ESSENTIAL p-DIMENSION OF SPLIT SIMPLE GROUPS OF TYPE A_n

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Abstract. We compute the essential p-dimension of split simple groups of type A_{n-1} in terms of the functor $\mathsf{Alg}(n, m)$ of central simple algebras of degree n and exponent dividing m.

1. Introduction

Let F be a field and let $\mathcal{F} : \mathsf{Fields}/F \to \mathsf{Sets}$ be a functor from the category Fields/F of field extensions over F to the category Sets of sets. Let $E \in \mathsf{Fields}/F$ and $K \subset E$ a subfield over F. We say that that K is a *field of definition of* $\alpha \in \mathcal{F}(E)$ if α belongs to the image of the map $\mathcal{F}(K) \to \mathcal{F}(E)$. The *essential dimension of* α, denoted $\operatorname{ed}^{\mathcal{F}}(\alpha)$, is the least transcendence degree $\operatorname{tr.deg}_F(K)$ over all fields of definition K of α. The *essential dimension of the functor* $\mathcal{F}$ is

$$\operatorname{ed}(\mathcal{F}) = \sup\{\operatorname{ed}^{\mathcal{F}}(\alpha)\},$$

where the supremum is taken over all fields $E \in \mathsf{Fields}/F$ and all $\alpha \in \mathcal{F}(E)$ (see [3, Def. 1.2] or [5, Sec.1]). Informally, the essential dimension of $\mathcal{F}$ is the smallest number of algebraically independent parameters required to define $\mathcal{F}$ and may be thought of as a measure of complexity of $\mathcal{F}$.

Let p be a prime integer. The *essential p-dimension of* α, denoted $\operatorname{ed}_p^{\mathcal{F}}(\alpha)$, is defined as the minimum of $\operatorname{ed}^{\mathcal{F}}(\alpha_{E'})$, where E' ranges over all finite field extensions of E of degree prime to p. The *essential p-dimension of* $\mathcal{F}$ is

$$\operatorname{ed}_p(\mathcal{F}) = \sup\{\operatorname{ed}_p^{\mathcal{F}}(\alpha)\},$$

where the supremum ranges over all fields $E \in \mathsf{Fields}/F$ and all $\alpha \in \mathcal{F}(E)$. By definition, $\operatorname{ed}(\mathcal{F}) \geq \operatorname{ed}_p(\mathcal{F})$ for all p.

For every integer $n \geq 1$, a divisor m of n and any field extension E/F, let $\mathsf{Alg}_E(n, m)$ denote the set of isomorphism classes of central simple E-algebras of degree n and exponent dividing m. We can identify $\mathsf{Alg}_E(n, m)$ with the subset of the m-torsion part $\operatorname{Br}_m(E)$ of the Brauer group of E consisting of all elements a such that the index $\operatorname{ind}(a)$ of a divides n. We view $\mathsf{Alg}(n, m)$

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as a functor $\mathit{Fields}/F \to \mathit{Sets}$. Upper and lower bounds for the essential p-dimension $\mathrm{ed}_p\big(\mathit{Alg}(n,m)\big)$ for a prime integer p different from $\mathrm{char}(F)$ can be found in [2].

Let G be an algebraic group scheme over F. Write $\mathcal{F}_G$ for the functor taking a field extension E/F to the set $H^1(E,G)$ of isomorphism classes of principal homogeneous G-spaces (G-torsors) over E. The essential (p-)dimension of $\mathcal{F}_G$ is called the *essential (p-)dimension of* G and is denoted by $\mathrm{ed}(G)$ and $\mathrm{ed}_p(G)$.

A split simple algebraic group G of type A_{n-1} is isomorphic to $\mathbf{SL}_n/\boldsymbol{\mu}_m$ for a divisor m of n. In the present paper we compute the essential p-dimension of G in terms of the integer $\mathrm{ed}_p\big(\mathit{Alg}(n,m)\big)$.

Theorem 1.1. *Let n be a natural number, m a divisor of n and p a prime integer. Let p^r and p^s be the largest powers of p dividing n and m respectively and let $G = \mathbf{SL}_n/\boldsymbol{\mu}_m$ be the algebraic group defined over a field F of the characteristic not p. Then*

$$ed_p(G) = \begin{cases} 0, & \text{if } s = 0; \\ \mathrm{ed}_p\big(\mathit{Alg}(p^r,p^r)\big), & \text{if } s = r; \\ \mathrm{ed}_p\big(\mathit{Alg}(p^r,p^s)\big) + 1, & \text{if } 0 < s < r. \end{cases}$$

Using lower bounds for $\mathrm{ed}_p\big(\mathit{Alg}(p^r,p^s)\big)$ obtained in [1] and [2], we get:

Corollary 1.2. *If p is a prime integer then*
(1) $p^{2r-2} + p^{r-s} + 1 \ge \mathrm{ed}_p(\mathbf{SL}_{p^r}/\boldsymbol{\mu}_{p^s}) \ge (r-1)p^r + p^{r-s} + 1$
if $0 < s < r$ and p is odd in the case $s = 1$,
(2) $2^{2r-4} + 2^{r-1} + 1 \ge \mathrm{ed}_2(\mathbf{SL}_{2^r}/\boldsymbol{\mu}_2) \ge (r-1)2^{r-1} + 1$ *if $r \ge 3$.*
(3) $\mathrm{ed}_p(\mathbf{SL}_{p^2}/\boldsymbol{\mu}_p) = p^2 + p + 1$ *if p is odd,*
(4) $\mathrm{ed}_2(\mathbf{SL}_4/\boldsymbol{\mu}_2) = 5$,
(5) $\mathrm{ed}_2(\mathbf{SL}_8/\boldsymbol{\mu}_2) = 9$,
(6) $\mathrm{ed}_2(\mathbf{SL}_8/\boldsymbol{\mu}_4) = 19$,
(7) $\mathrm{ed}_2(\mathbf{SL}_{16}/\boldsymbol{\mu}_2) = 25$.

2. Unramified torsors

Let R be a commutative ring and let G be a group scheme defined over R. There is a bijection between the set of isomorphism classes of G-torsors over R and the pointed set $H^1(R,G)$ of the first cohomology of G for the flat topology (see [4, Exp. XXIV]). If G is smooth, one can use the étale topology instead of flat topology.

Let K be a discrete valued field with valuation ring $R \subset K$ and residue field $\overline{K}$. We write $H^1(K,G)_{nr}$ for the image of the map

$$H^1(R,G) \to H^1(K,G).$$

Let $\alpha \in H^1(K,G)$. If $\alpha \in H^1(K,G)_{nr}$ we say that α is *unramified*. Otherwise α is *ramified*.

If K is complete and G is smooth, the canonical map $H^1(R,G) \to H^1(\overline{K},G)$ is a bijection [4, Exp. XXIV, Prop. 8.1], hence $H^1(K,G)_{nr} \simeq H^1(\overline{K},G)$.

If G is commutative, then $H^1(K,G)_{nr}$ is a subgroup of $H^1(K,G)$. We write $H^1(K,G)_{ram}$ for the factor group.

Example 2.1. We have $H^1(K,\boldsymbol{\mu}_k)=K^\times/K^{\times k}$, $H^1(K,\boldsymbol{\mu}_k)_{nr}=R^\times/R^{\times k}$ and $H^1(K,\boldsymbol{\mu}_k)_{ram}=\mathbb{Z}/k\mathbb{Z}$.

Suppose that K is complete. Let T' be a torus over R. We write T for $T'\otimes_R K$ and $\overline{T}$ for $T'\otimes_R\overline{K}$. Clearly, T and $\overline{T}$ are tori over K and $\overline{K}$ respectively. The character group $\overline{T}^*$ of $\overline{T}$ is a module over the absolute Galois group $\Gamma_{\overline{K}}$. The character group T^* coincides with $\overline{T}^*$, and Γ_K acts on T^* via the canonical surjective homomorphism $\Gamma_K\to\Gamma_{\overline{K}}$.

We have the split exact sequence of Galois $\Gamma_{\overline{K}}$-modules

$$1\to R_{nr}^\times\to K_{nr}^\times\xrightarrow{v}\mathbb{Z}\to 0,$$

where R_{nr} and K_{nr} are maximal unramified extensions of R and K respectively. Tensoring this sequence with the $\Gamma_{\overline{K}}$-module of co-characters $\overline{T}_*$, the dual of $\overline{T}^*$, and taking cohomology groups yields exact sequences

$$1\to T(R)\to T(K)\to\overline{T}_*(\overline{K})\to 0,$$

$$0\to H^1_{et}(R,T)\to H^1(K,T)\xrightarrow{v_*}H^1(\overline{K},\overline{T}_*)\to 0.$$

In particular, the group $H^1(K,T)_{ram}$ is canonically isomorphic to $H^1(\overline{K},\overline{T}_*)$.

3. Azumaya algebras and torsors

Let $n=km$ and $G=\mathbf{SL}_n/\boldsymbol{\mu}_m$ over a field F, so we have an exact sequence

$$1\to\boldsymbol{\mu}_k\to G\to\mathbf{PGL}_n\to 1. \tag{1}$$

Let R be a commutative local F-algebra. The exact sequence (1) yields an exact sequence of pointed sets

$$H^1(R,G)\xrightarrow{\alpha}H^1(R,\mathbf{PGL}_n)\xrightarrow{\partial}H^2(R,\boldsymbol{\mu}_k).$$

Moreover, the group $H^1(R,\boldsymbol{\mu}_k)=R^\times/R^{\times k}$ acts on the set $H^1(R,G)$ transitively in the fibers of the map α. For an element $r\in R^\times$ and $\xi\in H^1(R,G)$ we write $r\xi$ for the result of the action of $rR^{\times k}$ on ξ.

Recall that there is a canonical bijection between $H^1(R,\mathbf{PGL}_n)$ and the set of isomorphism classes $\mathsf{Alg}_R(n)$ of Azumaya R-algebras of degree n, so we have the map $H^1(R,G)\to\mathsf{Alg}_R(n)$ [6, Ch. IV].

The group $H^2(R,\boldsymbol{\mu}_k)$ is identified with the subgroup $\mathrm{Br}_k(R)$ of the Brauer group $\mathrm{Br}(R)=H^2(R,\mathbf{G}_{\mathrm{m}})$ of R and the map ∂ takes an algebra A to the class of $A^{\otimes m}$ in $\mathrm{Br}(R)$. Therefore, the image of an element $\xi\in H^1(R,G)$ in $H^1(R,\mathbf{PGL}_n)$ yields a class A_ξ in $\mathsf{Alg}_R(n,m)\subset\mathsf{Alg}_R(n)$ of algebras of exponent dividing m. Moreover, every class $A\in\mathsf{Alg}_R(n,m)$ is of the form $A=A_\xi$ for some $\xi\in H^1(R,G)$.

Twisting (1) by the class of an algebra $A\in\mathsf{Alg}_R(n)$ yields an exact sequence

$$1\to\boldsymbol{\mu}_k\to G'\to\mathbf{PGL}_1(A)\to 1.$$

The connecting homomorphism

$$A^\times/R^\times = \mathbf{PGL}_1(A)(R) \to H^1(R,\boldsymbol{\mu}_k) = R^\times/R^{\times k}$$

takes the class $aR^\times$ to $\mathrm{Nrd}(a)R^{\times k}$, where $\mathrm{Nrd} : A^\times \to R^\times$ is the reduced norm homomorphism. This yields:

Lemma 3.1. *Let R be a commutative local F-algebra and $A \in \mathsf{Alg}_R(n,m)$. Then the factor group $R^\times/\big(R^{\times k}\cdot \mathrm{Nrd}(A)\big)$ acts simply transitively on the fiber of the surjective map $H^1(R,G) \to \mathsf{Alg}_R(n,m)$ over A.*

Let K/F be a field extension with a discrete valuation v over F and a prime element π.

Lemma 3.2. *Let $\xi \in H^1(K,G)$ be an unramified element such that $A_\xi \neq 0$ in $\mathrm{Br}(K)$. If $\pi\xi$ is unramified then k and $\mathrm{ind}(A_\xi)$ are relatively prime.*

Proof. Let $R \subset K$ be the valuation ring. By assumption, there are $\zeta,\zeta' \in H^1(R,G)$ such that $\xi = \zeta_K$ and $\pi\xi = \zeta'_K$. We have $(A_\zeta)_K = A_\xi = A_{\pi\xi} = (A_{\zeta'})_K$. As the map $\mathrm{Br}(R) \to \mathrm{Br}(K)$ is injective by [6, Ch. IV, Cor. 2.6], we have $A_\zeta = A_{\zeta'}$. It follows from Lemma 3.1 that $\zeta' = \lambda\zeta$ for some $\lambda \in R^\times$. Then $\pi\xi = \zeta'_K = \lambda\zeta_K = \lambda\xi$, therefore by Lemma 3.1 again, $\pi \in \lambda(K^{\times k}\cdot \mathrm{Nrd}(A_\xi))$. Therefore, $1 = v(\pi) \in k\mathbb{Z} + \mathrm{ind}(A_\xi)\mathbb{Z}$ as $v(\mathrm{Nrd}(A_\xi)) \subset \mathrm{ind}(A_\xi)\mathbb{Z}$ by [7, Ch. XII, §2]. □

4. Tori

Let L/F be a separable field extension of degree $n = p^r$, where p is a prime integer and $m = p^s$ a divisor of n. Consider the torus of norm one elements $R^{(1)}_{L/F}\big(\mathbf{G}_{\mathrm{m},L}\big)$ for the extension L/F, the factor torus $T = R^{(1)}_{L/F}\big(\mathbf{G}_{\mathrm{m},L}\big)/\boldsymbol{\mu}_m$ and $S = R_{L/F}\big(\mathbf{G}_{\mathrm{m},L}\big)/\,\mathbf{G}_{\mathrm{m}}$. Then T and S can be viewed as maximal tori of G and $\mathbf{PGL}_n$ respectively and we have an exact sequence

$$1 \to \boldsymbol{\mu}_k \to T \to S \to 1. \tag{2}$$

Let R be a commutative local F-algebra. The group $H^1(R,S)$ is identified with the relative Brauer group $\mathrm{Br}(LR/R) := \mathrm{Ker}\big(\mathrm{Br}(R) \to \mathrm{Br}(LR)\big)$, where we write LR for $L \otimes_F R$. The composition $H^1(R,S) \to H^1(R,\mathbf{PGL}_n) \hookrightarrow \mathrm{Br}(R)$ is identified with the inclusion of $\mathrm{Br}(LR/R)$ into $\mathrm{Br}(R)$. Comparing the exact sequences (1) and (2) we have:

Lemma 4.1. *The image of $H^1(R,T) \to H^1(R,G)$ coincides with the set of all ξ such that $A_\xi \in \mathrm{Br}_m(LR/R)$.*

Let Γ be the Galois group of a normal closure L'/F of L/F, so Γ is the decomposition group of the tori T and S. Let X be the Γ-set of all F-homomorphisms $L \to L'$. We have $|X| = n$ and $R_{L/K}\big(\mathbf{G}_{\mathrm{m},L}\big)^* = \mathbb{Z}[X]$.

Choose a point $x_0 \in X$ and let Γ_0 be the stabilizer of x_0 in Γ. As Γ acts transitively on X, we have, $X \simeq \Gamma/\Gamma_0$ and $[\Gamma : \Gamma_0] = n$.

Let I be the augmentation ideal in $\mathbb{Z}[\Gamma]$. Write I_X for the kernel of the augmentation map $\varepsilon : \mathbb{Z}[X] \to \mathbb{Z}$. We have $I_X = I \cdot \mathbb{Z}[X]$.

Write $N_X = \sum_{x\in X} x \in \mathbb{Z}[X]$, so $\varepsilon(N_X) = n$.

Let $V = R_{L/K}\big(\mathbf{G}_{\mathrm{m},L}\big)/\boldsymbol{\mu}_m$. The character group J_X of V is identified with the subgroup of elements $w \in \mathbb{Z}[X]$ with $\varepsilon(w) \in m\mathbb{Z}$. Note that $I_X \subset J_X$.

Lemma 4.2. *Suppose that $r > s$. Then $N_X \in pJ_X + I \cdot J_X$.*

Proof. The map

$$\Gamma \to I/I^2, \quad \gamma \mapsto (\gamma - 1) + I^2$$

is a group homomorphism. It follows that if γ belongs to the commutator subgroup $[\Gamma, \Gamma]$ of Γ, then

$$\gamma - 1 \in I^2. \tag{3}$$

Set $\Delta := [\Gamma,\Gamma]\Gamma_0$. Suppose first that Δ contains Γ_0 properly. Consider the sum u in $\mathbb{Z}[\Gamma]$ of all representatives of the set of left cosets Δ/Γ_0 chosen in $[\Gamma,\Gamma]$. It follows from (3) that u is congruent to $[\Delta : \Gamma_0]$ modulo I^2.

The element N_X is divisible by u, i.e., there is $M \in \mathbb{Z}[X]$ such that $N_X = uM$. It follows that N_X is congruent to $[\Delta : \Gamma_0]M$ modulo $I \cdot I_X$. As $[\Delta : \Gamma_0]$ is divisible by p, we have $[\Delta : \Gamma_0]M = pR$ for some $R \in \mathbb{Z}[X]$ with $\varepsilon(R) = n/p$. Since $r > s$, n/p is divisible by m, hence we have $R \in J_X$. Overall $N_X \in pJ_X + I \cdot I_X \subset pJ_X + I \cdot J_X$.

Now suppose that $\Delta = \Gamma_0$, i.e., Γ_0 is normal in Γ. It follows that $\Gamma_0 = 1$ and Γ is an abelian p-group of order n. Let Γ' be a subgroup of Γ of order p and $v = \sum_{\gamma\in\Gamma'} \gamma$ in $\mathbb{Z}[\Gamma]$. Then N_X is divisible by v, i.e., there is $M' \in \mathbb{Z}[X]$ such that $N_X = vM'$. Since $\varepsilon(M') = n/p$, we have $M' \in J_X$. As v is congruent to p modulo I, N_X is congruent to pM' modulo $I \cdot J_X$, hence $N_X \in pJ_X + I \cdot J_X$. □

The exact sequence of tori

$$1 \to T \to V \to \mathbf{G}_{\mathrm{m}} \to 1$$

yields an exact sequence of Γ-modules of co-characters

$$0 \to T_* \to V_* \to \mathbb{Z} \to 0. \tag{4}$$

Write θ_T for the image of 1 under the connecting homomorphism $\mathbb{Z} \to H^1(\Gamma, T_*) = H^1(F, T_*)$.

Proposition 4.3. *Suppose that $r > s$. Then θ_T is not divisible by p in $H^1(F, T_*)$.*

Proof. Consider the exact sequence of Γ-modules

$$0 \to \mathbb{Z} \xrightarrow{f} J_X \to T^* \to 0 \tag{5}$$

dual to (4). The image ν of θ_T under the canonical isomorphisms

$$H^1(F, T_*) \simeq \mathrm{Ext}^1_\Gamma(\mathbb{Z}, T_*) \simeq \mathrm{Ext}^1_\Gamma(T^*, \mathbb{Z})$$

is the class of the sequence (5).

Suppose that ν is divisible by p. Then the image of ν under the map $\mathrm{Ext}^1_\Gamma(T^*, \mathbb{Z}) \to \mathrm{Ext}^1_\Gamma(T^*, \mathbb{Z}/p\mathbb{Z})$ is trivial, i.e, the canonical homomorphism

$\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ factors as $\mathbb{Z} \xrightarrow{f} J_X \xrightarrow{h} \mathbb{Z}/p\mathbb{Z}$ for a Γ-homomorphism h. Note that $f(1) = N_X$, hence $h(N_X) = 1 + p\mathbb{Z}$.

The map h vanishes on $pJ_X + I \cdot J_X$, hence by Lemma 4.2, $h(N_X) = 0$, a contradiction. $\square$

5. The key proposition

Let K/F be a complete field with discrete valuation v over F and residue field $\overline{K}$. Let $\xi \in H^1(K, G)$ be an element and L/K an unramified (separable) field extension of degree n splitting A_ξ. Let $T = R^{(1)}_{L/K}\big(\mathbf{G}_{\mathrm{m},L}\big)/\boldsymbol{\mu}_m$ be the torus as defined in Section 4. Note that T is actually defined over the valuation ring, so the residue torus $\overline{T}$ is defined over $\overline{K}$. As $A_\xi \in \mathrm{Br}_m(L/K)$, the element ξ has a lifting to $H^1(K, T)$ by Lemma 4.1.

Lemma 5.1. *The image of the class $xK^{\times k}$ under the composition*

$$K^\times/K^{\times k} = H^1(K, \boldsymbol{\mu}_k) \to H^1(K, T) \xrightarrow{v_*} H^1(\overline{K}, \overline{T}_*)$$

is equal to $v(x)\theta_{\overline{T}}$.

Proof. The commutativity of the diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \boldsymbol{\mu}_k & \longrightarrow & \mathbf{G}_\mathrm{m} & \xrightarrow{k} & \mathbf{G}_\mathrm{m} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & T & \longrightarrow & V & \longrightarrow & \mathbf{G}_\mathrm{m} & \longrightarrow & 1
\end{array}$$

shows that the image of $xK^{\times k}$ in $H^1(K, T)$ coincides with the image of $xK^{\times k}$ under the connecting homomorphism induced by the bottom sequence in the diagram.

The result follows from the commutativity of the diagram

$$\begin{array}{ccc}
K^\times & \longrightarrow & H^1(K, T) \\
{\scriptstyle v}\downarrow & & \downarrow{\scriptstyle v_*} \\
\mathbb{Z} & \longrightarrow & H^1(\overline{K}, \overline{T}_*),
\end{array}$$

where the bottom map in the connecting homomorphism for the exact sequence (4). $\square$

Lemma 5.2. *Suppose that ξ is unramified and* $\mathrm{ind}(A_\xi) \in k\mathbb{Z}$. *Then every $\rho \in H^1(K, T)$ over ξ is unramified.*

Proof. By assumption, the class $A_\xi \in \mathsf{Alg}_K(n, m)$ is unramified. By Lemma 4.1, there is an unramified element $\eta \in H^1(K, T)$ over ξ. In view of Lemma 3.1, $\rho = x\eta$ for some $x \in K^\times$. It follows that $\xi = x\xi$.

By Lemma 3.1, $x \in \mathrm{Nrd}(A_\xi)K^{\times k}$. As $v\big(\mathrm{Nrd}(A_\xi)\big) \subset \mathrm{ind}(A)\mathbb{Z}$, by assumption, $v(x) \in k\mathbb{Z}$. Multiplying x by a kth power in $K^\times$ we may assume that x is a unit. Therefore, ρ is unramified. $\square$

Let M be a field extension of K and let w be an extension on M of the discrete valuation v. We assume that M is complete. Write e for the ramification index of M/K and $\overline{M}$ for the residue field of M.

Lemma 5.3. *Suppose that* $\operatorname{ind}((A_\xi)_M) \in k\mathbb{Z}$ *and* $\xi_M = x\xi'$, *where* $x \in M^\times$ *and* ξ' *is an unramified element in* $H^1(M,G)$. *Then the element* $w(x)\theta_{\overline{T}}$ *in* $H^1(\overline{M},\overline{T}_*)$ *is divisible by* e.

Proof. Choose an element $\rho \in H^1(K,T)$ over ξ. The image of $\rho' := x^{-1}\rho_M$ in $H^1(M,G)$ is equal to ξ' and hence is unramified. By Lemma 5.2, applied to the field M, ρ' is unramified.

Consider the following commutative diagram:

$$\begin{array}{ccc} H^1(K,T) & \xrightarrow{v_*} & H^1(\overline{K},\overline{T}_*) \\ \big\downarrow & & \big\downarrow{\scriptstyle e} \\ H^1(M,T) & \xrightarrow{w_*} & H^1(\overline{M},\overline{T}_*) \end{array}$$

where the right vertical map is e times the canonical map. Hence the image of ρ in $H^1(\overline{M},\overline{T}_*)$ is divisible by e. On the other hand, $\rho_M = x\rho'$ and ρ' is unramified. Hence by Lemma 5.1, the image of ρ in $H^1(\overline{M},\overline{T}_*)$ coincides with $w(x)\theta_{\overline{T}}$. $\square$

Proposition 5.4. *Let M/K be an extension of complete fields with discrete valuations, let $\xi \in H^1(K,G)$ be such that* $\operatorname{ind}\big((A_\xi)_M\big) = n = p^r$. *Suppose that $\xi_M = \pi\xi'$ for a prime element $\pi \in M$ and an unramified element $\xi' \in H^1(M,G)$. If $s < r$ then the ramification index of the extension M/K is not divisible by p.*

Proof. Let L/K be an unramified spitting field for A_ξ of degree n and let T be the torus as above. By Lemma 5.3, $\theta_{\overline{T}}$ in $H^1(\overline{M},\overline{T}_*)$ is divisible by the ramification index e and by Proposition 4.3 applied to the torus T_M over M, $\theta_{\overline{T}}$ is not divisible by p in $H^1(\overline{M},\overline{T}_*)$. Hence, p does not divide e. $\square$

6. Proof of the theorem

We prove Theorem 1.1. Write $n = p^r n'$, $m = p^s m'$ and $k' = n'/m'$. Consider the groups $H = \mathbf{SL}_{p^r}/\boldsymbol{\mu}_{p^s}$ and $G' = \mathbf{SL}_{n'}/\boldsymbol{\mu}_{m'}$. We have a natural group homomorphism $H \times G' \to G$. For a field extension E/F take algebras $B \in \mathsf{Alg}_E(p^r,p^s)$, $A' \in \mathsf{Alg}_E(n',m')$ and $A := B \otimes A' \in \mathsf{Alg}_E(n,m)$. By Lemma 3.1, the fiber of the natural surjection $H^1(F,H) \to \mathsf{Alg}_R(p^r,p^s)$ over B is a principal homogeneous space under $C := E^\times/\big(E^{\times p^{r-s}} \cdot \operatorname{Nrd}(B)\big)$. Similarly, the fibers of the natural surjections $H^1(E,G') \to \mathsf{Alg}_E(n',m')$ and $H^1(E,G) \to \mathsf{Alg}_E(n,m)$ over A' and A are principal homogeneous spaces under $D' := E^\times/\big(E^{\times k'} \cdot \operatorname{Nrd}(A')\big)$ and $D := E^\times/\big(E^{\times k} \cdot \operatorname{Nrd}(A)\big)$ respectively.

The tensor product yields a bijection

$$\mathsf{Alg}_R(p^r,p^s) \times \mathsf{Alg}_R(n',m') \to \mathsf{Alg}_R(n,m).$$

There is a natural isomorphism $C \times D' \to D$. It follows that the natural map

$$H^1(E,H) \times H^1(E,G') \to H^1(E,G)$$

is a bijection.

This bijection yields a surjection $\mathcal{F}_G \to \mathcal{F}_H$ and a p-surjective map $\mathcal{F}_H \to \mathcal{F}_G$. By [5, Sec. 1.3], $\operatorname{ed}_p(G) = \operatorname{ed}_p(H)$.

Replacing G by H we may assume that $n = p^r$ and $m = p^s$. If $s = 0$ then $G = \mathbf{SL}_n$ and $\operatorname{ed}_p(G) = 0$ as G is special, i.e., all G-torsors over fields are trivial. If $s = r$, $G = \mathbf{PGL}_n$ and $\mathcal{F}_G = \mathsf{Alg}(p^r, p^r)$, hence $\operatorname{ed}_p(G) = \operatorname{ed}_p\big(\mathsf{Alg}(p^r,p^r)\big)$.

We may assume that $0 < s < r$. By Lemma 3.1, for any field E, the natural map $H^1(E,G) \to \mathsf{Alg}(p^r,p^s)(E)$ is surjective and the fibers are homogeneous sets under $E^\times$. It follows that

$$\operatorname{ed}_p(G) \le \operatorname{ed}_p\big(\mathsf{Alg}(p^r,p^s)\big) + 1.$$

To prove the opposite inequality choose a field E/F and a (generic) algebra A in $\mathsf{Alg}(p^r,p^s)(E)$ such that

$$\operatorname{ed}_p\big(\mathsf{Alg}(p^r,p^s)\big) = \operatorname{ed}_p(A). \tag{6}$$

Note that as $s > 0$, the index of A is equal to p^r. Choose an element $\eta \in H^1(E,G)$ with $A_\eta = A$.

Consider the field of formal Laurent series $E(\!(t)\!)$ and set $\xi' := \eta_{E((t))} \in H^1(E(\!(t)\!),G)$. We have $A_{\xi'} = A_{E((t))}$. Choose a finite field extension $M/E(\!(t)\!)$ of degree prime to p and a subfield $K \subset M$ over F such that $\operatorname{tr.deg}_F(K) = \operatorname{ed}_p(t\xi')$ and there is an element $\xi \in H^1(K,G)$ with $\xi_M = t\xi'_M$.

Let w be the extension of the discrete valuation of $E(\!(t)\!)$ on M. The ramification index of $M/E(\!(t)\!)$ is not divisible by p. The degree of the residue field $\overline{M}$ over E is also not divisible by p.

Note that the element ξ_M is ramified by Lemma 3.2, hence the restriction on K of the discrete valuation of M is nontrivial. We have $\operatorname{tr.deg}_F(K) \ge \operatorname{tr.deg}_F(\overline{K}) + 1$, therefore,

$$\operatorname{ed}_p(t\xi') \ge \operatorname{tr.deg}_F(\overline{K}) + 1. \tag{7}$$

Write $\hat{K}$ for the completion of K. As M is complete we may assume that $\hat{K}$ is a subfield of M. Since $0 < s < r$, by Proposition 5.4, the ramification index e of $M/\hat{K}$ is not divisible by p.

As $\operatorname{char}(F)$ is not equal to p, there is the residue homomorphism [7, Ch. XII]

$$\partial : \operatorname{Br}_n(\hat{K}) \to H^1(\overline{K}, \mathbb{Z}/n\mathbb{Z}).$$

Let $\bar\chi = \partial(A_\xi) \in H^1(\overline{K}, \mathbb{Z}/n\mathbb{Z})$. As $(A_\xi)_M = A_{t\xi'} = A_{\xi'} = A_M$ is unramified, we have $e \cdot \bar\chi_{\overline{M}} = 0$ and hence $\bar\chi_{\overline{M}} = 0$ as e is not divisible by p. Hence we can view the cyclic extension $\overline{K}(\bar\chi)$ of $\overline{K}$ given by $\bar\chi$ as a subfield of $\overline{M}$.

Let $\chi \in H^1(\hat{K}, \mathbb{Z}/n\mathbb{Z})$ be the lift of $\bar\chi$. The field $\hat{K}(\chi)$ is a subfield of M. Therefore, the algebra $B := (A_\xi)_{\hat{K}(\chi)}$ is unramified and its residue $\overline{B}$ satisfies $\overline{B} \in \mathsf{Alg}(p^r,p^s)\big(\overline{K}(\bar\chi)\big)$ and $(\overline{B})_{\overline{M}} = A_{\overline{M}}$.

Thus, the algebra $A_{\overline{M}}$ is defined over $\overline{K}(\bar{\chi})$, hence

$$\operatorname{tr.deg}_F(\overline{K}) = \operatorname{tr.deg}_F\big(\overline{K}(\bar{\chi})\big) \geq \operatorname{ed}_p(A). \tag{8}$$

We have by (6), (7) and (8):

$$\operatorname{ed}_p(G) \geq \operatorname{ed}_p(t\xi') \geq \operatorname{tr.deg}_F(\overline{K}) + 1 \geq \operatorname{ed}_p(A) + 1 = \operatorname{ed}_p\big(\mathsf{Alg}(p^r, p^s)\big) + 1.$$

References

[1] Baek, S.: Essential dimension of simple algebras with involutions. Preprint.

[2] Baek, S., Merkurjev, A.S.: Essential dimension of central simple algebras. To appear in Acta Mathematica.

[3] Berhuy, G., Favi, G.: Essential dimension: a functorial point of view (after A. Merkurjev). Doc. Math. **8**, 279–330 (electronic) (2003).

[4] Demazure, M., Grothendieck, A.: Schémas en groupes. III: Structure des schémas en groupes réductifs. Springer-Verlag, Berlin, 1962/1964, Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3). Dirigé par M. Demazure et A. Grothendieck. Lecture Notes in Mathematics, Vol. 153

[5] Merkurjev, A.S.: Essential dimension, Quadratic forms—algebra, arithmetic, and geometry. Contemp. Math., Amer. Math. Soc. Providence, RI. 493, 299–325 (2009)

[6] Milne, J.S.: Étale cohomology. Princeton University Press, Princeton, N.J. (1980)

[7] Serre, J.-S.: Local fields. Springer-Verlag, New York, Translated from the French by Marvin Jay Greenberg (1979)

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