D \otimes_F F(Y)$ is division, that is,* $\operatorname{ind}(D \otimes_F F(Y)) = \operatorname{ind} D$.

Proof. The proof is similar to the proof of [5, Lemma 6]. Assume that $\operatorname{ind}(D \otimes_F F(Y)) < \operatorname{ind} D$. Then we could prove as in [5, Lemma 6], that the upper indecomposable motivic summand of X is a motivic summand of Y. This implies (because the variety X is 2-incompressible) that the complete motivic decomposition of the variety $Y_{F(X)}$ contains the Tate summand $\mathbb{F}_2(\dim X)$. By Lemma 6.3 and Remark 6.4 we get a contradiction. □

We have checked condition (2) of Proposition 6.2. To check the remaining condition (1), we will need the same property for the variety Y as in [5, Lemma 7]. We can prove it for more general class of varieties. Let $\mathcal{Z}$ be a projective homogeneous variety under an arbitrary absolutely simple adjoint affine algebraic group G of type A_n over a field k (we can replace "absolutely simple of type A_n" by the condition, that G is semisimple and becomes of inner type over some quadratic separable field extension of k). In other words, $\mathcal{Z}$ is a variety of flags of isotropic right ideals of a central simple algebra over a quadratic separable field extension of k endowed (the algebra) with a unitary k-linear involution.

Lemma 6.6. *Let k'/k be a finite odd degree field extension and let $\bar{k}$ be an algebraic closure of k containing k'. Then* $\operatorname{Im}(\operatorname{Ch}(\mathcal{Z}) \to \operatorname{Ch}(\mathcal{Z}_{\bar{k}})) = \operatorname{Im}(\operatorname{Ch}(\mathcal{Z}_{k'}) \to \operatorname{Ch}(\mathcal{Z}_{\bar{k}}))$.

Proof. For any field extension $E \subset \bar{k}$ of k, we write I_E for the image of $\operatorname{Ch}(\mathcal{Z}_E) \to \operatorname{Ch}(\mathcal{Z}_{\bar{k}}))$. We only need to show that $I_{k'} \subset I_k$ because, clearly, $I_k \subset I_{k'}$.

If G is of inner type, the variety $\mathcal{Z}$ is a variety of flags of right ideals of a central simple k-algebra. Therefore the group $\operatorname{Aut}(\bar{k}/k)$ acts trivially on $\operatorname{Ch}(\mathcal{Z}_{\bar{k}})$. It follows that $[k' : k] \cdot I_{k'} \subset I_k$ and therefore $I_{k'} \subset I_k$.

Now we assume that G is of outer type. Let $K \subset \bar{k}$ be the separable quadratic field extension of k such that G_K is of inner type. Consider two subgroups $\operatorname{Aut}(\bar{k}/K)$ and $\operatorname{Aut}(\bar{k}/k')$ of the group $\operatorname{Aut}(\bar{k}/k)$. Acting on $\operatorname{Ch}(\mathcal{Z}_{\bar{k}})$, they act trivially on $I_{k'}$. The index of the first subgroup is 2 while the index of the second one is odd (a divisor of $[k' : k]$). Indeed,

$$\mathrm{Aut}(\bar{k}/k) = \mathrm{Aut}(k_{\mathrm{sep}}/k), \ \ \mathrm{Aut}(\bar{k}/K) = \mathrm{Aut}(k_{\mathrm{sep}}/K),$$

where k_{sep} is the separable closure of k in $\bar{k}$, so that $\mathrm{Aut}(\bar{k}/k)/\mathrm{Aut}(\bar{k}/K) = \mathrm{Aut}(K/k)$; if k'' is the separable closure of k in k', then $\mathrm{Aut}(\bar{k}/k') = \mathrm{Aut}(\bar{k}/k'')$, so that the index of $\mathrm{Aut}(\bar{k}/k')$ in $\mathrm{Aut}(\bar{k}/k)$ is $[k'' : k]$.

It follows that $\mathrm{Aut}(\bar{k}/k)$ acts trivially on $I_{k'}$. Therefore we still have the inclusion $[k' : k] \cdot I_{k'} \subset I_k$ giving $I_{k'} \subset I_k$. □

Corollary 6.7. *$U(X)(n)$ is a motivic summand of Y.*

Proof. As planned, we apply Proposition 6.2 to $p = 2$, $\mathfrak{X} = X$, $\mathcal{Y} = Y$, and $E = F(Y)$. Since $E(X) \subset L(Y)$, we have the commutative diagram

$$\begin{array}{ccccc} \mathrm{CH}(Y_{E(X)}) & \longrightarrow & \mathrm{CH}(Y_{L(Y)}) & \longrightarrow & \mathrm{CH}(Y_{\overline{L(Y)}}) \\ \uparrow & & \uparrow & & \\ \mathrm{CH}(Y_{F(X)}) & \longrightarrow & \mathrm{CH}(Y_L) & & \end{array}$$

where the maps are the change of field homomorphisms and where $\overline{L(Y)}$ is an algebraic closure of $L(Y)$. We check condition (1) for $\overline{E(\mathfrak{X})} = \overline{L(Y)}$. For any field extension $\mathcal{F} \subset \overline{L(Y)}$ of F, we write $I_{\mathcal{F}}$ for the image of $\mathrm{Ch}(\mathcal{Y}_{\mathcal{F}}) \to \mathrm{Ch}(\mathcal{Y}_{\overline{L(Y)}})$. We only need to show that $I_{E(X)} \subset I_{F(X)}$. We have $I_{E(X)} \subset I_{L(Y)}$. Since $Y(L) \neq \emptyset$, the field extension $L(Y)/L$ is purely transcendental. Therefore $\mathrm{res}_{L(Y)/L}$ is surjective and $I_{L(Y)} = I_L$. Finally, by Lemma 6.6, $I_L = I_{F(X)}$. We obtain the necessary inclusion $I_{E(X)} \subset I_{L(Y)} = I_L = I_{F(X)}$.

As already pointed out, condition (2) is satisfied by Lemma 6.5, and condition (3) is satisfied by Lemma 6.3. Therefore, by Proposition 6.2, a shift of $U(X)$ is a motivic summand of Y. By Remark 6.4, it follows that the shifting number of this motivic summand $U(X)$ is equal to n. □

As in [5] we need the following enhancement of Corollary 6.7.

Proposition 6.8. *There exists a symmetric projector π on Y such that the motive (Y, π) is isomorphic to $U(X)(n)$.*

Proof. We can follow the lines of the proof of [5, Proposition 9] if we know that the complete motivic decomposition of $Y_{F(X)}$ could not contain two copies of $\mathbb{F}_2(n)$. This is true by Lemma 6.3 and Remark 6.4. □

The following proposition finishes the proof of Theorem 6.1.

Proposition 6.9. *Let F be a field of characteristic $\neq 2$. Let K/F be a separable quadratic field extension. Let D be a central division K algebra of degree 2^r with some $r \geq 1$ admitting a K/F-unitary involution. Let X be the Weil transfer of the generalized Severi-Brauer variety $X(2^{r-1}, D)$. Let A be a central simple K-algebra Brauer-equivalent to D endowed with a K/F-unitary involution. Let Y be the variety of isotropic rank 2^r right ideals in A. Assume that there is a symmetric projector $\pi \in \mathrm{Ch}_{\dim Y}(Y \times Y)$ such that the motive (Y, π) is isomorphic to a shift of the upper motive of X. Then Y has a closed point of odd degree.*

Proof. By Lemma 2.10, it is enough to show that $\mathrm{sq}(\pi) \neq \mathrm{st}(\pi)$. Computing sq and st, we may go over any field extension of F. There exists a field extension $\tilde{F}/F$ over which D is split, but $\tilde{K} := K \otimes_F \tilde{F}$ is still a field. Since the motive of X over $\tilde{F}$ is a sum of shifts of the motives of $\operatorname{Spec} \tilde{F}$ and $\operatorname{Spec} \tilde{K}$, π decomposes in a sum of two orthogonal projectors α and β such that $(Y_{\tilde{F}}, \alpha)$ is a sum of shifts of the motive of $\operatorname{Spec} \tilde{F}$ and $(Y_{\tilde{F}}, \beta)$ is a sum of shifts of the motive of $\operatorname{Spec} \tilde{K}$.

The projectors α and β are symmetric. Indeed, the subspace $\mathrm{Ch}_*(Y_{\tilde{K}}, \beta) \subset \mathrm{Ch}_*(Y_{\tilde{K}}, \pi)$ coincides with

$$(1+\sigma)\,\mathrm{Ch}_*(Y_{\tilde{K}}, \pi) = \mathrm{Im}\,\big(\mathrm{Ch}_*(Y_{\tilde{K}}, \pi) \to \mathrm{Ch}_*(Y_{\tilde{F}}, \pi) \to \mathrm{Ch}_*(Y_{\tilde{K}}, \pi)\big).$$

Since π is also a sum of the orthogonal projectors α^t, β^t and $(X_{\tilde{F}}, \alpha^t)$ is still a sum of shifts of the motive of $\operatorname{Spec} \tilde{F}$ and $(Y_{\tilde{F}}, \beta^t)$ is still a sum of shifts of the motive of $\operatorname{Spec} \tilde{K}$, we also have that the subspace $\mathrm{Ch}_*(Y_{\tilde{K}}, \beta^t) \subset \mathrm{Ch}_*(Y_{\tilde{K}}, \pi)$ coincides with $(1+\sigma)\,\mathrm{Ch}_*(Y_{\tilde{K}}, \pi)$. This shows that β is symmetric. Therefore $\alpha = \pi - \beta$ also is symmetric.

Now we have (see Lemmas 2.2 and 2.5): $\mathrm{sq}(\alpha) = \mathrm{rk}_F(Y, \pi) \pmod 4 = \mathrm{rk}_F\, U(X) \pmod 4 = 2$ and $\mathrm{sq}(\beta) = 2\,\mathrm{rk}_K(Y, \pi) \pmod 4 = 2\,\mathrm{rk}_K\, U(X) \pmod 4 = 2$ by Proposition 5.2. On the other hand, $\mathrm{st}(\alpha) = 0$. Indeed, α over $\tilde{F}$ is a sum of $a \times b$ with $a, b \in \mathrm{Ch}_{\geq d}(Y_{\tilde{F}})$, where $d := (\dim Y - \dim X)/2 = \big(k(2n-3k) - k^2/2\big)/2$ with $k := 2^r = \operatorname{ind} A$ and $n := \deg A$. Since $d > k(n-2k)$, $\deg S(b) = 0$ by Theorem 4.5. Therefore $pr_*\, S(a \times b) = 0$. It follows that $pr_*(\mathfrak{a})$ is divisible by 2 for an integral representative $\mathfrak{a}$ of $S(a \times b)$. Therefore $pr_*(\mathfrak{a})$ is divisible by 2 if now $\mathfrak{a}$ is an integral representative of $S(\alpha)$. It follows that $pr_*(\mathfrak{a})^2$ is divisible by 4 and consequently $\mathrm{st}(\alpha) = 0$.

Finally, let us check that $\mathrm{st}(\beta) = 2$. The point is that β is in

$$(1+\sigma)\,\mathrm{Ch}(Y_{\tilde{K}}) = \mathrm{Im}\,\big(\mathrm{Ch}(Y_{\tilde{K}}) \to \mathrm{Ch}(Y_{\tilde{F}}) \to \mathrm{Ch}(Y_{\tilde{K}})\big).$$

Therefore β over $\tilde{K}$ is rational even if h is anisotropic (in which case the variety Y has no odd degree closed points). By Lemma 2.10, this shows that $\mathrm{st}(\beta) = \mathrm{sq}(\beta)$.

We have calculated the values of the operations sq and st on α and β. We have by Lemmas 2.2, 2.4, and 2.9 that $\mathrm{sq}(\pi) = \mathrm{sq}(\alpha) + \mathrm{sq}(\beta) = 0$ and $\mathrm{st}(\pi) = \mathrm{st}(\alpha) + \mathrm{st}(\beta) = 2$. In particular, $\mathrm{sq}(\pi) \neq \mathrm{st}(\pi)$. ☐

Theorem 6.1 is proved. ☐

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UPMC Univ Paris 06, Institut de Mathématiques de Jussieu (boite courrier 247), 4 place Jussieu, F-75252 Paris CEDEX 05, FRANCE

Web page: www.math.jussieu.fr/~karpenko
E-mail address: karpenko *at* math.jussieu.fr
E-mail address: zhykhovich *at* math.jussieu.fr